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Bifurcation Results for Periodic Third-Order Ambrosetti-Prodi-Type Problems

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Abstract: This paper presents sufficient conditions for the existence of a bifurcation point for nonlinear periodic third-order fully differential equations. In short, the main discussion on the parameter s about the existence, non-existence, or the multiplicity of solutions, states that there are some critical numbers σ_0 and σ_1 such that the problem has no solution, at least one or at least two solutions if $s < \sigma_0$, $s = \sigma_0$ or $\sigma_0 > s > \sigma_1$, respectively, or with reversed inequalities. The main tool is the different speed of variation between the variables, together with a new type of (strict) lower and upper solutions, not necessarily ordered. The arguments are based in the Leray–Schauder's topological degree theory. An example suggests a technique to estimate for the critical values σ_0 and σ_1 of the parameter.

Keywords: higher-order periodic problems; lower and upper solutions; Nagumo condition; degree theory

MSC: 34B15; 34B08; 34C25

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1. Introduction

This work deals with a third-order nonlinear fully differential equation with a weighted parameter

$$v'''(t) + f\big(t, v(t), v'(t), v'(t)\big) = s\, h(t),\ t \in [0, T],\ T > 0, \tag{1}$$

where $f : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ and $h : [0, T] \to (0, +\infty)$ are continuous functions, $\sigma \in \mathbb{R}$, together with the usual periodic boundary conditions

$$v^{(i)}(0) = v^{(i)}(T),\ i = 0, 1, 2. \tag{2}$$

Third-order equations, known in the literature as *jerk* equations, have been studied by many authors, not only from a purely mathematical approach but also in several fields where the study of the *jerk* dynamics is relevant. As examples, we mention: the Lorenz–Dirac equation for one of a pair of interacting electrons when radiation reaction is included [1]; the model of the transverse motion of a piano string to simulate the effect of a frequency-dependent decay [2]; the global dynamics of some jerk dynamical systems studying necessary and sufficient requirements in a time-continuous, autonomous dynamical system, to exhibit chaos [3]; the existence of attractors in chaotic flows in three dimensions dissipative and conservative dynamical systems [4,5].

The study of the periodic orbits of differential equations is an important line of research, namely: to obtain sufficient conditions for the non-existence and multiplicity for strongly nonlinear differential equations [6]; the existence of periodic orbits as limit cycles [7], or as solutions of the ϕ-Laplacian generalized Liénard equations [8]; solvability of higher-order periodic problems with fully differential equations [9], and singular third order problems via cones theory [10]; equations with asymptotically sign-changed nonlinearities [11], or with anti-periodic boundary conditions [12]; oscillations of nonlinear even order differential equations [13].

Equations with parameters, as in (1), are called Ambrosetti–Prodi type equations, as they have been introduced in [14]. Since then, they have been studied in several boundary value problems, such as two-point boundary conditions [15], Neuman's type [16], three-point problems [17], the periodic case [18–20], analysis for parametric problems driven by the nonlinear Robin (p, q)-Laplace operator [21], with different asymptotically behaviours [22] or with the fractional Laplacian [23], among others.

In short, the main discussion on the parameter s about the existence, non-existence, or the multiplicity of solutions, is given by the so-called Alternative by Ambrosetti–Prodi: there are real numbers σ_0 and σ_1 such that the problem has no solution, one or two solutions if $s < \sigma_0$, $s = \sigma_0$ or $\sigma_0 > s > \sigma_1$, respectively, or with reversed inequalities.

In [24], for a particular case of the problem (1) and (2), the authors prove the existence of solutions for the values of the parameter s such that there are lower and upper solutions for the problem. This paper completes the discussion of the non-existence and multiplicity of periodic solutions of (1) and (2), on the weighted parameter s.

These new discussions were possible due to a condition that requires different speeds of variation between the variables (see (11) in Theorem 2 and (18) in Theorem 3). A new type of (strict) lower and upper solutions, not necessarily ordered, plays a key role in the periodic structure of the problem, together with a Nagumo-type growth condition, which implies a subquadratic growth on the nonlinear part. The main tool for the multiplicity discussion is the Leray–Schauder's topological degree theory.

Moreover, for the first time, it exemplified a method to have approximations of the critical values of the parameter. This is particularly useful in applications, as in the thyroid-pituitary homeostatic mechanism studied in [25–27], where the various parameters have well-defined biological and physiological meanings, as it is shown in [24].

The paper is organized in the following way: in Section 2 we have the definition of lower and upper solutions, an *a priori* bound for the second derivative via Nagumo's condition, and an already known existence theorem. Section 3 contains a first existence and non-existence discussion on the parameter s, and in Section 4 it is obtained sufficient conditions for the existence of a bifurcation point. In the last section, we present an example and a technique that allows estimates for the critical values σ_0 and σ_1 of the parameter.

2. Definitions and *a Priori* Estimations

In higher-order periodic boundary value problems, the order between lower and upper solutions is an issue that should be avoided. The next definition follows a method to overcome it, shifting upper and lower solutions by perturbation with the sup norm,

$$\|w\| := \sup_{t\in[0,T]} |w(t)|,$$

as it is suggested in [9]:

Definition 1. *The function $\gamma \in C^3[0, T]$ is a lower solution of problem (1) and (2) if:*

(i) $\gamma'''(t) + f(t, \gamma_0(t), \gamma'(t), \gamma''(t)) \geq s\, h(t)$

where

$$\gamma_0(t) := \gamma(t) - \|\gamma\|; \tag{3}$$

(ii) $\gamma'(0) = \gamma'(T), \gamma''(0) \geq \gamma''(T)$.

