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On Some Generalizations of Reverse Dynamic Hardy Type Inequalities on Time Scales

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Abstract: In the present paper, we prove some new reverse type dynamic inequalities on $\mathbb{T}$. Our main inequalities are proved by using the chain rule and Fubini's theorem on time scales $\mathbb{T}$. Our results extend some existing results in the literature. As special cases, we obtain some new discrete inequalities, quantum inequalities and integral inequalities.

Keywords: reverse Hardy's inequality; dynamic inequality; time scale

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1. Introduction

In 1920, the renowned English mathematician Godfrey Harold Hardy [1] proved the following result.

Theorem 1. *Assume that $\{f_n\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers. If $r > 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n^r}\left(\sum_{k=1}^{n} f_k\right)^r \leq \left(\frac{r}{r-1}\right)^r \sum_{n=1}^{\infty} f_n^r. \tag{1}$$

Inequality (1) is known in the literature as discrete Hardy' inequality.

In 1925, Hardy himself [2] gave the integral analogous of inequality (1) in the following form.

Theorem 2. *Suppose that f is a nonnegative continuous function defined on $[0,\infty)$. If $r > 1$, then*

$$\int_0^{\infty} \frac{1}{\lambda^r}\left(\int_0^{\lambda} f(\zeta)d\zeta\right)^r d\lambda \leq \left(\frac{r}{r-1}\right)^r \int_0^{\infty} f^r(\lambda)d\lambda. \tag{2}$$

In 1927, Littlewood and Hardy [3] proved the reversed version of inequality (2) in the following manner:

Theorem 3. *Let f be a nonnegative function on $[0,\infty)$. If $0 < r < 1$, then*

$$\int_0^{\infty} \frac{1}{\lambda^r}\left(\int_{\lambda}^{\infty} f(\zeta)d\zeta\right)^r d\lambda \geq \left(\frac{r}{1-r}\right)^r \int_0^{\infty} f^r(\lambda)d\lambda. \tag{3}$$

In 1928, Hardy [4] established a generalization of inequality (2). He proved that:

Theorem 4. *Suppose that f is a nonnegative continuous function defined on $[0,\infty)$. Then,*

$$\int_0^{\infty} \frac{1}{\lambda^{\gamma}}\left(\int_0^{\lambda} f(\zeta)d\zeta\right)^r d\lambda \leq \left(\frac{r}{\gamma-1}\right)^r \int_0^{\infty} \lambda^{r-\gamma} f^r(\lambda)d\lambda, \qquad \textit{for} \quad r \geq \gamma > 1, \tag{4}$$

and

$$\int_0^\infty \frac{1}{\lambda^\gamma}\left(\int_\lambda^\infty f(\zeta)d\zeta\right)^r d\lambda \le \left(\frac{r}{1-\gamma}\right)^r \int_0^\infty \lambda^{r-\gamma} f^r(\lambda)d\lambda, \quad \text{for} \quad r > 1 > \gamma \ge 0. \tag{5}$$

In 1928, Copson [5] gave the next two discrete inequalities as generalizations of inequality (1).

Theorem 5. *Let $\{f_n\}_{n=1}^\infty$ and $\{\theta_n\}_{n=1}^\infty$ be sequences of nonnegative real numbers. Then,*

$$\sum_{n=1}^\infty \frac{\theta_n\left(\sum_{k=1}^n \theta_k f_k\right)^r}{\left(\sum_{k=1}^n \theta_k\right)^\gamma} \le \left(\frac{r}{\gamma-1}\right)^r \sum_{n=1}^\infty \theta_n f_n^r \left(\sum_{k=1}^n \theta_k\right)^{r-\gamma}, \quad \text{for} \quad r \ge \gamma > 1, \tag{6}$$

and

$$\sum_{n=1}^\infty \frac{\theta_n\left(\sum_{k=n}^\infty \theta_k f_k\right)^r}{\left(\sum_{k=1}^n \theta_k\right)^\gamma} \le \left(\frac{r}{1-\gamma}\right)^r \sum_{n=1}^\infty \theta_n f_n^r \left(\sum_{k=1}^n \theta_k\right)^{r-\gamma}, \quad \text{for} \quad r > 1 > \gamma \ge 0. \tag{7}$$

In 1970, Leindler [6] explored some discrete Hardy inequality versions (1) and was able to demonstrate that:

Theorem 6. *Let $\{f_n\}_{n=1}^\infty$ and $\{\theta_n\}_{n=1}^\infty$ be sequences of real numbers that are not negative and $r > 1$, then*

$$\sum_{n=1}^\infty \theta_n\left(\sum_{k=1}^n f_k\right)^r \le r^r \sum_{n=1}^\infty \theta_n^{1-r} f_n^r \left(\sum_{k=n}^\infty \theta_k\right)^r, \tag{8}$$

and

$$\sum_{n=1}^\infty \theta_n\left(\sum_{k=n}^\infty f_k\right)^r \le r^r \sum_{n=1}^\infty \theta_n^{1-r} f_n^r \left(\sum_{k=1}^n \theta_k\right)^r. \tag{9}$$

In 1976, Copson [7] gave the inequalities' continuous versions (6) and (7). He arrived at the following conclusion specifically:.

Theorem 7. *Let f and θ be continuous functions that are not negative on $[0, \infty)$. Then,*

$$\int_0^\infty \frac{\theta(\lambda)\left(\int_0^\lambda \theta(\zeta)f(\zeta)d\zeta\right)^r}{\left(\int_0^\lambda \theta(\zeta)d\zeta\right)^\gamma} d\lambda \le \left(\frac{r}{\gamma-1}\right)^r \int_0^\infty \theta(\lambda) f^r(\lambda)\left(\int_0^\lambda \theta(\zeta)d\zeta\right)^{r-\gamma} d\lambda, \quad \text{for} \quad r \ge \gamma > 1, \tag{10}$$

and

$$\int_0^\infty \frac{\theta(\lambda)\left(\int_\lambda^\infty \theta(\zeta)f(\zeta)d\zeta\right)^r}{\left(\int_0^\lambda \theta(\zeta)d\zeta\right)^\gamma} d\lambda \le \left(\frac{r}{1-\gamma}\right)^r \int_0^\infty \theta(\lambda) f^r(\lambda)\left(\int_0^\lambda \theta(\zeta)d\zeta\right)^{r-\gamma} d\lambda, \quad \text{for} \quad r > 1 > \gamma \ge 0. \tag{11}$$

In 1982, Lyon [8] discovered a reverse version of the discrete Hardy inequality (1) for the special case when $r = 2$. According to his conclusion:

Theorem 8. *Let $\{f_n\}_{n=0}^\infty$ be a nonincreasing sequence of real numbers that are nonnegative. Then,*

$$\sum_{n=0}^\infty \left(\frac{1}{n+1}\sum_{k=0}^n f_k\right)^2 \ge \frac{\pi^2}{6}\sum_{n=0}^\infty f_n^2. \tag{12}$$

In 1986, Renaud [9] proved the following two results.

Theorem 9. *Assume that* $\{f_n\}_{n=1}^{\infty}$ *is a nonincreasing sequence of nonnegative real numbers. If* $r > 1$*, then,*

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} f_k\right)^r \geq \sum_{n=1}^{\infty} n^r f_n^r. \tag{13}$$

Theorem 10. *Assume that* f *is a nonincreasing nonnegative function defined on* $[0,\infty)$*. If* $r > 1$*, then,*

$$\int_0^{\infty}\left(\int_{\lambda}^{\infty} f(\zeta)d\zeta\right)^p dx \geq \int_0^{\infty} \lambda^p f^r(\lambda)d\lambda. \tag{14}$$

In 1990, the reverses of inequalities (8) and (9) were demonstrated by Leindler in [10] as the following:

Theorem 11. *If* $\{f_n\}_{n=1}^{\infty}$ *and* $\{\theta_n\}_{n=1}^{\infty}$ *are sequences of nonnegative real numbers and* $0 < r \leq 1$*, then,*

$$\sum_{n=1}^{\infty}\theta_n\left(\sum_{k=1}^{n} f_k\right)^r \geq r^r \sum_{n=1}^{\infty} \theta^{1-r}(n) f_n^r \left(\sum_{k=n}^{\infty}\theta_k\right)^r, \tag{15}$$

and

$$\sum_{n=1}^{\infty}\theta_n\left(\sum_{k=n}^{\infty} f_k\right)^r \geq r^r \sum_{n=1}^{\infty} \theta_n^{1-r} f_n^r \left(\sum_{k=1}^{n}\theta_k\right)^r. \tag{16}$$

Hilger, in his Ph.D. thesis [11], was the first one to accomplish the unification and extension of differential equations, difference equations, q-difference equations, and so on to the encompassing theory of dynamic equations on time scales.

