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Sharp Bounds for the Second Hankel Determinant of Logarithmic Coefficients for Strongly Starlike and Strongly Convex Functions

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Abstract: The logarithmic coefficients are very essential in the problems of univalent functions theory. The importance of the logarithmic coefficients is due to the fact that the bounds on logarithmic coefficients of f can transfer to the Taylor coefficients of univalent functions themselves or to their powers, via the Lebedev–Milin inequalities; therefore, it is interesting to investigate the Hankel determinant whose entries are logarithmic coefficients. The main purpose of this paper is to obtain the sharp bounds for the second Hankel determinant of logarithmic coefficients of strongly starlike functions and strongly convex functions.

Keywords: logarithmic coefficient; Hankel determinant; strongly starlike; strongly convex

MSC: 30C45; 30C50

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1. Introduction

Let $\mathcal{A}$ stand for the standard class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}, \tag{1}$$

and let $\mathcal{S}$ be the class of functions in $\mathcal{A}$, which are univalent in $\mathbb{U}$.

A function f of the form (1) is said to be *starlike of order* α in $\mathbb{U}$ if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathbb{U}).$$

The set of all such functions is denoted by $\mathcal{S}^*(\alpha)$.

Next, by $\mathcal{K}(\alpha)$, we denote the class of *convex functions of order* α in $\mathbb{U}$ that satisfy the following inequality:

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathbb{U}).$$

A function f of the form (1) is said to be *strongly starlike of order* α, $(0 < \alpha \leq 1)$, in $\mathbb{U}$ if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi\alpha}{2} \qquad (z \in \mathbb{U}). \tag{2}$$

The set of all such functions is denoted by $\mathcal{S}_s^*(\alpha)$. Moreover, a function f of the form (1) is said to be *strongly convex of order* α, $(0 < \alpha \leq 1)$, in $\mathbb{U}$ if

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi\alpha}{2} \qquad (z \in \mathbb{U}). \tag{3}$$

The set of all such functions is denoted by $\mathcal{K}_c(\alpha)$.

The class $\mathcal{S}_s^*(\alpha)$ was independently introduced by Brannan and Kirwan [1] and Stankiewicz [2] (see also [3]). Clearly, $\mathcal{S}_s^*(1) = \mathcal{S}^*$ is the class of starlike functions and $\mathcal{K}_c^*(1) = \mathcal{K}$ is the class of convex functions in $\mathbb{U}$. We should observe that as α increases the sets $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ become smaller; however as α increases the sets $\mathcal{S}_s^*(\alpha)$ and $\mathcal{K}_c(\alpha)$ become larger. Furthermore, although the sharp coefficient bounds of the functions in the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ are known, sharp coefficient bounds for the functions in the sets $\mathcal{S}_s^*(\alpha)$ and $\mathcal{K}_c(\alpha)$ are much harder to obtain, and only partial results are known [1,4].

Let $\mathcal{P}$ denote the class of analytic functions $p(z)$ in $\mathbb{U}$ satisfying $p(0) = 1$ and $\Re\big(p(z)\big) > 0$. Thus, if $p \in \mathcal{P}$, then have the following form:

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \qquad z \in \mathbb{U}. \tag{4}$$

Functions in $\mathcal{P}$ are called *Carathedory functions*.

Associated with each $f \in \mathcal{S}$, is a well-defined logarithmic function

$$\mathcal{F}_f := \log\frac{f(z)}{z} = 2\sum_{k=1}^{\infty} \gamma_k z^k, \qquad z \in \mathbb{U}. \tag{5}$$

The numbers γ_k are called the *logarithmic coefficients of* f. The logarithmic coefficients are very essential in the problems of univalent functions coefficients. The importance of the logarithmic coefficients is due to the fact that the bounds on logarithmic coefficients of f can transfer to the Taylor coefficients of univalent functions themselves or to their powers, via the Lebedev–Milin inequalities.

Relatively little exact information is known about the logarithmic coefficients of f when $f \in \mathcal{S}$. The logarithmic coefficients of the Koebe function $\mathcal{K}(z) = z(1-z)^{-2}$ are $\gamma_k = 1/k$. Because of the extremal properties of the Koebe function, one could expect that $\gamma_k \leq 1/k$, for each $f \in \mathcal{S}$; however, this conjecture is false even in the case $k = 2$. For the whole class $\mathcal{S}$, the sharp estimates of single logarithmic coefficients are known only for

$$|\gamma_1| \leq 1 \qquad \text{and} \qquad |\gamma_2| \leq \frac{1}{2} + \frac{1}{e^2} = 0.6353\ldots$$

and are unknown for $k \geq 3$. Recently, logarithmic coefficients have been studied by various authors and upper bounds of logarithmic coefficients of functions in some important subclasses of $\mathcal{S}$ have been obtained (e.g., [5–10]). For a summary of some of the significant results concerning the logarithmic coefficients for univalent functions, we refer to [11].