The function $\Gamma \in C^3[0, T]$ is an upper solution of problem (1) and (2) if:

(iii) $\Gamma'''(t) + f(t, \Gamma_0(t), \Gamma'(t), \Gamma''(t)) \leq s\, h(t)$

where

$$\Gamma_0(t) := \Gamma(t) + \|\Gamma\|; \tag{4}$$

(iv) $\Gamma'(0) = \Gamma'(T), \Gamma''(0) \leq \Gamma''(T)$.

We underline that although γ and Γ are not necessarily ordered, the auxiliary functions γ_0 and Γ_0 are well ordered, as

$$\gamma_0(t) \leq 0 \leq \Gamma_0(t), \text{ for every } t \in [0, T].$$

The unique growth condition to require on the nonlinearity in (1) is given by a bilateral Nagumo-type condition, following [15]:

Definition 2. *A continuous function $\varphi : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ verifies a Nagumo-type condition relatively to some continuous functions $\gamma_i, \Gamma_i, i = 0, 1$, such that $\gamma_i(t) \leq \Gamma_i(t)$, for every $t \in [0, T]$, in the set*

$$S = \left\{(t, x_0, x_1, x_2) \in [0, T] \times \mathbb{R}^3 : \gamma_i(t) \leq x_i \leq \Gamma_i(t),\ i = 0, 1\right\},$$

if there is a continuous function $\psi_S : [0, +\infty[\to]0, +\infty[$ such that

$$|\varphi(t, x_0, x_1, x_2)| \leq \psi_S(|x_2|),\ \forall (t, x_0, x_1, x_2) \in S, \tag{5}$$

with

$$\int_0^{+\infty} \frac{z}{\psi_S(z)}\, dz = +\infty\,. \tag{6}$$

From this condition, it is possible to estimate the second derivative as it was proved in [28]:

Lemma 1. *Let $\varphi : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function verifying the Nagumo-type conditions (5) and (6) in S. Then there is $\rho > 0$ such that every solution $y(t)$ of (1) verifying*

$$\gamma_0(t) \leq y(t) \leq \Gamma_0(t),\ \gamma_1(t) \leq y'(t) \leq \Gamma_1(t)$$

for every $t \in [0, T]$, satisfies

$$\|y''\| < \rho.$$

Remark 1. *The radius ρ depends only on the parameter s and on the functions ψ_S, γ_1 and Γ_1 and it can be taken independent of s as long as it belongs to a bounded set.*

For the values of the parameter s such that there are upper and lower solutions of (1) and (2) where the first derivatives are well ordered, we refer the following theorem in [24], defined for $t \in [0, 1]$, but easily adapted to a more general interval $[0, T]$:

Theorem 1. *Let $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ and $h : [0, 1] \to \mathbb{R}^+$ be continuous functions. Assume that there are lower and upper solutions to Equations (1) and (2), $\gamma(t)$ and $\Gamma(t)$, respectively, accordingly Definition 1, such that*

$$\gamma'(t) \leq \Gamma'(t), \text{ for } t \in [0, 1],$$

and f verifies the Nagumo-type conditions (5) and (6) in S.

If

$$f(t, \gamma_0(t), x_1, x_2) \leq f(t, x_0, x_1, x_2) \leq f(t, \Gamma_0(t), x_1, x_2), \tag{7}$$

for fixed $(t, x_1, x_2) \in [0, 1] \times \mathbb{R}^2$ and $\gamma_0(t) \leq x_0 \leq \Gamma_0(t)$, then (1) and (2) has at least one solution $v(t) \in C^3([0, 1])$ such that $\gamma_0(t) \leq v(t) \leq \Gamma_0(t), \gamma'(t) \leq v'(t) \leq \Gamma'(t), \forall t \in [0, 1]$.

3. Existence and Non-Existence Theorem

The first discussion on s about the existence and nonexistence of a solution will be done in this section, for nonlinearities verifying an adequate speed growth condition.

Theorem 2. *Consider* $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ *a continuous function satisfying a Nagumo-type condition such that:*

(**i**) *for* $(t,y,z) \in [0,T] \times \mathbb{R}^2$

$$x_1 \geq x_2 \Rightarrow f(t,x_1,y,z) \geq f(t,x_2,y,z); \tag{8}$$

(**ii**) *for* $(t,x,z) \in [0,T] \times \mathbb{R}^2$

$$y_1 \geq y_2 \Rightarrow f(t,x,y_1,z) \leq f(t,x,y_2,z); \tag{9}$$

(**iii**) *there are* $\sigma_1 \in \mathbb{R}$ *and* $r > 0$ *such that*

$$\frac{f(t,0,0,0)}{h(t)} < \sigma_1 < \frac{f(t,x,-r,0)}{h(t)}, \tag{10}$$

for every $t \in [0,T]$ *and every* $x \leq -r$;

(**iv**) *for* $\nu > 0$ *such that, for every* $(t,x,y,z) \in [0,T] \times \mathbb{R}^3$ *and* $T \leq \xi \leq 2T$,

$$f(t,x+\xi\nu,y+\nu,z) \leq f(t,x,y,z). \tag{11}$$

Then there is $\sigma_0 < \sigma_1$ *(eventually* $\sigma_0 = -\infty$*) such that:*

(1) *for* $s < \sigma_0$, *(1) and (2) has no solution;*

(2) *for* $\sigma_0 < s \leq \sigma_1$, *(1) and (2) has at least one solution.*

Proof.