Throughout this work, a knowledge and understanding of time scales and time-scale notation is assumed; for an excellent introduction to the calculus on time scales, see Bohner and Peterson [12,13].

In 2005, Řehák [14] was a forerunner in extending Hardy-type inequalities to time scales. He expanded the original Hardy inequalities (1) and (2) to a time scale of our choosing, and so, he combined them into a single form, as illustrated below.

Theorem 12. *Suppose* $\mathbb{T}$ *is a time scale, and* $f \in C_{rd}([a,\infty)_{\mathbb{T}}, [0,\infty))$*. If* $r \geq 1$*, then,*

$$\int_a^{\infty}\left(\frac{\int_a^{\sigma(\eta)} f(\zeta)\Delta\zeta}{\sigma(\eta)-a}\right)^r \Delta\eta < \left(\frac{r}{r-1}\right)^r \int_a^{\infty} f^r(\eta)\Delta\eta, \tag{17}$$

unless $f \equiv 0$*.*

In 2017, Agarwal et al. [15] presented the next dynamic inequality.

Theorem 13. *Let* $\mathbb{T}$ *be a time scale such that* $0 \in \mathbb{T}$*. Moreover, assume* f *is a nonincreasing nonnegative function on* $[0,\infty)_{\mathbb{T}}$*. If* $r > 1$*, then,*

$$\int_0^{\infty} \frac{1}{\eta^r}\left(\int_0^{\eta} f(\zeta)\Delta\zeta\right)^r \Delta\eta \geq \frac{r}{r-1}\int_0^{\infty} f^r(\eta)\Delta\eta. \tag{18}$$

Very recently, El-Deeb et al. [16] established the next dynamic inequalities.

Theorem 14. *Suppose* $\mathbb{T}$ *is a time scale with* $a \in [0,\infty)_{\mathbb{T}}$*. Additionally, suppose that* $f > 0$ *and* $\theta > 0$ *are rd-continuous functions on* $[a,\infty)_{\mathbb{T}}$ *and* f *is nonincreasing.*

(i) *If* $r \geq 1$ *and* $\gamma \geq 0$, *then*

$$\int_a^\infty \frac{\theta(\eta)\left(\int_\eta^\infty \theta(\zeta)f(\zeta)\Delta\zeta\right)^r}{\left(\int_a^{\sigma(\eta)} \theta(\zeta)\Delta\zeta\right)^\gamma}\Delta\eta \geq \int_a^\infty \frac{\theta(\eta)\left(\int_a^\eta \theta(\zeta)\Delta\zeta\right)^r f^r(\eta)}{\left(\int_a^{\sigma(\eta)} \theta(\zeta)\Delta\zeta\right)^\gamma}\Delta\eta. \tag{19}$$

(ii) *If* $r \geq 1$ *and* $\gamma > 1$, *then*

$$\int_a^\infty \frac{\theta(\eta)\left(\int_\eta^\infty \theta(\zeta)f(\zeta)\Delta\zeta\right)^r}{\left(\int_\eta^\infty \theta(\zeta)\Delta\zeta\right)^\gamma}\Delta\eta \geq \frac{r}{\gamma-1}\int_a^\infty \theta(\eta)\left(\int_\eta^\infty \theta(\zeta)\Delta\zeta\right)^{r-\gamma} f^r(\eta)\Delta\eta. \tag{20}$$

(iii) *If* $r \geq 1$ *and* $\gamma > 1$, *then*

$$\int_a^\infty \frac{\theta(\eta)\left(\int_a^\eta \theta(\zeta)f(\zeta)\Delta\zeta\right)^r}{\left(\int_a^\eta \theta(\zeta)\Delta\zeta\right)^\gamma}\Delta\eta \geq \frac{r}{\gamma-1}\int_a^\infty \theta(\eta)\left(\int_a^\eta \theta(\zeta)\Delta\zeta\right)^{r-\gamma} f^r(\eta)\Delta\eta. \tag{21}$$

(iv) *If* $r \geq 1$ *and* $0 \leq \gamma < 1$, *then*

$$\int_a^\infty \frac{\theta(\eta)\left(\int_a^\eta \theta(\zeta)f(\zeta)\Delta\zeta\right)^r}{\left(\int_{\sigma(\eta)}^\infty \theta(\zeta)\Delta\zeta\right)^\gamma}\Delta\eta \geq \frac{r}{1-\gamma}\int_a^\infty \theta(\eta)\left(\int_a^\eta \theta(\zeta)\Delta\zeta\right)^{r-1}\left(\int_\eta^\infty \theta(\zeta)\Delta\zeta\right)^{1-\gamma} f^r(\eta)\Delta\eta. \tag{22}$$

For more details on Hardy-type inequalities and other types on time scales, we suggest [17–29] for the reader.

Theorem 15 (Fubini's Theorem, see [Theorem 1.1, Page 300] [30]). *Assume that* $(\lambda, \Sigma_1, \mu_\Delta)$ *and* $(Y, \Sigma_2, \nu_\Delta)$ *are two finite-dimensional time scales measure spaces. Moreover, suppose that* $f : \lambda \times Y \to \mathbb{R}$ *is a delta integrable function and define the functions*

$$\Phi(y) = \int_\lambda f(\lambda, y)d\mu_\Delta(\lambda), \quad y \in Y,$$

and

$$\hat{\Psi}(\lambda) = \int_Y f(\lambda, y)d\nu_\Delta(y), \quad \lambda \in \lambda.$$

Then, Φ *is delta integrable on* Y *and* $\hat{\Psi}$ *is delta integrable on* λ *and*

$$\int_\lambda d\mu_\Delta(\lambda)\int_Y f(\lambda, y)d\nu_\Delta(y) = \int_Y d\nu_\Delta(y)\int_\lambda f(\lambda, y)d\mu_\Delta(\lambda).$$

The basic theorems that will be required in the proof of our results are presented next.

Theorem 16 (Chain rule on time scales, see [Theorem 1.87, Page 31] [12]). *Assume* $g : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable on* $\mathbb{T}^\kappa$, *and* $f : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable. Then, there exists* $c \in [\eta, \sigma(\eta)]$ *with*

$$(f \circ g)^\Delta(\eta) = f'(g(c))g^\Delta(\eta). \tag{23}$$

Theorem 17 (**Chain rule on time scales, see [Theorem 1.90, Page 32] [12]**)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then, $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(\eta) = \left\{ \int_0^1 \left[f'(hg^{\sigma}(\eta) + (1-h)g(\eta)) \right] dh \right\} g^{\Delta}(\eta),$$

holds.

In this manuscript, we show and prove some new dynamic Hardy-type which are reverse inequalities on time scales. The dynamic Hardy-type inequalities we obtained are entirely original, and as a result, we could obtain some integral and discrete inequalities of Hardy-type that are new. Furthermore, our findings generalize inequities (19)–(22). This paper is organized in the following way: Some basic concepts of the calculus on time scales and useful lemmas are introduced in Section 1. In Section 2, we state and prove the main results. In Section 3, we state the conclusion.

2. Main Results

The version of inequality (14) on time scales is given as a special case of the following theorem.