For $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of $f \in \mathcal{A}$ of form (1) is defined as

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

The Hankel determinant $H_{2,1}(f) = a_3 - a_2^2$ is the well-known Fekete–Szegö functional. The second Hankel determinant $H_{2,2}(f)$ is given by $H_{2,2}(f) = a_2 a_4 - a_3^2$.

The problem of computing the upper bound of $H_{q,n}$ over various subfamilies of $\mathcal{A}$ is interesting and widely studied in the literature on the geometric function theory of complex analysis. The upper bounds of $H_{2,2}$, $H_{3,1}$ and higher-order Hankel determinants for subclasses of analytic functions were obtained by various authors [12–24].

Very recently, Kowalczyk and Lecko [25] introduced the Hankel determinant $H_{q,n}(F_f/2)$, which are logarithmic coefficients of f, i.e.,

$$H_{q,n}(F_f/2) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

For a function $f \in \mathcal{S}$ given in (1), by differentiating (5) one can obtain the following:

$$\gamma_1 = \frac{1}{2}a_2, \ \gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \ \gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \tag{6}$$

Therefore, the second Hankel determinant of $F_f/2$ can be obtained by

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right). \tag{7}$$

Furthermore, if $f \in \mathcal{S}$, then for

$$f_\theta(z) = e^{-i\theta} f(e^{i\theta}z) \qquad (\theta \in \mathbb{R}),$$

we find that (see [26])

$$H_{2,1}\left(\frac{F_{f_\theta}}{2}\right) = e^{4i\theta} H_{2,1}\left(\frac{F_f}{2}\right).$$

Kowalczyk and Lecko [26] obtained sharp bounds for $H_{2,1}(F_f/2)$ for the classes of starlike and convex functions of order α. The problem of computing the sharp bounds of $H_{2,1}(F_f/2)$ for starlike and convex functions with respect to symmetric points in the open unit disk has been considered by Allu and Arora [27].

In this paper, we calculate the sharp bounds for $H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2$ for the classes $\mathcal{S}_s^*(\alpha)$ and $\mathcal{K}_c(\alpha)$.

To establish our main results, we will require the following Lemmas:

Lemma 1 ([28] (see also [26])). *If $p \in \mathcal{P}$ is of the form (4) with $c_1 \geq 0$, then*

$$\begin{aligned} c_1 &= 2d_1, \\ c_2 &= 2d_1^2 + 2(1-d_1^2)d_2, \\ c_3 &= 2d_1^3 + 4(1-d_1^2)d_1d_2 - 2(1-d_1^2)d_1d_2^2 + 2(1-d_1^2)(1-|d_2|^2)d_3 \end{aligned} \tag{8}$$

for some $d_1 \in [0,1]$ and $d_2, d_3 \in \overline{\mathbb{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$.

For $d_1 \in \mathbb{U}$ and $d_2 \in \partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (8), namely

$$p(z) = \frac{1 + (\overline{d_1}d_2 + d_1)z + d_2z^2}{1 + (\overline{d_1}d_2 - d_1)z - d_2z^2}, \qquad z \in \mathbb{U}.$$

Lemma 2 ([29]). *Given real numbers A, B, C, let*

$$Y(A,B,C) = \max\left\{\left|A + Bz + Cz^2\right| + 1 - |z|^2 : \ z \in \overline{\mathbb{U}}\right\}.$$

I. *If* $AC \geq 0$, *then*

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1-|C|), \\ 1 + |A| + \frac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|). \end{cases}$$

II. *If* $AC < 0$, *then*

$$Y(A,B,C) = \begin{cases} 1 - |A| + \frac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1) \leq B^2 \ \wedge |B| < 2(1-|C|), \\ 1 + |A| + \frac{B^2}{4(1+|C|)}, & B^2 < \min\{4(1+|C|)^2, \ -4AC(C^{-2}-1)\}, \\ R(A,B,C), & \textit{otherwise}. \end{cases}$$

where

$$R(A,B,C) = \begin{cases} |A| + |B| - |C|, & |C|(|B| + 4|A|) \leq |AB|, \\ -|A| + |B| + |C|, & |AB| \leq |C|(|B| - 4|A|), \\ (|A| + |C|)\sqrt{1 - \frac{B^2}{4AC}}, & \textit{otherwise}. \end{cases}$$

2. Second Hankel Determinant of Logarithmic Coefficients for the Class $\mathcal{S}_s^*(\alpha)$

Theorem 1. *Let* $\alpha \in (0,1]$. *If* $f \in \mathcal{S}_s^*(\alpha)$, *then*

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \frac{\alpha^2}{4}. \tag{9}$$

This inequality is sharp. Equality holds for the function

$$f(z) = z exp \int_0^z \frac{(1-u^2)^{-2\alpha} - 1}{u}\, \mathrm{d}u, \qquad z \in \mathbb{U}. \tag{10}$$

