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All Traveling Wave Exact Solutions of the Kawahara Equation Using the Complex Method

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Abstract: In this article, we prove that the $\langle p, q \rangle$ condition holds, first by using the Fuchs index of the complex Kawahara equation, and then proving that all meromorphic solutions of complex Kawahara equations belong to the class W . Moreover, the complex method is employed to get all meromorphic solutions of complex Kawahara equation and all traveling wave exact solutions of Kawahara equation. Our results reveal that all rational solutions $u_r(x + vt)$ and simply periodic solutions $u_{s,1}(x + vt)$ of Kawahara equation are solitary wave solutions, while simply periodic solutions $u_{s,2}(x + vt)$ are not real-valued. Finally, computer simulations are given to demonstrate the main results of this paper. At the same time, we believe that this method is a very effective and powerful method of looking for exact solutions to the mathematical physics equations, and the search process is simpler than other methods.

Keywords: Kawahara equation; complex method; exact solution; elliptic function

MSC: 30D35; 34A05



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1. Introduction and Main Results

In this article, we study all traveling wave exact solutions of the Kawahara equation [1]:

$$u_t + auu_x + cu_{xxx} - du_{xxxxx} = 0, \quad (1)$$

where a, c, d are constants, x and t represent spatial and time variables, uu_x is the nonlinear disturbance term, and u_{xxx} and u_{xxxxx} are the dispersion terms of order three and five, respectively. The solutions of Equation (1) could be used to analyze and interpret a lot of nonlinear dispersive phenomena that can arise in optical fiber, ocean, plasma physics, etc. [2,3]. Given the importance of this equation, it remains the subject of study for many researchers. Many different techniques were devoted to studying various solutions of Equation (1) and its family [4–13]. For example, Kudryashov [14], obtained exact meromorphic solutions of the Kawahara equation using the Laurent series. Wazwaz [15], found some different solutions (compacton and solitons) to Equation (1) in terms of trigonometric functions. Khan [16], analyzed the Kawahara equation using the variational approach and derived new conditions for obtaining solitary wave solutions. Using the traveling wave ansatz, Baiswas [17], studied the generalized Kawahara equation and derived a solitary wave solution for the family of the Kawahara equation. A lot of effective methods are applied to study the exact solutions, which makes the research more abundant [18–23]. Wang [24] used ansatz method to derive the exact solitary wave solution for the generalized Korteweg–de Vries–Kawahara (GKdV-K) equation. Aiman Zara [25] studied numerical approximation of the modified Kawahara equation using the Kernel smoothing method. El-Tantawy [26] derived a set of novel exact and approximate analytic solutions to the family of the forced damped Kawahara equation (KE) using the ansatz method.

In recent years, Yuan et al. [27,28] introduced the $\langle p, q \rangle$ condition to study a class of constant-coefficient complex algebraic differential equations with dominant terms, and obtained all possible meromorphic solutions which belong to the class W . That is to say, the general representation of rational function solutions, finitely growing simply periodic solutions and elliptic solutions, and both rational function solutions and simply periodic solutions can be obtained by degenerating elliptic general solutions. Furthermore, a complex method for obtaining exact solutions in the mathematical physics is presented. Using the complex method [27,28], we prove that all meromorphic solutions of complex Kawahara equation belong to the class W , and then get all meromorphic solutions of complex Kawahara equation and all traveling wave exact solutions of Equation (1). This method is simpler than other methods in the process of finding solutions, and can also be applied to the solution of other nonlinear differential equations.

Substitute the traveling wave transform

$$u = u(x, t) = w(z), \quad z = x + vt \tag{2}$$

into Equation (1); integrate it to get

$$C_0 + vw + \frac{1}{2}aw^2 + cw'' - dw^{(4)} = 0, \tag{3}$$

where C_0 is an integral constant.

Multiply Equation (3) by w' , and integrate it; thus, we get the complex ordinary differential Kawahara equation

$$w'w''' - \frac{1}{2}(w'')^2 - Aw^3 + B(w')^2 + Cw^2 + Dw + E = 0, \tag{4}$$

where $A = \frac{a}{6d}$, $B = -\frac{c}{2d}$, $C = -\frac{v}{2d}$, $D = -\frac{C_0}{d}$, $E = -\frac{C_1}{d}$, C_0 and C_1 are integral constants.

Now, we give the main results in our paper.

Theorem 1. *Let $A \neq 0$, then all meromorphic solutions $w(z)$ of Equation (4) belong to the class W . Here, class W consists of elliptic functions and their degenerations with the form $R(z)$ or $R(e^{\alpha z})$, $\alpha \in \mathbb{C}$, where R is a rational function.*

In addition, the solutions of Equation (4) have the forms:

(1) *All elliptic function solutions*

$$w_d(z) = \frac{35}{2A} \left[\frac{\wp'(z) + M}{\wp(z) - N} \right]^4 - \left[\frac{140(\wp(z) + N)}{A} - \frac{70B}{39A} \right] \left[\frac{\wp'(z) + M}{\wp(z) - N} \right]^2 + \frac{280}{A} [\wp(z) + N]^2 - \frac{280B}{39A} [\wp(z) + N] + \frac{C}{3A} - \frac{62B^2}{1521A} - \frac{28}{A} g_2, \tag{5}$$

where $g_3 = \frac{31B^3}{593190} - \frac{7B}{390}g_2$, $M^2 = 4N^3 - g_2N - g_3$,

$$g_2^2 - \frac{2B^2}{507}g_2 + \frac{1457}{41384889}B^4 - \frac{AD}{1932} - \frac{C^2}{5796} = 0,$$

$$E = \frac{1}{A^2} \left[\left(\frac{1148}{69}AD + \frac{1148}{207}C^2 - \frac{661248}{656903}B^4 \right) g_2 + \frac{23808}{4826809}B^6 \right.$$

$$\left. + \left(\frac{20}{1521}AD + \frac{20}{4563}C^2 \right) B^2 - \frac{1}{3}ACD - \frac{2}{27}C^3 \right],$$

D and N are constants.