Theorem 18. *Assume that $\mathbb{T}$ is a time scale with $0 \le a \in \mathbb{T}$. Additionally, let f, g, $\check{\xi}$ and θ be nonnegative functions defined on $[0, \infty)_{\mathbb{T}}$ such that f and g are nonincreasing. Moreover, let $\check{\Psi} : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function such that $\check{\Psi}'$ is nondecreasing and $\check{\Psi}'(xy) = \check{\Psi}'(x)\check{\Psi}'(y)$ for all $x, y \in \mathbb{R}_+$. If $\gamma \ge 0$, then*

$$\int_a^{\infty} \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_\eta^{\infty} \check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_a^{\sigma(\eta)} \theta(\zeta)\Delta\zeta\right)^{\gamma}} \Delta\eta \ge \int_a^{\infty} \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_a^{\eta} \check{\xi}(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\eta))f(\eta)}{\left(\int_a^{\sigma(\eta)} \theta(\zeta)\Delta\zeta\right)^{\gamma}} \Delta\eta. \tag{24}$$

Proof. Owing to nonincreasity of f, we have for $\lambda \ge \eta \ge a$

$$\int_\eta^{\lambda} \check{\xi}(\zeta)f(\zeta)\Delta\zeta \ge f(\lambda) \int_\eta^{\lambda} \check{\xi}(\zeta)\Delta\zeta,$$

then, since $\check{\Psi}'$ is nondecreasing,

$$\check{\Psi}'\left(\int_\eta^{\lambda} \check{\xi}(\zeta)f(\zeta)\Delta\zeta\right) \ge \check{\Psi}'\left(f(\lambda)\int_\eta^{\lambda} \check{\xi}(\zeta)\Delta\zeta\right) = \check{\Psi}'(f(\lambda))\check{\Psi}'\left(\int_\eta^{\lambda} \check{\xi}(\zeta)\Delta\zeta\right). \tag{25}$$

Applying the chain rule (23), there exists $c \in [\lambda, \sigma(\lambda)]$ such that

$$\left[\check{\Psi}\left(\int_\eta^{\lambda} \check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)\right]^{\Delta_\lambda} = \check{\Psi}'\left(\int_\eta^{c} \check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)\left(\int_\eta^{\lambda} \check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)^{\Delta_\lambda}.$$

Since $c \ge \lambda$, $\check{\Psi}'$ is nondecreasing, and $\left(\int_\eta^{\lambda} \check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)^{\Delta_\lambda} = \check{\xi}(\lambda)f(\lambda) \ge 0$, we have

$$\left[\check{\Psi}\left(\int_\eta^{\lambda} \check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)\right]^{\Delta_\lambda} \ge \check{\xi}(\lambda)f(\lambda)\check{\Psi}'\left(\int_\eta^{\lambda} \check{\xi}(\zeta)f(\zeta)\Delta\zeta\right). \tag{26}$$

Combining (25) with (26) yields

$$\left[\check{\Psi}\left(\int_\eta^{\lambda} \check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)\right]^{\Delta_\lambda} \ge \check{\xi}(\lambda)\check{\Psi}'\left(\int_\eta^{\lambda} \check{\xi}(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda),$$

and so

$$\frac{\check{\xi}(\eta)g(\eta)\left[\check{\Psi}\left(\int_{\eta}^{\lambda}\check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)\right]^{\Delta_\lambda}}{\left(\int_{a}^{\sigma(\eta)}\theta(\zeta)\Delta\zeta\right)^{\gamma}} \geq \frac{\check{\xi}(\eta)g(\eta)\check{\xi}(\lambda)\check{\Psi}'\left(\int_{\eta}^{\lambda}\check{\xi}(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_{a}^{\sigma(\eta)}\theta(\zeta)\Delta\zeta\right)^{\gamma}}.$$

Considering that $\lambda \geq \eta$ implies: (i) $\sigma(\lambda) \geq \sigma(\eta)$ and hence $\int_{a}^{\sigma(\lambda)}\theta(\zeta)\Delta\zeta \geq \int_{a}^{\sigma(\eta)}\theta(\zeta)\Delta\zeta$; (ii) $g(\lambda) \leq g(\eta)$, we obtain

$$\frac{\check{\xi}(\eta)g(\eta)\left[\check{\Psi}\left(\int_{\eta}^{\lambda}\check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)\right]^{\Delta_\lambda}}{\left(\int_{a}^{\sigma(\eta)}\theta(\zeta)\Delta\zeta\right)^{\gamma}} \geq \frac{\check{\xi}(\eta)\check{\xi}(\lambda)g(\lambda)\check{\Psi}'\left(\int_{\eta}^{\lambda}\check{\xi}(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_{a}^{\sigma(\lambda)}\theta(\zeta)\Delta\zeta\right)^{\gamma}}.$$

If we integrate both sides with respect to λ over $[\eta,\infty)_{\mathbb{T}}$, we obtain

$$\frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_{a}^{\sigma(\eta)}\theta(\zeta)\Delta\zeta\right)^{\gamma}} \geq \int_{\eta}^{\infty}\frac{\check{\xi}(\eta)\check{\xi}(\lambda)g(\lambda)\check{\Psi}'\left(\int_{\eta}^{\lambda}\check{\xi}(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_{a}^{\sigma(\lambda)}\theta(\zeta)\Delta\zeta\right)^{\gamma}}\Delta\lambda.$$

If we integrate both sides once more, but with respect to η over $[a,\infty)_{\mathbb{T}}$, we obtain

$$\int_{a}^{\infty}\frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_{a}^{\sigma(\eta)}\theta(\zeta)\Delta\zeta\right)^{\gamma}}\Delta\eta \geq \int_{a}^{\infty}\check{\xi}(\eta)\left(\int_{\eta}^{\infty}\frac{\check{\xi}(\lambda)g(\lambda)\check{\Psi}'\left(\int_{\eta}^{\lambda}\check{\xi}(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_{a}^{\sigma(\lambda)}\theta(\zeta)\Delta\zeta\right)^{\gamma}}\Delta\lambda\right)\Delta\eta. \tag{27}$$

By Using Fubini's theorem on time scales, (27) can be rewritten as

$$\int_{a}^{\infty}\frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_{a}^{\sigma(\eta)}\theta(\zeta)\Delta\zeta\right)^{\gamma}}\Delta\eta \geq \int_{a}^{\infty}\frac{\check{\xi}(\lambda)g(\lambda)\check{\Psi}'(f(\lambda))f(\lambda)\left(\int_{a}^{\lambda}\check{\xi}(\eta)\check{\Psi}'\left(\int_{\eta}^{\lambda}\check{\xi}(\zeta)\Delta\zeta\right)\Delta\eta\right)}{\left(\int_{a}^{\sigma(\lambda)}\theta(\zeta)\Delta\zeta\right)^{\gamma}}\Delta\lambda. \tag{28}$$

Now, from the chain rule (23), one can see that there exists $c \in [\eta,\sigma(\eta)]$ with

$$\left[-\check{\Psi}\left(\int_{\eta}^{\lambda}\check{\xi}(\zeta)\Delta\zeta\right)\right]^{\Delta_\eta} = -\check{\Psi}'\left(\int_{c}^{\lambda}\check{\xi}(\zeta)\Delta\zeta\right)\left(\int_{\eta}^{\lambda}\check{\xi}(\zeta)\Delta\zeta\right)^{\Delta_\eta}.$$

Since $c \geq \eta$, $\check{\Psi}'$ is nondecreasing, $r \geq 1$ and $\left(\int_{\eta}^{\lambda}\check{\xi}(\zeta)\Delta\zeta\right)^{\Delta_\eta} = -\check{\xi}(\eta) \leq 0$, we have

$$\left[-\check{\Psi}\left(\int_{\eta}^{\lambda}\check{\xi}(\zeta)\Delta\zeta\right)\right]^{\Delta_\eta} \leq \check{\xi}(\eta)\check{\Psi}'\left(\int_{\eta}^{\lambda}\check{\xi}(\zeta)\Delta\zeta\right). \tag{29}$$

Substituting (29) into (28) leads to

$$\int_a^\infty \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_\eta^\infty \check{\xi}(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_a^{\sigma(\eta)}\theta(\zeta)\Delta\zeta\right)^\gamma}\Delta\eta$$

$$\geq \int_a^\infty \frac{\check{\xi}(\lambda)g(\lambda)\check{\Psi}'(f(\lambda))f(\lambda)\left(\int_a^\lambda\left[-\check{\Psi}\left(\int_\eta^\lambda \check{\xi}(\zeta)\Delta\zeta\right)\right]^{\Delta_\eta}\Delta\eta\right)}{\left(\int_a^{\sigma(\lambda)}\theta(\zeta)\Delta\zeta\right)^\gamma}\Delta\lambda$$

$$= \int_a^\infty \frac{\check{\xi}(\lambda)g(\lambda)\check{\Psi}\left(\int_a^\lambda \check{\xi}(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_a^{\sigma(\lambda)}\theta(\zeta)\Delta\zeta\right)^\gamma}\Delta\lambda.$$

This shows the validity of (24). □

Remark 1. *In Theorem 18, if we take $\check{\Psi}(\eta) = \eta^r$, $r \geq 1$, $\check{\xi}(\eta) = \theta(\eta)$ and $g(\eta) = 1$, then inequality (24) reduces to inequality (19).*

Corollary 1. *In Theorem 18, if we take $\check{\Psi}(\eta) = \eta^r$, $\check{\xi}(\eta) = g(\eta) = 1$ and $a = \gamma = 0$, then inequality (24) reduces to*

$$\int_0^\infty \left(\int_\eta^\infty f(\zeta)\Delta\zeta\right)^r \Delta\eta \geq \int_0^\infty \left(\int_0^\eta \check{\xi}(\zeta)\Delta\zeta\right)^r f^r(\eta)\Delta\eta,$$

which is the time scales version of (14).