Proof. Let $\alpha \in (0,1]$ and $f \in \mathcal{S}_s^*(\alpha)$ be of the form (1). Then by (2) we have

$$\frac{zf'(z)}{f(z)} = \big(p(z)\big)^\alpha, \qquad z \in \mathbb{U}, \tag{11}$$

for some function $p \in \mathcal{P}$ of the form (4). Since the class $\mathcal{P}$ and the functional $|H_{2,1}(F_f/2)|$ are rotationally invariant, we may assume that $c_1 \in [0,2]$ (i.e., in view of (8) that $d_1 \in [0,1]$). Equating the coefficients, we obtain

$$\begin{aligned} a_2 &= \alpha c_1 \\ a_3 &= \frac{\alpha}{2}\left(c_2 - \frac{1-3\alpha}{2}c_1^2\right) \\ a_4 &= \frac{\alpha}{3}\left(c_3 + \frac{5\alpha-2}{2}c_1c_2 + \frac{17\alpha^2-15\alpha+4}{12}c_1^3\right). \end{aligned} \tag{12}$$

Hence by using (6)–(8) we obtain

$$
\begin{aligned}
\gamma_1\gamma_3-\gamma_2^2 &= \frac{1}{4}\left(a_2a_4-a_3^2+\frac{1}{12}a_2^4\right)\\
&= \frac{\alpha^2}{576}\left[(7+\alpha)(1-\alpha)c_1^4-12(1-\alpha)c_1^2c_2+48c_1c_3-36c_2^2\right]\\
&= \frac{\alpha^2}{36}\bigg[(4-\alpha^2)d_1^4+6\alpha(1-d_1^2)d_1^2d_2-(1-d_1^2)\left[12d_1^2+9(1-d_1^2)\right]d_2^2\\
&\qquad +12(1-d_1^2)(1-|d_2|^2)d_1d_3\bigg].
\end{aligned} \tag{13}
$$

Now, we may have the following cases on d_1:
Case 1. Suppose that $d_1=1$. Then by (13) we obtain

$$\left|\gamma_1\gamma_3-\gamma_2^2\right|=\frac{\alpha^2}{36}(4-\alpha^2)$$

Case 2. Suppose that $d_1=0$. Then by (13) we obtain

$$\left|\gamma_1\gamma_3-\gamma_2^2\right|=\frac{\alpha^2}{4}|d_2|^2\leq\frac{\alpha^2}{4}.$$

Case 3. Suppose that $d_1\in(0,1)$. By the fact that $|d_3|\leq 1$, applying the triangle inequality to (13) we can write

$$
\begin{aligned}
\left|\gamma_1\gamma_3-\gamma_2^2\right| &= \left|\frac{\alpha^2(1-d_1^2)}{3}\left[\frac{4-\alpha^2}{12(1-d_1^2)}d_1^4+\frac{\alpha}{2}d_1^2d_2-\frac{12d_1^2+9(1-d_1^2)}{12}d_2^2+(1-|d_2|^2)d_1d_3\right]\right|\\
&\leq \frac{\alpha^2d_1(1-d_1^2)}{3}\left[\left|\frac{4-\alpha^2}{12(1-d_1^2)}d_1^3+\frac{\alpha}{2}d_1d_2-\frac{12d_1^2+9(1-d_1^2)}{12d_1}d_2^2\right|+1-|d_2|^2\right]\\
&= \frac{\alpha^2d_1(1-d_1^2)}{3}\left[\left|A+Bd_2+Cd_2^2\right|+1-|d_2|^2\right]
\end{aligned} \tag{14}
$$

where

$$A=\frac{4-\alpha^2}{12(1-d_1^2)}d_1^3 \qquad B=\frac{\alpha}{2}d_1 \qquad C=-\frac{d_1^2+3}{4d_1}.$$

Since $AC<0$, we apply Lemma 2 only for the case II.

We consider the following sub-cases.

3 (a) Since

$$-4AC\left(\frac{1}{C^2}-1\right)-B^2=\frac{(4-\alpha^2)d_1^2(d_1^2+3)}{12(1-d_1^2)}\left(\frac{16d_1^2}{(d_1^2+3)^2}-1\right)-\frac{\alpha^2d_1^2}{4}\leq 0$$

equivalent to $(1-\alpha^2)d_1^2\leq 9$, which evidently holds for $d_1\in(0,1)$. Further, the inequality $|B|<2(1-|C|)$ is equivalent to $3+(1+\alpha)d_1^2-4d_1<0$ which is false for $d_1\in(0,1)$.

