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A Fuzzy Random Boundary Value Problem

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Abstract: A system of generalized fuzzy random differential equations with boundary conditions is investigated, which is a fuzzy version of a system of general random differential equations. We first present random fixed point (RFP) theorems in fuzzy metric space (FM). In the sequel, we define the operators that are of integral type. Furthermore, these operators are related to a part of random differential equations (RDE). For the desired system with boundary conditions, we study the suitable integral operators associated with a large family of random differential equations. Finally, we prove the existence of a unique random solution (EURS).

Keywords: measurable space; random ODE; random FPT; fuzzy; vector-valued normed space

MSC: Primary 35R60, 47H10

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1. Introduction

Inspired by the basic notions of random normed spaces introduced by Šerstnev [1], and studied by Muštari [2] and Radu [3], Cheng and Mordeson [4] defined fuzzy normed spaces. In this paper, using iterative methods from the random fixed point theory in a fuzzy normed space, together with the theory of measurable spaces and monotone operators, we study a nonlinear boundary value problem (BVP) for a system of random differential equations. The motivation for such equations is as follows (see also the books of Bharucha-Reid [5] and Skorohod [6]): the mathematical model representations of natural phenomena arising in biology, physics, and engineering processes deal with specific parameters that may be unknown values in general. Furthermore, our results and methods can develop further research in the area and investigate uncertain cases of random equations.

The paper is structured as follows: In Section 2, we explain the notions of the triangular norm (t-norm), FM space, FN space, fuzzy Carathéodory function, and random fixed point. Section 3 introduces a general system of fuzzy random equations under nonlinear boundary conditions (2), which is a fuzzy version of a system of general random differential equations from [7]. Section 4 deals with some contraction results in FM space. In Section 5, we prove EURS for BVP (2).

2. Preliminaries

Following [8–10], we denote I as the unit interval $[0,1]$. Let the binary operation $* : I \times I \to I$ be a topological commutative monoid with unit 1 such that $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d \quad (a,b,c,d \in I)$. In this case, $*$ is said to be a continuous t-norm. For some examples, $a * b = a \times b$ and $a * b = \min(a,b)$ are continuous t-norms.

Assume that U is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $U^2\times]0,\infty[$. Then $(U,M,*)$ is called a fuzzy metric space (FM space for short), when for every $u,v,w\in U$ and $t,s>0$,

(FM1) $M(u,v,t)=1$ for every $t>0$ if and only if $u=v$;
(FM2) $M(u,v,t)=M(v,u,t)$;
(FM3) $M(u,w,t+s)\geqslant M(u,v,t)*M(v,w,s)$;
(FM4) $M(u,v,.):]0,\infty[\to]0,1]$ is continuous.

Assume that V is a linear space, $*$ is a continuous t-norm and N is a fuzzy set on $V\times]0,\infty[$. Then, $(V,N,*)$ is called a fuzzy normed space (FN space for short), when for every $u,v\in X$ and $t,s>0$, we have

(FN1) $N(u,t)=1$ for every $t>0$ if and only if $u=0$;
(FN2) $N(au,t)=N\left(u,\frac{t}{|a|}\right)$ for any $a\neq 0$;
(FN3) $N(u+v,t+s)\geqslant N(u,t)*N(v,s)$;
(FN4) $N(u,.):]0,\infty[\to]0,1]$ is continuous.

Let $(V,N,*)$ be a FN space. We define $M(u,v,t)=N(u-v,t)$. Then M is a fuzzy metric on V, which is called the fuzzy metric induced by the fuzzy norm N.

We assume that there exists an $n(\varepsilon)\in\mathbb{N}$ such that $0<\varepsilon<1$. Therefore, for $n\geqslant n(\varepsilon)$, we have $M(x_n,x,t)>1-\varepsilon$. Then the sequence $\{x_n\}$ converges to x. If, for every $0<\varepsilon<1$, there is an $n(\varepsilon)\in\mathbb{N}$ such that $M(x_n,x_m,t)>1-\varepsilon$ for every $n,m\geqslant n(\varepsilon)$, then we say that $\{x_n\}$ is a Cauchy sequence. A FM space is complete if every Cauchy sequence is convergent in it. It is well known that for a metric space (X,d) we say it is separable if there exists a countable dense subset Y of X. Let (X,d) be a metric space and $x_0\in X$ and let $f:X\to X$ be a given mapping. The sequence $\{x_n\}$ with initial point x_0 is a Picard sequence (PS) where $x_n=f^n(x_0)=f(x_{n-1})$ and $n\in\mathbb{N}$.

Assume that for every $\varepsilon\in]0,1[$, there exists a $\delta\in]0,1[$ (which does not depend on k) such that the following inequality holds

$$\overbrace{(1-\delta)*\cdots*(1-\delta)}^{k}>1-\varepsilon,\quad \text{for each } k\in\{2,3,\ldots\}. \tag{1}$$

Lemma 1 ([8]). *Let $(U,M,*)$ be a complete fuzzy metric space such that $*$ satisfies (1). If in U there exits a sequence $\{x_n\}$ such that, for all $n\in\mathbb{N}$, we have $M(x_n,x_{n+1},t)>M(x_0,x_1,knt)$ for all $k>1$, then the sequence $\{x_n\}$ is a Cauchy sequence in U.*

We consider the FM space X. We define a Borel σ-algebra (B-σA) on this space and denote it by $\mathcal{B}(X)$. For a given measurable space (Γ,Λ), we denote by $\Lambda\otimes\mathcal{B}(X)$ the smallest σ-algebra on $\Gamma\times X$ containing all sets $A\times B$ (for $A\in\Lambda$ and $B\in\mathcal{B}(X)$).

Definition 1. *We consider two FM spaces X and Y. We also assume that (Γ,Λ) is a measurable space ((Γ,Λ)-MS). The function $\gamma\to G(\gamma,x)$ is a $(\Lambda,\mathcal{B}(Y))$-measurable function ($(\Lambda,\mathcal{B}(Y))$-MF) where $x\in X$ and for any $\gamma\in\Gamma$. Moreover, the continuous function $x\to G(\gamma,x)$ for $\gamma\in\Gamma$ is called fuzzy Carathéodory (F-C) function.*

For the completeness of the paper, we prove the following theorem obtained in [11].