Corollary 2. *If $\mathbb{T} = \mathbb{R}$ in Theorem 18, then inequality (24) reduces to*

$$\int_a^\infty \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_\eta^\infty \check{\xi}(\zeta)f(\zeta)d\zeta\right)}{\left(\int_a^\eta \theta(\zeta)d\zeta\right)^\gamma}d\eta \geq \int_a^\infty \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_a^\eta \check{\xi}(\zeta)d\zeta\right)\check{\Psi}'(f(\eta))f(\eta)}{\left(\int_a^\eta \theta(\zeta)d\zeta\right)^\gamma}d\eta.$$

Remark 2. *In Corollary 2, if we take $\check{\Psi}(\eta) = \eta^r$, $\check{\xi}(\eta) = g(\eta) = 1$, $a = \gamma = 0$, then we reclaim inequality (14).*

Corollary 3. *If $\mathbb{T} = h\mathbb{Z}$ in Theorem 18, then inequality (24) is reduced to*

$$\sum_{n=\frac{a}{h}}^\infty \frac{\check{\xi}(nh)g(nh)\check{\Psi}\left(h\sum_{m=\frac{n}{\theta}}^\infty \check{\xi}(mh)f(mh)\right)}{\left(\sum_{m=\frac{a}{h}}^{\frac{n}{h}}\theta(mh)\right)^\gamma} \geq \sum_{n=\frac{a}{h}}^\infty \frac{\check{\xi}(nh)g(nh)\check{\Psi}\left(h\sum_{m=\frac{a}{h}}^{\frac{n}{\theta}-1}\check{\xi}(mh)\right)\check{\Psi}'(f(nh))f(nh)}{\left(\sum_{m=\frac{a}{h}}^{\frac{n}{h}}\theta(mh)\right)^\gamma}.$$

Corollary 4. *In Corollary 3, if we take $h = 1$, then, inequality (24) will be reduced to*

$$\sum_{n=a}^\infty \frac{\check{\xi}(n)g(n)\check{\Psi}\left(\sum_{m=n}^\infty \check{\xi}(m)f(m)\right)}{\left(\sum_{m=a}^n \theta(m)\right)^\gamma} \geq \sum_{n=a}^\infty \frac{\check{\xi}(n)g(n)\check{\Psi}\left(\sum_{m=a}^{n-1}\check{\xi}(m)\right)\check{\Psi}'(f(n))f(n)}{\left(\sum_{m=a}^n \theta(m)\right)^\gamma}.$$

Remark 3. *In Corollary 4, if we take* $\check{\Psi}(\eta) = \eta^r$, $\check{\xi}(\eta) = g(\eta) = 1$, $a = 1$ *and* $\gamma = 0$, *then we reclaim inequality* (13).

Corollary 5. *If* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Theorem 18, then*

$$\sum_{n=(\log_q a)}^{\infty} \frac{q^n \check{\xi}(q^n) g(q^n) \check{\Psi}\Big((q-1)\sum_{m=(\log_q n)}^{\infty} q^m \check{\xi}(q^m) f(q^m)\Big)}{\Big(\sum_{m=(\log_q a)}^{(\log_q qn)-1} q^m h(q^m)\Big)^{\gamma}}$$
$$\geq \sum_{n=(\log_q a)}^{\infty} \frac{q^n \check{\xi}(q^n) g(q^n) \check{\Psi}\Big((q-1)\sum_{m=(\log_q a)}^{(\log_q n)-1} q^m \check{\xi}(q^m)\Big) \check{\Psi}'(f(q^n)) f(q^n)}{\Big(\sum_{m=(\log_q a)}^{(\log_q qn)-1} q^m h(q^m)\Big)^{\gamma}}.$$

Now, as a new result, we are interested in discussing the inequality (24) in the case of the extrema of integration $\int_a^{\eta} \theta(s)\Delta s$ being replaced to be from η to ∞. In fact, that is what we will do in the following theorem.

Theorem 19. *Assume that* $\mathbb{T}$ *is a time scale with* $0 \leq a \in \mathbb{T}$. *Additionally, let* f, g, θ *and* $\check{\xi}$ *be nonnegative functions defined on* $[0,\infty)_{\mathbb{T}}$ *such that* f *and* g *are nonincreasing. Furthermore, let* $\check{\Psi} : \mathbb{R}_+ \to \mathbb{R}_+$ *be a differentiable function such that* $\check{\Psi}'$ *is nondecreasing and* $\check{\Psi}'(xy) = \check{\Psi}'(x)\check{\Psi}'(y)$ *for all* $x, y \in \mathbb{R}_+$. *If* $\gamma > 1$, *then*

$$\int_a^{\infty} \frac{\check{\xi}(\eta) g(\eta) \check{\Psi}\Big(\int_{\eta}^{\infty} \theta(\zeta) f(\zeta)\Delta\zeta\Big)}{\Big(\int_{\eta}^{\infty} \check{\xi}(\zeta)\Delta\zeta\Big)^{\gamma}} \Delta\eta \geq \frac{1}{\gamma-1} \int_a^{\infty} \frac{\theta(\eta) g(\eta) \check{\Psi}'\Big(\int_{\eta}^{\infty} \theta(\zeta)\Delta\zeta\Big) \check{\Psi}'(f(\eta)) f(\eta)}{\Big(\int_{\eta}^{\infty} \check{\xi}(\zeta)\Delta\zeta\Big)^{\gamma-1}} \Delta\eta. \tag{30}$$

Proof. Because of nonincreasity of f, we have for $\eta \geq \lambda \geq a$

$$\int_{\lambda}^{\infty} \theta(\zeta) f(\zeta)\Delta\zeta \leq f(\lambda) \int_{\lambda}^{\infty} \theta(\zeta)\Delta\zeta,$$

therefore, because $\check{\Psi}'$ is nondecreasing,

$$\check{\Psi}'\Big(\int_{\lambda}^{\infty} \theta(\zeta) f(\zeta)\Delta\zeta\Big) \geq \check{\Psi}'\Big(f(\lambda)\int_{\lambda}^{\infty} \theta(\zeta)\Delta\zeta\Big) = \check{\Psi}'(f(\lambda))\check{\Psi}'\Big(\int_{\lambda}^{\infty} \theta(\zeta)\Delta\zeta\Big). \tag{31}$$

From the chain rule (23), we see that there is $c \in [\lambda, \sigma(\lambda)]$ with

$$\Big[\check{\Psi}\Big(\int_{\lambda}^{\infty} \theta(\zeta) f(\zeta)\Delta\zeta\Big)\Big]^{\Delta} = \check{\Psi}'\Big(\int_{c}^{\infty} \theta(\zeta) f(\zeta)\Delta\zeta\Big)\Big(\int_{\lambda}^{\infty} \theta(\zeta) f(\zeta)\Delta\zeta\Big)^{\Delta}.$$

Since $c \geq \lambda$, $\check{\Psi}'$ is nondecreasing, $r \geq 1$ and $\Big(\int_{\lambda}^{\infty} \theta(\zeta) f(\zeta)\Delta\zeta\Big)^{\Delta} = -\theta(\lambda) f(\lambda) \leq 0$, we have

$$\Big[\check{\Psi}\Big(\int_{\lambda}^{\infty} \theta(\zeta) f(\zeta)\Delta\zeta\Big)\Big]^{\Delta} \geq -\theta(\lambda) f(\lambda) \check{\Psi}'\Big(\int_{\lambda}^{\infty} \theta(\zeta) f(\zeta)\Delta\zeta\Big) \tag{32}$$

