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Towards Strong Convergence and Cauchy Sequences in Binary Metric Spaces

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Abstract: A Kuratowski topology is a topology specified in terms of closed sets rather than open sets. Recently, the binary metric was introduced as a symmetric, distributive-lattice-ordered magma-valued function of two variables satisfying a "triangle inequality" and subsequently proved that every Kuratowski topology can be induced by such a binary metric. In this paper, we define the strong convergence of a sequence in a binary metric space and prove that strong convergence implies convergence. We state the conditions under which strong convergence is equivalent to convergence. We define a strongly Cauchy sequence and a strong complete binary metric space. Finally, we give the strong completion of all binary metric spaces with a countable indexing set.

Keywords: binary metric; generalized metric; convergence in binary metric

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1. Introduction

Topology generalizes the theory of metric spaces. Numerous attempts [1–5] have been made to generalize the notion of a metric so that larger classes of topologies can be induced from such a generalized metric, which would benefit from the quantifiable and computable nature of such functions. Moreover, with the notion of a generalized metric, one may be able to introduce a version of a Cauchy sequence (and ultimately, the completeness) beyond metric spaces. Recall that such notions are central in several existence theorems via the completeness condition. In [3], Kopperman showed that any topology can be induced by a generalized metric having values in an additive semigroup.

In 2020, Assaf et al. [1] proved that every (Kuratowski) topology can be induced by a binary metric. However, they left open the question of topological completeness in terms of binary metric spaces, among other questions. Note that even though a topological space can always be equipped with a binary metric, the indirect construction via Kuratowski topology makes it difficult to derive the binary metrical definition of convergence. This is clearly the obstacle of defining a Cauchy sequence as consequently the binary metric completeness. Our task, then, is to find a workaround to this.

With this in mind, in Section 2, we cover the preliminaries required to work with binary metric spaces. This section covers all the main results and the constructs used in [1]. The convergence of a sequence is usually defined in terms of open neighbourhoods. In Section 3, we state equivalent conditions for convergence in terms of closed sets. We also present an example of a binary metric space where a particular sequence (x_n) converges or diverges irrespective of the values of $\xi(x_n, x)$. Then, we present the concept of "strong convergence" as an alternative to convergence. We also give a condition under which convergence and strong convergence are equivalent. We then define a "strongly Cauchy sequence" and a "strong complete binary metric space" as a natural extension of strong convergence. Finally, we give a condition under which the strong completion of a binary metric space exists.

2. Preliminaries

2.1. Lattice-Ordered Magma

A binary metric space has values in $\{0,1\}^{\mathcal{I}}$ for some indexing set $\mathcal{I}$. Some form of algebra on $\{0,1\}^{\mathcal{I}}$ is needed to work on the binary metric. Assaf et al. [1] used a lattice-ordered magma for this role.

Let $\mathcal{I}$ be a indexing set. We denote

$$\Gamma = \{-1,0,1\},\ \Gamma^{\mathcal{I}} = \{-1,0,1\}^{\mathcal{I}},\ \Gamma^{\mathcal{I}+} = \{0,1\}^{\mathcal{I}}$$

For any element $a \in \Gamma^{\mathcal{I}}, b \in \Gamma, i \in \mathcal{I}$, we denote:

1. $a_i = \pi_i(a)$, the projection of a along the ith coordinate;
2. $-a = \prod_{i \in \mathcal{I}}(-a_i)$;
3. $\bar{b} = \prod_{i \in \mathcal{I}}(b)$.

In line with the above notations, we also write $(a_n|_i)$ to denote the sequence $(\pi_i(a_n))$ for any sequence (a_n) in Γ.

Definition 1 (Def. 2.2, [1])**.** *For $a, b \in \Gamma^{\mathcal{I}}$, we define $\preccurlyeq^{\mathcal{I}}$ as*

$$a \preccurlyeq^{\mathcal{I}} b \iff a_i \leq b_i,\ \forall i \in \mathcal{I}$$

Definition 2 (Def. 2.3, [1])**.** *We define a binary operation $\oplus$ on Γ by Table 1*

Table 1. Operation table for $\ominus$.

$\oplus$	-1	0	1
-1	-1	-1	0
0	-1	0	1
1	0	1	1

For $a, b \in \Gamma^{\mathcal{I}}$, we define $a \oplus^{\mathcal{I}} b$ by $(a \oplus^{\mathcal{I}} b)_i = a_i \oplus b_i$.

Definition 3 (Def. 2.4, [1])**.** *We define a corresponding "subtraction" operation $\ominus^{\mathcal{I}}$ on $\Gamma^{\mathcal{I}}$ by*

$$a \ominus^{\mathcal{I}} b = a \oplus^{\mathcal{I}} (-b)$$

The following are some results concerning $(\Gamma^{\mathcal{I}+}, \oplus^{\mathcal{I}}, \preccurlyeq^{\mathcal{I}})$.

Proposition 1 (Prop. 2.5, [1])**.** *Consider the lattice-ordered magma $(\Gamma^{\mathcal{I}+}, \preccurlyeq^{\mathcal{I}}, \oplus^{\mathcal{I}})$. If $a, b, c, d \in \Gamma^{\mathcal{I}+}$, then:*

1. $a \oplus^{\mathcal{I}} a = a$;
2. $a \oplus^{\mathcal{I}} b = b \oplus^{\mathcal{I}} a$;
3. $a \oplus^{\mathcal{I}} \bar{0} = a$, $a \oplus^{\mathcal{I}} \bar{1} = \bar{1}$;
4. $a \preccurlyeq^{\mathcal{I}} b$ *and* $b \preccurlyeq^{\mathcal{I}} a \implies a = b$;
5. $a \preccurlyeq^{\mathcal{I}} b$ *and* $c \preccurlyeq^{\mathcal{I}} d \implies a \oplus^{\mathcal{I}} c \preccurlyeq^{\mathcal{I}} b \oplus^{\mathcal{I}} d$;
6. $a \preccurlyeq^{\mathcal{I}} b \implies a \ominus^{\mathcal{I}} c \preccurlyeq^{\mathcal{I}} b \ominus^{\mathcal{I}} c$;
7. $(a \oplus^{\mathcal{I}} b) \oplus^{\mathcal{I}} c = a \oplus^{\mathcal{I}} (b \oplus^{\mathcal{I}} c)$;
8. $(a \ominus^{\mathcal{I}} b) \oplus^{\mathcal{I}} (c \ominus^{\mathcal{I}} d) = (a \ominus^{\mathcal{I}} d) \oplus^{\mathcal{I}} (c \ominus^{\mathcal{I}} b)$;
9. $(a \oplus^{\mathcal{I}} b) \ominus^{\mathcal{I}} c \preccurlyeq^{\mathcal{I}} a \oplus^{\mathcal{I}} (b \ominus^{\mathcal{I}} c)$;
10. $a \oplus^{\mathcal{I}} (b \ominus^{\mathcal{I}} c) = (a \ominus^{\mathcal{I}} c) \oplus^{\mathcal{I}} b$.

2.2. Binary Metric

Let us first recall the notion of a Kuratowski topology.

Definition 4 (Def. 1.1, [1]). *Consider X and let $\mathcal{C}$ be a family of subsets of X. We say that $(X, \mathcal{C})$ is a Kuratowski topology if:*

1. $\emptyset, X \in \mathcal{C}$;
2. *Any arbitrary intersection of elements of $\mathcal{C}$ is an element of $\mathcal{C}$;*
3. *Any finite union of elements of $\mathcal{C}$ is an element of $\mathcal{C}$.*

For any Kuratowski topology $(X, \mathcal{C})$, the collection

$$\tau = \{X \setminus A \mid A \in \mathcal{C}\}$$

forms a topology on X. We will call this the corresponding topology of $(X, \mathcal{C})$. Similarly, given a topology (X, τ), the collection

$$\mathcal{C} = \{X \setminus U \mid U \in \tau\}$$

forms a Kuratowski topology on X. We will call this the corresponding Kuratowski topology of (X, τ).