Theorem 1. *We consider (Γ,Λ), $(X,M,*)$, $(Y,M,*)$, and $G:\Gamma\times X\to Y$, which are respectively MS, separable FMS, FMS, and F-CF. With these conditions, G is a $\Lambda\otimes\mathcal{B}(X)$-measurable ($\Lambda\otimes\mathcal{B}(X)$-M).*

Proof. Assume that D is a countable dense subset of X and C is a closed subset of Y. We consider the subset $C_n=\left\{y\in Y: M_y(y,C,t)>1-\frac{1}{n}\right\}$, where M_y is a fuzzy metric on Y. Then, $G(\gamma,x)\in C$ if and only if for every $n>1$ there exists $v\in D$ such that

$M_X(x,v,t) > 1 - \frac{1}{n}$ and $G(\gamma,v) \in C_n$ (here M_X is a fuzzy metric on X). Hence, we conclude that

$$\begin{aligned} & G^{-1}(C) \\ = & \bigcap_{n\geqslant 1} \bigcup_{v\in D} \{\gamma \in \Gamma : G(\gamma,v) \in C_n\} \times \left\{x \in X : M_X(x,v,t) > 1 - \frac{1}{n}\right\} \\ \in & \Lambda \otimes \mathcal{B}(X), \end{aligned}$$

which implies that G is $\Lambda \otimes \mathcal{B}(X)$-measurable. □

Corollary 1. *We consider (Γ,Λ), X, Y, $G : \Gamma \times X \to Y$, and $u : \Gamma \to X$ which are, respectively MS, separable FMS, FMS, F-CF, and Λ-measurable map. With these conditions,$\gamma \to G(\gamma,u(\gamma))$ from Γ into Y is a Λ-measurable mapping ($\Lambda \otimes \mathcal{B}(X)$-MM).*

Proof. Let $k : \Gamma \to \Gamma \times X$ be given by $k(\gamma) = (\gamma,u(\gamma))$. Therefore k is $\Lambda \otimes \mathcal{B}(X)$-measurable and $G(\gamma,u(\gamma)) = (if\ G\ o\ k)(\gamma)$. Additionally, by Theorem 1 G is $\Lambda \otimes \mathcal{B}(X)$-measurable. Therefore, it follows readily that $\gamma \to (if\ G\ o\ k)(\gamma)$ is Λ-measurable. □

Consider a measurable space (Γ,Λ), a separable FM space X and a FM space Y. If the mapping $\gamma \to H(\gamma,u(\gamma))$ from Γ to Y for every Λ-MM $u : \Gamma \to X$ is Λ-M, then we say $H : \Gamma \times X \to Y$ is superpositionally measurable (SUP-M). According to the definition of the SUP-M, we have the following results (see Denkowski-Migórski-Papageorgiou ([11], Remark 2.5.26)):

- Considering Corollary 1, we conclude that a F-C mapping is SUP-M;
- Every $\Lambda \otimes \mathcal{B}(X)$-MM is SUP-M.

Furthermore, if $\gamma \to F(\gamma,x)$ is Λ-M, then we say that $F : \Gamma \times X \to X$ is a random operator (RO) for every $x \in X$. Therefore, every fixed point of the random operator (RO) F is also a random point (RP) such as the Λ-MM $z : \Gamma \to X$ ($z(\gamma) = F(\gamma,z(\gamma))$ for every $\gamma \in \Gamma$). Our results can extend some recent ones and improve them to obtain new results (see [12–17]).

3. A General System of Fuzzy Random Equations

Let $C(I,\mathbb{R})$ be all continuous scalar-valued mappings from (S-VM) onto I. For these mappings, the following partial order relation for $f,g \in C(I,\mathbb{R})$ and every $p \in I$ always holds

$$f \precsim g \text{ if and only if} f(p) \leqslant g(p).$$

We extend this relation on $C(I,\mathbb{R}) \times C(I,\mathbb{R})$ as follows:

$$(f,g),(h,k) \in C(I,\mathbb{R}) \times C(I,\mathbb{R}), \quad (f,g) \precsim (h,k) \Longleftrightarrow f \precsim h,\ g \precsim k.$$

In this work, we consider a nonlinear boundary value problem (N-BVP) for a system of random differential equations. This problem is given below for two functions $q_1,q_2 : \Gamma \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with regular properties

$$\begin{aligned} f''(\gamma,p) &= q_1(\gamma,p,f(\gamma,p),g(\gamma,p)), \quad 0 < p < 1, \gamma \in \Gamma, \\ g''(\gamma,p) &= q_2(\gamma,p,f(\gamma,p),g(\gamma,p)), \quad 0 < p < 1, \gamma \in \Gamma, \\ f(\gamma,0) = 0, \quad f(\gamma,1) &= \psi_1\left(\int_0^1 f(\gamma,p)dp\right), \quad \gamma \in \Gamma, \quad \psi_1 \in C(\mathbb{R},\mathbb{R}), \\ g(\gamma,0) = 0, \quad g(\gamma,1) &= \psi_2\left(\int_0^1 g(\gamma,p)dp\right), \quad \gamma \in \Gamma, \quad \psi_2 \in C(\mathbb{R},\mathbb{R}). \end{aligned} \tag{2}$$

Two measurable functions $(f, g) : \Gamma \to C(I, \mathbb{R}) \times C(I, \mathbb{R})$ are a pair of random solutions (RS) for system (2). If we want to take into account of this uncertainty, a way to model it is based on the parameter $\gamma \in \Gamma$.

In particular, under the absence of the parameter γ, we obtain the following system from (2) for $q_1, q_2 \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$

$$\begin{aligned} f''(p) &= q_1(p, f(p), g(p)), \quad 0 < p < 1, \\ g''(p) &= q_2(p, f(p), g(p)), \quad 0 < p < 1, \\ f(0) = 0, \quad f(1) &= \psi_1\left(\int_0^1 f(p)dp\right), \quad \psi_1 \in C(\mathbb{R}, \mathbb{R}), \\ g(0) = 0, \quad g(1) &= \psi_2\left(\int_0^1 g(p)dp\right), \quad \psi_2 \in C(\mathbb{R}, \mathbb{R}). \end{aligned} \tag{3}$$

Two measurable functions $(f, g) \in C(I, \mathbb{R}) \times C(I, \mathbb{R})$ are a pair of solutions (RS) for system (3). According to Green's function, these two functions are as follows

$$f(p) = \int_0^1 T(p, s)q_1(s, f(s), g(s))ds + \psi_1\left(\int_0^1 f(s)ds\right)p, \quad 0 < p < 1,$$
$$g(p) = \int_0^1 T(p, s)q_2(s, f(s), g(s))ds + \psi_2\left(\int_0^1 g(s)ds\right)p, \quad 0 < p < 1,$$

where

$$T(p, s) = \begin{cases} -p(1-s), & 0 \leqslant p \leqslant s \leqslant 1, \\ -s(1-p), & 0 \leqslant s \leqslant p \leqslant 1. \end{cases} \tag{4}$$

Our main goal in this article is to investigate the three general RFP theorems in FM space. Finally, we prove the existence of a unique random solution (EURS) for system (2). In fact, our work is in a fuzzy state of [7] (see also [18–21]).