Combining (31) with (32) yields

$$\Big[\check{\Psi}\Big(\int_{\lambda}^{\infty} \theta(\zeta) f(\zeta)\Delta\zeta\Big)\Big]^{\Delta} \geq -\theta(\lambda) \check{\Psi}'\Big(\int_{\lambda}^{\infty} \theta(\zeta)\Delta\zeta\Big) \check{\Psi}'(f(\lambda)) f(\lambda),$$

which implies

$$\frac{\check{\xi}(\eta)g(\eta)\left[\check{\Psi}\left(\int_{\lambda}^{\infty}\theta(\zeta)f(\zeta)\Delta\zeta\right)\right]^{\Delta}}{\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}} \geq \frac{-\check{\xi}(\eta)g(\eta)\theta(\lambda)\check{\Psi}'\left(\int_{\lambda}^{\infty}\theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}}.$$

As g is nonincreasing and $\lambda \leq \eta$, we have $g(\lambda) \geq g(\eta)$ and hence,

$$\frac{\check{\xi}(\eta)g(\eta)\left[\check{\Psi}\left(\int_{\lambda}^{\infty}\theta(\zeta)f(\zeta)\Delta\zeta\right)\right]^{\Delta}}{\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}} \geq \frac{-\check{\xi}(\eta)\theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_{\lambda}^{\infty}\theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}}.$$

Now, after both sides are integrated with respect to λ over $[a,\eta]_{\mathbb{T}}$, we could have

$$\begin{aligned}&\frac{\check{\xi}(\eta)g(\eta)\left[\check{\Psi}\left(\int_{\eta}^{\infty}\theta(\zeta)f(\zeta)\Delta\zeta\right)-\check{\Psi}\left(\int_{a}^{\infty}\theta(\zeta)f(\zeta)\Delta\zeta\right)\right]}{\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}}\\ &\geq \int_{a}^{\eta}\frac{-\check{\xi}(\eta)\theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_{\lambda}^{\infty}\theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}}\Delta\lambda.\end{aligned}$$

Since $\check{\Psi}\left(\int_{\eta}^{\infty}\theta(\zeta)f(\zeta)\Delta\zeta\right) \geq \check{\Psi}\left(\int_{\eta}^{\infty}\theta(\zeta)f(\zeta)\Delta\zeta\right)-\check{\Psi}\left(\int_{a}^{\infty}\theta(\zeta)f(\zeta)\Delta\zeta\right)$, we have

$$\frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_{\eta}^{\infty}\theta(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}} \geq -\int_{a}^{\eta}\check{\xi}(\eta)\theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_{\lambda}^{\infty}\theta(\zeta)\Delta\zeta\right)\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}\check{\Psi}'(f(\lambda))f(\lambda)\Delta\lambda.$$

Afterwards, if both sides are integrated with respect to η over $[a,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned}&\int_{a}^{\infty}\frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_{\eta}^{\infty}\theta(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}}\Delta\eta\\ &\geq -\int_{a}^{\infty}\check{\xi}(\eta)\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}\left(\int_{a}^{\eta}\theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_{\lambda}^{\infty}\theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)\Delta\lambda\right)\Delta\eta.\end{aligned} \tag{33}$$

Using Fubini's theorem on time scales, (33) can be rewritten as

$$\begin{aligned}&\int_{a}^{\infty}\frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_{\eta}^{\infty}\theta(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}}\Delta\eta\\ &\geq -\int_{a}^{\infty}\theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_{\lambda}^{\infty}\theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)\left(\int_{\lambda}^{\infty}\check{\xi}(\eta)\left(\int_{\eta}^{\infty}\check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}\Delta\eta\right)\Delta\lambda.\end{aligned} \tag{34}$$

If we take a look at the chain rule, (23), we could say that there exists $c \in [\eta, \sigma(\eta)]$ such that

$$\left[-\left(\int_\eta^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{1-\gamma}\right]^\Delta = -(1-\gamma)\left(\int_c^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}\left(\int_\eta^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\Delta.$$

Since $c \geq \eta$, $\gamma > 1$ and $\left(\int_\eta^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\Delta = -\check{\xi}(\eta) \leq 0$, we get

$$\left[-\left(\int_\eta^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{1-\gamma}\right]^\Delta \geq -(\gamma-1)\check{\xi}(\eta)\left(\int_\eta^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}. \tag{35}$$

Substituting (35) into (34) leads to

$$\begin{aligned}
&\int_a^\infty \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_\eta^\infty \theta(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_\eta^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\gamma}\Delta\eta \\
&\geq \frac{1}{\gamma-1}\int_a^\infty \theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_\lambda^\infty \theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)\left(\int_\lambda^\infty\left[-\left(\int_\eta^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{1-\gamma}\right]^\Delta \Delta\eta\right)\Delta\lambda \\
&= \frac{1}{\gamma-1}\int_a^\infty \theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_\lambda^\infty \theta(\zeta)\Delta\zeta\right)\left(\int_\lambda^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{1-\gamma}\check{\Psi}'(f(\lambda))f(\lambda)\Delta\lambda,
\end{aligned}$$

from which inequality (30) follows. □

Remark 4. *In Theorem 19, if we take* $\check{\Psi}(\eta) = \eta^r$, $\check{\xi}(\eta) = \theta(\eta)$ *and* $g(\eta) = 1$, *then inequality (30) reduces to inequality (20).*

Corollary 6. *If* $\mathbb{T} = \mathbb{R}$ *in Theorem 19, then, inequality (30) will be reduced to*

$$\int_a^\infty \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_\eta^\infty \theta(\zeta)f(\zeta)d\zeta\right)}{\left(\int_\eta^\infty \check{\xi}(\zeta)d\zeta\right)^\gamma}d\eta \geq \frac{1}{\gamma-1}\int_a^\infty \frac{\theta(\eta)g(\eta)\check{\Psi}'\left(\int_\eta^\infty \theta(\zeta)d\zeta\right)\check{\Psi}'(f(\eta))f(\eta)}{\left(\int_\eta^\infty \check{\xi}(\zeta)d\zeta\right)^{\gamma-1}}d\eta.$$

Corollary 7. *If* $\mathbb{T} = h\mathbb{Z}$ *in Theorem 19, then inequality (30) is reduced to*

$$\sum_{n=\frac{a}{h}}^\infty \frac{\check{\xi}(nh)g(nh)\check{\Psi}\left(h\sum_{m=\frac{n}{h}}^\infty \theta(mh)f(mh)\right)}{\left(h\sum_{m=\frac{n}{h}}^\infty \check{\xi}(mh)\right)^\gamma} \geq \frac{1}{\gamma-1}\sum_{n=\frac{a}{h}}^\infty \frac{\theta(nh)g(nh)\check{\Psi}'\left(h\sum_{m=\frac{n}{h}}^\infty \theta(mh)\right)\check{\Psi}'(f(nh))f(nh)}{\left(h\sum_{m=\frac{n}{h}}^\infty \check{\xi}(mh)\right)^{\gamma-1}}.$$

Corollary 8. *In Corollary 7, if we take* $h = 1$, *then inequality (30) reduces to*

$$\sum_{n=a}^\infty \frac{\check{\xi}(n)g(n)\check{\Psi}\left(\sum_{m=n}^\infty \theta(m)f(m)\right)}{\left(\sum_{m=n}^\infty \check{\xi}(m)\right)^\gamma} \geq \frac{1}{\gamma-1}\sum_{n=a}^\infty \frac{\theta(n)g(n)\check{\Psi}'\left(\sum_{m=n}^\infty \theta(m)\right)\check{\Psi}'(f(n))f(n)}{\left(\sum_{m=n}^\infty \check{\xi}(m)\right)^{\gamma-1}}.$$

Corollary 9. *If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ in Theorem 19, then inequality (30) will be reduced to*

$$\sum_{n=(\log_q a)}^{\infty} \frac{q^n \check{\xi}(q^n) g(q^n) \check{\Psi}\left((q-1) \sum_{m=(\log_q n)}^{\infty} q^m h(q^m) f(q^m) \right)}{\left((q-1) \sum_{m=(\log_q n)}^{\infty} q^m \check{\xi}(q^m) \right)^{\gamma}}$$

$$\geq \frac{1}{\gamma - 1} \sum_{n=(\log_q a)}^{\infty} \frac{q^n h(q^n) g(q^n) \check{\Psi}'\left((q-1) \sum_{m=(\log_q n)}^{\infty} q^m h(q^m) \right) \check{\Psi}'(f(q^n)) f(q^n)}{\left((q-1) \sum_{m=(\log_q n)}^{\infty} q^m \check{\xi}(q^m) \right)^{\gamma-1}}.$$

In the next theorem, we make a broad popularization of Theorem 13.