3 (b) Since

$$4(1+|C|)^2=\frac{(d_1^2+4d_1+3)^2}{4d_1^2}>0$$

and

$$-4AC\left(\frac{1}{C^2}-1\right)=\frac{(4-\alpha^2)d_1^2(d_1^2-9)}{12(d_1^2+3)}<0,$$

we see that the inequality

$$\frac{\alpha^2 d_1^2}{4} < \min\left\{4(1+|C|)^2,\ -4AC(\frac{1}{C^2}-1)\right\}$$

is false for $d_1 \in (0,1)$.

3 (c) The inequality

$$|C|\Big(|B|+4|A|\Big)-|AB| = \frac{(d_1^2+3)}{4d_1}\left(\frac{\alpha d_1}{2}+\frac{(4-\alpha^2)d_1^3}{3(1-d_1^2)}\right)-\frac{\alpha(4-\alpha^2)d_1^4}{24(1-d_1^2)} \leq 0,$$

is equivalent to

$$d^4(8+\alpha^3-2\alpha^2-7\alpha)+d^2(24-6\alpha^2-6\alpha)+9\alpha \leq 0.$$

It is easy to verify that

$$\begin{aligned} &d^4(8+\alpha^3-2\alpha^2-7\alpha)+d^2(24-6\alpha^2-6\alpha)+9\alpha \\ &\qquad > d^4(32+\alpha^3-8\alpha^2-13\alpha)+9\alpha > 0. \end{aligned}$$

for $d_1 \in (0,1)$. Thus, the inequality $|C|\Big(|B|+4|A|\Big) \leq |AB|$ does not hold for $\alpha \in (0,1]$ and $d_1 \in (0,1)$.

3 (d) We can write

$$\begin{aligned} |AB|-|C|\Big(|B|-4|A|\Big) &= \frac{\alpha(4-\alpha^2)d_1^4}{24(1-d_1^2)}-\frac{(d_1^2+3)}{4d_1}\left(\frac{\alpha d_1}{2}-\frac{(4-\alpha^2)d_1^3}{3(1-d_1^2)}\right) \\ &= \frac{1}{24(1-t)}\big(K_1t^2+L_1t+M_1\big) \end{aligned}$$

where $t = d_1^2 \in (0,1)$ and

$$\begin{aligned} K_1 &= -\alpha^3-2\alpha^2+7\alpha+8 \\ L_1 &= 6(4+\alpha-\alpha^2) \\ M_1 &= -9\alpha. \end{aligned}$$

It is easy to see that $K_1 > 0$, $L_1 > 0$ and $M_1 < 0$, for $\alpha \in (0,1]$.

For the equation $K_1t^2+L_1t+M_1$, we have $\Delta = 144(4+4\alpha-\alpha^3) > 0$. Since $K_1 > 0$, $\frac{M_1}{K_1} < 0$ and $K_1+L_1+M_1 = 32-\alpha^3-8\alpha^2+4\alpha > 0$, for $\alpha \in (0,1]$, the equation $K_1t^2+L_1t+M_1$ has positive unique root such that

$$0 < t_1 = \frac{-L_1+\sqrt{\Delta}}{2K_1} < 1,$$

Therefore, for $d_1^* = \sqrt{t_1}$, it follows that $|AB| = |C|\Big(|B|-4|A|\Big)$.

Moreover, $|AB| \leq |C|\Big(|B|-4|A|\Big)$, when $d_1 \in (0,d_1^*]$, and $|AB| \geq |C|\Big(|B|-4|A|\Big)$, when $d_1 \in [d_1^*,1)$.

Then for $d_1 \in (0,d_1^*]$, we can write from (14) and Lemma 2, we obtain

$$\left|\gamma_1\gamma_3-\gamma_2^2\right| \leq \frac{\alpha^2 d_1(1-d_1^2)}{3}\Big(-|A|+|B|+|C|\Big) = \Phi(d_1)$$

where

$$\Phi(d_1) = \frac{\alpha^2}{36}\Big(-(4-\alpha^2)d_1^4 + 3(1+2\alpha)d_1^2(1-d_1^2) + 9(1-d_1^2)\Big).$$

Since

$$\Phi'(d_1) = \frac{-\alpha^2 d_1}{9}\Big[(7+6\alpha-\alpha^2)d_1^2 + 3(1-\alpha)\Big] < 0,$$

for $d_1 \in [0, d_1^*]$, Φ is a decreasing function on $[0, d_1^*]$. This implies that

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \Phi(0) = \frac{\alpha^2}{4}.$$

3 (e) Next consider the case $d_1 \in [d_1^*, 1]$. Using the last case of Lemma 2,

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \frac{\alpha^2 d_1(1-d_1^2)}{3}\Big((|A|+|C|)\sqrt{1-\frac{B^2}{4AC}}\Big) = \Psi(d_1)$$

where

$$\Psi(d_1) = \frac{\alpha^2}{18}[9 + (1-\alpha^2)d_1^4 - 6d_1^2]\sqrt{\frac{(1-\alpha^2)d_1^2+3}{(4-\alpha^2)(d_1^2+3)}}.$$

