axioms

MDPI

Article

Dirichlet Problem with $L^1(S)$ Boundary Values

Alexander G. Ramm

Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA; ramm@ksu.edu

Abstract: Let D be a connected bounded domain in $\mathbb{R}^2$, S be its boundary, which is closed and C^2-smooth. Consider the Dirichlet problem $\Delta u = 0$ in D, $u|_S = h$, where $h \in L^1(S)$. The aim of this paper is to prove that the above problem has a solution for an arbitrary $h \in L^1(S)$, and this solution is unique. The result is new. The method of its proof is new. The definition of the $L^1(S)$-boundary value of a harmonic in the D function is given. No embedding theorems are used. The history of the Dirichlet problem goes back to 1828. The result in this paper is, to the author's knowledge, the first result in the 194 years of research (since 1828) that yields the existence and uniqueness of the solution to the Dirichlet problem with the boundary values in $L^1(S)$.

Keywords: Dirichlet problem; $L^1(S)$ boundary values

MSC: 31A05; 35J25

check for updates

Citation: Ramm, A.G. Dirichlet Problem with $L^1(S)$ Boundary Values. *Axioms* **2022**, *11*, 371. https://doi.org/10.3390/axioms11080371

Academic Editor: Clemente Cesarano

Received: 20 June 2022
Accepted: 22 July 2022
Published: 28 July 2022

1. Introduction

Let D be a connected bounded domain on the complex plane, S be its boundary, which is closed and C^2-smooth.

The aim of this paper is to prove that an arbitrary $h \in L^1(S)$ can be the boundary value of a harmonic in the D function. The boundary value $h \in L^1(S)$ uniquely determines the harmonic function in D.

There is a large body of literature on the Dirichlet problem for elliptic equations going back to 1828; see references. There are three basic directions of research: non-smooth domains, non-smooth coefficients and non-smooth boundary values. This paper deals with smooth domains, the simplest elliptic operator, the Laplacean and non-smooth boundary values. In the published papers and books, the boundary values of harmonic functions were always assumed to be smoother than $L^1(S)$. For example, the maximal non-smoothness, allowed in [1], is bounded continuous function h on S with finitely many points of discontinuity of the first kind. In [2], the boundary conditions in $L^1(S)$ are not considered at all.

We deal with the smooth two-dimensional domains ($n = 2$) for definiteness. In the two-dimensional case, the kernel of the integral equation of the potential theory is continuous, and the corresponding integral operator A is compact in $L^1(S)$. The compactness of A in $L^1(S)$ holds for any finite dimension $n \geq 2$, but the kernel $A(t, s)$ of A, defined below formula (2), is not continuous for $n > 2$. This does not prevent A from being compact in $L^1(S)$. Our arguments are based on the new definition of the boundary values of a harmonic function in $L^1(S)$; see Definition 1 below. To our knowledge, in this paper, the $L^1(S)$-boundary values of harmonic functions are considered for the first time.

The problem we study is:

$$\Delta u = 0 \text{ in } D, \quad u|_S = h. \tag{1}$$

This problem has been studied in many papers and books for a long time. We mention only a few names: G. Green (1828), Gauss, Thomson, Dirichlet (1850), Hilbert (1900). One of the methods to solve this problem is based on the potential theory. Let us look for the solution in the form of the double-layer potential

$$u(x) = \int_S \frac{\partial g(x,s)}{\partial N} \mu(s) ds, \quad N := N_s. \tag{2}$$

Here $g = -\frac{1}{2\pi} \ln r$, $r := r_{xy} = |x-y|$, $x, y \in \mathbb{R}^2$, $A(t,s) := \frac{\partial g(t,s)}{\partial N} = -\frac{1}{2\pi} \frac{N_s \cdot r^0}{r_{ts}}$, $r^0 = \frac{s-t}{|s-t|}$, $N = N_s$ is the unit normal to S at the point s, N is directed out of D, $\mu = \mu(s)$ is the unknown function. The kernel $A(t,s)$ is a continuous function of t and s on $S \times S$ when $D \subset \mathbb{R}^2$ and S is C^2-smooth. We could assume S to be $C^{1,a}-$ smooth, $a \in (0,1]$, but this is not important in this paper.