(2) *All simply periodic solutions*

$$w_{s,1}(z) = \frac{2B^2}{169A} [35 \coth^4 \frac{\sqrt{26Bi}}{26} (z - z_0) - 70 \coth^2 \frac{\sqrt{26Bi}}{26} (z - z_0) + 23] + \frac{C}{3A}, \tag{6}$$

and

$$w_{s,2}(z) = \frac{B^2}{507A\gamma^2} [210 \coth^4 \sqrt{\frac{B}{26\gamma}} (z - z_0) - (140\gamma + 560) \coth^2 \sqrt{\frac{B}{26\gamma}} (z - z_0) + 166\gamma + 352] + \frac{C}{3A}, \tag{7}$$

where $z_0 \in \mathbf{C}$; $D = \frac{1}{85683} \frac{5184B^4 - 28561C^2}{A}$,

$$E = \frac{1}{130323843} \frac{746496B^6 - 2628288B^4C + 4826809C^3}{A^2}$$

in $w_{s,1}(z)$, and

$$D = \frac{1}{23905557} \frac{(217620\gamma - 59724)B^4}{\gamma^4 A} - \frac{C^2}{3A},$$

$$E = \frac{1}{A^2} [\frac{16B^6}{448893237} \frac{3001 - 7007\gamma}{\gamma^6} - \frac{B^4C}{71716671} \frac{217620\gamma - 59724}{\gamma^4} + \frac{C^3}{27}],$$

γ is a root of $31z^2 - 31z + 10 = 0$ in $w_{s,2}(z)$.

(3) All rational function solutions

$$w_r(z) = \frac{280}{A} \frac{1}{(z - z_0)^4} + \frac{C}{3A}, \tag{8}$$

where $B = 0$, $D = -\frac{C^2}{3A}$, $E = \frac{C^3}{27A^2}$ and $z_0 \in \mathbf{C}$.

All traveling wave exact solutions of Equation (1) are obtained by substituting (2) into meromorphic solutions w of Equation (4). So, we get the following Theorem.

Theorem 2. Let $ad \neq 0$, all traveling wave exact solutions $u(x, t)$ of Equation (1) have the following forms:

(1) All elliptic function solutions

$$u_d(x + vt) = \frac{105d}{a} [\frac{\wp'(x+vt)+M}{\wp(x+vt)-N}]^4 - (\frac{840d(\wp(x+vt)+N)}{a} + \frac{70c}{13a}) [\frac{\wp'(x+vt)+M}{\wp(x+vt)-N}]^2 + \frac{1680d}{a} [\wp(x + vt) + N]^2 + \frac{280c}{13a} [\wp(x + vt) + N] - \frac{168d}{a} g_2 - \frac{31c^2}{507ad} - \frac{v}{a}, \tag{9}$$

where $M^3 = 4N^3 - g_2N - g_3$, $g_3 = -\frac{31c^3}{4745520d^3} + \frac{7c}{780d}g_2$,

$$g_2^2 - \frac{c^2}{1014d^2}g_2 + \frac{1457c^4}{662158224d^4} + \frac{aC_0}{11592d^2} - \frac{v^2}{23184d^2} = 0,$$

and C_0, N are constants.

(2) All simply periodic solutions

$$u_{s,1}(x + vt) = \frac{3c^2}{169ad} [35 \coth^4 (\frac{1}{2} \sqrt{\frac{c}{13ad}} (x - x_0 + v(t - t_0))) - 70 \coth^2 (\frac{1}{2} \sqrt{\frac{c}{13ad}} (x - x_0 + v(t - t_0))) + 23] - \frac{v}{a}, \tag{10}$$

and

$$u_{s,2}(x + vt) = \frac{c^2}{338ad\gamma^2} [210 \coth^4(\frac{1}{2} \sqrt{-\frac{c}{13d\gamma}}(x - x_0 + v(t - t_0))) - (140\gamma + 560) \coth^2(\frac{1}{2} \sqrt{-\frac{c}{13d\gamma}}(x - x_0 + v(t - t_0))) + 166\gamma + 352] - \frac{v}{a}, \tag{11}$$

where γ is a root of $31z^2 - 31z + 10 = 0$.

(3) All rational function solutions

$$u_r(x + vt) = \frac{1680d}{a} \frac{1}{(x - x_0 + v(t - t_0))^4} - \frac{v}{a}, \tag{12}$$

where $c = 0, x_0$ and t_0 are arbitrary real constants.

The rest of the article is organized in the following order: In Section 2, we will present the relevant lemmas and methodology. Section 3 gives the detailed proof process of Theorem 1 and concise proof method of Theorem 2. Section 4 illustrates our main results using computer simulations. In the last section, some conclusions are given.

2. Preliminary Lemmas and the Complex Method

We need some definitions and lemmas in order to prove Theorem 1 and present the complex method.

Set $k, n \in \mathbf{N} := \{1, 2, 3, \dots\}, r_j \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}, r = (r_0, r_1, \dots, r_k), j = 0, 1, \dots, k$.

$$M_r[w](z) := [w(z)]^{r_0} [w'(z)]^{r_1} [w''(z)]^{r_2} \dots [w^{(k)}(z)]^{r_k}. \tag{13}$$

The degree of $M_r[w]$ defined by $d(r) := r_0 + r_1 + \dots + r_k$.

Definition 1 ([28]). A differential polynomial is defined by

$$P[w] := \sum_{r \in \Lambda} b_r M_r[w], \tag{14}$$

where Λ is a finite index set, and b_r are constants. $\deg P[w] := \max_{r \in \Lambda} \{d(r)\}$ is called the degree of $P[w]$.

Consider the differential equation

$$E(z, w) := P[w] - aw^n = 0, \tag{15}$$

where $a \neq 0$ is a constant.

The dominant part of $E(z, w)$ composes all dominant terms which can determine the multiplicity q of w in $E(z, w)$, and is denoted by $\hat{E}(z, w)$. $D(q)$ and $D_r(q)$ represent the multiplicity of pole of each term in $\hat{E}(z, w)$ and the multiplicity of pole of each monomial in $E(z, w) - \hat{E}(z, w)$, respectively.

Obviously

$$D_r(q) = qd(r) + r_1 + 2r_2 + \dots + kr_k < D(q). \tag{16}$$

Definition 2 ([28]). The derivative of $\hat{E}(z, w)$ with respect to w can be calculated by the following formula, for any χ ,

$$\hat{E}'(z, w)\chi = \lim_{\lambda \rightarrow 0} \frac{\hat{E}(z, w + \lambda\chi) - \hat{E}(z, w)}{\lambda}. \tag{17}$$

Definition 3 ([28]). Substituting Laurent series

$$w(z) = \sum_{l=-q}^{\infty} c_l z^l, \tag{18}$$

into Equation (15), where $c_{-q} \neq 0, q > 0$. Then, we can get p different principle

$$\sum_{l=-q}^{-1} c_l z^l$$

with pole of multiplicity q at $z = 0$, Equation (15) is said to satisfy weak $\langle p, q \rangle$ condition. If Equation (15) has p different meromorphic solutions with pole of multiplicity q at $z = 0$, Equation (15) satisfies $\langle p, q \rangle$ condition.