To find the maximum of the function $\Psi(d_1)$ on the interval $d_1 \in [d_1^*, 1]$, let us investigate the derivative of $\Psi(d_1)$:

$$\begin{aligned}\Psi'(d_1) = {} & \frac{-d_1^2\alpha^2}{18(4-\alpha^2)(d_1^2+3)^2}\sqrt{\frac{(4-\alpha^2)(d_1^2+3)}{(1-\alpha^2)d_1^2+3}} \\ & \times \Big[4(3-(1-\alpha^2)d_1^2)(d_1^2+3)((1-\alpha^2)d_1^2+3) + 3\alpha^2(9+(1-\alpha^2)d_1^4 - 6d_1^2))\Big] < 0,\end{aligned}$$

since

$$4(3-(1-\alpha^2)d_1^2 \ge 8 + 4\alpha^2 > 0$$

and

$$9 + (1-\alpha^2)d_1^4 - 6d_1^2 \ge 9 - d_1^2(6 - (1-\alpha^2)d_1^2) = 3 + (1-\alpha^2)d_1^2 > 0$$

for $\alpha \in (0,1]$ and $d_1 \in [d_1^*, 1]$. Thus Ψ is a decreasing function on $[d_1^*, 1]$.

Furthermore, $\Phi(d_1^*) = \Psi(d_1^*)$. This implies that

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \Psi(d_1) \le \Psi(d_1^*) = \Phi(d_1^*) \le \Phi(0) = \frac{\alpha^2}{4}.$$

Summarizing parts from Cases 1–3, it follows the desired inequality.

In order to show that the inequality is sharp, let us set $c_1 = 0$ and $d_2 = 1$ into (8). Then, we obtain $c_2 = 2$ and $c_3 = 0$. Hence by (12) we have $a_2 = a_4 = 0$ and $a_3 = \alpha$. This shows that equality is attained for the function given in (10).

This completes the proof of the theorem. □

For $\alpha = 1$ we obtain the bounds for the class $\mathcal{S}^*$ of starlike functions given in [25].

Corollary 1. *Let $f(z) \in \mathcal{S}^*$. Then*

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \frac{1}{4}.$$

The inequality is sharp.

3. Second Hankel Determinant of Logarithmic Coefficients for the Class $\mathcal{K}_c(\alpha)$

Theorem 2. *Let $\alpha \in (0,1]$. If $f \in \mathcal{K}_c(\alpha)$, then*

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \begin{cases} \frac{\alpha^2}{36}, & 0 < \alpha \leq \frac{1}{3} \\ \frac{\alpha^2(13\alpha^2+18\alpha+17)}{144(\alpha^2+6\alpha+4)}, & \frac{1}{3} < \alpha \leq 1. \end{cases} \tag{15}$$

The inequalities in (15) are sharp.

Proof. Let $\alpha \in (0,1]$ and $f \in \mathcal{K}_c(\alpha)$ be of the form (1). Then, by (3), we have

$$1 + \frac{zf''(z)}{f'(z)} = \left(p(z)\right)^\alpha, \qquad z \in \mathbb{U}, \tag{16}$$

for some function $p \in \mathcal{P}$ of the form (4). As in the proof of Theorem 1, we may assume that $c_1 \in [0,2]$ (i.e., in view of (8) that $d_1 \in [0,1]$). Equating the coefficients, we obtain

$$\begin{aligned} a_2 &= \frac{\alpha}{2}c_1 \\ a_3 &= \frac{\alpha}{6}\left(c_2 - \frac{1-3\alpha}{2}c_1^2\right) \\ a_4 &= \frac{\alpha}{144}\left((17\alpha^2 - 15\alpha + 4)c_1^3 + 6(5\alpha - 2)c_1c_2 + 12c_3\right). \end{aligned} \tag{17}$$

Hence, by using (6)–(8) we obtain

$$\begin{aligned} \gamma_1\gamma_3 - \gamma_2^2 &= \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) \\ &= \frac{\alpha^2}{2304}\left[(\alpha^2 - 6\alpha + 4)c_1^4 + 4(3\alpha - 2)c_1^2c_2 + 24c_1c_3 - 16c_2^2\right] \\ &= \frac{\alpha^2}{144}\Big[(2+\alpha^2)d_1^4 + 6\alpha(1-d_1^2)d_1^2d_2 - (1-d_1^2)\left[6d_1^2 + 4(1-d_1^2)\right]d_2^2 \\ &\qquad + 6(1-d_1^2)(1-|d_2|^2)d_1d_3\Big]. \end{aligned} \tag{18}$$

Now, we may have the following cases on d_1:

Case 1. Suppose that $d_1 = 1$. Then, by (18) we obtain

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \frac{\alpha^2}{144}(2+\alpha^2)$$