4. Fixed Point Theorems

We assume that Γ and X are two non-empty sets. We consider the mapping $f : \Gamma \times X \to X$. In the following, we propose and prove some theorems that show the existence of a unique random solution (EURS) for mapping f in a FMS. We denote all mappings from Γ to X by X^Γ, so that X is a FMS and (Γ, Λ) is a MS. A subset of this set is denoted by $\mathcal{U}(\Gamma, X)$, which contains all Λ-measurable mappings (Λ-MMs). We consider $g, h \in X^\Gamma$. We say that g and h are comparable, if we have

$$g(\gamma) \precsim h(\gamma),$$

or

$$h(\gamma) \precsim g(\gamma),$$

where $\precsim$ is a partial relationship in X. The sequence $\{h_n\}$ in which $h_n(\gamma) = f(\gamma, h_{n-1}(\gamma))$, for$\gamma \in \Gamma$ and $n \in \mathbb{N}$ and with starting point h_0 is called a Picard sequence.

Now we consider the following assumptions:

Hypothesis 0 (H0). *For each $\gamma \in \Gamma$ and considering (Γ, Λ), $(X, M, *, \precsim)$, and $F : \Gamma \times X \to X$, which are MS, separable complete ordered FM space (SCO-FMS), and random mapping (RM), respectively, the mapping $x \to F(\gamma, x)$ is a monotone operator. Additionally, $*$ in SCO-FMS applies to* (1).

Hypothesis 1 (H1)**.** *For the function* $\varphi_\gamma : [0,\infty[\to [0,\infty[$*, which is a non-decreasing function, we have*

$$\lim_{n\to\infty} \varphi_\gamma^n(t) = \infty,$$

for every $t > 0$ *and for each* $\gamma \in \Gamma$*. Moreover,*

$$M(F(\gamma,x),F(\gamma,y),t) \geqslant M(x,y,\varphi_\gamma(t)),$$

for every $\gamma \in \Gamma$*,* $x,y \in X$*,* $x \precsim y$ *and* $t > 0$*;*

Hypothesis 2 (H2)**.** *there exists a mapping* $x_0 \in \mathcal{U}(\Gamma,X)$ *with "*$x_0(\gamma) \precsim F(\gamma,x_0(\gamma))$*, for each* $\gamma \in \Gamma$*" or "*$x_0(\gamma) \succsim F(\gamma,x_0(\gamma))$*, for each* $\gamma \in \Gamma$*";*

Hypothesis 3 (H3)**.** *if* $\{x_n\}$ *is a monotone sequence in* X *and* $x_n \to x$*, then* x_n *and* x *are comparable for every* $n \in \mathbb{N}$*.*

Remark 1. (i) *(H0) describes the space we use as well as the monotonicity of the mapping* $x \to F(\gamma,x)$*.*
(ii) *(H1) states the Matkowski-contraction condition (see* [22]*).*
(iii) *Assumption (H3) shows the existence of the partial order relation in the case that F is not F-Carathéodory.*

We first assume that F is F-Carathéodory mapping (F-CM), then we prove the theorem with this assumption. Then, we further show our theorem without this assumption.

Theorem 2. *We consider the assumptions of* (H0)–(H2)*. We assume that G is a F-CM. Therefore, G has a RFP such as a* $z \in \mathcal{U}(\Gamma,X)$*. Furthermore, if there exists* $u \in X^\Gamma$ *such that for every* $x,y \in \mathcal{U}(\Gamma,X)$*, u is comparable to x and y, then u is a unique random solution (URS) of G.*

Proof. We consider the Picard sequences $\{x_n\}$ and $\{u_n\}$ with starting points $x_0,u_0 \in X^\Gamma$ such that x_0 and u_0 are comparable. Then, we prove

$$\lim_{n\to\infty} M(x_n(\gamma),u_n(\gamma),t) = 1, \quad \text{for every } \gamma \in \Gamma, t > 0. \tag{5}$$

Considering that the function $x \to G(\gamma,x)$ is a monotone operator, for every $n \in \mathbb{N}$ and every constant $\gamma \in \Gamma$, we have the comparability of $x_n(\gamma)$ and $u_n(\gamma)$. Now, if $x_n(\gamma) = u_n(\gamma)$, it is clear that (5) is established and the proof is complete. Then for any $n \in \mathbb{N}$, we assume $x_n(\gamma) \neq u_n(\gamma)$. Then by (H1), we have

$$\begin{aligned} & M(x_n(\gamma),u_n(\gamma),t) \\ = \; & M(G(\gamma,x_{n-1}(\gamma)),G(\gamma,u_{n-1}(\gamma)),t) \\ \geqslant \; & M\Big(x_0(\gamma),u_0(\gamma),\varphi_\gamma^n(t)\Big), \end{aligned} \tag{6}$$

for every $n \in \mathbb{N}$, $t > 0$. Now, when $n \to \infty$, according to (6) and considering the property of φ_γ, we have

$$\lim_{n\to\infty} M(x_n(\gamma),u_n(\gamma),t) = 1.$$

Clearly, this holds for every $\gamma \in \Gamma$, $t > 0$. We assume that $x_0 \in \mathcal{U}(\Gamma,X)$ is a mapping like the one introduced in hypothesis (H2). If, for each $\gamma \in \Gamma$, $G(\gamma,x_0(\gamma)) = x_0(\gamma)$, then x_0 is a RFP of G. Suppose that, for some $\gamma \in \Gamma$, $G(\gamma,x_0(\gamma)) \neq x_0(\gamma)$. By (H2), we have that x_0 and x_1 are two comparable elements of X^Γ. Then, by (5), if we choose $u_0 = x_1$, we obtain

$$\lim_{n\to\infty} M(x_n(\gamma),x_{n+1}(\gamma),t) = 1, \quad \text{for every } \gamma \in \Gamma, t > 0. \tag{7}$$

Now, we show that $\{x_n(\gamma)\}$ is a Cauchy sequence for each $\gamma \in \Gamma$. Let $\gamma \in \Gamma$ be fixed. Using (7) and considering $\varphi_\gamma(0) = 0$ and $\varphi_\gamma(t) > t$ for$t > 0$, there is a $n(\varepsilon) \in \mathbb{N}$ for $0 < \varepsilon < 1$ such that

$$\begin{array}{rcl} M(x_n(\gamma), x_{n+1}(\gamma), t) & = & M(G(\gamma, x_{n-1}), G(\gamma, x_n), t) \\ & \geqslant & M(x_n(\gamma), x_{n+1}(\gamma), \varphi_\gamma(t)) \\ & \cdots & \\ & \geqslant & M(x_0(\gamma), x_1(\gamma), \varphi_\gamma^n(t)), \end{array}$$

for every $m \in \mathbb{N}$, $m \geqslant n(\varepsilon)$. By Lemma 1, we obtain that $\{x_n(\gamma)\}$ is a Cauchy sequence for every $\gamma \in \Gamma$. Then there exists $z \in X^\Gamma$ such that

$$z(\gamma) = \lim_{n\to\infty} x_n(\gamma), \quad \text{for every } \gamma \in \Gamma.$$