Definition 4 ([28]). Let T_1, T_2 be two given complex numbers, such that $\text{Im} \frac{T_1}{T_2} > 0, L = L[2T_1, 2T_2]$ is discrete subset $L[2T_1, 2T_2] = \{T | T = 2mT_1 + 2nT_2, m, n \in \mathbb{Z}\}$, which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. The discriminant $\Delta = \Delta(c_1, c_2) := c_1^3 - 27c_2^2$ and

$$s_n = s_n(L) := \sum_{T \in L \setminus \{0\}} \frac{1}{T^n}. \tag{19}$$

Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ is a meromorphic function with double periods $2T_1, 2T_2$, which satisfies the following formula

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{20}$$

where $g_2 = 60s_4, g_3 = 140s_6$ and $\Delta(g_2, g_3) \neq 0$.

Definition 5 ([29]). The Fuchs index of Equation (15) can be defined as the root of the equation

$$P(i) = \lim_{z \rightarrow 0} z^{-i+D(q)} \hat{E}'(z, c_{-q}z^{-q})z^{i-q} = 0 \tag{21}$$

Lemma 1 ([28,30–32]). Set $p, q, m, n \in \mathbb{N}, \deg P[w] < n$. If Equation (15) satisfies $\langle p, q \rangle$ condition, all meromorphic solutions $w(z)$ of Equation (15) belong to class W .

Any elliptic function solution with pole at $z = 0$ is given in the form

$$\begin{aligned} w(z) = & \sum_{i=1}^{m-1} \sum_{j=2}^q \frac{(-1)^j c_{-ij}}{(j-1)!} \frac{d^{j-2}}{dz^{j-2}} \left(\frac{1}{4} \left[\frac{\wp'(z) + M_i}{\wp(z) - N_i} \right]^2 - \wp(z) \right) \\ & + \sum_{i=1}^{m-1} \frac{c_{-i1}}{2} \frac{\wp'(z) + M_i}{\wp(z) - N_i} + \sum_{j=2}^q \frac{(-1)^j c_{-mj}}{(j-1)!} \frac{d^{j-2}}{dz^{j-2}} \wp(z) + k_0, \end{aligned} \tag{22}$$

where c_{-ij} can be determined by (18), c_{-ij} and k_0 are constants. $M_i^2 = 4N_i^3 - g_2N_i - g_3$ and $\sum_{i=1}^m c_{-i1} = 0$.

Any rational function solution $w := R(z)$ is expressed as

$$R(z) = \sum_{i=1}^m \sum_{j=1}^q \frac{c_{ij}}{(z - z_i)^j} + k_0, \tag{23}$$

which has $m(\leq p)$ different poles of multiplicity q .

Any simply periodic solution $w := R(\xi)$ is a rational function of $\xi = e^{\alpha z} (\alpha \in \mathbb{C})$ and can be given in the form

$$R(\xi) = \sum_{i=1}^m \sum_{j=1}^q \frac{c_{ij}}{(\xi - \xi_i)^j} + k_0, \tag{24}$$

which has $m(\leq p)$ different poles of multiplicity q .

Lemma 2 ([33,34]). Weierstrass elliptic functions $\wp(z)$ have the addition formula and two successive degeneracies, as shown below:

(I) Addition formula

$$\wp(z - z_0) = -\wp(z) - \wp(z_0) + \frac{1}{4} \left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)} \right]^2. \tag{25}$$

(II) If $\Delta(g_2, g_3) = 0$, Weierstrass elliptic functions degenerate to simply periodic functions, which can be expressed as

$$\wp(z, 3\delta^2, -\delta^3) = 2\delta - \frac{3\delta}{2} \coth^2 \sqrt{\frac{3\delta}{2}} z. \tag{26}$$

(III) If $g_2 = g_3 = 0$, Weierstrass elliptic functions degenerate to rational functions of z , which can be expressed as

$$\wp(z, 0, 0) = \frac{1}{z^2}. \tag{27}$$

Next, we give the **complex method**.

1. Substituting the transform $T : u(x, t) \rightarrow w(z), (x, t) \rightarrow z$ into a given PDE yields a nonlinear ODE: Equation (4) here.
2. Insert (18) into Equation (4) here to determine that weak $\langle p, q \rangle$ condition holds.
3. By (22)–(24), we obtain all meromorphic solutions $w(z)$ of Equation (4) here with pole at $z = 0$.
4. Get all meromorphic solutions $w(z - z_0)$ by Lemmas 1 and 2.
5. Inserting the inverse transform T^{-1} into $w(z - z_0)$, we obtain all exact solutions $u(x, t)$ of the given partial differential equation.

3. Proof of Theorem 1

From balance the order of the poles in Equation (4), yields

$$(q + 1) + (q + 3) = 2(q + 2) = 3q,$$

and we get $q = 4$.

Substitute (18) into Equation (4), and set the coefficients to zero, we have

$$\begin{aligned} c_{-4} &= \frac{280}{A}, c_{-3} = 0, c_{-2} = \frac{280B}{39A}, c_{-1} = 0, \\ c_0 &= -\frac{62B^2 - 507C}{1521A}, c_1 = 0, c_2 = \frac{62B^3}{59319A}, c_3 = 0, \\ c_4 &= \frac{1}{425673144} \frac{-11780B^4 + 257049AD + 85683C^2}{A}, \dots \end{aligned} \tag{28}$$

and then determine $p = 1$. Therefore, Equation (4) is said to satisfy weak $\langle 1, 4 \rangle$ condition. We also get that all meromorphic solutions of Equation (4) belong to W if Equation (4) satisfies $\langle 1, 4 \rangle$ condition by Lemma 1.

In fact, since (17) and $\hat{E}(z, w) = w'w''' - \frac{1}{2}(w'')^2 - Aw^3$ of Equation (4), we have $D(4) = 12$,

$$D_r(4) < 12, \quad Ac_{-4} = 280, \tag{29}$$

and

$$\begin{aligned} \hat{E}'(x, c_{-4}x^{-4})x^{i-4} &= ((c_{-4}x^{-4})' \frac{d^3}{dx^3} + (c_{-4}x^{-4})''' \frac{d}{dx} - (c_{-4}x^{-4})'' \frac{d^2}{dx^2} \\ &\quad - 3A(c_{-4}x^{-4})^2)x^{i-4} \end{aligned}$$

$$= -(4(i - 1)(i - 4)(i - 5) + 120i + 360)c_{-4}x^{i-12}. \tag{30}$$

Thus, from (29), (30) and (21), we infer that the Fuchs index of Equation (4) are zeros of the function

$$P(i) = -(4(i - 1)(i - 4)(i - 5) + 120i + 360)c_{-4}. \tag{31}$$

It is easy to prove that $f(x) := 4(x - 1)(x - 4)(x - 5) + 120x + 360 > 0$ if $x > -1$. Hence, if $i \in \mathbf{N}$, then $P(i) \neq 0$. That is to say, the Fuchs index of Equation (4) cannot be a positive integer. This implies that (ref. [29], p. 90): The principle part of the Laurent series of $w(z)$ determines the whole Laurent series of $w(z)$. Therefore, weak $\langle 1, 4 \rangle$ condition implies that $\langle 1, 4 \rangle$ condition holds.