Definition 5 (Def. 1.3, [1]). *For a Kuratowski topology $(X, \mathcal{C})$, $\mathcal{B} \subseteq \mathcal{C}$ is called a closed basis of $\mathcal{C}$ if, for each $x \in X$:*

1. *There exists $B \in \mathcal{B}$ such that $x \notin B$;*
2. *For all $A \in \mathcal{C}$ such that $x \notin A$, there exists $B \in \mathcal{B}$ such that $x \notin B \supseteq A$.*

Elements of $\mathcal{B}$ are called basic closed sets.

We say that $\mathcal{B}$ induces $\mathcal{C}$ since, for any closed set $A \in \mathcal{C}$, there is a collection of basic closed sets $\{B_j \mid j \in \mathcal{J} \subseteq \mathcal{I}\} \subseteq \mathcal{B}$ such that

$$A = \bigcap_{j \in \mathcal{J}} B_j$$

Given any set X, we require a collection of subsets of X whose arbitrary intersection generates a Kuratowski topology on X.

Definition 6 (Def. 1.4, [1]). *For set X, a collection $\mathcal{B} \subseteq \mathscr{P}(X)$ is called a closed basis if, for each $x \in X$:*

1. *There exists $B \in \mathcal{B}$ such that $x \notin B$;*
2. *For all $B_1, B_2 \in \mathcal{B}$ not containing x, there exists $B \in \mathcal{B}$ such that $x \notin B \supseteq B_1 \cup B_2$.*

The collection of arbitrary intersections of elements of a closed basis forms a Kuratowski topology, say $\mathcal{C}$. We say that $\mathcal{B}$ induces $\mathcal{C}$.

Definition 7 (Def. 3.2, [1]). *Consider a set X and an indexing set $\mathcal{I}$. Let $\xi : X \times X \to \Gamma^{\mathcal{I}+}$. Then, ξ is said to be a binary metric if it satisfies the following properties for all $x, y, z \in X$:*

1. $\xi(x, x) \preccurlyeq^{\mathcal{I}} \xi(x, y)$ *(also known as small self-distance axiom);*
2. $\xi(x, y) = \xi(y, x)$;
3. $\xi(x, y) \preccurlyeq^{\mathcal{I}} \xi(x, z) \oplus^{\mathcal{I}} [\xi(z, y) \ominus^{\mathcal{I}} \xi(z, z)]$.

We call the triplet $(X, \xi, \mathcal{I})$ a binary metric space (or BMS for short).

Definition 8 (Def. 3.7, [1])**.** *Let $(X, \mathcal{C})$ be a Kuratowski topology with a closed basis $\mathcal{B}$. Then, the canonical binary metric determined by $\mathcal{B}$ is given by $\xi : X \times X \to \Gamma^{\mathcal{B}+}$ such that for all $x, y \in X$ and all $A \in \mathcal{B}$:*

$$\xi(x,y)|_A = \begin{cases} 0 & \text{if } x, y \in A, \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

Definition 9 (Def. 3.8, [1])**.** *Let ξ be a binary metric on X with $\mathcal{I}$ as indexing set. For $x \in X$ and $\epsilon \in \Gamma^{\mathcal{I}+}$, the ξ-closed ball around x of radius ϵ is defined by*

$$B(x, \epsilon) = \{y \in X \mid \xi(x,y) \ominus^{\mathcal{I}} \xi(x,x) \preccurlyeq^{\mathcal{I}} \epsilon\}.$$

Note that the following forms a closed basis over X:

$$\mathcal{B} = \left\{ \bigcup_{j=1}^{m} B(y_j, \epsilon_j) \mid m \in \mathbb{N}, y_j \in X \text{ and } \epsilon_j \in \Gamma^{\mathcal{I}+} \right\} \bigcup \{\emptyset, X\}.$$

The fact that ξ defined by (1) is a binary metric and the following theorem form the main result in [1].

Theorem 1 (Thm. 4.1, [1])**.** *Let $(X, \mathcal{C})$ be a Kuratowski topology with $\mathcal{B}$ as closed basis. Then, the canonical binary metric determined by $\mathcal{B}$ induces $\mathcal{C}$.*

Remark 1. *Note that on any non-empty set X, a discrete metric d, given for any $x, y \in X$ by $d(x,y) = 0$ for $x = y \in X$ and $d(x,y) = 1$ otherwise, can be viewed as a binary metric with a singleton indexing set. The metric topology is clearly the discrete topology (every subset of X is hence open), while the topology generated in correspondence to Theorem 1 is the co-finite topology.*

3. Main Result

3.1. Convergence in Kuratowski Topology

Every Kuratowski topology $(X, \mathcal{C})$ has a corresponding topology $\tau = \{D^c \subseteq X | D \in \mathcal{C}\}$. We require the definition of convergence in the Kuratowski topology to be such that the convergence is equivalent in a Kuratowski topology and its corresponding topology, and thus by extension in a topology and its corresponding Kuratowski topology. The definition of convergence for a Kuratowski topology can be written simply in terms of complements of closed sets. This would achieve our objective. However, as we see in the following sections, it would be beneficial for us to define convergence directly in terms of closed sets by using contrapositives of the usual definition. We present these definitions next. Although they are mere contrapositives of their usual counterparts, the complexity of the statements compel us to present them as lemmas.

Definition 10. *Let (X, τ) be a topology. We say that the sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ if, for each $\mathcal{U} \in \tau$ with $x \in U$, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. We denote this as $x_n \to x$.*

Simply writing the above definition in terms of complements of closed sets rather than open sets gives the following definition.

Definition 11. *Let $(X, \mathcal{C})$ be a Kuratowski topology. We say that the sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ if, for each $D \in \mathcal{C}$ with $x \notin D$, there exists $N \in \mathbb{N}$ such that $x_n \notin D$ for all $n \geq N$. We denote this by $x_n \to x$.*

Lemma 1. *Let $(X, \mathcal{C})$ be a Kuratowski topology. The sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ if and only if, for all $D \in \mathcal{C}$, $x_n \in D$ for infinitely many $n \in \mathbb{N} \implies x \in D$.*

Proof. Let $x_n \to x$ and let $D \in \mathcal{C}$ such that $x_n \in D$ for infinitely many n. If $x \notin D$, then $x_n \to x$ implies the existence of $N \in \mathbb{N}$ such that $x_n \notin D \; \forall n \geq N$. Hence, $\{n \in \mathbb{N} \mid x_n \in D\}$ is finite, which is contradictory to the choice of D.

For the converse part, let (x_n) be a sequence in X and $x \in X$ such that for each $D \in \mathcal{C}, x_n \in D$ for infinitely many $n \in \mathbb{N} \implies x \in D$.

Let $F \in \mathcal{C}$ and $x \notin F$. The given hypothesis implies that $\{m \in \mathbb{N} \mid x_m \in F\}$ must be finite, i.e., there is an $N \in \mathbb{N}$ for which $x_n \notin F \; \forall n \geq N$. Thus by the definition of convergence in a Kuratowski topology, $x_n \to x$. □

Lemma 2. *Let $(X, \mathcal{C})$ be a Kuratowski topology with closed basis $\mathcal{B}$. A sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ if and only if $\forall B \in \mathcal{B}, x_n \in B$ for infinitely many $n \in \mathbb{N} \implies x \in B$.*

Proof. The necessary part follows from the fact that $\mathcal{B} \subseteq \mathcal{C}$.

For the converse, let $(x_n)_{n\in\mathbb{N}}$ and $x \in X$ satisfy $\forall B \in \mathcal{B}$, $x_n \in B$ for infinitely many $n \implies x \in B$. Let $D \in \mathcal{C}$ such that $x_n \in D$ for infinitely many n. If $x \notin D$, then by the definition of a closed basis, $\exists B_0 \in \mathcal{B}$ such that $x \notin B_0 \supseteq D$. However, $x_n \in D \subseteq B_0$ for infinitely many n implies $x \in B_0$ by the hypothesis. This is contradictory to the definition of B_0. □

3.2. Convergence in Binary Metric

In a metric space (X, d), we know that a sequence (x_n) converges to $x \in X$ if and only if $\lim\limits_{n\to 0} d(x_n, x) = 0$. Similarly, for a partial metric space (X, p), a sequence (x_n) converges to $x \in X$ in the topology induced by the partial metric if and only if $\lim\limits_{n\to\infty} p(x, x_n) = p(x, x)$ [6,7]. In both these cases, the problem of convergence of a sequence in the associated topology is transferred to the problem of the convergence of values of the metric (the convergence of $d(x_n, x)$ to 0 in a metric space and $p(x_n, x)$ to $p(x, x)$ in a partial metric space). Our aim is to do the same for a binary metric space. We wish to transfer the problem of convergence of the sequence in a topology generated by a binary metric to a problem of convergence of the values of the binary metric. For this, we require the convergence in the co-domain of the binary metric in the first place, i.e., we need a topology on $\Gamma^{\mathcal{I}}$.