Theorem 20. *Let $\mathbb{T}$ be a time scale with $0 \leq a \in \mathbb{T}$. Moreover, suppose that f, g, θ and $\check{\xi}$ are nonnegative functions defined on $[0, \infty)_{\mathbb{T}}$ such that f is nonincreasing and g is nondecreasing. In addition, let $\check{\Psi} : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function such that $\check{\Psi}'$ is nondecreasing and $\check{\Psi}'(xy) = \check{\Psi}'(x)\check{\Psi}'(y)$ for all $x, y \in \mathbb{R}_+$. If $\gamma > 1$, then*

$$\int_a^{\infty} \frac{\check{\xi}(\eta) g(\eta) \check{\Psi}\left(\int_0^{\eta} \theta(\zeta) f(\zeta) \Delta\zeta \right)}{\left(\int_0^{\eta} \check{\xi}(\zeta) \Delta\zeta \right)^{\gamma}} \Delta\eta \geq \frac{1}{\gamma - 1} \int_a^{\infty} \frac{\theta(\eta) g(\eta) \check{\Psi}'\left(\int_0^{\eta} \theta(\zeta) \Delta\zeta \right) \check{\Psi}'(f(\eta)) f(\eta)}{\left(\int_0^{\eta} \check{\xi}(\zeta) \Delta\zeta \right)^{\gamma-1}} \Delta\eta. \tag{36}$$

Proof. As a result of of the nonincreasity of f, we have for $\eta \geq \lambda \geq 0$

$$\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \geq f(\lambda) \int_0^{\lambda} \theta(\zeta) \Delta\zeta,$$

then, since $\check{\Psi}'$ is nondecreasing,

$$\check{\Psi}'\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right) \geq \check{\Psi}'\left(f(\lambda) \int_0^{\lambda} \theta(\zeta) \Delta\zeta \right) = \check{\Psi}'(f(\lambda)) \check{\Psi}'\left(\int_0^{\lambda} \theta(\zeta) \Delta\zeta \right). \tag{37}$$

Using the chain rule (23), there exists $c \in [\lambda, \sigma(\lambda)]$ such that

$$\left[\check{\Psi}\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right) \right]^{\Delta} = \check{\Psi}'\left(\int_0^{c} \theta(\zeta) f(\zeta) \Delta\zeta \right) \left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right)^{\Delta}.$$

Since $c \geq \lambda$, $\check{\Psi}'$ is nondecreasing, $r \geq 1$ and $\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right)^{\Delta} = \theta(\lambda) f(\lambda) \geq 0$, we have

$$\left[\check{\Psi}\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right) \right]^{\Delta} \geq \theta(\lambda) f(\lambda) \check{\Psi}'\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right). \tag{38}$$

By using (37) and (38) together we could have

$$\left[\check{\Psi}'\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right) \right]^{\Delta} \geq \theta(\lambda) \check{\Psi}'\left(\int_0^{\lambda} \theta(\zeta) \Delta\zeta \right) \check{\Psi}'(f(\lambda)) f(\lambda).$$

and thus

$$\frac{\check{\xi}(\eta) g(\eta) \left[\check{\Psi}\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right) \right]^{\Delta}}{\left(\int_0^{\eta} \check{\xi}(\zeta) \Delta\zeta \right)^{\gamma}} \geq \frac{\check{\xi}(\eta) g(\eta) \theta(\lambda) \check{\Psi}'\left(\int_0^{\lambda} \theta(\zeta) \Delta\zeta \right) \check{\Psi}'(f(\lambda)) f(\lambda)}{\left(\int_0^{\eta} \check{\xi}(\zeta) \Delta\zeta \right)^{\gamma}}.$$

As g is nondecreasing and $\lambda \leq \eta$, we have $g(\lambda) \leq g(\eta)$ and hence,

$$\frac{\check{\xi}(\eta)g(\eta)\left[\check{\Psi}\left(\int_0^{\lambda}\theta(\zeta)f(\zeta)\Delta\zeta\right)\right]^{\Delta}}{\left(\int_0^{\eta}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}} \geq \frac{\check{\xi}(\eta)g(\lambda)\theta(\lambda)\check{\Psi}'\left(\int_0^{\lambda}\theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_0^{\eta}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}}.$$

Integrating both sides of the last inequality with respect to λ over $[0,\eta]_{\mathbb{T}}$ gives

$$\begin{aligned}&\frac{\check{\xi}(\eta)g(\eta)\left[\check{\Psi}\left(\int_0^{\eta}\theta(\zeta)f(\zeta)\Delta\zeta\right)-\check{\Psi}\left(\int_0^{a}\theta(\zeta)f(\zeta)\Delta\zeta\right)\right]}{\left(\int_0^{\eta}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}} \\ &\geq \int_a^{\eta}\frac{\check{\xi}(\eta)\theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_0^{\lambda}\theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_0^{\eta}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}}\Delta\lambda.\end{aligned}$$

Since $\check{\Psi}\left(\int_0^{\eta}\theta(\zeta)f(\zeta)\Delta\zeta\right) \geq \check{\Psi}\left(\int_0^{\eta}\theta(\zeta)f(\zeta)\Delta\zeta\right)-\check{\Psi}\left(\int_0^{a}\theta(\zeta)f(\zeta)\Delta\zeta\right)$, we obtain

$$\frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_0^{\eta}\theta(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_0^{\eta}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}} \geq \int_a^{\eta}\check{\xi}(\eta)\theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_0^{\lambda}\theta(\zeta)\Delta\zeta\right)\left(\int_0^{\eta}\check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}\check{\Psi}'(f(\lambda))f(\lambda)\Delta\lambda,$$

After integrating both sides with respect to η over $[a,\infty)_{\mathbb{T}}$,

$$\begin{aligned}&\int_a^{\infty}\frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_0^{\eta}\theta(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_0^{\eta}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}}\Delta\eta \\ &\geq \int_a^{\infty}\check{\xi}(\eta)\left(\int_0^{\eta}\check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}\left(\int_a^{\eta}\theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_0^{\lambda}\theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)\Delta\lambda\right)\Delta\eta. \quad (39)\end{aligned}$$

Employing Fubini's theorem on time scales, (39) can be rewritten as

$$\begin{aligned}&\int_a^{\infty}\frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_0^{\eta}\theta(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_0^{\eta}\check{\xi}(\zeta)\Delta\zeta\right)^{\gamma}}\Delta\eta \\ &\geq \int_a^{\infty}\theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_0^{\lambda}\theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)\left(\int_{\lambda}^{\infty}\check{\xi}(\eta)\left(\int_0^{\eta}\check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}\Delta\eta\right)\Delta\lambda. \quad (40)\end{aligned}$$

Additionally, by taking a look at the chain rule (23), we can say that there exists $c \in [\eta,\sigma(\eta)]$ such that

$$\left[-\left(\int_0^{\eta}\check{\xi}(\zeta)\Delta\zeta\right)^{1-\gamma}\right]^{\Delta} = -(1-\gamma)\left(\int_0^{c}\check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}\left(\int_0^{\eta}\check{\xi}(\zeta)\Delta\zeta\right)^{\Delta}.$$

Since $c \geq \eta$, $\gamma > 1$ and $\left(\int_0^\eta \check{\xi}(\zeta)\Delta\zeta\right)^\Delta = \check{\xi}(\eta) \geq 0$, we get

$$\left[-\left(\int_0^\eta \check{\xi}(\zeta)\Delta\zeta\right)^{1-\gamma}\right]^\Delta \leq (\gamma-1)\check{\xi}(\eta)\left(\int_0^\eta \check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}. \tag{41}$$