Claim 1. *There is* $\sigma^* < \sigma_1$ *such that* (E_{σ^*})*-(2)* has a solution.

Defining

$$\sigma^* = \max\left\{\frac{f(t,0,0,0)}{h(t)}, t \in [0,T]\right\},$$

by (10), there exist $t^* \in [0,T]$ such that

$$\frac{f(t,0,0,0)}{h(t)} \leq \sigma^* = \frac{f(t^*,0,0,0)}{h(t^*)} < \sigma_1, \forall t \in [0,T]. \tag{12}$$

Thus $\Gamma(t) \equiv 0$ is a trivial upper solution of (E_{σ^*})-(2).

The function $\gamma(t) = -rt$ is a lower solution of (E_{σ^*})-(2) with $\gamma_0(t) = -rt - rT$, as by (8) and (10)

$$\begin{aligned} \gamma'''(t) &= 0 > \sigma_1 h(t) - f(t,-rt-r,-r,0) \\ &> \sigma^* h(t) - f(t,-rt-r,-r,0). \end{aligned}$$

So, by Theorem 1, there is at least a solution of (E_{σ^*})-(2) with $\sigma^* < \sigma_1$.

Claim 2. *If (1) and (2) has a solution for* $s < \sigma_1$, *then it has at least one solution for* $\sigma \in [s,\sigma_1]$.

Suppose that (1) and (2) has a solution $v_s(t)$. For σ such that $s \leq \sigma \leq \sigma_1$, $R > 0$, and, by (11),

$$\begin{aligned} v_s'''(t) &= sh(t) - f\big(t,v_s(t),v_s'(t),v_s''(t)\big) \\ &\leq sh(t) - f\big(t,v_s(t)+R(t+T),v_s'(t)+R,v_s''(t)\big) \\ &\leq \sigma h(t) - f\big(t,v_s(t)+R(t+T),v_s'(t)+R,v_s''(t)\big), \end{aligned}$$

and so $v_s(t) + Rt$ is an upper solution of (1) and (2), for every $\sigma \in [s, \sigma_1]$, with $\Gamma_0(t) = v_s(t) + \|v_s\|_\infty + R(t+T)$.

For $r > 0$ given by (10), take R large enough such that $RT \geq r$,

$$v_s'(0) \geq -R, v_s'(T) \geq -R \text{ and } \min_{t\in[0,T]} v_s(t) \geq -R \tag{13}$$

By (8) and (10) ,for $\sigma \leq \sigma_1$,

$$0 > \sigma_1 h(t) - f(t, -R(t+T), -r, 0) \geq \sigma h(t) - f(t, -R(t+T), -R, 0).$$

Then $\gamma(t) = -Rt$ is a lower solution of (1) and (2) for $\sigma \leq \sigma_1$, with $\gamma_0(t) = -Rt - RT$. To apply Theorem 2, the condition

$$-R \leq v_s'(t) + R, \forall t \in [0, T], \tag{14}$$

must be verified. Suppose that (14) is not true. Therefore there is $t \in [0, T]$ such that $v_s'(t) < -2R$.

Defining

$$\min_{t\in[0,T]} v_s'(t) := v_s'(t_0) \tag{15}$$

then, by (13), $t_0 \in [0, T]$ and, by (15), $v_\sigma''(t_0) = 0$ and $v_\sigma'''(t_0) > 0$.

By (9), (10) and (13), the following contradiction holds

$$\begin{aligned} 0 &\leq v_s'''(t_0) = sh(t_0) - f\left(t_0, v_s(t_0), v_s'(t_0), v_s''(t_0)\right) \\ &\leq sh(t_0) - f(t_0, v_s(t_0), -R, 0) \\ &\leq \sigma_1 h(t_0) - f(t_0, -R, -R, 0) < 0. \end{aligned}$$

So $-R \leq v_s'(t)$,for every $t \in [0, T]$, and, by Theorem 2, problem (1) and (2) has at least one solution $v(t)$ for every σ such that $s \leq \sigma \leq \sigma_1$.

Claim 3. *There is $\sigma_0 \in \mathbb{R}$ such that:*

- *for $s < \sigma_0$, (1) and (2) has no solution;*
- *for $s \in]\sigma_0, \sigma_1]$, (1) and (2) has at least one solution.*

Let $C = \{\sigma \in \mathbb{R} : (1) \text{ and } (2) \text{ has at least a solution}\}$.

As, by Claim 1, $\sigma^* \in C$ then $C \neq \emptyset$.

Defining $\sigma_0 = \inf C$, by Claim 1, $\sigma_0 \leq \sigma^* < \sigma_1$ and, by Claim 2, (1) and (2) has at least a solution for $s \in [\sigma_0, \sigma_1]$ and (1) and (2) has no solution for $s < \sigma_0$.