However before we do this, we consider an example in which such a condition for convergence is not possible. Note that the convergence of a sequence does not change with the removal of a finite number of terms of the sequence but depends on all the terms after a particular index. That is, we can always exclude the first n terms of the sequence, but each of the remaining ones must be taken into consideration for convergence. Recall from Remark 1 that the topology generated by Theorem 1 does not correspond to the metric topology (in this case the discrete topology) of X. The following example illustrates that the (binary) metric fails to communicate the convergence in the sense of Theorem 1 with the values $d(x_n, x)$.

Example 1. *Over a non-empty set X, we take a singleton indexing set. We define $\xi : X \times X \to \{0, 1\}$ such that for $x, y \in X$*

$$\xi(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

Note that for each $x \in X$, we have $B(x, 0) = \{x\}$ and $B(x, 1) = X$. The finite union of these closed balls forms a closed basis over X. Thus, the closed sets in this topology comprise all finite subsets of X and X itself. Note that the corresponding topology has, as open sets, sets with finite complements, i.e., the corresponding topology is the co-finite topology.

Let $(a_n)_{n\in\mathbb{N}}$ be any sequence in X. Note that $\xi(a_n, a_n) = 0$ and $\xi(a, a) = 0$ holds for all $a \in X$. Thus, the condition for convergence of the sequence, if it exists, depends exclusively on $\xi(a_n, a)$.

The convergence of a sequence in a co-finite topology is a well-known result. For the sake of illustration, we will present the same result for the Kuratowski topology before discussing the values of $\xi(a_n, a)$. There are three possible cases:

Case 1: (a_n) *has no infinitely repeating terms.*
Since every closed set is finite and $\mathcal{B} \subseteq \mathcal{C}$, *for any* $B \in \mathcal{B}$, *there exists* $N \in \mathbb{N}$ *such that* $a_n \notin B$ *for each* $n \geq N$. *Now, for any arbitrary* $a \in X$ *and for* $B \in \mathcal{B}$ *such that* $a \notin B$, *there must exist* $N \in \mathbb{N}$ *for which* $a_n \notin B$ *for all* $n \geq N$, *i.e.,* $a_n \to a$. *The sequence converges to each point of* X. *Note that for each* $a \in X$, *there is an* $M \in \mathbb{N}$ *such that* $a_n \neq a$ *for all* $n \geq M$. *Thus we obtain*

$$\xi(a_n, a) = 1$$

for all $n \geq M$.

Case 2: (a_n) *has only one infinitely repeating term, say* a_0.
Take $B \in \mathcal{B}$ *with* $a_0 \notin B$. *Since* B *is finite and* (a_n) *has no repeating term other that* a_0, *there exists* $N \in \mathbb{N}$ *such that* $a_n \notin B$ *for any* $n \geq N$, *i.e.,* $a_n \to a_0$. *For* $a \neq a_0$, *the set* $\{a_0\}$ *is closed and contains infinitely many terms of the sequence, but* $a \notin \{a_0\}$. *Thus,* (a_n) *does not converge to* a. *For this case, we have*

$$\xi(a_n, a_0) = \begin{cases} 0 & \text{if } a_n = a_0, \\ 1 & \text{otherwise.} \end{cases}$$

Since $a_n = a_0$ *for infinitely many* n, $a \neq a_0$, *and* a_0 *is the only repeating term, there exists* $N \in \mathbb{N}$ *such that* $a_n \neq a$ *when every* $n \geq N$. *This yields*

$$\xi(a_n, a) = 1$$

for $n \geq N$.

Case 3: (a_n) *has more than one infinitely repeating term.*
For each $a \in X$, *there is an infinitely repeating term, say* $b \neq a$. *Thus the closed set* $\{b\}$ *contains infinitely many terms of the sequence, but* $a \notin \{b\}$. *Thus, the sequence does not converge to* a. *Since* a *was arbitrary, the sequence converges nowhere in* X.
Thus, if a *is non-repeating, then there exists* $N \in \mathbb{N}$ *such that* $a_n \neq a$ *whenever* $n \geq N$. *Then, for any* $n \geq N$, *we have*

$$\xi(a_n, a) = 1.$$

For an infinitely repeating term a_0, *following the same reasoning as for case 2, we obtain*

$$\xi(a_n, a_0) = \begin{cases} 0 & \text{if } a_n = a_0, \\ 1 & \text{otherwise.} \end{cases}$$

As discussed earlier, the convergence of a sequence depends on the behaviour of all the terms of the sequence except for a finite few. Case 1, *the second part of Case* 2, *and the first part of Case* 3 *illustrate the condition where the value of* $\xi(a_n, a) = 1$ *after a certain* N; *however, the former corresponds to a convergent sequence while the latter two correspond to divergent sequences. Similarly, the first part of Case* 2 *and the second part of Case* 3 *illustrate the condition where the* $\xi(a_n, a)$ *never attain a constant value for increasing* n, *with the former corresponding to a convergent sequence and the latter to a divergent one.*

3.3. Strong Convergence

At the beginning of Section 3.2, we explained the need to specify a topology on $\Gamma^{\mathcal{I}}$. However, as Example 1 illustrates, a condition for convergence that covers all binary metric spaces is not possible. We address this by introducing a stronger form of convergence in binary metric spaces. This is partly motivated by Ge and Lin [6] who distinguished convergence in partial metric spaces from convergence in the topology induced by it, to remedy the fact that a partial metric induces a very coarse topology which makes practical use of convergence in the context of complete partial metric spaces difficult.

We begin by considering the lattice-ordered magma $(\Gamma, \oplus, \preccurlyeq)$. We define $d : \Gamma \times \Gamma \to \mathbb{R}$ as follows:

$$d(a, b) = |a \ominus b| \quad \text{for all } a, b \in \Gamma. \tag{3}$$

To see that d is a metric on Γ, one needs to only refer to Table 1. It is easy to see that d can be alternatively defined as

$$d(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{otherwise.} \end{cases}$$

In other words d is the discrete metric on Γ.

We call a sequence eventually constant if it is a constant sequence but for a finite few terms. It is easy to see that a sequence is convergent in a discrete metric space if and only if it is eventually constant. This metric induces a topology on Γ which can be extended to $\Gamma^{\mathcal{I}}$ via product topology. We now have a topology on $\Gamma^{\mathcal{I}}$. Furthermore, a sequence is convergent in a product topology if and only if the projections of the sequence are convergent themselves.

We summarize what we have discussed so far as follows.
For the lattice-ordered magma $(\Gamma^{\mathcal{I}}, \oplus^{\mathcal{I}}, \preccurlyeq^{\mathcal{I}})$, consider the metric d as defined by (3) and extend the topology induced by it to $\Gamma^{\mathcal{I}}$ by considering its product topology. Then, a sequence $(a_n)_{n\in\mathbb{N}}$ in $\Gamma^{\mathcal{I}}$ converges to a if and only if, for all $i \in \mathcal{I}$, $(a_n|_i)$ converges to a_i. Adopting the usual notation for convergence, this can be written as

$$\lim_{n\to\infty} a_n = a \iff \lim_{n\to\infty} a_n|_i = a_i \quad \forall i \in \mathcal{I}$$

In addition, d being a discrete metric implies that $a_n|_i$ converges to a_i if and only if $a_n|_i$ is eventually constant to a_i.

The following are some results that will aid us in working with $(\Gamma^{\mathcal{I}}, \oplus^{\mathcal{I}}, \preccurlyeq^{\mathcal{I}})$. The proof follows directly from the fact that a sequence converges in $\Gamma^{\mathcal{I}}$ if and only if its projections to each i are eventually constant.