Using Corollary 1 and considering that $z \in \mathcal{U}(\Gamma, X)$, we have $x_n \in \mathcal{U}(\Gamma, X)$ for every $n \in \mathbb{N}$. In the following, considering the assumption that G is a F-CM, we prove that for every $\gamma \in \Gamma$, $z(\gamma) = G(\gamma, z(\gamma))$. With the assumption considered, we have

$$M(z(\gamma), G(\gamma, z(\gamma)), t) = \lim_{n\to+\infty} M(x_n(\gamma), G(\gamma, x_n(\gamma)), t), \quad \text{for every } \gamma \in \Gamma, t > 0.$$

From

$$\begin{aligned} M(x_n(\gamma), G(\gamma, x_n(\gamma)), t) &= M(G(\gamma, x_{n-1}(\gamma)), G(\gamma, x_n(\gamma)), t) \\ &\geqslant M(x_{n-1}(\gamma), x_n(\gamma), \varphi_\gamma(t)) \\ &\geqslant \ldots \geqslant M(x_0(\gamma), x_1(\gamma), \varphi_\gamma^n(t)), \end{aligned}$$

letting $n \to \infty$, we obtain $M(z(\gamma), G(\gamma, z(\gamma)), t) = 1$ for every $\gamma \in \Gamma$, $t > 0$. Thus $z(\gamma) = G(\gamma, z(\gamma))$ for each $\gamma \in \Gamma$, that is, z is a RFP of G. Next, we prove the uniqueness of the solution. To begin this proof, first we consider another RFP such as $v \in \mathcal{U}(\Gamma, X)$ for G and assume that these two solutions are comparable; then, using (H1), we obtain

- $z = v$;
- $M(z(\gamma), G(\gamma, z(\gamma)), t) = 1$ when $n \to \infty$.

Therefore, assuming that z and v are comparable, we have nothing to prove. Now we assume that z and v are not comparable and further we assume that $u \in X^\Gamma$ is an element comparable to z and v. Considering the Picard sequence $\{u_n\}$ with starting point $u_0 = u$ and also $x_0 = z$ and $x_0 = v$, we have

$$\lim_{n\to\infty} M(z(\gamma), u_n(\gamma), t) = \lim_{n\to\infty} M(v(\gamma), u_n(\gamma), t) = 1. \tag{8}$$

Therefore, we conclude that z is a unique random solution (URS) for G because using (8) we have $z = v$. □

Now, we change the considered assumptions and with these new assumptions we prove another theorem, which also shows the existence of a RBF for G. We consider condition (H3) instead of condition F-Carathéodory (F-C) and also assume that H is a sup-measurable mapping (SUP-MM).

Theorem 3. *We consider assumptions* (H0)–(H3) *as well as SUP-M mapping H. Then, there is a RFP of H like a mapping $z \in \mathcal{U}(\Gamma, X)$.*

Proof. We consider $\{x_n\}$ and $z \in \mathcal{U}(\Gamma, X)$ as introduced in Theorem 2. According to the assumption that we considered a SUP-MM like H and according to condition (H3), we conclude that for each $n \in \mathbb{N}$ and $\gamma \in \Gamma$, $z \in \mathcal{U}(\Gamma, X)$, and $x_n(\gamma)$ and $z(\gamma)$ are comparable.

and $\gamma \in \Gamma$. From (H1) we obtain

$$\begin{aligned} M(z(\gamma), H(\gamma, z(\gamma)), 2t) &\geqslant M(z(\gamma), H(\gamma, x_n(\gamma)), t) * M(f(\gamma, x_n(\gamma), H(\gamma, z(\gamma)), t) \\ &\geqslant M(z(\gamma), H(\gamma, x_n(\gamma)), t) * M(x_n(\gamma), z(\gamma), \varphi_\gamma(t)) \\ &\geqslant M(z(\gamma), x_{n+1}(\gamma), t) * M(x_n(\gamma), z(\gamma), \varphi_\gamma(t)). \end{aligned}$$

Letting $n \to \infty$, we obtain $M(z(\gamma), H(\gamma, z(\gamma)), 2t) = 1$ for every $\gamma \in \Gamma$. This means that z is a RFP of H. $\square$

Considering the generalized FM space (GFMS), we consider the previous assumptions to solve the above RFP problem corresponding to this space.

Definition 2. *Let $*$ be a continuous t-norm on I. Let $\boldsymbol{a} = (a_1, \ldots, a_k)$ and $\boldsymbol{b} = (b_1, \ldots, b_k)$ be in I^k. We define a binary operation $\star$ on I^k by*

$$\boldsymbol{a} \star \boldsymbol{b} = (a_1 * b_1, \ldots, a_k * b_k).$$

Then we call $\star$ a continuous t-norm on I^k.

Definition 3 ([23])**.** *Let $\mathbb{R}^k_+ := \{\boldsymbol{a} \in \mathbb{R}^k : a_j \geqslant 0$ for every $j = 1, \ldots, k\}$, where $\mathbb{R}^k$ is equipped with the following partial order*

$$\boldsymbol{a} = (a_1, \ldots, a_k),\ \boldsymbol{b} = (b_1, \ldots, b_k) \in \mathbb{R}^k, \quad \boldsymbol{a} \preceq \boldsymbol{b} \Longleftrightarrow a_j \leqslant b_j \text{ for every } j = 1, \ldots, k.$$

Additionally, $\boldsymbol{a} \prec \boldsymbol{b}$ denote that $\boldsymbol{a} \preceq \boldsymbol{b}$ and $\boldsymbol{a} \neq \boldsymbol{b}$; $\boldsymbol{a} \ll \boldsymbol{b}$ and $a_j < b_j$ for every $j = 1, \ldots, k$.

We define $\bar{a} := (\overbrace{a, \ldots, a}^{k})$ in $\mathbb{R}^k$. Specifically, the zero vector is denoted by $\bar{0} := (\overbrace{0, \ldots, 0}^{k})$.