From, (23), we get the indeterminant form of rational function solutions of Equation (4) at pole $z = 0$

$$R_1(z) = \frac{c_{-4}}{z^4} + \frac{c_{-2}}{z^2} + k_0 = \frac{280}{A} \frac{1}{z^4} + \frac{280B}{39A} \frac{1}{z^2} + k_0. \tag{32}$$

Inserting (32) into Equation (4), and then setting the coefficients to zero, we get the system of Equations (1) which are given in Appendix A. Computing the system of Equations (1), we get

$$k_0 = \frac{C}{3A}, B = 0, D = -\frac{C^2}{3A}, E = \frac{C^3}{27A^2}.$$

So we find that the rational function solutions of Equation (4) are

$$w_{r0}(z) = \frac{280}{A} \frac{1}{z^4} + \frac{C}{3A}, \tag{33}$$

at pole $z = 0$, here $B = 0, D = -\frac{C^2}{3A}, E = \frac{C^3}{27A^2}$.

Hence, all rational function solutions of Equation (4) are

$$w_r(z) = \frac{280}{A} \frac{1}{(z - z_0)^4} + \frac{C}{3A}, \tag{34}$$

where $B = 0, D = -\frac{C^2}{3A}, E = \frac{C^3}{27A^2}$ and $z_0 \in \mathbf{C}$.

By (22) and (26), we get the indeterminant form of simply periodic solutions of Equation (4) at pole $z = 0$

$$\begin{aligned} w_{s0}(z) &= \frac{1}{6}c_{-4}\wp''(z, 3\delta^2, -\delta^3) + c_{-2}\wp(z, 3\delta^2, -\delta^3) + k_0 \\ &= \frac{280}{6A}\wp''(z, 3\delta^2, -\delta^3) + \frac{280B}{39A}\wp(z, 3\delta^2, -\delta^3) + k_0, \end{aligned} \tag{35}$$

where $\wp(z, 3\delta^2, -\delta^3) = 2\delta - \frac{3}{2}\delta \coth^2 \sqrt{\frac{3\delta}{2}}z$.

Expanding $\wp(z, 3\delta^2, -\delta^3)$ at $z_0 = 0$, we have

$$\wp(z, 3\delta^2, -\delta^3) = \frac{1}{z^2} + \frac{3\delta^2}{20}z^2 - \frac{\delta^3}{28}z^4 + o(z^4) \tag{36}$$

Substituting (36) into (35), we get

$$\begin{aligned} w(z) &= \frac{280}{A} \frac{1}{z^4} + \frac{280B}{39A} \frac{1}{z^2} + \frac{14\delta^2}{A} + k_0 + \left(\frac{14B\delta^2}{13A} - \frac{20\delta^3}{A}\right)z^2 \\ &\quad + \left(-\frac{10B\delta^3}{39A} + \frac{21\delta^4}{2A}\right)z^4 + o(z^4) \end{aligned} \tag{37}$$

Comparing coefficients of (37) and (28), we obtain the system of Equations (2) which are showed in Appendix A. Solving the system of Equations (2), we have

$$\delta = -\frac{B}{39}, k_0 = -\frac{76B^2}{1521A} + \frac{C}{3A}, D = \frac{1}{85683} \frac{5184B^4 - 28561C^2}{A}, \tag{38}$$

and

$$\delta = \frac{B}{39\gamma}, k_0 = \frac{B^2}{1521A} \frac{6 - 62\gamma}{\gamma^2} + \frac{C}{3A},$$

$$D = \frac{1}{23905557} \frac{(217620\gamma - 59724)B^4}{\gamma^4 A} - \frac{C^2}{3A}. \tag{39}$$

Substituting (35), (38) and (39) into Equation (4), from the correlation of coefficients we can obtain, respectively,

$$E = \frac{1}{130323843} \frac{746496B^6 - 2628288B^4C + 4826809C^3}{A^2}$$

and

$$E = \frac{1}{A^2} \left[\frac{16B^6}{448893237} \frac{3001 - 7007\gamma}{\gamma^6} - \frac{B^4C}{71716671} \frac{217620\gamma - 59724}{\gamma^4} + \frac{C^3}{27} \right],$$

where γ is a root of $31z^2 - 31z + 10 = 0$.

Hence, we obtain the simply periodic solutions of Equation (4) at pole $z = 0$ are

$$w_{s,1}(z) = \frac{2B^2}{169A} \left[35 \coth^4 \frac{\sqrt{26Bi}}{26} z - 70 \coth^2 \frac{\sqrt{26Bi}}{26} z + 23 \right] + \frac{C}{3A}, \tag{40}$$

and

$$w_{s,2}(z) = \frac{B^2}{507A\gamma^2} \left[210 \coth^4 \sqrt{\frac{B}{26\gamma}} z - (140\gamma + 560) \coth^2 \sqrt{\frac{B}{26\gamma}} z + 166\gamma + 352 \right] + \frac{C}{3A}, \tag{41}$$

where $\delta = -\frac{B}{39}, k_0 = -\frac{76B^2}{1521A} + \frac{C}{3A}, D = \frac{1}{85683} \frac{5184B^4 - 28561C^2}{A},$

$$E = \frac{1}{130323843} \frac{746496B^6 - 2628288B^4C + 4826809C^3}{A^2}$$

in $w_{s,1}(z)$, and $D = \frac{1}{23905557} \frac{(217620\gamma - 59724)B^4}{\gamma^4 A} - \frac{C^2}{3A},$

$$E = \frac{1}{A^2} \left[\frac{16B^6}{448893237} \frac{3001 - 7007\gamma}{\gamma^6} - \frac{B^4C}{71716671} \frac{217620\gamma - 59724}{\gamma^4} + \frac{C^3}{27} \right],$$

γ is a root of $31z^2 - 31z + 10 = 0$ in $w_{s,2}(z)$.