Lemma 3. *Let $\alpha_n, \beta_n \in \Gamma^{\mathcal{I}}$. Then:*

1. *If $\alpha_n \preccurlyeq^{\mathcal{I}} \beta_n$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \alpha_n$, $\lim_{n\to\infty} \beta_n$ exist, then*

$$\lim_{n\to\infty} \alpha_n \preccurlyeq^{\mathcal{I}} \lim_{n\to\infty} \beta_n.$$

2. *If $\lim_{n\to\infty} \alpha_n$, $\lim_{n\to\infty} \beta_n$ exist, then*

$$\lim_{n\to\infty} (\alpha_n \oplus^{\mathcal{I}} \beta_n) = \lim_{n\to\infty} \alpha_n \oplus^{\mathcal{I}} \lim_{n\to\infty} \beta_n,$$
$$\lim_{n\to\infty} (\alpha_n \ominus^{\mathcal{I}} \beta_n) = \lim_{n\to\infty} \alpha_n \ominus^{\mathcal{I}} \lim_{n\to\infty} \beta_n.$$

Definition 12. *For a binary metric space $(X, \xi, \mathcal{I})$, a sequence $(x_n)_{n\in\mathbb{N}}$ is said to strongly converge to $x \in X$ if*

$$\lim_{n\to\infty} \xi(x_n, x) \ominus^{\mathcal{I}} \xi(x_n, x_n) = \bar{0}.$$

Equivalently, we say $(x_n)_{n\in\mathbb{N}}$ strongly converges to x if $\forall i \in \mathcal{I}$, and we have

$$\lim_{n\to\infty} \xi(x_n, x)|_i \ominus \xi(x_n, x_n)|_i = 0.$$

We denote this as $x_n \xrightarrow{s} x$.

Before we proceed with an example illustrating a strongly convergent sequence, note that since $\xi(x_n, x_n) \preccurlyeq^{\mathcal{I}} \xi(x, x_n)$, in order to prove for some $i \in \mathcal{I}$ that $\lim_{n\to\infty} \xi(x, x_n)|_i \ominus \xi(x_n, x_n)|_i = 0$ it is sufficient to show that for some N, we have the following implication: $\forall n \geq N,\ \xi(x_n, x)|_i = 1 \implies \xi(x_n, x_n)|_i = 1$.

Example 2. *Consider the usual topology on* $\mathbb{R}$. *Let* $O(q,k)$ *denote the open interval* $\left(q - \frac{1}{k}, q + \frac{1}{k}\right)$. *We know that* $\mathcal{B} = \{O(q,k)^c \mid q \in \mathbb{Q}, k \in \mathbb{N}\}$ *forms a closed basis for the corresponding Kuratowski topology. Let,* ξ *denote the canonical binary metric determined by* $\mathcal{B}$.

We claim that the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}} \xrightarrow{s} 0$. *To prove this, note that for any* $A \in \mathcal{B}$, *there are three possible cases:*

Case 1: $A = O(0,k)^c$ *for some* $k \in \mathbb{N}$.
$0 \in O(0,k)$ *and for* $n > k$, $\frac{1}{n} \in O(0,k)$. *Thus,* $\xi\left(0, \frac{1}{n}\right)|_A = 1$ *and* $\xi\left(\frac{1}{n}, \frac{1}{n}\right)|_A = 1$ *for all* $n > k$.

Case 2: $A = O(q,k)^c$ *where* $0 \notin O(q,k)$ *but* $0 = q - \frac{1}{k}$. *It is easy to see that for some* $N \in \mathbb{N}$, $\frac{1}{n} \in O(q,k)$ *holds for all* $n \geq N$. *Hence, we obtain* $\xi\left(0, \frac{1}{n}\right)|_A = 1$ *and* $\xi\left(\frac{1}{n}, \frac{1}{n}\right)|_A = 1$ *for* $n \geq N$.

Case 3: $A = O(q,k)^c$ *such that* $0 \notin O(q,k)$ *and* $0 \neq q - \frac{1}{k}$.
Again, it is easy to see that for some $N \in \mathbb{N}$, $\frac{1}{n} \notin O(q,k)$ *holds whenever* $n \geq N$. *Thus,* $\xi\left(0, \frac{1}{n}\right)|_A = 0$ *and* $\xi\left(\frac{1}{n}, \frac{1}{n}\right)|_A = 0$ *for every* $n \geq N$.

From the three cases above, we conclude that

$$\lim_{n\to\infty} \xi\left(0, \frac{1}{n}\right)|_A \ominus \xi\left(\frac{1}{n}, \frac{1}{n}\right)|_A = 0.$$

Therefore $\left(\frac{1}{n}\right)_{n\in\mathbb{N}} \xrightarrow{s} 0$.

Let us also consider $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ *as follows:*
Cases 1 and 3 work out in a similar fashion as above, however, for the case when $A = O(q,k)^c$ *where* $0 \notin O(q,k)$ *but* $0 = q - \frac{1}{k}$ *or* $0 = q + \frac{1}{k}$. *The values* $\xi\left(0, \frac{(-1)^n}{n}\right)|_A$ *and* $\xi\left(\frac{(-1)^n}{n}, \frac{(-1)^n}{n}\right)|_A$ *keep alternating between* 0 *and* 1, *but* $\xi\left(0, \frac{1}{n}\right)|_A \ominus \xi\left(\frac{1}{n}, \frac{1}{n}\right)|_A = 0$ *for sufficiently large values of* n. *Thus, even though the individual limits do not exist, the limit of their difference exists and is* 0. *Thus,* $(\frac{(-1)^n}{n}) \xrightarrow{s} 0$.

Theorem 2. *In any BMS, strong convergence implies convergence.*

Proof. Let $(X, \xi, \mathcal{I})$ be a BMS and let the sequence $x_n \xrightarrow{s} x \in X$. Let, $x_n \in B(y, \epsilon)$ for infinitely many values of n. Therefore, for infinitely many n, we have

$$\xi(x_n, y) \ominus^{\mathcal{I}} \xi(y, y) \preccurlyeq^{\mathcal{I}} \epsilon \tag{4}$$

The triangle inequality gives

$$\xi(x, y) \ominus^{\mathcal{I}} \xi(y, y) \preccurlyeq^{\mathcal{I}} \left[\xi(x, x_n) \ominus^{\mathcal{I}} \xi(x_n, x_n)\right] \oplus^{\mathcal{I}} \left[\xi(x_n, y) \ominus^{\mathcal{I}} \xi(y, y)\right].$$

Passing to the limit as $n \to \infty$, since (4) is satisfied for infinitely many n and $x_n \xrightarrow{s} x$, we obtain

$$\xi(x, y) \ominus^{\mathcal{I}} \xi(y, y) \preccurlyeq^{\mathcal{I}} 0 \oplus^{\mathcal{I}} \epsilon = \epsilon$$

i.e., $x \in B(y, \epsilon)$. Thus, $x_n \to x$. $\square$

The above theorem dictates that every strongly convergent sequence in a BMS is convergent. However, not every convergent sequence is strongly convergent, as is evident from Example 1. The theorem gives rise to some important corollaries with regard to properties of a strongly convergent sequence.

Corollary 1. *Let $(X, \xi, \mathcal{I})$ be a binary metric space and let $Y \subseteq X$. For any sequence (x_n) in Y, $x_n \xrightarrow{s} x \implies x \in \bar{Y}$, the closure of Y.*

Since the complement of a pre-image is the pre-image of complement, the definition of a continuous function for a Kuratowski topology can be stated as follows.

Definition 13. *A function $f : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is said to be continuous if $\forall D \in \mathcal{C}_Y, f^{-1}(D) \in \mathcal{C}_X$.*

Corollary 2. *Let $f : (X, \xi, \mathcal{I}) \to (Y, \gamma, \mathcal{J})$ be a continuous function. For $x \in X$ and any sequence $(x_n)_{n\in\mathbb{N}}$ in X, $x_n \xrightarrow{s} x \implies f(x_n) \to f(x)$.*

We now state the condition under which the strong convergence and convergence are equivalent.