In our case, operator

$$A\mu = \int_S A(t,s)\mu(s) ds \tag{3}$$

is well defined as an operator in $L^1(S)$ and is compact in this space, see [3–5] for the compactness test of $L^1(S)$.

Let us check that the kernel $A(t,s)$ is continuous on $S \times S$ if $n = 2$ and $S \in C^2$. For $|t-s| > \epsilon$ this kernel is C^1-smooth. Therefore, only its behavior as $t \to s$ should be considered. This behavior is determined by the function $\frac{N_s \cdot r^0}{-2\pi r_{ts}}$. Choose the coordinate system in which the $y-$axis is directed along N_t, so $N_t = j$, where i and j are the orthogonal unit vectors of the coordinate system. The equation of S in a neighborhood of t in this system is $y = f(x)$, $f(0) = f'(0) = 0$, the vector $t = (0,0)$, the vector $s = xi + jf(x)$, the vector $s - t = xi + jf(x)$, $r^0 = \frac{xi + f(x)j}{(x^2 + f^2(x))^{1/2}}$, $N_s = \frac{f'(x)i - j}{(1+(f')^2)^{1/2}}$, $N_s \cdot r^0 = \frac{xf'(x) - f(x)}{(x^2+f^2(x))^{1/2}(1+(f')^2)^{1/2}}$. Denote $\frac{N_s \cdot r^0}{r_{ts}} := J$. In our coordinate system $f(0) = f'(0) = 0$, so $f(x) \sim \frac{f''(0)x^2}{2}$ as $x \to 0$. Therefore, one gets $J(0) = -\lim_{x\to 0} \frac{f''(0)x^2}{2x^2} = -\frac{f''(0)}{2}$. Thus, the kernel $A(t,s)$ is continuous as $t \to s$. Therefore, it is continuous on $S \times S$.

If $D \subset \mathbb{R}^n$, $n > 2$, and S is smooth, then the kernel $A(t,s)$ is $O(\frac{1}{r_{ts}^{n-2}})$. Therefore, if $n > 2$, operator A is compact in $L^1(S)$, but the kernel is not continuous on $S \times S$.

If one looks for the solution to Equation (1) of the form (2) and $\mu \in C^1(S)$, then the integral equation for μ is:

$$h(t) = -\frac{\mu(t)}{2} + \int_S A(t,s)\mu(s) ds. \tag{4}$$

Equation (4) holds everywhere with respect to the Lebesgue measure on S if $A(t,s)$ is continuous. See, for example, [6], where the derivation of Equation (4) under the assumption $\mu \in C^1(S)$ is given. It is well known that the set $C^1(S)$ is dense in $L^1(S)$ in the norm of $L^1(S)$. Equation (4) holds almost everywhere with respect to Lebesgue's measure on S if $h \in L^1(S)$.

Let us recall M. Riesz's compactness criterion for sets in $L^1(S)$:

Proposition 1. *For a bounded set $M \subset L^1(S)$ to be compact in $L^1(S)$, it is necessary and sufficient that for an arbitrary small $\epsilon > 0$ there exists a $\delta > 0$ such that if $|\sigma| \le \delta$, then for any $h \in M$ one has $\|h(s+\sigma) - h(s)\| < \epsilon$, where $s + \sigma \in S$.*

Here and below, the norm is the $L^1(S)$ norm, $\|h\| = \int_S |h(s)| ds$. Proofs of Proposition 1 can be found in [3,5].

Lemma 1. *If $A(t,s)$ is continuous on $S \times S$ and a set $M \in L^1(S)$ is bounded, then the set AM is compact in $L^1(S)$.*

Proof. By Proposition 1, it is sufficient to check that $\|(Ah)(t+\sigma)-(Ah)(t)\| \le \epsilon$ for $|\sigma| \le \delta$, where $h \in M$ is arbitrary. Let $|S|$ denote the length of S. One has

$$\|(Ah)(t+\sigma)-(Ah)(t)\| \le |S| \sup_{s,t\in S} |A(t+\sigma,s)-A(t,s)|\|h\| \le c\epsilon, \tag{5}$$

provided that $|\sigma| \le \delta$. Here $c > 0$ does not depend on δ, it comes from the bound $\|h\| \le c_1$ for all $h \in M$, $c = |S|c_1$. We have used the continuity of $A(t,s)$ on $S \times S$ to conclude that

$$\sup_{s,t} |A(t+\sigma,s)-A(t,s)| \le \epsilon, \tag{6}$$

if $|\sigma|$ is sufficiently small. Lemma 1 is proven. □

We want to make sense of the method of integral equation for solving the Dirichlet problem (1), assuming that $h \in L^1(S)$.