Furthermore, all simply periodic solutions of Equation (4) are given by

$$w_{s,1}(z) = \frac{2B^2}{169A} \left[35 \coth^4 \frac{\sqrt{26Bi}}{26} (z - z_0) - 70 \coth^2 \frac{\sqrt{26Bi}}{26} (z - z_0) + 23 \right] + \frac{C}{3A}, \tag{42}$$

and

$$w_{s,2}(z) = \frac{B^2}{507A\gamma^2} \left[210 \coth^4 \sqrt{\frac{B}{26\gamma}} (z - z_0) - (140\gamma + 560) \coth^2 \sqrt{\frac{B}{26\gamma}} (z - z_0) + 166\gamma + 352 \right] + \frac{C}{3A}, \tag{43}$$

where $z_0 \in \mathbf{C}, D = \frac{1}{85683} \frac{5184B^4 - 28561C^2}{A},$

$$E = \frac{1}{130323843} \frac{746496B^6 - 2628288B^4C + 4826809C^3}{A^2}$$

in $w_{s,1}(z)$, and $D = \frac{1}{23905557} \frac{(217620\gamma - 59724)B^4}{\gamma^4 A} - \frac{C^2}{3A}$,

$$E = \frac{1}{A^2} \left[\frac{16B^6}{448893237} - \frac{3001 - 7007\gamma}{\gamma^6} - \frac{B^4 C}{71716671} \frac{217620\gamma - 59724}{\gamma^4} + \frac{C^3}{27} \right],$$

γ is a root of $31z^2 - 31z + 10 = 0$ in $w_{s,2}(z)$.

From (22), we infer the indeterminant forms of elliptic function solutions of Equation (4) are

$$\begin{aligned} w_{d0}(z) &= \frac{c-4}{6} \wp''(z) + c_{-2} \wp(z) + k_0 \\ &= \frac{280}{6A} \wp'' + \frac{280B}{39A} \wp + k_0. \end{aligned} \tag{44}$$

with pole at $z = 0$.

Expanding $\wp(z, g_2, g_3)$ at $z = 0$, we have

$$\wp(z, g_2, g_3) = \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + o(z^4). \tag{45}$$

Put (45) into (44) and apply (20), we get

$$\begin{aligned} w(z) &= \frac{280}{A} \frac{1}{z^4} + \frac{280B}{39A} \frac{1}{z^2} + \frac{14g_2}{3A} + k_0 + \left(\frac{14Bg_2}{39A} + \frac{20g_3}{A} \right) z^2 \\ &\quad + \left(\frac{10Bg_3}{39A} + \frac{7g_2^2}{6A} \right) z^4 + o(z^4). \end{aligned} \tag{46}$$

Comparing coefficients of $w(z)$ and (28), we obtain the system of Equations (3). which are shown in the Appendix. Computing the system of Equations (3), we derive

$$k_0 = \frac{C}{3A} - \frac{14}{3A} g_2 - \frac{62B^2}{1521A}, g_3 = \frac{31B^3}{593190} - \frac{7B}{390} g_2. \tag{47}$$

and g_2 satisfies

$$g_2^2 - \frac{2B^2}{507} g_2 + \frac{1457B^4}{41384889} - \frac{AD}{1932} - \frac{C^2}{5796} = 0.$$

Substituting (44) and (47) into Equation (4), from the correlation of coefficients, we can get

$$\begin{aligned} E &= \frac{1}{A^2} \left\{ \left(\frac{1148}{69} AD + \frac{1148}{207} C^2 - \frac{661248}{656903} B^4 \right) g_2 + \frac{23808}{4826809} B^6 \right. \\ &\quad \left. + \left(\frac{20}{1521} AD + \frac{20}{4563} C^2 \right) B^2 - \frac{1}{3} ACD - \frac{2}{27} C^3 \right\}, \end{aligned}$$

then we have

$$w_{d0}(z) = \frac{280}{A} \wp^2(z) + \frac{280B}{39A} \wp(z) + \frac{C}{3A} - \frac{62B^2}{1521A} - \frac{28}{A} g_2. \tag{48}$$

Thus, all elliptic function solutions of Equation (4) are

$$w_d(z) = \frac{280}{A} \wp^2(z - z_0) + \frac{280B}{39A} \wp(z - z_0) + \frac{C}{3A} - \frac{62B^2}{1521A} - \frac{28}{A} g_2, \tag{49}$$

where $z_0 \in \mathbf{C}, g_3 = \frac{31B^3}{593190} - \frac{7B}{390} g_2$,

$$\begin{aligned} E &= \frac{1}{A^2} \left\{ \left(\frac{1148}{69} AD + \frac{1148}{207} C^2 - \frac{661248}{656903} B^4 \right) g_2 + \frac{23808}{4826809} B^6 \right. \\ &\quad \left. + \left(\frac{20}{1521} AD + \frac{20}{4563} C^2 \right) B^2 - \frac{1}{3} ACD - \frac{2}{27} C^3 \right\}, \end{aligned}$$

and g_2 satisfies

$$g_2^2 - \frac{2B^2}{507}g_2 + \frac{1457B^4}{41384889} - \frac{AD}{1932} - \frac{C^2}{5796} = 0.$$

By using the addition formula, we can get another representation of it as

$$w_d(z) = \frac{35}{2A} \left[\frac{\wp'(z) + M}{\wp(z) - N} \right]^4 - \left[\frac{140(\wp(z) + N)}{A} - \frac{70B}{39A} \right] \left[\frac{\wp'(z) + M}{\wp(z) - N} \right]^2 + \frac{280}{A} [\wp(z) + N]^2 - \frac{280B}{39A} [\wp(z) + N] + \frac{C}{3A} - \frac{62B^2}{1521A} - \frac{28}{A} g_2^2, \tag{50}$$

where

$$M^2 = 4N^3 - g_2N - g_3, \quad g_3 = \frac{31B^3}{593190} - \frac{7B}{390}g_2,$$

$$g_2^2 - \frac{2B^2}{507}g_2 + \frac{1457}{41384889}B^4 - \frac{AD}{1932} - \frac{C^2}{5796} = 0,$$

$$E = \frac{1}{A^2} \left[\left(\frac{1148}{69}AD + \frac{1148}{207}C^2 - \frac{661248}{656903}B^4 \right) g_2 + \frac{23808}{4826809}B^6 \right.$$

$$\left. + \left(\frac{20}{1521}AD + \frac{20}{4563}C^2 \right) B^2 - \frac{1}{3}ACD - \frac{2}{27}C^3 \right],$$

D and N are constants.

So far, we have completed the proof of Theorem 1.

Substituting (2) into all meromorphic solutions $w(z)$ of Equation (4), we obtained Theorem 2. According to the theorem in ref. [35], we can get the following corollary.

Corollary. All rational solutions $u_r(x + vt)$ and simply periodic solutions $u_{s,1}(x + vt)$ of Equation (1) are real valued, while simply periodic solutions $u_{s,2}(x + vt)$ are not real valued.