Theorem 3. *Let $\mathcal{B}$ be a closed basis for the Kuratowski topology $(X, \mathcal{C})$. Then, for the binary metric ξ induced by $\mathcal{B}$, convergence in $\mathcal{C}$ is equivalent to strong convergence in $(X, \xi, \mathcal{B})$.*

Proof. As a consequence of Theorem 2, we only have to prove that a convergent sequence in this BMS is also strongly convergent.

Let the sequence $(x_n)_{n\in\mathbb{N}}$ in X converge to $x \in X$. For each $A \in \mathcal{B}$, the following implications hold:

$$\xi(x_n, x)|_A = 0 \implies \xi(x_n, x_n)|_A = 0 \implies \xi(x_n, x)|_A \ominus \xi(x_n, x_n)|_A = 0,$$

$$\xi(x_n, x)|_A = 1 \implies x_n \notin A \text{ or } x \notin A \text{ or both.}$$

Now, $x_n \notin A \implies \xi(x_n, x_n)|_A = 1 \implies \xi(x, x_n) = 1 \implies \xi(x, x_n)|_A \ominus \xi(x_n, x_n)|_A = 0$. Finally, if $x \notin A$ then $x_n \to x \implies \exists N$ such that $x_n \notin A$ for every $n \geq N$. Therefore, $\xi(x_n, x_n)|_A = 1$ for $n \geq N$, which in turn gives $\xi(x_n, x_n)|_A \ominus \xi(x_n, x_n)|_A = 0$ for $n \geq N$. Combining all the possible cases, we have $\lim_{n\to\infty} \xi(x_n, x_n)|_A \ominus \xi(x_n, x_n)|_A = 0$. Therefore $x_n \xrightarrow{s} x$. □

The above theorem, when taken together with Corollary 2, gives the following result.

Corollary 3. *Let $f : (X, \xi, \mathcal{I}) \to (Y, \gamma, \mathcal{B})$ be a continuous function such that γ is a binary metric induced by a closed basis $\mathcal{B}$ on Y. Then, $x_n \xrightarrow{s} x \in X \implies f(x_n) \xrightarrow{s} f(x) \in Y$.*

3.4. Strongly Complete Binary Metric Spaces

We now turn our attention to introducing Cauchy sequences in binary metric spaces. As with convergence, we will take metric spaces and partial metric spaces as the basis for defining Cauchy sequences.

In a metric space (X, d), a sequence (x_n) is said to be Cauchy if and only if $\lim_{m,n\to\infty} d(x_m, x_n) = 0$. In a partial metric space (X, p), a sequence (x_n) is said to be Cauchy if and only if the limit $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and is finite. Thus, Cauchy sequences are defined in terms of simultaneous limits of the values of the respective metrics, which are in $\mathbb{R}$. We do not have a metric on the co-domain $\Gamma^{\mathcal{I}}$ of the binary metric, as $\mathcal{I}$ is not necessarily countable. We do, however, have a topology on the same, which is enough to define the simultaneous limits.

Definition 14. *Consider $(\alpha_{m,n})_{m,n\in\mathbb{N}}$ such that $\alpha_{m,n} \in \Gamma^{\mathcal{I}}$ for every $m, n \in \mathbb{N}$. We say that the simultaneous limit $\lim\limits_{m,n\to\infty} \alpha_{m,n}$ exists if there exists $\alpha \in \Gamma^{\mathcal{I}}$ such that every open set containing α contains all but finite members of $(\alpha_{m,n})_{m,n\in\mathbb{N}}$. We denote this as $\lim\limits_{m,n\to\infty} \alpha_{m,n} = \alpha$.*

Since we are considering product topology on $\Gamma^{\mathcal{I}}$, we have

$$\lim_{m,n\to\infty} \alpha_{m,n} = \alpha \iff \lim_{m,n\to\infty} d(\alpha_{m,n}|_i, \alpha|_i) = 0 \quad \text{for all } i \in \mathcal{I}.$$

Once again, since d is the discrete metric, one can easily show that

$$\lim_{m,n\to\infty} d(\alpha_{m,n}|_i, \alpha|_i) = \lim_{\substack{m\to\infty\\ n\to\infty}} d(\alpha_{m,n}|_i, \alpha|_i),$$

i.e., the simultaneous limits can be interchanged with the iterated limits.

Lemma 4. *For $(\alpha_{m,n})_{m,n\in\mathbb{N}}$ such that $\alpha_{m,n} \in \Gamma^{\mathcal{I}}$ $\forall m, n \in \mathbb{N}$, if any of the following limits exist then*

$$\lim_{m,n\to\infty} \alpha_{m,n} = \lim_{\substack{m\to\infty\\ n\to\infty}} \alpha_{m,n}.$$

This is important since this allows us to apply Lemma 3 to simultaneous limits.
We are now in a position to define a strongly Cauchy sequence in a BMS.

Definition 15. *A sequence (x_n) in a binary metric space $(X, \xi, \mathcal{I})$ is said to be strongly Cauchy if the limit $\lim\limits_{m,n\to\infty} \xi(x_m, x_n)$ exists.*

Note that the limit need not necessarily be 0. This is not needed since, whenever the limit exists, we have $\lim\limits_{m,n\to\infty} \xi(x_m, x_n)|_i \ominus \xi(x_n, x_n)|_i = 0$.

Unfortunately, not every strongly convergent sequence is strongly Cauchy as can be seen from the sequence $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ in Example 2. In spite of this, the parallels between strongly Cauchy sequences in binary metric spaces and Cauchy sequences in metric spaces make it worth investigating.

We consider an example of a sequence that is strongly Cauchy but not strongly convergent.

Example 3. *Let $X = \{x_n \mid n \in \mathbb{N}\}$ be any countable set. We use $\mathbb{N}$ as the indexing set for the binary metric. Before we proceed, we introduce the following notation.*
We denote by $\mathring{n} \in \Gamma^{\mathbb{N}}$ the following:

$$\pi_i\left(\mathring{n}\right) = \begin{cases} 1 & i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for $a = \mathring{m} \oplus^{\mathbb{N}} \mathring{n}$, we have

$$\pi_i(a_i) = \begin{cases} 1 & i = n \text{ or } i = m, \\ 0 & \text{otherwise.} \end{cases}$$

We define a binary metric ξ as $\xi(x_n, x_n) = \mathring{n}$ and $\xi(x_n, x_m) = \mathring{n} \oplus^{\mathbb{N}} \mathring{m}$ for each $m, n \in \mathbb{N}$. To prove that it is indeed a binary metric, we need only show that it satisfies the "triangle inequality"; the rest is trivial.

For $m, n, p \in \mathbb{N}$, we have $\xi(x_n, x_p) \ominus^{\mathbb{N}} \xi(x_p, x_p) = [\mathring{n} \oplus^{\mathbb{N}} \mathring{p}] \ominus^{\mathbb{N}} \mathring{p} = \mathring{n}$. Therefore, $\xi(x_m, x_n) = \mathring{m} \oplus^{\mathbb{N}} \mathring{n} \preccurlyeq^{\mathbb{N}} \mathring{m} \oplus^{\mathbb{N}} \mathring{p} \oplus^{\mathbb{N}} \mathring{n} = \xi(x_m, x_p) \oplus^{\mathbb{N}} [\xi(x_p, x_n) \ominus^{\mathbb{N}} \xi(x_p, x_p)]$.

Consider the sequence $(x_n)_{n\in\mathbb{N}}$. For each $i \in \mathbb{N}$, we have $\xi(x_m, x_n)|_i = 0$ whenever $m, n > i$. Therefore $\lim_{m,n\to\infty} \xi(x_m, x_n)|_i = 0$, i.e., the sequence is strongly Cauchy.

To test convergence, let $x_N \in X$ be any arbitrary element. Then, $\xi(x_n, x_N) \ominus^{\mathbb{N}} \xi(x_n, x_n) = \overset{\circ}{N}$ for all $n > N$. This means that $\lim_{n\to\infty} \xi(x_n, x_N)|_N \ominus \xi(x_n, x_n)|_N = 1 \neq 0$, i.e., (x_n) does not converge to x_N. Since the choice of x_N was arbitrary, (x_n) is not strongly convergent in X.