Definition 4. *Assume that $X \neq \varnothing$, $\star$ is a continuous t-norm on I^k and M^V is a vector-value fuzzy set on $X^2 \times]0, \infty[^k$. Then, $(X, M^V, \star)$ is called a fuzzy vector-valued metric space (FVVM space for short) when for every $u, v, w \in X$, $\boldsymbol{t} \gg \bar{0}$,*

(FVM1) $M^V(u, v, \boldsymbol{t}) \gg \bar{0}$;
(FVM2) $M^V(u, v, \boldsymbol{t}) = \bar{1}$ *for every* $\boldsymbol{t} \gg \bar{0}$ *if and only if* $u = v$;
(FVM3) $M^V(u, v, \boldsymbol{t}) = M^V(v, u, \boldsymbol{t})$;
(FVM4) $M^V(u, v, \boldsymbol{t} + \boldsymbol{s}) \succeq M^V(u, w, \boldsymbol{t}) \star M^V(w, v, \boldsymbol{s})$;
(FVM5) $M^V(u, v, .) :]0, \infty[^k \to]0, 1]^k$ *is continuous.*

Example 1. *Let (X, d) be a vector-valued metric space. Let $\boldsymbol{a} = (a_1, \ldots, a_k)$ and $\boldsymbol{b} = (b_1, \ldots, b_k)$ be in I^k. We define,*

$$\boldsymbol{a} \star \boldsymbol{b} = \mathbf{min}(\boldsymbol{a}, \boldsymbol{b}) := (\min(a_1, b_1), \min(a_2, b_2), \ldots, \min(a_k, b_k)),$$

and

$$M^V(x, y, \boldsymbol{t}) = \left(\frac{t_1}{t_1 + d(x, y)}, \frac{t_2}{t_2 + d(x, y)}, \ldots, \frac{t_k}{t_k + d(x, y)} \right),$$

for every $x, y \in X$ and $\boldsymbol{t} = (t_1, t_2, \ldots, t_k) \in]0, \infty[^k$. Then $(X, M^V, \star)$ is a FVVM space.

Definition 5. *Assume that X is a vector space, $\star$ is a continuous t-norm on I^k and N^V is a vector-valued fuzzy set on $X \times]0, \infty[^k$. Then, $(X, N^V, \star)$ is called a fuzzy vector-valued norm space (FVVN space for short) when for every $u, v \in X$, $\boldsymbol{t}, \boldsymbol{s} \gg \bar{0}$,*

(FVN1) $N^V(u, \boldsymbol{t}) \gg \bar{0}$;

(FVN2) $N^V(u, \boldsymbol{t}) = \bar{1}$ *for every* $\boldsymbol{t} \gg \bar{0}$ *if and only if* $u = 0$;
(FVN3) $N^V(au, \boldsymbol{t}) = N^V\left(u, \frac{1}{|a|}\boldsymbol{t}\right)$ *for every* $a \neq 0$;
(FVN4) $N^V(u + v, \boldsymbol{t} + \boldsymbol{s}) \succeq N^V(u, \boldsymbol{t}) \star N^V(v, \boldsymbol{s})$;
(FVN5) $N^V(u, .) :]0, \infty[^k \to (0, 1]^k$ *is continuous.*

We assume further that all t–norms satisfy (1). Let $\mathcal{R}_k$ be the family of all non-decreasing maps $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_k) : \mathbb{R}_+^k \to \mathbb{R}_+^k$ such that

(i) $\lim_{n \to +\infty} \boldsymbol{\varphi}^n(\boldsymbol{t}) = \bar{\infty}$ for every $\boldsymbol{t} \in \mathbb{R}_+^k$ with $\bar{0} \prec \boldsymbol{t}$;
(ii) $\boldsymbol{\varphi}(\bar{0}) = \bar{0}$ and $\boldsymbol{\varphi}(\boldsymbol{t}) \succ \boldsymbol{t}$ for $\boldsymbol{t} \in \mathbb{R}_+^k \setminus \{\bar{0}\}$;
(iii) $\bar{0} \ll \boldsymbol{t}$ implies $\boldsymbol{\varphi}(\boldsymbol{t}) \gg \boldsymbol{t}$.

Example 2. *We consider an* $A = \operatorname{diag}(a_1, \ldots, a_k)$*-like diagonal matrix. According to A, the function* $\boldsymbol{\varphi} : \mathbb{R}_+^k \to \mathbb{R}_+^k$ *is defined as follows*

$$\boldsymbol{\varphi}(\boldsymbol{t}) = A\boldsymbol{t}^T \quad \text{for every } \boldsymbol{t} = (t_1, \ldots, t_k) \in \mathbb{R}_+^k, \text{ and for } a_j \geqslant 1, \quad j = 1, \ldots, k,$$

then $\boldsymbol{\varphi} \in \mathcal{R}_k$.

Remark 2. *For the completeness, we consider hypotheses (H4) and (H6), which are completely similar to the introduced hypotheses (H0) and (H2), differing only in the* $\left(X, M^V, \star\right)$ *space.*

Next, we add the following conditions to complete the work.

Hypothesis 4 (H4)**.** *Considering* (Γ, Λ), $\left(X, M^V, \star, \precsim\right)$, *and* $F : \Gamma \times X \to X$, *which are MS, separable complete ordered FVVM space (SCO-FVVMS), and RO, respectively,* $x \to F(\gamma, x)$ *is a monotone operator for every* $\gamma \in \Gamma$. *In the assumed space of* $\left(X, M^V, \star, \precsim\right)$, $\star$ *in* (1) *satisfies.*

Hypothesis 5 (H5)**.** *There exists a function* $\varphi_\gamma = (\varphi_{\gamma,1}, \ldots, \varphi_{\gamma,k}) \in \mathcal{R}_k$ *such that for each* $\gamma \in \Gamma$,

$$M^V(F(\gamma, x), F(\gamma, y), \boldsymbol{t}) \succeq M^V(x, y, \varphi_\gamma(\boldsymbol{t})) \quad \text{for every } x, y \in X,\ x \precsim y, \boldsymbol{t} \gg \bar{0};$$

Hypothesis 6 (H6)**.** *There exists a mapping* $x_0 \in \mathcal{U}(\Gamma, X)$ *with "*$x_0(\gamma) \precsim F(\gamma, x_0(\gamma))$*, for every* $\gamma \in \Gamma$*" or "*$x_0(\gamma) \succsim F(\gamma, x_0(\gamma))$*, for every* $\gamma \in \Gamma$*".*

Now we define the complete FVVM space (CFVVMS). For this purpose, we consider a FVVM space such as $\left(X, M^V, \star\right)$ and a sequence such as $\{x_n\}$. If for every $x \in X$ and every $\bar{\varepsilon} \in \mathbb{R}_+^k$ with $\bar{0} \ll \bar{\varepsilon} \ll \bar{1}$, there exists an $n(\varepsilon) \in \mathbb{N}$ such that for every $n \geqslant n(\varepsilon)$ we have $M^V(x_n, x, \boldsymbol{t}) \gg \overline{1 - \varepsilon}$, then we say that the sequence $\{x_n\}$ converges to x and therefore it is a Cauchy sequence. A space X is a CFVVMS if every Cauchy sequence converges in it.