4. Computer Simulations

This subsection will show our results through computer simulations of $u_r(x + vt)$ and $u_{s,1}(x + vt)$, as demonstrated in the following figures.

1. By applying the complex method, we are able to achieve the rational solution $u_r(x + vt)$ of Equation (4). Figure 1 describes the 3D graphs of solution $u_r(x + vt)$ for $a = 1$, $v = 1$, and $d = \frac{1}{1680}$ within the interval $-5 \leq x, t \leq 5$. Figure 2 shows the 2D graphs of solution $u_r(x + vt)$ for $a = 1$, $v = 1$, and $d = \frac{1}{1680}$ within the interval $-10 \leq x \leq 10$ when $t = 0$. It could be observed that they have one generation pole, which is shown by Figures 1 and 2.

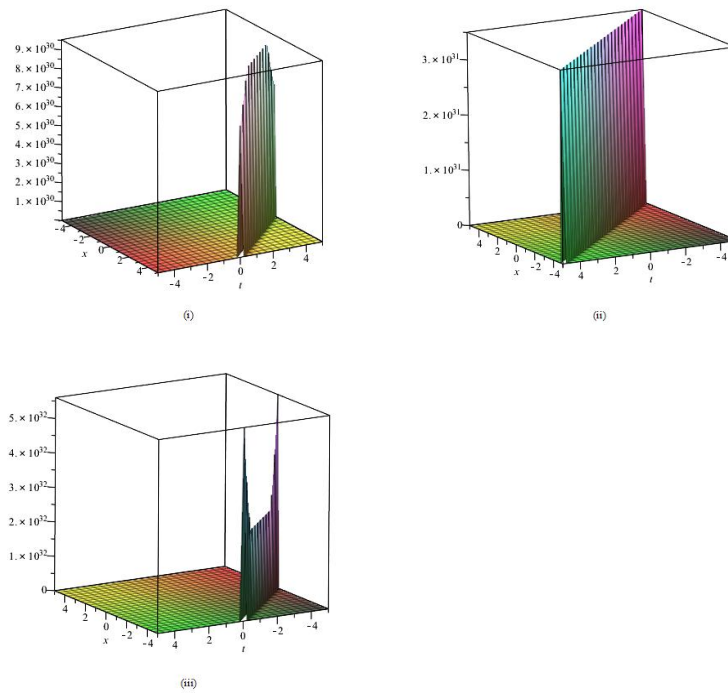


Figure 1. The 3D profiles of solution $u_r(x + vt)$ of Equation (1) corresponding to (i) $x_0 = 5, t_0 = 3$, (ii) $x_0 = 0, t_0 = 0$, (iii) $x_0 = -5, t_0 = -3$.

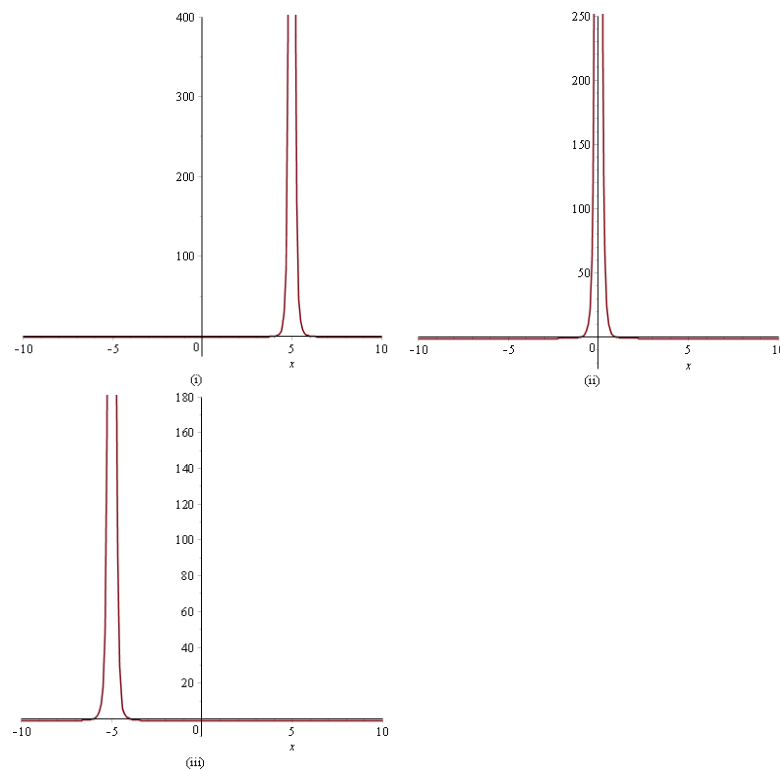


Figure 2. The 2D profiles of solution $u_r(x + vt)$ of Equation (1) corresponding to (i) $x_0 = 5, t_0 = 0$, (ii) $x_0 = 0, t_0 = 0$, (iii) $x_0 = -5, t_0 = 0$.

2. By employing the complex method, we are able to obtain the simply periodic solutions $u_{s,1}(x + vt)$ and $u_{s,2}(x + vt)$ of Equation (4). Figure 3 shows the 3D graphs of solution $u_{s,1}(x + vt)$ for $a = 3, c = 13, d = 1$, and $v = 3$ within the interval $-2\pi \leq x, t \leq 2\pi$.

Figure 4 describes the 2D graphs of solution $u_{s,1}(x + vt)$ for $a = 3, c = 13, d = 1,$ and $v = 3$ within the interval $-2\pi \leq x \leq 2\pi$ when $t = 0$.

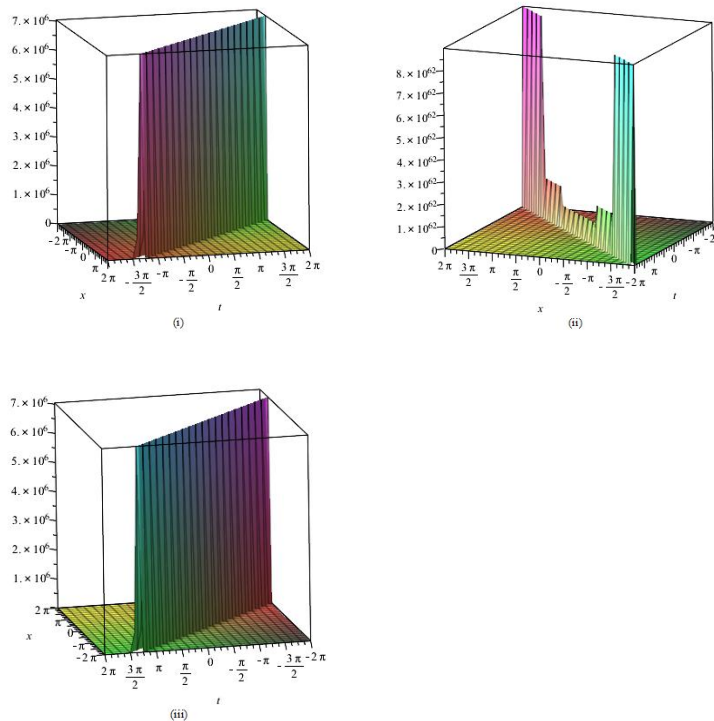


Figure 3. The 3D profiles of solution $u_{s,1}(x + vt)$ of Equation (1) corresponding to (i) $x_0 = 1, t_0 = 1,$ (ii) $x_0 = 0, t_0 = 0,$ (iii) $x_0 = -1, t_0 = -1.$