Definition 16. *A binary metric space $(X, \xi, \mathcal{I})$ is said to be strongly complete if every strongly Cauchy sequence is strongly convergent in X.*

Definition 17. *1. A subset $Y \subseteq X$ in a topology is said to be sequentially dense in X if, for $x \in X$, there exists a sequence in Y that converges to x. 2. For a BMS $(X, \xi, \mathcal{I})$, we say that $Y \subseteq X$ is strong sequentially dense in X if, for each $x \in X$, we have a sequence (y_n) in Y such that $y_n \xrightarrow{s} x$.*

Since strong convergence implies convergence, a strong sequentially dense subset is also sequentially dense. The converse, however, is not true. Consider the binary metric defined in Example 1 over the set $\mathbb{N}$. The sequence $(2n)_{n\in\mathbb{N}}$ converges to each and every point of $\mathbb{N}$; however, none of its subsequences (including itself) strongly converge to 1. Thus the set $Y = \{2n \mid n \in \mathbb{N}\}$ is sequentially dense but not a strong sequentially dense subset.

Lemma 5. *Let (x_n) and (y_n) be two strongly Cauchy sequences in $(X, \xi, \mathcal{I})$. Then, $\lim_{n\to\infty} \xi(x_n, y_n)$ exists.*

Proof. The triangle inequality gives

$$\xi(x_n, y_n) \preccurlyeq^{\mathcal{I}} \Big[\xi(x_n, x_m) \ominus^{\mathcal{I}} \xi(x_m, x_m)\Big] \oplus^{\mathcal{I}} \xi(x_m, y_m) \oplus^{\mathcal{I}} \Big[\xi(y_m, y_n) \ominus^{\mathcal{I}} \xi(y_m, y_m)\Big]. \tag{5}$$

Now, since (x_n) and (y_n) are strongly Cauchy, we have for each $i \in \mathcal{I}$ the following limits:

$$\lim_{m,n\to\infty} \xi(x_n, x_m)|_i \ominus \xi(x_m, x_m)|_i = 0 \text{ and } \lim_{m,n\to\infty} \xi(y_n, y_m)|_i \ominus \xi(y_m, y_m)|_i = 0.$$

This, along with (5), gives for some $N_i \in \mathbb{N}$ the inequality

$$\xi(x_n, y_n)|_i \leq 0 \oplus \xi(x_m, y_m)|_i \oplus 0 = \xi(x_m, y_m)|_i \text{ whenever } m, n \geq N_i.$$

Since, m, n on both sides can be switched, we obtain $\xi(x_n, y_n)|_i = \xi(x_m, y_m)|_i = \xi(x_N, y_N)|_i$ for all $m, n \geq N_i$, i.e., $\xi(x_n, y_n)|_i$ is eventually constant for each $i \in \mathcal{I}$. In conclusion, the limit $\lim_{n\to\infty} \xi(x_n, y_n)$ exists. □

Theorem 4. *For any BMS $(X, \xi, \mathcal{I})$, there exists an isometry $f : (X, \xi, \mathcal{I}) \to (Z, \gamma, \mathcal{I})$ such that for each strongly Cauchy sequence (x_n) in X, $f(x_n) \xrightarrow{s} z$ for some $z \in Z$. In addition, $f(X)$ is strong sequentially dense in Z.*

Proof. Let

$$\mathcal{K} = \{(x_n) \mid (x_n) \text{ be a strongly Cauchy sequence in } X\}.$$

We define a relation $\sim$ on $\mathcal{K}$ as

$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} \xi(x_n, x_n) = \lim_{n\to\infty} \xi(x_n, y_n) = \lim_{n\to\infty} \xi(y_n, y_n). \tag{6}$$

Now, it is easy to see that $\sim$ is an equivalence relation.

Let Z be the set of equivalence classes in $\mathcal{K}$ for $\sim$. We define $\gamma : Z \times Z \to \Gamma^{\mathcal{I}}$ as follows. For $\tilde{x}, \tilde{y} \in Z$, since these are equivalence classes, we arbitrarily choose sequences $(x_n) \in \tilde{x}$ and $(y_n) \in \tilde{y}$. Then,

$$\gamma(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} \xi(x_n, y_n).$$

By the definition of $\tilde{x}$ and $\tilde{y}$, $(x_n), (y_n)$ are both strongly Cauchy sequences, and Lemma 5 implies that the limit in the RHS of the above equation exists. To prove that the function is indeed well defined, we need only show that the choice of strongly Cauchy sequences does not change the value of the said limit.
Let $(x_n), (x'_n) \in \tilde{x}$ and $(y_n), (y'_n) \in \tilde{y}$. Thus by definition we have, $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$, i.e.,

$$\begin{aligned} \lim_{n\to\infty} \xi(x_n, x'_n) &= \lim_{n\to\infty} \xi(x_n, x_n) = \lim_{n\to\infty} \xi(x'_n, x'_n), \\ \lim_{n\to\infty} \xi(y_n, y'_n) &= \lim_{n\to\infty} \xi(y_n, y_n) = \lim_{n\to\infty} \xi(y'_n, y'_n). \end{aligned} \tag{7}$$

The triangle inequality gives

$$\xi(x_n, y_n) \preccurlyeq^{\mathcal{I}} \left[\xi(x_n, x'_n) \ominus^{\mathcal{I}} \xi(x'_n, x'_n)\right] \oplus^{\mathcal{I}} \xi(x'_n, y'_n) \oplus^{\mathcal{I}} \left[\xi(y'_n, y_n) \ominus^{\mathcal{I}} \xi(y'_n, y'_n)\right].$$

Since the limit on each term exists and the number of terms is finite, Lemma 3 gives

$$\begin{aligned} &\lim_{n\to\infty} \xi(x_n, y_n) \preccurlyeq^{\mathcal{I}} \bar{0} \oplus^{\mathcal{I}} \lim_{n\to\infty} \xi(x'_n, y'_n) \oplus^{\mathcal{I}} \bar{0} \\ \implies &\lim_{n\to\infty} \xi(x_n, y_n) \preccurlyeq^{\mathcal{I}} \lim_{n\to\infty} \xi(x'_n, y'_n). \end{aligned}$$

Similarly, we can prove that

$$\lim_{n\to\infty} \xi(x'_n, y'_n) \preccurlyeq^{\mathcal{I}} \lim_{n\to\infty} \xi(x_n, y_n).$$

Combining the above two equations

$$\lim_{n\to\infty} \xi(x_n, y_n) = \lim_{n\to\infty} \xi(x'_n, y'_n).$$

Therefore $\gamma(\tilde{x}, \tilde{y})$ is well defined.

To prove that γ is a binary metric, we need only prove the triangle inequality, as the rest is trivial. For $\tilde{x}, \tilde{y}, \tilde{z} \in Z$, we choose $(x_n) \in \tilde{x}$, $(y_n) \in \tilde{y}$, and $(z_n) \in \tilde{z}$. Then,

$$\xi(x_n, z_n) \leq \xi(x_n, y_n) \oplus [\xi(y_n, z_n) \ominus \xi(y_n, y_n)].$$

Once again, since the limit on each term exists individually and there are only finite terms in the equation, Lemma 3 gives us

$$\begin{aligned} &\lim_{n\to\infty} \xi(x_n, z_n) \preccurlyeq^{\mathcal{I}} \lim_{n\to\infty} \xi(x_n, y_n) \oplus \lim_{n\to\infty} [\xi(y_n, z_n) \ominus \xi(y_n, y_n)] \\ \implies &\gamma(\tilde{x}, \tilde{z}) \leq \gamma(\tilde{x}, \tilde{y}) \oplus [\gamma(\tilde{y}, \tilde{z}) \ominus \gamma(\tilde{z}, \tilde{z})]. \end{aligned}$$

For every $x \in X$, the sequence $(x, x, x, x, \ldots)$ is strongly Cauchy. Let $\hat{x} \in Z$ be the equivalence class containing the sequence. We define the function $f : x \mapsto \hat{x}$. Thus, we have $\forall x, y \in X$

$$\gamma(f(x), f(y)) = \gamma(\hat{x}, \hat{y}) = \lim_{n\to\infty} \xi(x, y) = \xi(x, y).$$

Thus, f is an isometry.