In the following, we state a result similar to Theorem 2, which is the third theorem to prove the existence of a RFP for F.

Theorem 4. *Assume that the conditions* (H4)–(H6) *are fulfilled and G as a F-C mapping, then there is a RFP such as* $z \in \mathcal{U}(\Gamma, X)$ *for G.*

Proof. To prove this theorem, we consider the cases introduced in Theorem 2. For example, we consider the sequences $\{x_n\}$ and $\{u_n\}$ as well as $x_0, u_0 \in X^\Gamma$ such that these two comparable members are the starting point of the sequences $\{x_n\}$ and $\{u_n\}$. Now, we prove that

$$\lim_{n \to \infty} M^V(x_n(\gamma), u_n(\gamma), \boldsymbol{t}) = \bar{1} \quad \text{for every } \gamma \in \Gamma, \boldsymbol{t} \gg \bar{0}. \tag{9}$$

Therefore, considering that for each $n \in \mathbb{N}$, $x_n(\gamma)$ and $u_n(\gamma)$ are comparable and $x \to G(\gamma, x)$ is a monotone, then using (H5)

$$M^V(x_n(\gamma), u_n(\gamma), \boldsymbol{t}) \succeq M^V(x_{n-1}(\gamma), u_{n-1}(\gamma), \varphi_\gamma(\boldsymbol{t})), \succeq M^V(x_0(\gamma), u_0(\gamma), \varphi_\gamma^n(\boldsymbol{t})),$$

for each $\gamma \in \Gamma$, $\boldsymbol{t} \gg \bar{0}$, $n \in \mathbb{N}$.

Let $n \to \infty$ and property (i) hold for members of $\mathcal{R}_k$, then (9) is true. We consider $x_0 \in \mathcal{U}(\Gamma, X)$ as the mapping introduced in (H6). Therefore, the following condition holds for every $\gamma \in \Gamma$

$$G(\gamma, x_0(\gamma)) = x_0(\gamma),$$

and this means that x_0 is a RFT for G and the proof is finished. Now, we assume that $G(\gamma, x_0(\gamma)) \neq x_0(\gamma)$ for some $\gamma \in \Gamma$ and we also consider the Picard sequence $\{x_n\}$ with the starting point x_0. Considering (H6) and choosing two comparable members x_0 and x_1 and using (9) to choose $u_0 = x_1$, we have

$$\lim_{n\to\infty} M^V(x_n(\gamma), x_{n+1}(\gamma), \boldsymbol{t}) = \bar{1}, \quad \text{for each } \gamma \in \Gamma , \boldsymbol{t} \gg \bar{0}. \tag{10}$$

In the following, for the constant $\gamma \in \Gamma$, we show that $\{x_n(\gamma)\}$ is a Cauchy sequence.

$$\begin{aligned} M^V(x_n(\gamma), x_{n+1}(\gamma), \boldsymbol{t}) &= M^V(G(\gamma, x_{n-1}), G(\gamma, x_n), \boldsymbol{t}) \\ &\succeq M^V(x_n(\gamma), x_{n+1}(\gamma), \varphi_\gamma(\boldsymbol{t})) \\ &\succeq \ldots \succeq M^V(x_0(\gamma), x_1(\gamma), \varphi_\gamma^n(\boldsymbol{t})), \end{aligned}$$

and this means that $\{x_n(\gamma)\}$ is a Cauchy sequence. Then, we conclude that there exists $z \in X^\Gamma$ such that

$$z(\gamma) = \lim_{n\to\infty} x_n(\gamma), \quad \text{for every } \gamma \in \Gamma.$$

From Corollary 1, for every $n \in \mathbb{N}$, we have $x_n \in \mathcal{U}(\Gamma, X)$ and $z \in \mathcal{U}(\Gamma, X)$. Given that G is a FC-M, we obtain that

$$M^V(z(\gamma), G(\gamma, z(\gamma)), \boldsymbol{t}) = \lim_{n\to\infty} M^V(x_n(\gamma), G(\gamma, x_n(\gamma)), \boldsymbol{t}), \quad \text{for every } \gamma \in \Gamma, \boldsymbol{t} \gg \bar{0}.$$

From

$$\begin{aligned} M^V(x_n(\gamma), G(\gamma, x_n(\gamma)), \boldsymbol{t}) &= M^V(G(\gamma, x_{n-1}(\gamma)), G(\gamma, x_n(\gamma)), \boldsymbol{t}) \\ &\succeq M^V(x_{n-1}(\gamma), x_n(\gamma), \varphi_\gamma(\boldsymbol{t})) \\ &\succeq \ldots \succeq M^V(x_0(\gamma), x_1(\gamma), \varphi_\gamma^n(\boldsymbol{t})), \end{aligned}$$

letting $n \to \infty$, we obtain $M^V(z(\gamma), G(\gamma, z(\gamma)), \boldsymbol{t}) = \bar{1}$ for every $\gamma \in \Gamma$, $\boldsymbol{t} \gg \bar{0}$. Thus $z(\gamma) = G(\gamma, z(\gamma))$ for each $\gamma \in \Gamma$, $\boldsymbol{t} \gg \bar{0}$, that is, z is a RFP of G. □

Remark 3. *By adding the assumptions in Theorem 2, we can also prove Theorem 4. For this purpose, we prove the uniqueness of RBF by eliminating duplicate problems.*

5. Investigating the Solution for a BVP

In this section, we investigate EURS for (2). We state our investigations in the form of a theorem.

We consider the following

- The measurable space of (Γ, Λ);

- We consider functions $q_1, q_2 : \Gamma \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which are of F-C type. Then, for each $(p, u, v) \in I \times \mathbb{R} \times \mathbb{R}$ and $\gamma \in \Gamma$ where $i = 1, 2$, $\gamma \to q_i(\gamma, p, u, v)$ are measurable and $(p, u, v) \to q_i(\gamma, p, u, v)$ are continuous;
- For each $u \in C(I, \mathbb{R})$ and each $(\gamma, p) \in \Gamma \times I$, we consider a family of F-C functions such as $r : \Gamma \times I \times \mathbb{R} \to \mathbb{R}$, where $r_i, u : \Gamma \times I \to \mathbb{R}$ is defined as $r_i, u(\gamma, p) = r(\gamma, p, u(p))$. We denote this set of functions with the letter $\mathcal{G}$.