Case 2. Suppose that $d_1 = 0$. Then, by (18) we obtain

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \frac{\alpha^2}{36}|d_2|^2 \leq \frac{\alpha^2}{36}.$$

Case 3. Suppose that $d_1 \in (0,1)$. By the fact that $|d_3| \leq 1$, applying the triangle inequality to (18) we can write

$$
\begin{aligned}
\left|\gamma_1\gamma_3-\gamma_2^2\right| &= \left|\frac{\alpha^2}{144}\Big[(2+\alpha^2)d_1^4+6\alpha(1-d_1^2)d_1^2d_2\right.\\
&\quad\left.-(1-d_1^2)\big[6d_1^2+4(1-d_1^2)\big]d_2^2+6(1-d_1^2)(1-|d_2|^2)d_1d_3\Big]\right|\\
&\le \frac{\alpha^2 d_1(1-d_1^2)}{24}\left[\left|\frac{(2+\alpha^2)}{6(1-d_1^2)}d_1^3+\alpha d_1d_2-\frac{4+2d_1^2}{6d_1}d_2^2\right|+1-|d_2|^2\right]\\
&= \frac{\alpha^2 d_1(1-d_1^2)}{24}\left[\left|A+Bd_2+Cd_2^2\right|+1-|d_2|^2\right]
\end{aligned} \tag{19}
$$

where

$$
A=\frac{2+\alpha^2}{6(1-d_1^2)}d_1^3 \qquad B=\alpha d_1 \qquad C=-\frac{2+d_1^2}{3d_1}.
$$

Since $AC<0$, we apply Lemma 2 only for the case II.

We consider the following sub-cases.

3 (a) Note that

$$
\begin{aligned}
-4AC\left(\frac{1}{C^2}-1\right)-B^2 &= \frac{-d_1^2\big[d_1^2(7\alpha^2-4)+26\alpha^2+16\big]}{9(d_1^2+2)}\\
&= \frac{-d_1^2\big[\alpha^2(7d_1^2+26)+4(4-d_1^2)\big]}{9(d_1^2+2)} \le 0.
\end{aligned}
$$

for $d_1\in(0,1)$ and $\alpha\in(0,1]$. On the other hand, we have

$$
|B|-2(1-|C|)=\frac{d_1^2(3\alpha+2)-6d_1+4}{3d_1}.
$$

Since $\Delta=4(1-12\alpha)\le 0$ for $\frac{1}{12}\le\alpha<1$, we have

$$
d_1^2(3\alpha+2)-6d_1+4\ge 0.
$$

Further, since $\Delta=4(1-12\alpha)>0$ for $0<\alpha<\frac{1}{12}$, the equation

$$
d_1^2(3\alpha+2)-6d_1+4=0
$$

has the roots

$$
s_{1,2}=\frac{3\pm\sqrt{1-12\alpha}}{3\alpha+2}
$$

which are greater than 1. So

$$
d_1^2(3\alpha+2)-6d_1+4>0
$$

for $d_1\in(0,1)$ and $\alpha\in(0,1]$.

Consequently $|B|<2(1-|C|)$ does not hold for $d_1\in(0,1)$ and $\alpha\in(0,1]$.

3 (b) Since

$$
4(1+|C|)^2=\frac{4(d_1^2+3d_1+2)^2}{9d_1^2}>0
$$

and

$$
-4AC\left(\frac{1}{C^2}-1\right)=-\frac{2d_1^2(4-d_1^2)(\alpha^2+2)}{9(d_1^2+2)}<0,
$$

we see that the inequality

$$
\alpha^2 d_1^2<\min\left\{4(1+|C|)^2,\ -4AC\left(\frac{1}{C^2}-1\right)\right\}
$$

is false for $d_1 \in (0,1)$.

3 (c) We can write

$$|C|\Big(|B|+4|A|\Big)-|AB|=\frac{1}{18(1-d_1^2)}(K_2d_1^4+L_2d_1^2+M_2)$$

where

$$\begin{aligned}K_2&=-3\alpha^3+4\alpha^2-12\alpha+8\\L_2&=8\alpha^2-6\alpha+16\\M_2&=12\alpha.\end{aligned}$$

It is easy to see that $L_2>0$ and $M_2>0$, for $\alpha\in(0,1]$.