Let (x_n) be a strongly Cauchy sequence in X and let $\tilde{x} \in Z$ be the equivalence class containing it. Thus, $(x_n) \in \tilde{x}$. We have $f(x_m) = \hat{x}_m$. Since $\hat{x}_m$ contains the sequence $(x_m, x_m, x_m, \dots)$, we have

$$\gamma(f(x_m), f(x_m)) = \gamma(\hat{x}_m, \hat{x}_m) = \xi(x_m, x_m)$$
$$\gamma(f(x_m), \tilde{x}) = \gamma(\hat{x}_m, \tilde{x}) = \lim_{n\to\infty} \xi(x_m, x_n).$$

Finally, since (x_n) is a strongly Cauchy sequence, Lemmas 4 and 3 give us

$$\begin{aligned}\lim_{m\to\infty} \gamma(\hat{x}_m, \tilde{x}) &= \lim_{m\to\infty} \left[\lim_{n\to\infty} \xi(x_m, x_n)\right] = \lim_{m,n\to\infty} \xi(x_m, x_n) \\ &= \lim_{m\to\infty} \xi(x_m, x_m) = \lim_{m\to\infty} \gamma(\hat{x}_m, \hat{x}_m).\end{aligned}$$

Therefore, we have

$$\lim_{m\to\infty} \gamma(f(x_m), \tilde{x}) \ominus^{\mathcal{I}} \gamma(f(x_m), f(x_m)) = \bar{0},$$

i.e., $(f(x_m)) \xrightarrow{s} \tilde{x}$. We have proved that for every strongly Cauchy sequence in X, the sequence of its images under f strongly converges in Z.

For the last claim, let $\tilde{z} \in Z$ and let $(z_n)_{n\in\mathbb{N}} \in \tilde{z}$. Thus, $\lim_{m,n\to\infty} \xi(z_m, z_n)$ exists. The definition of f implies that $f(z_n) = \hat{z}_n$, where $\hat{z}_n$ contains the constant sequence $(z_n, z_n, z_n, \dots)$. Thus, $\gamma(\hat{z}_n, \hat{z}_n) = \xi(z_n, z_n)$. In addition, $\gamma(\hat{z}_n, \tilde{z}) = \lim_{m\to\infty} \xi(z_n, z_m)$. Finally, since (z_n) is a strongly Cauchy sequence,

$$\lim_{n\to\infty} \gamma(\hat{z}_n, \tilde{z}) = \lim_{m,n\to\infty} \xi(z_n, z_m) = \lim_{n\to\infty} \xi(z_n, z_n) = \lim_{n\to\infty} \gamma(\hat{z}_n, \hat{z}_n).$$

Therefore, we have

$$\lim_{n\to\infty} \gamma(f(z_n), \tilde{z}) \ominus^{\mathcal{I}} \lim_{n\to\infty} \gamma(f(z_n), f(z_n)) = \bar{0},$$

i.e., $(f(z_n)) \xrightarrow{s} \tilde{z}$. Therefore, $f(X)$ is strong sequentially dense in Z. □

Note that the space we constructed $(Z, \gamma, \mathcal{I})$ is not necessarily a strong complete BMS. We require an additional condition for this. The condition requires that the indexing set $\mathcal{I}$ is countable. Though this is restrictive for an arbitrary BMS, it still covers a wide range of spaces, most notable of which are the spaces induced by the closed basis of a second countable space. To prove this, we continue from where we left off in Theorem 4 and consider a lemma before the actual proof.

Lemma 6. *From the same construct as in the proof of Theorem 4, $\forall \tilde{z} \in Z$ and $\epsilon \in \Gamma^{\mathcal{I}+}$ for which the set $\{j \in \mathcal{I} \mid \epsilon_j = 0\}$ is finite $\exists \hat{y} \in f(X)$ such that $\tilde{z} \in B(\hat{y}, \epsilon)$ and $\hat{y} \in B(\tilde{z}, \epsilon)$.*

Proof. Let $\tilde{z} \in Z$ and $\epsilon \in \Gamma^{\mathcal{I}+}$ such that the set $\{j \in \mathcal{I} \mid \epsilon_j = 0\}$ is finite.

Choose a strongly Cauchy sequence $(z_n) \in \tilde{z}$. Now, $f(z_m) = \hat{z}_m = (z_m, z_m, z_m, \dots) \in f(X)$. Therefore,

$$\gamma(\tilde{z}, \tilde{z}) = \lim_{n\to\infty} \xi(z_n, z_n)$$
$$\gamma(\tilde{z}, \hat{z}_m) = \lim_{n\to\infty} \xi(z_n, z_m)$$
$$\gamma(\hat{z}_m, \hat{z}_m) = \xi(z_m, z_m).$$

Since (z_n) is strongly Cauchy, we obtain

$$\lim_{m\to\infty} \gamma(\hat{z}_m, \tilde{z}) \ominus^{\mathcal{I}} \gamma(\hat{z}_m, \hat{z}_m) = \bar{0}$$
$$\lim_{m\to\infty} \gamma(\hat{z}_m, \tilde{z}) \ominus^{\mathcal{I}} \gamma(\tilde{z}, \tilde{z}) = \bar{0}.$$

The above equations imply that $\forall i \in \mathcal{I},\ \exists N_i$ such that $\forall m \geq N_i$

$$\lim_{m\to\infty} \gamma(\hat{z}_m, \tilde{z})|_i \ominus^{\mathcal{I}} \gamma(\hat{z}_m, \hat{z}_m)|_i = 0$$
$$\lim_{m\to\infty} \gamma(\hat{z}_m, \tilde{z})|_i \ominus^{\mathcal{I}} \gamma(\tilde{z}, \tilde{z})|_i = 0.$$

Let $N = \max\{N_j \mid j \in \mathcal{I} \text{ and } \epsilon_j = 0\}$. Since $\epsilon_j = 0$ for only finitely many values of j, N exists and is finite. Thus, we have $\forall m \geq N$ and for $\epsilon_j = 0$,

$$\lim_{m\to\infty} \gamma(\hat{z}_m, \tilde{z})|_j \ominus^{\mathcal{I}} \gamma(\hat{z}_m, \hat{z}_m)|_j = 0 = \epsilon_j,$$
$$\lim_{m\to\infty} \gamma(\hat{z}_m, \tilde{z})|_j \ominus^{\mathcal{I}} \gamma(\tilde{z}, \tilde{z})|_j = 0 = \epsilon_j.$$

In other words, $\tilde{z} \in B(\hat{z}_m, \epsilon)$ and $\hat{z}_m \in B(\tilde{z}, \epsilon)$ whenever $m \geq N$. Let $\hat{y} = \hat{z}_m$ for some $m \geq N$, and we obtain the desired result. □

Theorem 5. *From the same construct as in the proof of Theorem 4, if $\mathcal{I}$ is countable, then $(Z, \gamma, \mathcal{I})$ is a strongly complete BMS.*

Proof. Let $(\tilde{z}_n)$ be a strongly Cauchy sequence in Z. In addition, choose a sequence $\epsilon_n \in \Gamma^{\mathcal{I}+}$ such that $\epsilon_n \to \bar{0}$ and for a fixed $n \in \mathbb{N}$, the set $\{j \in \mathcal{I} \mid \epsilon_n|_j = 0\}$ is finite.