If $\sigma_0 = -\infty$ then, by Claim 2, (1) and (2) has a solution for every $s \leq \sigma_1$. □

4. Existence of a Bifurcation Point

The existence of a bifurcation point will be proved in presence of strict lower and upper solutions, according to the next definition:

Definition 3. *The function $\gamma \in C^3[0, T]$ is a strict lower solution of problem (1) and (2) if*

(i) $\gamma'''(t) + f(t, \gamma_0(t), \gamma'(t), \gamma''(t)) > s\, h(t)$, *with γ_0 given by (3);*
(ii) $\gamma'(0) = \gamma'(T), \gamma''(0) \geq \gamma''(T)$.

The function $\Gamma \in C^3[0, T]$ is a strict upper solution of problem (1) and (2) if

(iii) $\Gamma'''(t) + f(t, \Gamma_0(t), \Gamma'(t), \Gamma''(t)) < s\, h(t)$, *with Γ_0 given by (4);*
(iv) $\Gamma'(0) = \Gamma'(T), \Gamma''(0) \leq \Gamma''(T)$.

The multiplicity of solutions is proven by the topological degree theory applied to a homotopic modified and perturbed problem. In short, the main assumptions require that f is bounded from below verifying some adequate growth conditions.

Theorem 3. *We assume that $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function verifying the assumptions of Theorem 2.*

If there are $B > 0$ such that every solution v of (1) and (2) with $s \le \sigma_1$, verifies

$$\left|v'(t)\right| \le \frac{B}{2}\,, \forall t \in [0,T], \tag{16}$$

and $b \in \mathbb{R}$ such that

$$f(t,x,y,z) \ge b\,h(t), \tag{17}$$

for every $(t,x,y,z) \in [0,T] \times [-rT + \gamma_0(0), BT + \Gamma_0(0)] \times [-r,B] \times \mathbb{R}$, with r given by (10), then σ_0, given by Theorem 2, is finite and:

(1) if $s < \sigma_0$, (1) and (2) has no solution;

(2) if $s = \sigma_0$, (1) and (2) has at least one solution.

Moreover, if condition (11) is replaced by,

$$f(t, x + \xi\nu_1 + \nu_2, y + \xi, z) \le f(t,x,y,z), \tag{18}$$

for every $(t,x,y,z) \in [0,T] \times [-C,C]^2 \times \mathbb{R}$, where $C := \max\{r, rT - \gamma_0(0), BT + \Gamma_0(0)\}$, and ν_1, ν_2, ξ are positive constants, then

(3) for $s \in]\sigma_0, \sigma_1]$, (1) and (2) has at least two solutions.

Proof. Consider the truncature functions

$$\delta_0(t,x) = \begin{cases} \Gamma_0(t) & , \quad x > \Gamma_0(t) \\ x & , \quad \gamma_0(t) \le x \le \Gamma_0(t) \\ \gamma_0(t) & , \quad x < \gamma_0(t), \end{cases}$$

$$\delta_1(t,y) = \begin{cases} \Gamma'(t) & , \quad y > \Gamma'(t) \\ y & , \quad \gamma'(t) \le y \le \Gamma'(t) \\ \gamma'(t) & , \quad y < \gamma'(t), \end{cases} \tag{19}$$

and the modified problem composed of the homotopic and perturbed differential equation

$$\begin{aligned} v'''(t) + \lambda f\big(t, \delta_0(t,v(t)), \delta_1(t,v'(t)), v''(t)\big) \\ -v'(t) = \lambda\,\big[s\,h(t) - \delta_1(t,v'(t))\big], \end{aligned} \tag{20}$$

for $\lambda \in [0,1]$, and the homotopic boundary conditions

$$\begin{aligned} v(0) &= \lambda\delta^*(v(T)), \\ v^{(j)}(0) &= v^{(j)}(T) \end{aligned} \tag{21}$$

with $j = 1,2$, and

$$\delta^*(w) = \begin{cases} \Gamma_0(0) & , \quad w > \Gamma_0(0) \\ w & , \quad \gamma_0(0) \le w \le \Gamma_0(0) \\ \gamma_0(0) & , \quad w < \gamma_0(0). \end{cases} \tag{22}$$

Consider the set

$$Y = \left\{y \in C^2([0,T]) : y^{(j)}(0) = y^{(j)}(T),\ j = 0,1,2\right\}.$$

Define the operators $\mathcal{L} : C^3([0,T]) \cap Y \to C^2([0,T]) \times \mathbb{R}^3$ given by

$$\mathcal{L}v = \left(v''' - v', v(0), v'(0), v''(0)\right)$$

and, for $s \in \mathbb{R}$, $\mathcal{F}_s : C^2([0,T]) \cap Y \to C^2([0,T]) \times \mathbb{R}^3$ by

$$\mathcal{F}_s v = \begin{pmatrix} \lambda[s\,h(t) - f(t,\delta_0(t,v(t)),\delta_1(t,v'(t)),v''(t)) - \delta_1(t,v'(t))], \\ \lambda\delta^*(v(T)),v'(T),v''(T) \end{pmatrix}.$$

As $\mathcal{L}^{-1}$ is compact then it can be defined the completely continuous operator $\mathcal{T}_\lambda : C^3([0,T]) \to C^3([0,T])$ by

$$\mathcal{T}_\lambda v = \mathcal{L}^{-1}(\mathcal{F}_s v).$$

Claim 4. *Problem (1) and (21) has a solution for* $\lambda = 1$.