Substituting (41) into (40) leads to

$$\begin{aligned}
&\int_a^\infty \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_0^\eta \theta(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_0^\eta \check{\xi}(\zeta)\Delta\zeta\right)^\gamma}\Delta\eta \\
&\geq \frac{1}{\gamma-1}\int_a^\infty \theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_0^\lambda \theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)\left(\int_\lambda^\infty \left[-\left(\int_0^\eta \check{\xi}(\zeta)\Delta\zeta\right)^{1-\gamma}\right]^\Delta \Delta\eta\right)\Delta\lambda \\
&= \frac{1}{\gamma-1}\int_a^\infty \theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_0^\lambda \theta(\zeta)\Delta\zeta\right)\left(\int_0^\lambda \check{\xi}(\zeta)\Delta\zeta\right)^{1-\gamma}\check{\Psi}'(f(\lambda))f(\lambda)\Delta\lambda.
\end{aligned}$$

This concludes the proof. □

Remark 5. *In Theorem 20, if we make $\check{\Psi}(\eta) = \eta^r$, $\check{\xi}(\eta) = \theta(\eta)$ and $g(\eta) = 1$, then inequality (36) reduces to inequality (21).*

Remark 6. *In Theorem 20, if we make $\check{\Psi}(\eta) = \eta^r$, $\check{\xi}(\eta) = \theta(\eta) = g(\eta) = 1$, $r = \gamma$ and $a = 0$, then we reclaim Theorem 13.*

Corollary 10. *If $\mathbb{T} = \mathbb{R}$ in Theorem 20, then, inequality (36) boils down to*

$$\int_a^\infty \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_0^\eta \theta(\zeta)f(\zeta)d\zeta\right)}{\left(\int_0^\eta \check{\xi}(\zeta)d\zeta\right)^\gamma}d\eta \geq \frac{1}{\gamma-1}\int_a^\infty \frac{\theta(\eta)g(\eta)\check{\Psi}'\left(\int_0^\eta \theta(\zeta)d\zeta\right)\check{\Psi}'(f(\eta))f(\eta)}{\left(\int_0^\eta \check{\xi}(\zeta)d\zeta\right)^{\gamma-1}}d\eta.$$

Corollary 11. *If $\mathbb{T} = h\mathbb{Z}$ in Theorem 20, then, inequality (36) boils down to*

$$\sum_{n=\frac{a}{h}}^\infty \frac{\check{\xi}(nh)g(nh)\check{\Psi}\left(h\sum_{m=0}^{\frac{n}{h}-1}\theta(mh)f(mh)\right)}{\left(h\sum_{m=0}^{\frac{n}{h}-1}\check{\xi}(mh)\right)^\gamma} \geq \frac{1}{\gamma-1}\sum_{n=\frac{a}{h}}^\infty \frac{\theta(nh)g(nh)\check{\Psi}'\left(h\sum_{m=0}^{\frac{n}{h}-1}\theta(mh)\right)\check{\Psi}'(f(nh))f(nh)}{\left(h\sum_{m=0}^{\frac{n}{h}-1}\check{\xi}(mh)\right)^{\gamma-1}}.$$

Corollary 12. *In Corollary 11, if we take $\mathbb{T} = \mathbb{Z}$, and inequality (36) abbreviates to*

$$\sum_{n=a}^\infty \frac{\check{\xi}(n)g(n)\check{\Psi}\left(\sum_{m=0}^{n-1}\theta(m)f(m)\right)}{\left(\sum_{m=0}^{n-1}\check{\xi}(m)\right)^\gamma} \geq \frac{1}{\gamma-1}\sum_{n=a}^\infty \frac{\theta(n)g(n)\check{\Psi}'\left(\sum_{m=0}^{n-1}\theta(m)\right)\check{\Psi}'(f(n))f(n)}{\left(\sum_{m=0}^{n-1}\check{\xi}(m)\right)^{\gamma-1}}.$$

Corollary 13. *If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ in Theorem 20, and inequality (36) abbreviates to*

$$\sum_{n=(\log_q a)}^{\infty} \frac{q^n \check{\xi}(q^n) g(q^n) \check{\Psi}\left((q-1) \sum_{m=0}^{(\log_q n)-1} q^m h(q^m) f(q^m) \right)}{\left((q-1) \sum_{m=0}^{(\log_q n)-1} q^m \check{\xi}(q^m) \right)^{\gamma}}$$
$$\geq \frac{1}{\gamma - 1} \sum_{n=(\log_q a)}^{\infty} \frac{q^n h(q^n) g(q^n) \check{\Psi}'\left((q-1) \sum_{m=0}^{(\log_q n)-1} q^m h(q^m) \right) \check{\Psi}'(f(q^n)) f(q^n)}{\left((q-1) \sum_{m=0}^{(\log_q n)-1} q^m \check{\xi}(q^m) \right)^{\gamma - 1}}.$$

Now, as a new result, we are interested in discussing the results in Theorem (20) in the case of the extrema of integration $\int_a^{\eta} \check{\xi} \Delta s$ being replaced to be from η to ∞. In fact, that is exactly what we shall accomplish in the next theorem.

Theorem 21. *Suppose that $\mathbb{T}$ is a time scale with $0 \leq a \in \mathbb{T}$. Moreover, assume that f, g, θ and $\check{\xi}$ are nonnegative functions defined on $[0, \infty)_{\mathbb{T}}$ such that f is nonincreasing and g is nondecreasing. Moreover, let $\check{\Psi} : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function such that $\check{\Psi}'$ is nondecreasing and $\check{\Psi}'(xy) = \check{\Psi}'(x)\check{\Psi}'(y)$ for all $x, y \in \mathbb{R}_+$. If $0 \leq \gamma < 1$, then*

$$\int_a^{\infty} \frac{\check{\xi}(\eta) g(\eta) \check{\Psi}\left(\int_0^{\eta} \theta(\zeta) f(\zeta) \Delta\zeta \right)}{\left(\int_{\sigma(\eta)}^{\infty} \check{\xi}(\zeta) \Delta\zeta \right)^{\gamma}} \Delta\eta \geq \frac{1}{1-\gamma} \int_a^{\infty} \frac{\theta(\eta) g(\eta) \check{\Psi}'\left(\int_0^{\eta} \theta(\zeta) \Delta\zeta \right) \check{\Psi}'(f(\eta)) f(\eta)}{\left(\int_{\eta}^{\infty} \check{\xi}(\zeta) \Delta\zeta \right)^{\gamma-1}} \Delta\eta. \tag{42}$$

Proof. Due to nonincreasity of f, we have for $\eta \geq \lambda \geq 0$

$$\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \geq f(\lambda) \int_0^{\lambda} \theta(\zeta) \Delta\zeta,$$

and thus,

$$\check{\Psi}\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right) \geq \check{\Psi}\left(f(\lambda) \int_0^{\lambda} \theta(\zeta) \Delta\zeta \right) = \check{\Psi}(f(\lambda)) \check{\Psi}\left(f(\lambda) \int_0^{\lambda} \theta(\zeta) \Delta\zeta \right). \tag{43}$$

Applying the chain rule (23), there exists $c \in [\lambda, \sigma(\lambda)]$ such that

$$\left[\check{\Psi}\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right) \right]^{\Delta} = \check{\Psi}'\left(\int_0^{c} \theta(\zeta) f(\zeta) \Delta\zeta \right) \left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right)^{\Delta}.$$

Since $c \geq \lambda$, $\check{\Psi}'$ is nondecreasing, $r \geq 1$ and $\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right)^{\Delta} = \theta(\lambda) f(\lambda) \geq 0$, we get

$$\left[\check{\Psi}\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right) \right]^{\Delta} \geq \theta(\lambda) f(\lambda) \check{\Psi}'\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right). \tag{44}$$