Now we define the integral operator $Q : \Gamma \times C(I, \mathbb{R}) \times C(I, \mathbb{R}) \to C(I, \mathbb{R}) \times C(I, \mathbb{R})$ as follows:

$$Q(\gamma, f, g)(p) = (Q_1(\gamma, f, g)(p), Q_2(\gamma, f, g)(p)), \quad f, g \in C(I, \mathbb{R}),\ p \in I,$$

with

$$Q_1(\gamma, f, g)(p) = \int_0^1 T(p, s) q_1(\gamma, s, f(s), g(s)) ds + r_{1,x}(\gamma, p), \tag{11}$$

$$Q_2(\gamma, f, g)(p) = \int_0^1 T(p, s) q_2(\gamma, s, f(s), g(s)) ds + r_{2,y}(\gamma, p), \tag{12}$$

where $T : I \times I \to \mathbb{R}$ is a continuous function such that $|T(p, s)| \leqslant 1$ for every $p, s \in I$ and $r_1, r_2 \in \mathcal{G}$.

Remark 4. *For $(f, g) \in C(I, \mathbb{R}) \times C(I, \mathbb{R})$, q_i and r_i $(i = 1, 2)$ are functions which are of F-C type. For constant $s \in I$, the F-C function, $h : \Gamma \times I \to \mathbb{R}$ is defined as follows*

$$h(\gamma, p) = T(p, s) q_i(\gamma, s, f(s), g(s)).$$

Functions $\gamma \to Q_1(\gamma, f, g)$ and $\gamma \to Q_2(\gamma, f, g)$ are measurable functions, because the functions in integrals (11) and (12) are measurable. In fact, these integrals are limits of a finite sum of measurable functions. Consequently, Q is a random operator (RO).

At this step, we consider the following assumptions

Hypothesis 7 (H7)**.** *There exists a non-decreasing function $\phi_\gamma = (\phi_{\gamma,1}, \phi_{\gamma,2}) : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ such that*

$$N^V(q_i(\gamma, p, x, y) - q_i(\gamma, p, u, v), \boldsymbol{t}) \succeq (N(x - u, \phi_{\gamma,i}(\boldsymbol{t})), N(y - v, \phi_{\gamma,i}(\boldsymbol{t}))), i = 1, 2,$$

for each $\gamma \in \Gamma$ and for every $p \in I$ and all $x, y, u, v \in \mathbb{R}$ with $(x, y) \preceq (u, v)$ and $\boldsymbol{t} \gg \bar{0}$;

Hypothesis 8 (H8)**.** *For each $\gamma \in \Gamma$, there exists a function $\varphi_\gamma = (\varphi_{\gamma,1}, \varphi_{\gamma,2}) \in \mathcal{R}_2$ such that*

$$\begin{aligned}
&\left(\inf_{p \in I} N((f-h)(p), \phi_{\gamma,1}(\boldsymbol{t})), \inf_{p \in I} N((g-k)(p), \phi_{\gamma,1}(\boldsymbol{t}))\right) \star N^V(r_{1,x}(\gamma, p) - r_{1,u}(\gamma, p), \boldsymbol{t}) \\
&\succeq \left(\inf_{p \in I} N((f-h)(p), \varphi_{\gamma,1}(\boldsymbol{t})), \inf_{p \in I} N((g-k)(p), \varphi_{\gamma,1}(\boldsymbol{t}))\right),
\end{aligned}$$

$$\begin{aligned}
&\left(\inf_{p \in I} N((f-h)(p), \phi_{\gamma,2}(\boldsymbol{t})), \inf_{p \in I} N((g-k)(p), \phi_{\gamma,2}(\boldsymbol{t}))\right) \star N^V(r_{2,x}(\gamma, p) - r_{2,u}(\gamma, p), \boldsymbol{t}) \\
&\succeq \left(\inf_{p \in I} N((f-h)(p), \varphi_{\gamma,2}(\boldsymbol{t})), \inf_{p \in I} N((g-k)(p), \varphi_{\gamma,2}(\boldsymbol{t}))\right),
\end{aligned}$$

for each $p \in I$, $f, g, h, k \in C(I, \mathbb{R})$ and $\boldsymbol{t} \gg \bar{0}$;

Hypothesis 9 (H9)**.** *For each $\gamma \in \Gamma$ fixed, $(x, y) \to q_i(\gamma, p, x, y)$ and $x \to r_i(\gamma, p, x)$, $i = 1, 2$, (for every $p \in I$), are all non-decreasing or all non-increasing operators;*

Hypothesis 10 (H10). *One of the following conditions holds:*

$$0 \leqslant q_i(\gamma, p, 0, 0), \quad 0 \leqslant r_i(\gamma, p, 0), \quad \textit{for every } p \in I,\ \gamma \in \Gamma,\ i = 1, 2,$$

or

$$0 \geqslant q_i(\gamma, p, 0, 0), \quad 0 \geqslant r_i(\gamma, p, 0), \quad \textit{for every } p \in I,\ \gamma \in \Gamma,\ i = 1, 2.$$

We consider the FVVM M^V on $C(I, \mathbb{R}) \times C(I, \mathbb{R})$ as follows

$$M^V((f, g), (h, k), \boldsymbol{t}) = \left(\inf_{p \in I} N((f-h)(p), \boldsymbol{t}), \inf_{p \in I} N((g-k)(p), \boldsymbol{t})\right),$$

for every $(f, g), (h, k) \in C(I, \mathbb{R}) \times C(I, \mathbb{R})$, $\boldsymbol{t} \gg \bar{0}$.

Theorem 5. *We consider assumptions* (H7)–(H10)*. We prove that the random integral operator Q has a unique RFP.*

Proof. We first show that $(f, g) \to Q(\gamma, f, g)$ is a continuous operator for constant $\gamma \in \Gamma$. For this purpose, we consider the sequence $\{(f_n, g_n)\}$ on $C^2(I, \mathbb{R})$ such that $(f_n, g_n) \to (f, g) \in C(I, \mathbb{R}) \times C(I, \mathbb{R})$ when $n \to \infty$. For $p \in I$, we have

$$\begin{aligned}
&N^V(Q_1(\gamma, f_n, g_n)(p) - Q_1(\gamma, f, g)(p), \boldsymbol{t}) \\
&= N^V\bigg(\int_0^1 T(p, s)[q_1(\gamma, s, f_n(s), g_n(s)) - q_1(\gamma, s, f(s), g(s))]ds \\
&+ [r_{1,f_n}(\gamma, p) + r_{1,f}(\gamma, p)], \boldsymbol{t}\bigg) \\
&\succeq N^V\left(\int_0^1 T(p, s)[q_1(\gamma, s, f_n(s), g_n(s)) - q_1(\gamma, s, f(s), g(s))]ds, \frac{1}{2}\boldsymbol{t}\right) \\
&\star N^V\left(r_{1,f_n}(\gamma, p) + r_{1,f}(\gamma, p)], \frac{1}{2}\boldsymbol{t}\right) \\
&= N^V\left(\lim_{\|\Delta s\| \to 0} \sum_{i=1}^k T(p, \eta_i)[q_1(\gamma, \eta_i, f_n(\eta_i), g_n(\eta_i)) - q_1(\gamma, \eta_i, f(\eta_i), g(\eta_i))]\Delta s_i, \frac{1}{2}\boldsymbol{t}\right) \\
&\star N^V\left(r_{1,f_n}(\gamma, p) - r_{1,f}(\gamma, p), \frac{1}{2}\boldsymbol{t}\right),
\end{aligned}$$

where $0 = \eta_1 < \eta_2 < \ldots < \eta_k = 1$, $\Delta s_i = \eta_i - \eta_{i-1} = \frac{1}{k}$, $i = 1, 2, \ldots, k$ and $\|\Delta s\| = \max\limits_{1 \leqslant i \leqslant k}(\Delta s_i)$.