Since all the conditions are satisfied, Lemma 6 implies the existence of a sequence $(\hat{y}_n)$ in $f(X)$ such that

$$\hat{y}_n \in B(\tilde{z}_n, \epsilon_n) \text{ and } \tilde{z}_n \in B(\hat{y}_n, \epsilon_n). \tag{8}$$

Successively applying the triangle inequality followed by (8) gives

$$\begin{aligned}\gamma(\hat{y}_n, \hat{y}_m) &\preccurlyeq^{\mathcal{I}} \left[\gamma(\hat{y}_n, \tilde{z}_n) \ominus^{\mathcal{I}} \gamma(\tilde{z}_n, \tilde{z}_n)\right] \oplus^{\mathcal{I}} \gamma(\tilde{z}_n, \tilde{z}_m) \oplus^{\mathcal{I}} \left[\gamma(\tilde{z}_m, \hat{y}_m) \ominus^{\mathcal{I}} \gamma(\tilde{z}_m, \tilde{z}_m)\right] \\ &\preccurlyeq^{\mathcal{I}} \epsilon_n \oplus^{\mathcal{I}} \gamma(\tilde{z}_n, \tilde{z}_m) \oplus^{\mathcal{I}} \epsilon_m.\end{aligned}$$

Since $\epsilon_n \to \bar{0}$, we have $\forall i \in \mathcal{I},\ \exists N_i$ such that $\forall m, n \geq N_i$

$$\gamma(\hat{y}_n, \hat{y}_m)|_i \leq \gamma(\tilde{z}_n, \tilde{z}_m)|_i. \tag{9}$$

Proceeding in a similar manner:

$$\begin{aligned}\gamma(\tilde{z}_n, \tilde{z}_m) &\preccurlyeq^{\mathcal{I}} \left[\gamma(\tilde{z}_n, \hat{y}_n) \ominus^{\mathcal{I}} \gamma(\hat{y}_n, \hat{y}_n)\right] \oplus^{\mathcal{I}} \gamma(\hat{y}_n, \hat{y}_m) \oplus^{\mathcal{I}} \left[\gamma(\hat{y}_m, \tilde{z}_m) \ominus^{\mathcal{I}} \gamma(\hat{y}_m, \hat{y}_m)\right] \\ &\preccurlyeq^{\mathcal{I}} \epsilon_n \oplus^{\mathcal{I}} \gamma(\tilde{z}_n, \tilde{z}_m) \oplus^{\mathcal{I}} \epsilon_m.\end{aligned}$$

Again, since $\epsilon_n \to \bar{0}$, we have $\forall i \in \mathcal{I},\ \exists N_i'$ such that $\forall m, n \geq N_i'$

$$\gamma(\tilde{z}_n, \tilde{z}_m)|_i \leq \gamma(\hat{y}_n, \hat{y}_m)|_i. \tag{10}$$

From (9) and (10), we have $\forall m, n \geq N = \max\{N_i, N_i'\}$,

$$\gamma(\tilde{z}_n, \tilde{z}_m)|_i = \gamma(\hat{y}_n, \hat{y}_m)|_i.$$

Therefore, since $(\tilde{z}_n)$ is strongly Cauchy, so is $(\hat{y}_n)$.

Finally, from Theorem 4, $\exists \tilde{z} \in Z$ such that $\hat{y}_n \xrightarrow{s} \tilde{z}$ i.e.,

$$\lim_{n\to\infty} \gamma(\tilde{z}, \hat{y}_n) \ominus^{\mathcal{I}} \gamma(\hat{y}_n, \hat{y}_n) = \bar{0}.$$

Now,

$$\begin{aligned}\gamma(\tilde{z}, \tilde{z}_n) \ominus^{\mathcal{I}} \gamma(\tilde{z}_n, \tilde{z}_n) &\preccurlyeq^{\mathcal{I}} \left[\gamma(\tilde{z}, \hat{y}_n) \ominus^{\mathcal{I}} \gamma(\hat{y}_n, \hat{y}_n)\right] \oplus^{\mathcal{I}} \left[\gamma(\hat{y}_n, \tilde{z}_n) \ominus^{\mathcal{I}} \gamma(\tilde{z}_n, \tilde{z}_n)\right] \\ &\preccurlyeq^{\mathcal{I}} \left[\gamma(\tilde{z}, \hat{y}_n) \ominus^{\mathcal{I}} \gamma(\hat{y}_n, \hat{y}_n)\right] \oplus^{\mathcal{I}} \epsilon_n.\end{aligned}$$

Hence, $\lim_{n\to\infty} \gamma(\tilde{z}, \tilde{z}_n) \ominus^{\mathcal{I}} \gamma(\tilde{z}_n, \tilde{z}_n) = \bar{0}$, i.e., $\tilde{z}_n \xrightarrow{s} \tilde{z}$. $(Z, \gamma, \mathcal{I})$ is strongly complete. □

Now that we have a completion at hand, we revisit Example 3 and consider its completion.

Example 3 (continued)**.** *Let (y_n) be a strongly Cauchy sequence in X. Then there are two possible cases:*

Case 1: *If there exists $j \in \mathbb{N}$ for which $\lim_{n\to\infty} \xi(y_n, y_n)|_j = 1 \implies$, there exists $N \in \mathbb{N}$ such that $y_n = x_j$ for $n \geq N$, i.e., the sequences are eventually constant. In addition, since $x_i \neq x_j$ for $i \neq j$, $\lim_{n\to\infty} \xi(y_n, y_n)|_i = 0$ for all $i \neq j$.*

Case 2: *For any $i \in \mathbb{N}$, $\lim_{n\to\infty} \xi(y_n, y_n)|_i = 0$. This means that each x_i appears only finitely many times in the sequence.*

These two are the only types of strongly Cauchy sequences in this BMS.

We now consider the sequences that belong to the same equivalence class as that constructed in Theorem 4. We note that since for each $y, z \in X$, $\xi(y, z) = \xi(y, y) \oplus^{\mathbb{N}} \xi(z, z)$, we have, for the strongly Cauchy sequences $(y_n), (z_n)$ in X, $(y_n) \sim (z_n)$ if and only if the following condition holds for each $i \in \mathbb{N}$:

$$\lim_{n\to\infty} \xi(y_n, y_n)|_i = 1 \iff \lim_{n\to\infty} \xi(z_n, z_n)|_i = 1.$$

In other words, two strongly Cauchy sequences belong to the same equivalence class if and only if any one of the sequences being eventually constant implies that the other one is eventually constant with the same term. This also means that all the strongly Cauchy sequences that are not eventually constant (those that belong to Case 2) vacuously satisfy the condition and thus are all in the same equivalence class. This can also be seen from the fact that if (x_n) and (y_n) belong to Case 2, then $\lim_{n\to\infty} \xi(x_n, x_n) = \lim_{n\to\infty} \xi(x_n, y_n) = \lim_{n\to\infty} \xi(y_n, y_n) = 0 \implies (x_n)\tilde{(}y_n)$.

Now, it is easy to see that the strongly Cauchy sequences that are eventually constant to, say x_N, belong to $\hat{x}_N$. For the others, we introduce the class $\tilde{x}_\infty$. Thus the set $Z = \{\hat{x}_N | N \in \mathbb{N}\} \cup \{\tilde{x}_\infty\}$. Furthermore, the binary metric γ is given by

$$\begin{aligned}\gamma(\hat{x}_m, \hat{x}_n) &= \mathring{m} \oplus^{\mathbb{N}} \mathring{n} \\ \gamma(\hat{x}_n, \tilde{x}_\infty) &= \lim_{k\to\infty} \xi(x_n, x_k) = \mathring{n} \\ \gamma(\tilde{x}_\infty, \tilde{x}_\infty) &= \lim_{k\to\infty} \xi(x_k, x_k) = \bar{0}.\end{aligned}$$

Therefore, the completion requires the addition of only one new point $\tilde{x}_\infty$.

4. Conclusions and Further Work

We presented the definition of strong convergence in a binary metric space as an alternative to the usual concept of convergence. We showed that both these concepts coincide in the case of a metric induced by the closed basis of a Kuratowski topology. We presented the definitions of a strongly Cauchy sequence and a strongly complete binary metric space and gave a condition under which the completion exists. The following are some of the questions that should be answered to better understand binary metric spaces.

Problem 1. *We have given a sufficient condition under which strong convergence and convergence are equivalent in a BMS. Under what other conditions does the equivalence hold?*

Problem 2. *The fact that a strongly convergent sequence is not necessarily strongly Cauchy prevents us from determining the uniqueness of the strong completion. Is the strong completion unique? If not, under what conditions is it unique?*

Problem 3. *What other topologies on* $\Gamma^{\mathcal{I}}$ *can be considered to define other forms of convergence on a BMS? How drastically does varying this topology vary the form of convergence?*

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