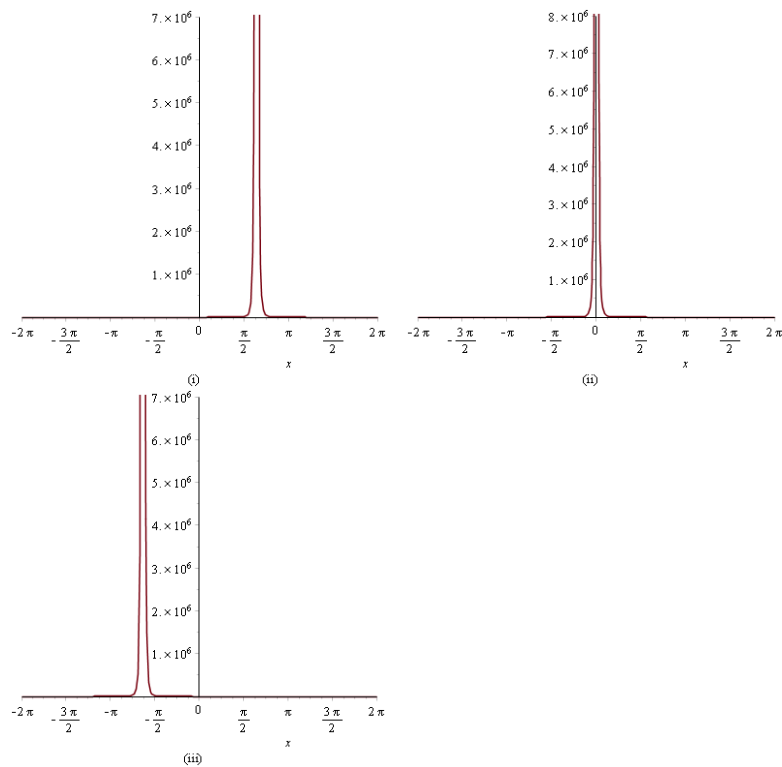


Figure 4. The 2D profiles of solution $u_{s,1}(x + vt)$ of Equation (1) corresponding to (i) $x_0 = 1, t_0 = 0,$ (ii) $x_0 = 0, t_0 = 0,$ (iii) $x_0 = -1, t_0 = 0.$

5. Conclusions

In this paper, we are the first to utilize the complex method to prove that all meromorphic solutions of complex Kawahara equation belong to the class W , and then we get all meromorphic solutions of complex Kawahara equation and all traveling wave exact solutions of Equation (1). In addition, we find that all rational solutions $u_r(x + vt)$ and simply periodic solutions $u_{s,1}(x + vt)$ of Equation (1) are solitary wave solutions, which could be used to analyze and interpret a lot of nonlinear dispersive phenomena. This research enriches the methods of solving differential equations. Our results also reveal that the complex method of looking for traveling wave exact solutions is general and feasible, and can be applied to other nonlinear partial differential equations.

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Appendix A

1. The system of Equations (1):

$$\begin{aligned}
 & -Ak_0^3 + Ck_0^2 + Dk_0 + E = 0, \\
 & -\frac{280}{13}Bk_0^2 + \frac{560}{39}\frac{BC}{A}k_0 + \frac{280BD}{39A} = 0, \\
 & -840k_0^2 + \left(\frac{560C}{A} - \frac{78400B^2}{507A}\right)k_0 + \frac{78400B^2C}{1521A^2} + \frac{280D}{A} = 0, \\
 & -\frac{156800B}{13A}k_0 - \frac{9721600B^3}{59319A^2} + \frac{156800BC}{39A^2} = 0, \\
 & -\frac{235200}{A}k_0 - \frac{4860800B^2}{507A^2} + \frac{78400C}{A^2} = 0.
 \end{aligned}$$

2. The system of Equations (2):

$$\begin{aligned}
 & \frac{14\delta^2}{A} + k_0 = -\frac{62B^2 - 507C}{1521A}, \\
 & \frac{14B\delta^2}{13A} - \frac{20\delta^3}{A} = \frac{62B^3}{59319A}, \\
 & -\frac{10B\delta^3}{39A} + \frac{21\delta^4}{2A} = \frac{-11780B^4 + 257049AD + 85683C^2}{425673144}.
 \end{aligned}$$

3. The system of Equations (3):

$$\frac{14g_2}{3A} + k_0 = -\frac{62B^2 - 507C}{1521A},$$

$$\frac{14Bg_2}{39A} + \frac{20g_3}{A} = \frac{62B^3}{59319A},$$

$$\frac{10Bg_3}{39A} + \frac{7g_2^2}{6A} = \frac{-11780B^4 + 257049AD + 85683C^2}{425673144}.$$