Lemma 2. *Operator A is compact in $L^1(S)$. Operator $-\frac{I}{2}+A$ is Fredholm-type, where I is the identity operator. The null-space of operator $-\frac{I}{2}+A$ is trivial.*

Proof. Operator $A : L^1(S) \to L^1(S)$ is compact by Lemma 1. This is also true if $n > 2$ and $A(t,s) = O(|t-s|^{-(n-2)})$. Operator $-\frac{I}{2}+A$, where I is the identity operator, is bounded and continuous as an operator from $L^1(S)$ into itself. It is known (see, e.g., ref. [1]) that the homogeneous problem (1) has only the trivial solution in the space $C(S)$. We claim that the same is true in the space $L^1(S)$. Indeed, if μ solves the homogeneous Equation (4) and $n = 2$, then $\mu \in C(S)$ because $(A\mu) \in C(S)$ if $\mu \in L^1(S)$ since the kernel $A(t,s) \in C(S \times S)$. Therefore, the null-space of operator $-\frac{I}{2}+A$ is trivial in $L^1(S)$ as well.

Since A is compact and the null-space of operator $-\frac{I}{2}+A$ is trivial, the Fredholm alternative holds: the inverse operator $(-\frac{I}{2}+A)^{-1}$ exists, is bounded, and it maps $L^1(S)$ onto itself. This means not only that Equation (4) makes sense for $\mu \in L^1(S)$ and $h \in L^1(S)$, but also that μ depends continuously on h in the norm of $L^1(S)$.

Lemma 2 is proved. □

Remark 1. *It follows from Lemma 2 that the only solution in $L^1(S)$ of the homogeneous problem (1) is $u = 0$. This result is new because $L^1(S)$ boundary values of harmonic functions were not considered earlier.*

Remark 2. *One can find a harmonic function u in the circle $D = \{x,y : (x-1)^2+y^2 < 1\}$, which is zero on $S = \{x,y : (x-1)^2+y^2 = 1\}$, except at one point $x = 0$, $y = 0$, and which is not zero in D. For example, $u = 1 - 2Re\, z^{-1}$, $z = x+iy$. Of course, $u|_S$ in this example does not belong to $L^1(S)$.*

To check this, write $u = \frac{(x-y)^2}{x^2+y^2}$ and use the polar coordinates $x - 1 = r\cos\phi$, $y = r\sin\phi$. Then S has representation $x = 1+\cos\phi$, $y = \sin\phi$ and the point $(0,0)$ has coordinates $r = 1$, $\phi = \pi$. One has

$$\int_0^{2\pi} \frac{(x-y)^2}{x^2+y^2} d\phi = \int_0^{2\pi} \frac{1-\sin(2\phi)}{2+2\cos\phi} d\phi.$$

The integrand in the above integral is not absolutely integrable in a neighborhood of the point $\phi = \pi$. The function $(x-y)^2 = 0$ on S because the equation $(x-1)^2+y^2 = 1$ of S is equivalent to the equation $(x-y)^2 = 0$.

This example shows that the assumption $h \in L^1(S)$ is necessary for the uniqueness of the solution to the Dirichlet problem (1).

Our next step is to define the $\lim_{x\to t} A_x\mu$ *for* $\mu \in L^1(S)$*, where*

$$A_x\mu := \int_S A(x,s)\mu(s)ds, \tag{7}$$

and the kernel of operator A_x *is* $A(x,s) := \frac{\partial g(x,s)}{\partial N_s}$.

By $x \to t$ we mean *a non-tangential limit* $x \to t$, where $x \in D$ and $t \in S$.