Following the arguments in [24] (steps 1 and 2 in the proof of Theorem 1), we have *a priori* estimations, that is, for every solution of the problem (1) and (21) exist $r_i > 0$, $i = 0,1,2$, such that

$$\left\| v^{(i)} \right\| < r_i,\ i = 0,1,2.$$

Consider the set

$$\Omega_1 = \left\{ v \in C^2([0,T]) : \left\| v^{(i)} \right\| < r_i,\ i = 0,1,2 \right\}.$$

By homotopic invariance of the degree

$$d(\mathcal{T}_0,\Omega_1) = d(\mathcal{T}_1,\Omega_1). \tag{23}$$

The equation $\mathcal{T}_0(v) = 0$, corresponds to the linear problem

$$\begin{gathered} v'''(t) - v'(t) = 0, \\ v(0) = 0, \\ v'(0) = v'(T), \\ v''(0) = v''(T), \end{gathered}$$

which has only a trivial solution. Therefore, by degree theory,

$$d(\mathcal{T}_0,\Omega_1) = \pm 1. \tag{24}$$

By (23) and (24), $d(\mathcal{T}_1,\Omega_1) \neq 0$, that is the equation, $\mathcal{T}_1(v) = v$, corresponding to the problem

$$\begin{gathered} v'''(t) - v'(t) = s\,h(t) - f(t,\delta_0(t,v(t)),\delta_1(t,v'(t)),v''(t)) - \delta_1(t,v'(t)), \\ v(0) = \delta^*(v(T)), \\ v'(0) = v'(T), \\ v''(0) = v''(T), \end{gathered}$$

has at least a solution v_0 in Ω_1.

Define the set

$$\Omega = \left\{ v \in dom\mathcal{L} : \gamma_0(t) < v(t) < \Gamma_0(t), \gamma'(t) < v'(t) < \Gamma'(t), \|v''\| < r_2 \right\}.$$

Claim 5. *If* $v_0 \in \Omega_1$ *is a solution of* $T_1(v) = v$ *then* $v_0 \in \Omega$.

Suppose, by contradiction, that exists $t \in [0,T]$ such that

$$v_0'(t) \le \gamma'(t),$$

and

$$\min_{t\in[0,T]} \left[v_0'(t) - \gamma'(t) \right] := v_0'(t_1) - \gamma'(t_1) \le 0.$$

If $t_1 \in \,]0,T[$, then

$$\begin{array}{c} v_0''(t_1) = \gamma''(t_1), \\ v_0'''(t_1) \geq \gamma'''(t_1). \end{array} \tag{25}$$

By (8) and Definition 3, we have the following contradiction with (25):

$$\begin{array}{rcl} v_0'''(t_1) & = & s\,h(t_1) - f\big(t_1, \delta_0(t_1, v_0(t_1)), \delta_1(t_1, v_0'(t_1)), v_0''(t_1)\big) \\ & & + v_0'(t_1) - \delta_1(t_1, v_0'(t_1)) \\ & \leq & s\,h(t_1) - f\big(t_1, \gnsim_0(t_1), \gnsim'(t_1), \gnsim''(t_1)\big) + v_0'(t_1) - \gnsim'(t_1) \\ & \leq & s\,h(t_1) - f\big(t_1, \gnsim_0(t_1), \gnsim'(t_1), \gnsim''(t_1)\big) < \gamma'''(t_1). \end{array}$$

Then $v_0'(t) > \gamma'(t), \forall t \in \,]0,T[$.

If $t_1 = 0$ or $t_1 = T$ we have, by (2) and Definition 3 (ii),

$$\min_{t\in[0,T]} \left[v_0'(t) - \gamma'(t)\right] := v_0'(0) - \gamma'(0) = v_0'(T) - \gamma'(T) \leq 0,$$

and

$$0 \leq v_0''(0) - \gamma''(0) \leq v_0''(T) - \gamma''(T) \leq 0.$$

Therefore

$$v_0''(0) - \gamma''(0) = 0 \text{ and } v_0'''(0) - \gamma'''(0) \geq 0\,.$$

Applying an analogous technique to the previous case, it can be proved that $v_0'(0) > \gamma'(0)$ and $v_0'(T) > \gamma'(T)$. Then

$$\gamma'(t) < v_0'(t),\ \forall t \in [0,T].$$

Applying an analogous technique, we obtain $v_0'(t) < \Gamma'(t), \forall t \in [0,T]$, and so

$$\gamma'(t) < v_0'(t) < \Gamma'(t),\ \forall t \in [0,T]. \tag{26}$$

By integration of the second inequality of (26) on $[0,t]$, we get, by (22) and Definition 3,

$$\begin{array}{rcl} v_0(t) & < & \Gamma(t) - \Gamma(0) + v_0(0) = \Gamma(t) - \Gamma(0) + \delta^*(v(T)) \\ & \leq & \Gamma(t) - \Gamma(0) + \Gamma_0(0) \\ & = & \Gamma(t) + \|\Gamma\| = \Gamma_0(t),\ \forall t \in [0,T]. \end{array}$$