There are two cases according to the sign of K_2:

(i) If $K_2\geq 0$, then we have

$$|C|\Big(|B|+4|A|\Big)-|AB|=\frac{1}{18(1-d_1^2)}(K_2d_1^4+L_2d_1^2+M_2)>0.$$

(ii) If $K_2<0$, then using the fact that $\alpha\in(0,1]$ and $d_1\in(0,1)$, we can write

$$\begin{aligned}|C|\Big(|B|+4|A|\Big)-|AB|&=\frac{1}{18(1-d_1^2)}\big(K_2d_1^4+L_2d_1^2+M_2\big)\\&>\frac{1}{18(1-d_1^2)}\big(K_2+L_2d_1^2+M_2\big)\\&=\frac{1}{18(1-d_1^2)}\big(L_2d_1^2-3\alpha^3+4\alpha^2+8\big)\\&\geq\frac{1}{18(1-d_1^2)}\big(L_2d_1^2+5+4\alpha^2\big)>0.\end{aligned}$$

Therefore, the inequality $|C|\Big(|B|+4|A|\Big)\leq|AB|$ does not hold for $\alpha\in(0,1]$ and $d_1\in(0,1)$.

3 (d) We can write

$$\begin{aligned}|AB|-|C|\Big(|B|-4|A|\Big)&=\frac{\alpha(\alpha^2+2)}{6(1-d_1^2)}d_1^4-\frac{d_1^2+2}{3d_1}\left(\alpha d_1-4\frac{\alpha^2+2}{6(1-d_1^2)}d_1^3\right)\\&=\frac{1}{18(1-t)}\big(K_3t^2+L_3t+M_3\big)\end{aligned}$$

where $t=d_1^2\in(0,1)$ and

$$\begin{aligned}K_3&=3\alpha^3+4\alpha^2+12\alpha+8\\L_3&=8\alpha^2+6\alpha+16\\M_3&=-12\alpha.\end{aligned}$$

It is easy to see that $K_3>0$, $L_3>0$ and $M_3<0$, for $\alpha\in(0,1]$.

For the equation $K_3t^2+L_3t+M_3=0$, we have $\Delta>0$. Since $\frac{M_3}{K_3}<0$ and $K_3+L_3+M_3>0$, for $\alpha\in(0,1]$, the equation $K_3t^2+L_3t+M_3=0$ has a unique positive root $t_1<1$.

Thus, the inequality $|AB| - |C|\Big(|B| - 4|A|\Big) \leq 0$ holds for $(0, d_1^{**}]$, where $d_1^{**} = \sqrt{t_1}$. So we can write from (19) and Lemma 2,

$$\begin{aligned}\left|\gamma_1\gamma_3 - \gamma_2^2\right| &\leq \frac{\alpha^2 d_1(1-d_1^2)}{24}\Big(-|A| + |B| + |C|\Big) \\ &= \frac{\alpha^2}{144}\Phi_1(d_1)\end{aligned}$$

where

$$\Phi_1(d_1) = \Big(Dd_1^4 + Ed_1^2 + 4\Big),$$

and

$$\begin{aligned}D &= -(\alpha^2 + 6\alpha + 4) \\ E &= 6\alpha - 2.\end{aligned}$$

If $\Phi_1'(d_1) = 2d_1\big(2Dd_1^2 + E\big) = 0$, then $d_1^2 = -\frac{E}{2D}$. So if $E = 6\alpha - 2 > 0$, i.e., $\frac{1}{3} < \alpha \leq 1$, then we have a critical point:

$$\xi = \sqrt{-\frac{E}{2D}} = \sqrt{\frac{3\alpha - 1}{\alpha^2 + 6\alpha + 4}}. \tag{20}$$

Since

$$\begin{aligned}K_3\xi^4 + L_3\xi^2 + M_3 &= K_3\left(\frac{3\alpha-1}{\alpha^2+6\alpha+4}\right)^2 + L_3\left(\frac{3\alpha-1}{\alpha^2+6\alpha+4}\right) + M_3 \\ &= \frac{39\alpha^5 + 28\alpha^4 - 243\alpha^3 - 296\alpha^2 - 156\alpha - 56}{(\alpha^2+6\alpha+4)^2} \\ &\leq \frac{-243\alpha^3 - 296\alpha^2 - 89\alpha - 56}{(\alpha^2+6\alpha+4)^2} \\ &< 0,\end{aligned}$$

we have $0 < \xi < d_1^{**}$; therefore, we obtain

$$\begin{aligned}\left|\gamma_1\gamma_3 - \gamma_2^2\right| &\leq \frac{\alpha^2}{144}\Phi_1(\xi) \\ &= \frac{\alpha^2(13\alpha^2 + 18\alpha + 17)}{144(\alpha^2+6\alpha+4)},\end{aligned}$$

for $\frac{1}{3} < \alpha \leq 1$.