Combining (43) with (44) gives

$$\left[\check{\Psi}\left(\int_0^{\lambda} \theta(\zeta) f(\zeta) \Delta\zeta \right) \right]^{\Delta} \geq \theta(\lambda) \check{\Psi}'\left(\int_0^{\lambda} \theta(\zeta) \Delta\zeta \right) \check{\Psi}'(f(\lambda)) f(\lambda),$$

and then

$$\frac{\check{\xi}(\eta)g(\eta)\left[\check{\Psi}\left(\int_0^\lambda \theta(\zeta)f(\zeta)\Delta\zeta\right)\right]^\Delta}{\left(\int_{\sigma(\eta)}^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\gamma} \geq \frac{\check{\xi}(\eta)g(\eta)\theta(\lambda)\check{\Psi}'\left(\int_0^\lambda \theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_{\sigma(\eta)}^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\gamma}.$$

Since g is nondecreasing and $\lambda \leq \eta$, we have $g(\lambda) \leq g(\eta)$ and thus,

$$\frac{\check{\xi}(\eta)g(\eta)\left[\check{\Psi}\left(\int_0^\lambda \theta(\zeta)f(\zeta)\Delta\zeta\right)\right]^\Delta}{\left(\int_{\sigma(\eta)}^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\gamma} \geq \frac{\check{\xi}(\eta)g(\lambda)\theta(\lambda)\check{\Psi}'\left(\int_0^\lambda \theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_{\sigma(\eta)}^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\gamma}.$$

Therefore,

$$\frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_0^\eta \theta(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_{\sigma(\eta)}^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\gamma} \geq \int_a^\eta \frac{\check{\xi}(\eta)\theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_0^\lambda \theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)}{\left(\int_{\sigma(\eta)}^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\gamma}\Delta\lambda.$$

Hence,

$$\begin{aligned}&\int_a^\infty \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_0^\eta \theta(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_{\sigma(\eta)}^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\gamma}\Delta\eta \\ &\geq \int_a^\infty \check{\xi}(\eta)\left(\int_{\sigma(\eta)}^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}\left(\int_a^\eta \theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_0^\lambda \theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)\Delta\lambda\right)\Delta\eta. \quad (45)\end{aligned}$$

Equation (45) can be reformulated as follows by using Fubini's theorem on time scales:

$$\begin{aligned}&\int_a^\infty \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_0^\eta \theta(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_{\sigma(\eta)}^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\gamma}\Delta\eta \\ &\geq \int_a^\infty \theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_0^\lambda \theta(\zeta)\Delta\zeta\right)\check{\Psi}'(f(\lambda))f(\lambda)\left(\int_\lambda^\infty \check{\xi}(\eta)\left(\int_{\sigma(\eta)}^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}\Delta\eta\right)\Delta\lambda. \quad (46)\end{aligned}$$

By recalling the chain rule (23), we can say there exists $c \in [\eta, \sigma(\eta)]$ such that

$$\left[-\left(\int_\eta^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{1-\gamma}\right]^\Delta = -(1-\gamma)\left(\int_c^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}\left(\int_\eta^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\Delta.$$

Since $c \leq \sigma(\eta)$, $0 \leq \gamma < 1$ and $\left(\int_\eta^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\Delta = -\check{\xi}(\eta) \leq 0$, we get

$$\left[-\left(\int_\eta^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{1-\gamma}\right]^\Delta \leq (1-\gamma)\check{\xi}(\eta)\left(\int_{\sigma(\eta)}^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{-\gamma}. \quad (47)$$

Substituting (47) into (46) leads to

$$\int_a^\infty \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_0^\eta \theta(\zeta)f(\zeta)\Delta\zeta\right)}{\left(\int_{\sigma(\eta)}^\infty \check{\xi}(\zeta)\Delta\zeta\right)^\gamma}\Delta\eta$$
$$\geq \frac{1}{1-\gamma}\int_a^\infty \theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_0^\lambda \theta(\zeta)\Delta\zeta\right)\Psi'(f(\lambda))f(\lambda)\left(\int_\lambda^\infty \left[-\left(\int_\eta^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{1-\gamma}\right]^\Delta \Delta\eta\right)\Delta\lambda$$
$$= \frac{1}{1-\gamma}\int_a^\infty \theta(\lambda)g(\lambda)\check{\Psi}'\left(\int_0^\lambda \theta(\zeta)\Delta\zeta\right)\Psi'(f(\lambda))f(\lambda)\left(\int_\lambda^\infty \check{\xi}(\zeta)\Delta\zeta\right)^{1-\gamma}\Delta\lambda,$$

which is our desired inequality (42). □

Remark 7. *In Theorem 21, if we take* $\check{\Psi}(\eta) = \eta^r$, $\check{\xi}(\eta) = \theta(\eta)$ *and* $g(\eta) = 1$, *then inequality (42) reduces to inequality (22).*

Corollary 14. *If* $\mathbb{T} = \mathbb{R}$ *in Theorem 21, and by considering, inequality (42) abbreviates to*

$$\int_a^\infty \frac{\check{\xi}(\eta)g(\eta)\check{\Psi}\left(\int_0^\eta \theta(\zeta)f(\zeta)d\zeta\right)}{\left(\int_\eta^\infty \check{\xi}(\zeta)d\zeta\right)^\gamma}d\eta \geq \frac{1}{1-\gamma}\int_a^\infty \frac{\theta(\eta)g(\eta)\check{\Psi}'\left(\int_0^\eta \theta(\zeta)d\zeta\right)\Psi'(f(\eta))f(\eta)}{\left(\int_\eta^\infty \check{\xi}(\zeta)d\zeta\right)^{\gamma-1}}d\eta.$$

Corollary 15. *If* $\mathbb{T} = h\mathbb{Z}$ *in Theorem 21, and by considering, inequality (42) abbreviates to*

$$\sum_{n=\frac{a}{h}}^\infty \frac{\check{\xi}(nh)g(nh)\check{\Psi}\left(h\sum_{m=0}^{\frac{n}{h}-1}\theta(mh)f(mh)\right)}{\left(h\sum_{m=\frac{n}{h}+1}^\infty \check{\xi}(mh)\right)^\gamma} \geq \frac{1}{1-\gamma}\sum_{n=\frac{a}{h}}^\infty \frac{\theta(nh)g(nh)\check{\Psi}'\left(h\sum_{m=0}^{\frac{a}{h}-1}\theta(mh)\right)\Psi'(f(nh))f(nh)}{\left(h\sum_{m=\frac{n}{h}}^\infty \check{\xi}(mh)\right)^{\gamma-1}}.$$

Corollary 16. *In Corollary 15, if we take* $h = 1$, *then, inequality (42) boils down to*

$$\sum_{n=a}^\infty \frac{\check{\xi}(n)g(n)\check{\Psi}\left(\sum_{m=0}^{n-1}\theta(m)f(m)\right)}{\left(\sum_{m=n+1}^\infty \check{\xi}(m)\right)^\gamma} \geq \frac{1}{1-\gamma}\sum_{n=a}^\infty \frac{\theta(n)g(n)\check{\Psi}'\left(\sum_{m=0}^{n-1}\theta(m)\right)\Psi'(f(n))f(n)}{\left(\sum_{m=n}^\infty \check{\xi}(m)\right)^{\gamma-1}}.$$

Corollary 17. *If* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Theorem 21, and by considering, inequality (42) abbreviates to*

$$\sum_{n=(\log_q a)}^\infty \frac{q^n\check{\xi}(q^n)g(q^n)\check{\Psi}\left((q-1)\sum_{m=0}^{(\log_q n)-1}q^m h(q^m)f(q^m)\right)}{\left((q-1)\sum_{m=(\log_q n)+1}^\infty q^m\check{\xi}(q^m)\right)^\gamma}$$
$$\geq \frac{1}{1-\gamma}\sum_{n=(\log_q a)}^\infty \frac{q^n h(q^n)g(q^n)\check{\Psi}'\left((q-1)\sum_{m=0}^{(\log_q n)-1}q^m h(q^m)\right)\Psi'(f(q^n))f(q^n)}{\left((q-1)\sum_{m=(\log_q n)}^\infty q^m\check{\xi}(q^m)\right)^{\gamma-1}}.$$

3. Conclusions

In this paper, with the help of Fubini's theorem as well as a straightforward outcome of Keller's chain rule on time scales, we generalized some reverse Hardy-type inequalities to a general time scale. Moreover, we generalized a number of other inequalities to a general time scale. We obtained the discrete and the continuous inequalities as special cases of our main results.

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