Similarly, we have

$$\gamma_0(t) < v_0(t),\ \forall t \in [0,T].$$

Therefore $v_0 \in \Omega$, and by the excision property of the topological degree

$$d(\mathcal{T}_1, \Omega) = d(\mathcal{T}_1, \Omega_1) = \pm 1.$$

Claim 6. *Every solution v of problem (1) and (2) for $s \in [\sigma_0, \sigma_1]$, satisfies*

$$-r < v'(t) < \frac{B}{2} \text{ and } -rT + \gamma_0(0) < v(t) < \frac{B}{2}T + \Gamma_0(0), \forall t \in [0,T],$$

with r given by (10) and B by (16).

Assume, by contradiction, that there are $s \in \,]\sigma_0, \sigma_1]$, a solution, v, of (1) and (2) and $\tau \in [0,T]$ such that

$$v'(\tau) := \min_{t\in[0,T]} v'(t) \leq -r.$$

If $\tau \in]0,T[$, then $v''(\tau) = 0$ and $v'''(\tau) \geq 0$. From (9),

$$0 \leq v'''(\tau) = s\, h(\tau) - f\big(\tau, v(\tau), v'(\tau), v''(\tau)\big) \leq \sigma_1\, h(\tau) - f(\tau, v(\tau), -r, 0).$$

For $v(\tau) < -r$, from (10), we have the contradiction

$$0 \leq \sigma_1 h(\tau) - f(\tau, v(\tau), -r, 0) < 0.$$

In the case $v(\tau) \geq -r$, from (8) and (10), a similar contradiction is achieved

$$0 \leq \sigma_1 h(\tau) - f(\tau, v(\tau), -r, 0) \leq \sigma_1 h(\tau) - f(\tau, -r, -r, 0) < 0.$$

If $\tau = 0$ or $\tau = T$,

$$\min_{t\in[0,T]} v'(t) = v'(0) = v'(T).$$

Then $0 \leq v''(0) = v''(T) \leq 0$ and, therefore,

$$v''(0) = v''(T) = 0,\ v'''(0) \geq 0,\ v'''(T) \geq 0.$$

Applying an analogous technique to the previous case it can be proved similar contradictions.

So, every solution v of (1) and (2), with $\sigma_0 < s \leq \sigma_1$, verifies

$$v'(t) > -r, \forall t \in [0,T],$$

and, therefore,

$$-r < v'(t) < \frac{B}{2}, \forall t \in [0,T].$$

Integrating on $[0,t]$, we obtain

$$-rT + \gamma_0(0) < v(t) < \frac{B}{2}T + \Gamma_0(0), \forall t \in [0,T].$$

Claim 7. *σ_0 is finite.*

Assume that $\sigma_0 = -\infty$. So, by Theorem 2, for every $s \leq \sigma_1$ problem (1) and (2) has at least a solution.

Define $h_1 := \min_{t\in[0,T]} h(t) > 0$, and take s sufficiently small such that

$$b - s > 0 \text{ and } \frac{T(b-s)h_1}{16} > B. \tag{27}$$

For v solution of (1) and (2), we have, by (17),

$$v'''(t) = s\, h(t) - f\big(t, v(t), v'(t), v''(t)\big) \leq (s-b)h(t)$$

and, by (2), there exists $\xi \in]0,T[$ such that $v''(\xi) = 0$. For $t < \xi$

$$v''(t) = -\int_t^{\xi} v'''(\tau)d\tau \geq \int_t^{\xi} (b-s)h(\tau)d\tau \geq (b-s)(\xi - t)h_1.$$

For $t \geq \xi$

$$v''(t) = -\int_{\xi}^{t} v'''(\tau)d\tau \leq (s-b)(t-\xi)h_1.$$

Choose $I = \left[0, \frac{T}{4}\right]$,or $I = \left[\frac{3}{4}T, T\right]$, such that $|\xi - t| \geq \frac{T}{4}$, for every $t \in I$. If $I = \left[0, \frac{T}{4}\right]$, then

$$v''(t) \geq \frac{T(b-s)}{4} h_1, \forall t \in I.$$

If $I = \left[\frac{3}{4}T, T\right]$, we have

$$v''(t) \leq \frac{T(b-s)}{4} h_1, \forall t \in I.$$

In the first case, by (27) and (16), we have the contradiction

$$\begin{aligned} 0 &= \int_0^T v''(\tau)d\tau = \int_0^{\frac{T}{4}} v''(\tau)d\tau + \int_{\frac{T}{4}}^T v''(\tau)d\tau \\ &\geq \int_0^{\frac{T}{4}} \frac{T(b-s)}{4} h_1 d\tau + v'(T) - v'\left(\frac{T}{4}\right) \\ &> B + v'(T) - v'\left(\frac{T}{4}\right) \geq 0. \end{aligned}$$

For $I = \left[\frac{3}{4}T, T\right]$ a similar contradiction is achieved, and, therefore, σ_0 is finite.

Claim 8. *For $s \in]\sigma_0, \sigma_1]$ (1) and (2) has at least two solutions.*

By Claim 7 and Theorem 2, for $\sigma_{-1} < \sigma_0$, (1) and (2), has no solution.