Let $h \in L^1(S)$ be arbitrary. Choose any sequence $h_\delta \in C^1(S)$ such that

$$\lim_{\delta\to 0} \|h - h_\delta\| = 0. \tag{8}$$

By Lemma 2, Equation (8) implies

$$\lim_{\delta\to 0} \|\mu - \mu_\delta\| = 0, \tag{9}$$

where μ_δ is the unique solution to the equation:

$$-\frac{\mu_\delta(t)}{2} + \int_S \frac{\partial g(t,s)}{\partial N_s}\mu_\delta(s)ds := h_\delta. \tag{10}$$

Definition 1. *We define*

$$A_x\mu := \lim_{\delta\to 0} A_x\mu_\delta, \quad x \in D, \tag{11}$$

and

$$A_t\mu := \lim_{\delta\to 0}\lim_{x\to t} \int_S \frac{\partial g(x,s)}{\partial N_s}\mu_\delta(s)ds, \quad t \in S. \tag{12}$$

This definition gives meaning to the boundary condition in Equation (1) if $h \in L^1(S)$. The existence of the limit

$$\lim_{\delta\to 0} \int_S \frac{\partial g(x,s)}{\partial N}\mu_\delta(s)ds = \int_S \frac{\partial g(x,s)}{\partial N}\mu(s)ds$$

is obvious for $x \in D$ because of relation (9) and because kernel $\frac{\partial g(x,s)}{\partial N}$ is smooth when $x \in D$.

The existence of the limit

$$\lim_{x\to t} \int_S \frac{\partial g(x,s)}{\partial N}\mu_\delta(s)ds = -\frac{\mu_\delta(t)}{2} + A\mu_\delta \tag{13}$$

is known from the potential theory if $\mu_\delta \in C^1(S)$, see, for example, ref. [6], pp. 148–152. The existence of the limit

$$\lim_{\delta\to 0}\Big(-\frac{\mu_\delta(t)}{2} + A\mu_\delta\Big) = -\frac{\mu(t)}{2} + A\mu \tag{14}$$

is clear from relation (9) and Lemma 2.

For the convenience, of the reader we sketch a proof of Equation (13) following [6]. The proof is shorter than in [6] because the kernel $\frac{\partial g(t,s)}{\partial N}$ is continuous if $n = 2$.

Note that $J(x) := \lim_{x\to t} \int_S \frac{\partial g(x,s)}{\partial N}ds = -1$ if $x \in D$; $J(x) = 0$ if $x \in D'$, where D' is defined by the formula: $D' := \mathbb{R}^3 \setminus \bar{D}$; $J(x) = -\frac{1}{2}$ if $x = t \in S$. This result is well known and is proven by applying Green's formula and the equation $\Delta g(x,y) = -\delta(x-y)$, where $\delta(x)$ is the delta function.

Let $\mu_\delta \in C^1(S)$. Then,

$$M := \int_S \frac{\partial g(x,s)}{\partial N}\mu_\delta(s)ds = J(x)\mu_\delta(t) + \int_S \frac{\partial g(x,s)}{\partial N}[\mu_\delta(s) - \mu_\delta(t)]ds := J(x)\mu_\delta(t) + K.$$

One has (the $+$ sign denotes the non-tangential to S limit when $x \in D$, $x \to t \in S$ and the $-$ sign denotes the similar limit when $x \in D'$, $x \to t \in S$):

$$M_+ = -\mu_\delta(t) + \lim_{x \to t, x \in D} K := J_+, \quad M_- = 0 + \lim_{x \to t, x \in D'} K := J_-. \quad M_0 = -\frac{1}{2}\mu_\delta(t) + K.$$

If one proves that K is continuous when x passes t along the normal N_t, then $M_0 = -\frac{1}{2}\mu_\delta + K(t)$ and the desired statement is proven. Here $K(t) = \int_S \frac{\partial g(t,s)}{\partial N}[\mu_\delta(s) - \mu_\delta(t)]ds$.