References

1. Kawahara, T. Oscillatory solitary waves in dispersive media. *J. Phys. Soc. Jpn.* **1972**, *33*, 260–264. [\[CrossRef\]](#)
2. Wazwaz, A.M. New solitary wave solutions to the Kuramoto-Sivashinsky and the Kawahara equations. *Appl. Math. Comput.* **2006**, *182*, 1642–1650. [\[CrossRef\]](#)
3. Yusufoglu, E.; Bekir, A.; Alp, M. Periodic and solitary wave solutions of Kawahara and modified Kawahara equations by using sine-cosine method. *Chaos. Soliton. Fract.* **2008**, *37*, 1193–1197. [\[CrossRef\]](#)
4. Kaya, D.; Al-Khaled, K. A numerical comparison of a Kawahara equation. *Phys. Lett. A* **2007**, *363*, 433–439. [\[CrossRef\]](#)
5. Wazwaz, A.M. *Partial Differential Equations and Solitary Waves Theory*; Higher Education Press: Beijing, China, 2009.
6. Jang, B. New exact travelling wave solutions of Kawahara type equations. *Nonlinear Anal.* **2009**, *70*, 510–515. [\[CrossRef\]](#)
7. Kudryashov, N.A. A note on new exact solutions for the Kawahara equation using exp-function method. *J. Comput. Appl. Math.* **2010**, *234*, 3511–3512. [\[CrossRef\]](#)
8. Kaur, L.; Gupta, R.K. Kawahara equation and modified Kawahara equation with time dependent coefficients: Symmetry analysis and generalized (G'/G)-expansion method. *Math. Methods Appl. Sci.* **2012**, *36*, 584–600. [\[CrossRef\]](#)
9. Pinar, Z.; Öziş, Z.T. The periodic solutions to Kawahara equation by means of the auxiliary equation with a sixth-degree nonlinear term. *J. Math.* **2013**, *2013*, 106349. [\[CrossRef\]](#)
10. Mahmood, B.A.; Yousif, M.A. A novel analytical solution for the modified Kawahara equation using the residual power series method. *Nonlinear Dyn.* **2017**, *89*, 1233–1238. [\[CrossRef\]](#)
11. El-Tantawy, S.A.; Salas, A.H.; Alharthi, M.R. Novel analytical cnoidal and solitary wave solutions of the extended Kawahara equation. *Chaos. Soliton. Fract.* **2021**, *147*, 110965. [\[CrossRef\]](#)
12. Ghanbari, B.; Kumar, S.; Niwas, M.; Baleanu, D. The life symmetry analysis and exact Jacobi elliptic solutions for the Kawahara-KDV type equations. *Results. Phys.* **2021**, *23*, 104006. [\[CrossRef\]](#)
13. Dascioglu, A.; Cuiha, S. Unal New exact solutions for the space-time fractional Kawahara equation. *Appl. Math. Model.* **2021**, *89*, 952–965. [\[CrossRef\]](#)
14. Demina, M.V.; Kudryashov, N.A. From Laurent series to exact meromorphic solutions: The Kawahara equation. *Phys. Lett. A* **2010**, *374*, 4023–4029. [\[CrossRef\]](#)
15. Wazwaz, A.M. Compacton solutions of the Kawahara-type nonlinear dispersive equation. *Appl. Math. Comput.* **2003**, *145*, 133–150. [\[CrossRef\]](#)
16. Khan, Y. A new necessary condition of soliton solutions for Kawahara equation arising in physics. *Optik* **2018**, *155*, 273–275. [\[CrossRef\]](#)
17. Biswas, A. Solitary wave solution for the generalized Kawahara equation. *Appl. Math. Lett.* **2009**, *22*, 208–210. [\[CrossRef\]](#)
18. Khan, K.; Akbar, M. The $\exp(-\phi(\zeta))$ -expansion method for finding traveling wave solutions of Vakhnenko-Parkes equation. *Int. J. Dyn. Syst. Differ. Equ.* **2014**, *5*, 72–83.
19. Khan, K.; Akbar, M.; Koppelaar, H. Study of coupled nonlinear partial differential equations for finding exact analytical solutions. *R. Soc. Open Sci.* **2015**, *2*, 140406. [\[CrossRef\]](#)
20. Jafari, H.; Kadkhoda, N.; Baleanu, D. Fractional Lie group method of the time-fractional Boussinesq equation. *Nonlinear Dyn.* **2015**, *81*, 1569–1574. [\[CrossRef\]](#)
21. Jafari, H.; Kadkhoda, N.; Azadi, M.; Yaghoubi, M. Group classification of the time-fractional Kaup-Kupershmidt equation. *Sci. Iran.* **2017**, *24*, 302–307. [\[CrossRef\]](#)
22. Sahoo, S.; Ray, S. Solitary wave solutions for time fractional third order modified kdv equation using two reliable techniques G'/G-expansion method and improved G'/G-expansion method. *Phys. Lett. A* **2010**, *374*, 4023–4029. [\[CrossRef\]](#)
23. Özkan, E.M.; Özkan, A. The soliton solutions for some nonlinear fractional differential equations with Beta-derivative. *Axioms* **2021**, *10*, 203. [\[CrossRef\]](#)
24. Wang, X.F.; Cheng, H. Solitary wave solution and a linear mass-conservative difference scheme for the generalized Korteweg-de Vries Kawahara equation. *Comput. Appl. Math.* **2021**, *40*, 273. [\[CrossRef\]](#)
25. Rehman, S.U.; Ahmad, F.; Kouser, S.; Pervaiz, A. Numerical approximation of modified Kawahara equation using Kernel smoothing method. *Math. Comput. Simul.* **2022**, *194*, 169–184.
26. El-Tantawy, S.A.; Salas, A.H.; Alyousef, H.A.; Alharthi, M.R. Novel exact and approximate solutions to the family of the forced damped Kawahara equation and modeling strong nonlinear waves in a plasma. *Chin. J. Phys.* **2022**, *77*, 2454–2471. [\[CrossRef\]](#)

27. Yuan, W.J.; Li, Y.Z.; Lin, J.M. Meromorphic solutions of an auxiliary ordinary differential equation using complex method. *Math. Method Appl. Sci.* **2013**, *36*, 1776–1782. [[CrossRef](#)]
28. Yuan, W.J.; Wu, Y.H.; Chen, Q.H.; Huang, Y. All meromorphic solutions for two forms of odd order algebraic differential equations and its applications. *Appl. Math. Comput.* **2014**, *240*, 240–251.
29. Conte, R. The Painlevé approach to nonlinear ordinary differential equations. In *The Painlevé Property, One Century Later*, CRM Series in Mathematical Physics; Conte, R., Ed.; Springer: New York, NY, USA, 1999; pp. 77–180.
30. Yuan, W.J.; Huang, Y.; Shang, Y.D. All travelling wave exact solutions of two nonlinear physical models. *Appl. Math. Comput.* **2013**, *219*, 6212–6223.
31. Yuan, W.J.; Shang, Y.D.; Huang, Y.; Wang, H. The representation of meromorphic solutions of certain ordinary differential equations and its applications. *Sci. Sin. Math.* **2013**, *43*, 563–575.
32. Yuan, W.J.; Meng, F.N.; Huang, Y.; Wu, Y.H. All traveling wave exact solutions of the variant Boussinesq equations. *Appl. Math. Comput.* **2015**, *268*, 865–872. [[CrossRef](#)]
33. Lang, S. *Elliptic Functions*, 2nd ed.; Springer: New York, NY, USA, 1987.
34. Conte, R.; Musette, M. Elliptic general analytic solutions. *Stud. Appl. Math.* **2009**, *123*, 63–81. [[CrossRef](#)]
35. Gu, Y.Y.; Wu, C.F.; Yuan, W.J. Characterizations of all real solutions for the KdV equation and $W_{\mathbb{R}}$. *Appl. Math. Lett.* **2020**, *107*, 106446.