From Lemma 1 and Remark 1, we can take $r_2 > 0$ large enough such that $\|v''\| < r_2$, for every solution v of (1) and (2), with $s \in [\sigma_{-1}, \sigma_1]$.

Let $B_1 := \max\{r, B\}$ and define the set

$$\Omega_2 = \{y \in dom\mathcal{L} : \|y'\| < B_1, \|y''\| < \rho_2\}.$$

Then, by degree theory,

$$d\left(\mathcal{L}^{-1}\mathcal{F}_{\sigma_{-1}}, \Omega_2\right) = 0. \tag{28}$$

By Claim 8, if v is a solution of (1) and (2), with $s \in]\sigma_{-1}, \sigma_1]$, then $v \notin \partial\Omega_2$.

Consider the convex combination of σ_{-1} and σ_1, as $\mathcal{H}(\lambda) = (1-\lambda)\sigma_{-1} + \lambda\sigma_1$ and the corresponding homotopic problems $(E_{\mathcal{H}(\lambda)})$-(2). So, the topological degree $d\left(\mathcal{L}^{-1}\mathcal{F}_{\mathcal{H}(\lambda)}, \Omega_2\right)$ is well defined for $\lambda \in [0,1]$ and for every $s \in]\sigma_{-1}, \sigma_1]$.

Therefore, by (28) and the invariance of the degree under homotopy,

$$0 = d\left(\mathcal{L}^{-1}\mathcal{F}_{\sigma_{-1}}, \Omega_2\right) = d\left(\mathcal{L}^{-1}\mathcal{F}_s, \Omega_2\right), \tag{29}$$

for $s \in]\sigma_{-1}, \sigma_1]$.

Take $s \in]\sigma_0, \sigma_1] \subset]\sigma_{-1}, \sigma_1]$ and, by Theorem 2, let v_s be the corresponding solution of $(E_{\mathcal{H}(\lambda)})$-(2).

Consider $\delta > 0$, small enough, such that

$$\left|v_s'(t) + \delta\right| < B_1, \forall t \in [0, T]. \tag{30}$$

Then $v_* := v_s(t) + \delta t$ is a strict upper solution of (1) and (2), with $s < \sigma \leq \sigma_1$. Indeed, by (9) and (18) with $\xi = \delta, v_1 = t + T$ and $v_2 = \|v_\sigma\|$, for such σ,

$$\begin{aligned} v_*'''(t) &= v_s'''(t) = sh(t) - f\left(t, v_s(t), v_s'(t), v_s''(t)\right) \\ &< \sigma h(t) - f\left(t, v_s(t), v_s'(t), v_*''(t)\right) \\ &\leq \sigma h(t) - f\left(t, v_s(t) + \delta(t+T) + \|v_s\|, v_s'(t) + \delta, v_*''(t)\right) \\ &= \sigma h(t) - f\left(t, v_s(t) + \delta(t+T) + \|v_s\|, v_*'(t), v_*''(t)\right), \end{aligned}$$

and, for the boundary conditions

$$\begin{aligned} v_*'(0) &= v_s'(0) + \delta = v_s'(T) + \delta = v_*'(T), \\ v_*''(0) &= v_*''(T). \end{aligned}$$

Following the arguments as in Claim 7 of Theorem 2, it can be shown that $\gamma(t) := -rt$ is a strict lower solution of (1) and (2), for $\sigma \leq \sigma_1$.

By Claim 8, $-r < v_s'(t)$, for every $t \in [0, T]$ and therefore $-r < v_s'(t) + \delta, \forall t \in [0, T]$. So, $\gamma'(t) < v_*'(t), \forall t \in [0, T]$, and integrating in $[0, t]$ we have

$$-rt < v_s(t) + \delta t - v_s(0) \leq v_s(t) + \|v_s\| + \delta t, \forall t \in [0, T].$$

Remark that, as long as there are strict lower and upper solutions of (1) and (2), accordingly Definition 3, and σ belongs to a bounded set, it can be defined as a set independently of σ.

So, there exist $\rho_2^* > 0$, not dependent from σ, and the set

$$\Omega_\delta = \left\{ \begin{array}{c} y \in dom\mathcal{L} : -rt - r < y < v_s(t) + \|v_s\| + \delta t, -r < y' < v_s'(t) + \delta, \\ \|y''\| < \bar{\rho}_2^* \end{array} \right\}$$

such that, by Claim 8,

$$d\left(\mathcal{L}^{-1}\mathcal{F}_s, \Omega_\delta\right) = \pm 1, \text{ for } \sigma \in]\sigma, \sigma_1].$$

Considering ρ_2 in Ω_2 sufficiently large such that $\Omega_\delta \subset \Omega_2$, by (29) and (30) and the additivity of the degree, we have

$$d\left(\mathcal{L}^{-1}\mathcal{F}_s, \Omega_2 - \Omega_\delta\right) = \pm 1, \text{ for } \sigma \in]\sigma, \sigma_1].$$

So, (1) and (2) has at least two solutions u and v such that $u \in \Omega_\delta$ and $v \in \Omega_2 - \overline{\Omega_\delta}$ for $\sigma \in]s, \sigma_1]$, as s is arbitrary in $]\sigma_0, \sigma_1]$.