If $\mu_\delta \in C^1(S)$, then $|\mu_\delta(s) - \mu_\delta(t)| \leq c|s-t|$. Therefore,

$$|K(x) - K(t)| \leq c \int_S |\frac{N \cdot r^0_{xs}}{|x-s|} - \frac{N \cdot r^0_{ts}}{|t-s|}||t-s|ds := L.$$

The function r^0_{xs} is continuous with respect to x. The function $|t-s|/|x-s|$ is continuous with respect to x when $x \to t$ along the normal N_t. This implies continuity of L when x crosses t along N_t. Therefore, M is a continuous function of x when x crosses t along N_t, as we claimed.

Since operator A is compact in $L^1(S)$, the Fredholm alternative yields the unique solution to Equation (4) with an arbitrary $h \in L^1(S)$, because Equation (4) with $h = 0$ has only the trivial solution in $L^1(S)$. Given an arbitrary $h \in L^1(S)$, one finds $h_\delta \in C^1(S)$ such that

$$\lim_{\delta \to 0} \|h_\delta - h\| = 0.$$

If $\lim_{\delta \to 0} \|h_\delta - h\| = 0$, then $\lim_{\delta \to 0} \|\mu_\delta - \mu\| = 0$ since the inverse operator $\left(-\frac{I}{2} + A\right)^{-1}$ is continuous and defined on all of $L^1(S)$. The function $u(x) = A_x\mu$, where μ is the unique solution to Equation (4), solves the Dirichlet problem (1). We have proven the following result:

Theorem 1. *Assume that $h \in L^1(S)$ is arbitrary. Then there exists a unique harmonic in the D function $u = A_x\mu$, $x \in D$, such that $u = h$ on S. The boundary value of u on S is defined by formula* (12).

2. Conclusions

The history of the Dirichlet problem goes back to 1828. The result in this paper is, to the author's knowledge, the first result in the 194 years of research since 1828 that yields the existence and uniqueness of the solution to the Dirichlet problem with the boundary values in $L^1(S)$.

It is proven that the Dirichlet problem (1) with the boundary function $h \in L^1(S)$ has a solution, and this solution is unique.

Open problem. Let us keep our assumption about D. Given a harmonic function $u(x,y)$ in D; one can use the Schwarz operator to construct the conjugate harmonic function $v(x,y)$ (up to an additive constant) and to get the corresponding analytic function $f(z) = u + iv$, $z = x + iy$, in D. The open problem is:

What is the set of boundary values of $f(z)$ on S when the values h of u on S run through all of $L^1(S)$?

The Schwarz operator is known explicitly if, for example, D is the unit disc; see, for example, ref. [7]. In [8,9], one can find information about the action of singular integral operators in Lebesgue's spaces $L^p(S)$, $1 < p < \infty$.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Lavrent'ev M.A.; Shabat B.V. *Methods of the Theory of Functions of Complex Variable*; GIFML: Moscow, Russia, 1958. (In Russian)
2. Gilbarg, D.; Trudinger, N. *Elliptic Partial Differential Equations of Second Order*; Springer: New York, NY, USA, 1983.
3. Hanche-Olsen, H.; Holden, H. The Kolmogorov-Riesz compactness theorem. *Expo. Math.* **2010**, *28*, 385–394. [CrossRef]
4. Kolmogorov, A.N. Über Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel. *Nachr. Ges. Wiss. Göttingen* **1931**, *9*, 60–63.
5. Riesz, M. Sur les ensembles compacts de fonctions sommables. *Acta Szeged Sect. Math.* **1933**, *6*, 136–142. (In French)
6. Ramm, A.G. *Scattering of Acoustic and Electromagnetic Waves by Small Bodies of Arbitrary Shapes: Applications to Creating New Engineered Materials*; Momentum Press: New York, NY, USA, 2013.
7. Gahov, F. *Boundary Value Problems*; Nauka: Moscow, Russia, 1977. (In Russian)
8. Khvedelidze, B. Linear discontinuous boundary value problems of function theory, singular integral equations and some applications. *Tr. Tbil. Math. Instituta Akad. Nauk Grusinskoi SSR* **1956**, *23*, 3–158.
9. Mikhlin, S.; Prössdorf, S. *Singular Integral Operators*; Springer: New York, NY, USA, 1986.