Furthermore, if $0 < \alpha \leq \frac{1}{3}$, then the function $\Phi_1(d_1)$ is decreasing on $(0, d_1^{**}]$. Thus we have

$$\begin{aligned}\left|\gamma_1\gamma_3 - \gamma_2^2\right| &\leq \frac{\alpha^2}{144}\Phi_1(d_1) \\ &\leq \frac{\alpha^2}{36}.\end{aligned}$$

3 (e) Next consider the case $d_1 \in [d_1^{**}, 1]$. Using the last case of the Lemma 2,

$$\left|\gamma_1\gamma_3-\gamma_2^2\right| \leq \frac{\alpha^2 d_1(1-d_1^2)}{24}\Big((|A|+|C|)\sqrt{1-\frac{B^2}{4AC}}\Big)$$
$$= \frac{\alpha^2}{144}\Psi_1(d_1)$$

where

$$\Psi_1(d_1) = (\alpha^2 d_1^4 - 2d_1^2 + 4)\sqrt{1+\frac{9\alpha^2(1-d_1^2)}{2(\alpha^2+2)(d_1^2+2)}}.$$

To find the maximum of the function $\Psi_1(d_1)$ on the interval $d_1 \in [d_1^{**}, 1]$, let us investigate the derivative of $\Psi_1(d_1)$:

$$\Psi_1'(d_1) = \frac{-d_1}{(\alpha^2+2)(d_1^2+2)^2}\sqrt{\frac{(\alpha^2+2)(d_1^2+2)}{(4-7\alpha^2)d_1^2+13\alpha^2+8}} \times$$
$$\left\{4(d_1^2+2)\left(1-\alpha^2 d_1^2\right)\left[\left(4-7\alpha^2\right)d_1^2+13\alpha^2+8\right]+\left(\alpha^2 d_1^4-2d_1^2+4\right)27\alpha^2\right\}.$$

Since for $d_1 \in [d_1^{**}, 1]$

$$\left(4-7\alpha^2\right)d_1^2+13\alpha^2+8 = \alpha^2(13-7d_1^2)+4(d_1^2+2) > 0$$

and

$$\left(\alpha^2 d_1^4 - 2d_1^2 + 4\right) = 4 - d_1^2(2-\alpha^2 d_1^2) \geq 4-(2-\alpha^2 d_1^2) = 2+\alpha^2 d_1^2 > 0,$$

for $\alpha \in (0,1]$ and $d_1 \in [d_1^{**}, 1]$. Thus $\Psi_1(d_1)$ is a decreasing function on the interval $[d_1^{**}, 1]$. This implies that

$$\left|\gamma_1\gamma_3-\gamma_2^2\right| \leq \frac{\alpha^2}{144}\Psi_1(d_1) \leq \frac{\alpha^2}{144}\Psi_1(d_1^{**}) = \frac{\alpha^2}{144}\Phi_1(d_1^{**}).$$

Summarizing parts from Cases 1–3, it follows the desired inequalities.

To show the sharpness for the case $0 < \alpha \leq \frac{1}{3}$, consider the function

$$p_1(z) = \frac{1-z^2}{1+z^2}, \qquad (z \in \mathbb{U}).$$

It is obvious that the function p_1 is in $\mathcal{P}$ with $c_1 = c_3 = 0$ and $c_2 = -2$. The corresponding function f_1 can be obtained from (16). Hence, by (17) we have $a_2 = a_4 = 0$ and $a_3 = -\frac{\alpha}{3}$. From (18) we obtain

$$\left|\gamma_1\gamma_3-\gamma_2^2\right| = \frac{\alpha^2}{36},$$

for $0 < \alpha \leq \frac{1}{3}$.

For the case $\frac{1}{3} < \alpha \leq 1$, consider the function

$$p_2(z) = \frac{1-z^2}{1-2\xi z+z^2}, \qquad (z \in \mathbb{U})$$

where ξ is given in (20). From Lemma 1, it is obvious that the function p_2 is in $\mathcal{P}$. The corresponding function f_2 can be obtained from (16), having the following coefficients:

$$\begin{aligned} a_2 &= \alpha\xi, \\ a_3 &= \frac{1}{3}\alpha\big((1+3\alpha)\xi^2 - 1\big), \\ a_4 &= \frac{1}{18}\alpha\xi\big((17\alpha^2+15\alpha+4)\xi^2 - 15\alpha - 3\big). \end{aligned}$$

Hence from (18) we obtain

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \frac{\alpha^2(13\alpha^2+18\alpha+17)}{144(\alpha^2+6\alpha+4)}.$$

This completes the proof.
□

For $\alpha = 1$ we obtain the bounds for the class $\mathcal{K}$ of convex functions given in [25].

Corollary 2. *Let $f(z) \in \mathcal{K}$. Then*

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \frac{1}{33}.$$

The inequality is sharp.

4. Discussion

In this work, we have obtained the sharp bounds for the second Hankel determinant of logarithmic coefficients of strongly starlike functions and strongly convex functions. Because of the importance of the logarithmic coefficients of univalent functions, our results provide a basis for research on the Hankel determinant of the logarithmic coefficients of the class of strongly starlike and strongly convex functions and other classes associated with these classes. Furthermore, our results could also inspire further studies taking other subclasses of $\mathcal{S}$ into consideration and/or obtaining the bounds for higher-order Hankel determinants.

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