Claim 9. *For $s = \sigma_0$, the problem (1) and (2) has at least one solution.*

Take a sequence (σ_n) with $\sigma_n \in]\sigma_0, \sigma_1]$ and $\lim \sigma_n = \sigma_0$. By Theorem 2, for each σ_n, (E_{σ_n})-(2) has a solution v_n. Applying the bounds given by Claim 9, we have $\|v_n\| < B_1$, $\|v_n'\| < B_1$, independently of n, and, there exists $\tilde{r}_2 > 0$ sufficiently large such that $\|v_n''\| < \tilde{r}_2$, independently of n. Therefore, sequences (v_n) and (v_n'), $n \in \mathbb{N}$, are bounded in $C([0, T])$. By the Arzèla-Ascoli Theorem, consider a subsequence of (v_n) that converges in $C^2([0, 1])$ to a solution $\tilde{v}_0(t)$ of (E_{σ_0})-(2).

So, there is at least a solution for $\sigma = \sigma_0$. $\square$

5. Example

Consider the problem composed of the nonlinear third order equation with the parameter $\sigma \in \mathbb{R}$,

$$v'''(t) + (3 + \arctan(v(t)))e^{-v'(t)} = \sigma, \; t \in [0, 1], \tag{31}$$

together with the periodic boundary conditions

$$v^{(i)}(0) = v^{(i)}(1),\ i = 0,1,2. \tag{32}$$

It can be easily verified that the functions $\gamma(t) \equiv 0$ and $\Gamma(t) = t$ are, respectively, strict lower and upper solutions of problem (31) and (32), according to Definition 3, with $\gamma_0(t) \equiv 0$ and $\Gamma_0(t) = t + 1$, for σ such that

$$\frac{3 + \frac{\pi}{2}}{e} < \sigma < 3. \tag{33}$$

The problem (31) and (32) is a particular case of (1) and (2) with

$$f(t, x, y, z) = (3 + \arctan x)e^{-y}, \text{ and } h(t) \equiv 1,$$

which verifies the local monotony given by (7) and the Nagumo condition in

$$C = \left\{ \begin{array}{c} (t, x, y, z) \in [0,1] \times \mathbb{R}^3 : 0 \le x \le t + 1, \\ 0 \le y \le 1 \end{array} \right\}, \tag{34}$$

as

$$\left|(3 + \arctan x)e^{-y}\right| \le 3 + \frac{\pi}{2}$$

and

$$\int_0^{+\infty} \frac{z}{3 + \frac{\pi}{2}}\, dz = +\infty\,.$$

Therefore, by Theorem 1, there is a periodic solution $v_0(t)$ of the problem (31) and (32) for σ given by (33), and

$$0 \le v_0(t) \le t + 1, \forall t \in [0,1].$$

Remark that this solution $v_0(t)$ is not a trivial periodic one, that is a constant function, because if we have $v_0(t) \equiv k \in [0.1]$, then

$$(3 + \arctan k) = \sigma,$$

contradicts (33).

For the existence of a bifurcation parameter σ_0, the assumptions (8) and (9) of Theorem 2 are trivially verified, (10) holds for

$$3 < \sigma_1 < \left(3 - \frac{\pi}{2}\right)e^r \tag{35}$$

with $r \ge 0.75$, and (11) is verified for $\nu = 1$ and $1 \le \xi \le 2$.

Therefore, by Theorem 2 there is $\sigma_0 < \sigma_1$ such that the problem (31) and (32) has no solution for $\sigma < \sigma_0$, and at least one solution for $\sigma_0 < \sigma \le \sigma_1$.

Let us restrict the search of solutions for (31) and (32) on the set σ, given by (34).

So, every solution v_0 of (31) and (32) satisfies (16) with $B = 2$. The condition (17) holds with $b = 0$ for $(t, x, y, z) \in [0,1] \times \mathbb{R}^3$ and (18) is verified with $\xi = 1, v_1 = 2$ and $v_2 = 1$ for $(t, x, y, z) \in C$.

Therefore, from (3), σ_0 is finite, the problem (31) and (32) at least a solution for $\sigma = \sigma_0$, and, for $\sigma \in]\sigma_0, \sigma_1]$, (31) and (32) has at least two solutions.

Remark that, by (33),

$$\sigma_0 > \frac{3 + \frac{\pi}{2}}{e} \simeq 1.681$$

and, by (35),

$$\sigma_1 < \left(3 - \frac{\pi}{2}\right)e^{0.75} \simeq 3.025.$$

6. Conclusions

This work presents sufficient conditions for third-order differential equations, with nonlinearities depending on all derivatives of the unknown function, have no solution, at least one or at least two solutions, associated with adequate values of some real parameter s.

More precisely, it was proved for the first time in third order periodic problems, that a speed-growth condition type, that is, the nonlinearity must have different growth velocities on the unknown function and its derivative is a key point to discuss the non-existence or the multiplicity of the solutions.

As a consequence, the lower and upper solutions techniques applied in this paper allows some estimations on the critical values of the parameter, which may be an important issue in studying periodic real phenomena modeled by third order problems.

Future research in this direction may rely on studying some methods and/or techniques to avoid the speed growth condition or replacing it with a more general assumption.

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