For any given $p \in I$ and $\boldsymbol{t} \gg \bar{0}$, then we have

$$
\begin{aligned}
&N^V(Q_1(\gamma, f_n, g_n)(p) - Q_1(\gamma, f, g)(p), \boldsymbol{t}) \\
&\succeq \lim_{\|\Delta s\| \to 0} N^V\left(\sum_{i=1}^{k} T(p, \eta_i)[q_1(\gamma, \eta_i, f_n(\eta_i), g_n(\eta_i)) - q_1(\gamma, \eta_i, f(\eta_i), g(\eta_i))]\Delta s_i, \frac{1}{2}\boldsymbol{t}\right) \\
&\star N^V\left(r_{1,f_n}(\gamma, p) - r_{1,f}(\gamma, p), \frac{1}{2}\boldsymbol{t}\right) \\
&\succeq \lim_{\|\Delta s\| \to 0} \left[N^V\left(T(p, \eta_1)[q_1(\gamma, \eta_1, f_n(\eta_1), g_n(\eta_1)) - q_1(\gamma, \eta_1, f(\eta_1), g(\eta_1))], \frac{1}{2|\Delta s_1|}\boldsymbol{t}\right)\right. \\
&\left.\star \cdots \star N^V\left(T(p, \eta_k)[q_1(\gamma, \eta_k, f_n(\eta_k), g_n(\eta_k)) - q_1(\gamma, \eta_k, f(\eta_k), g(\eta_k))], \frac{1}{2|\Delta s_k|}\boldsymbol{t}\right)\right] \\
&\star N^V\left(r_{1,f_n}(\gamma, p) - r_{1,f}(\gamma, p), \frac{1}{2}\boldsymbol{t}\right) \\
&\succeq \lim_{\|\Delta s\| \to 0} \left[N^V\left(q_1(\gamma, \eta_1, f_n(\eta_1), g_n(\eta_1)) - q_1(\gamma, \eta_1, f(\eta_1), g(\eta_1)), \frac{1}{2|\Delta s_1||T(p, \eta_1)|}\boldsymbol{t}\right)\right. \\
&\left.\star \cdots \star N^V\left(q_1(\gamma, \eta_k, f_n(\eta_k), g_n(\eta_k)) - q_1(\gamma, \eta_k, f(\eta_k), g(\eta_k), \frac{1}{2|\Delta s_k||T(p, \eta_k)|}\boldsymbol{t}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\star N^V\left(r_{1,f_n}(\gamma, p) - r_{1,f}(\gamma, p), \frac{1}{2}\boldsymbol{t}\right) \\
&\succeq \lim_{\|\Delta s\| \to 0} \min N^V\left(q_1(\gamma, \eta_i, f_n(\eta_i), g_n(\eta_i)) - q_1(\gamma, \eta_i, f(\eta_i), g(\eta_i)), \frac{1}{2}\boldsymbol{t}\right) \\
&\star N^V\left(r_{1,x_n}(\gamma, p) - r_{1,x}(\gamma, p), \frac{1}{2}\boldsymbol{t}\right) \\
&\succeq \inf_{s \in I} N^V\left(q_1(\gamma, s, f_n(s), g_n(s)) - q_1(\gamma, s, f(s), g(s)), \frac{1}{2}\boldsymbol{t}\right) \\
&\star N^V\left(r_{1,f_n}(\gamma, p) - r_{1,x}(\gamma, p), \frac{1}{2}\boldsymbol{t}\right) \\
&\succeq \left(\inf_{s \in I} N\left((f_n - f)(s), \phi_{\gamma,i}\left(\frac{1}{2}\boldsymbol{t}\right)\right), \inf_{s \in I} N\left((g_n - g)(s), \phi_{\gamma,i}\left(\frac{1}{2}\boldsymbol{t}\right)\right)\right) \\
&\star N^V\left(r_{1,f_n}(\gamma, p) - r_{1,f}(\gamma, p), \frac{1}{2}\boldsymbol{t}\right) \\
&\succeq \left(\inf_{s \in I} N\left((f_n - f)(s), \varphi_{\gamma,i}\left(\frac{1}{2}\boldsymbol{t}\right)\right), \inf_{s \in I} N\left((g_n - g)(s), \varphi_{\gamma,i}\left(\frac{1}{2}\boldsymbol{t}\right)\right)\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\inf_{s \in I} N^V(Q_1(\gamma, f_n, g_n)(s) - Q_1(\gamma, f, g)(s), \boldsymbol{t}) \\
\succeq \quad &\left(\inf_{s \in I} N((f_n - f)(s), \varphi_{\gamma,1}(\boldsymbol{t})), \inf_{s \in I} N((g_n - g)(s), \varphi_{\gamma,1}(\boldsymbol{t}))\right),
\end{aligned}
$$

by (H8). By analogous reasoning, one has

$$
\begin{aligned}
&\inf_{s \in I} N^V(Q_2(\gamma, f_n, g_n)(s) - Q_2(\gamma, f, g)(s), \boldsymbol{t}) \\
\succeq \quad &\left(\inf_{s \in I} N((f_n - f)(s), \varphi_{\gamma,2}(\boldsymbol{t})), \inf_{s \in I} N((g_n - g)(s), \varphi_{\gamma,2}(\boldsymbol{t}))\right).
\end{aligned}
$$

Then $M^V(Q(\gamma, f_n, g_n), Q(\gamma, f, g), \boldsymbol{t}) \to \bar{1}$ as $n \to \infty$, implies that $(f, g) \to Q(\gamma, f, g)$ is a continuous operator, for each fixed $\gamma \in \Gamma$ and for every $\boldsymbol{t} \gg \bar{0}$. In addition, for each $\gamma \in \Gamma$ and $(f, g) \to Q(\gamma, f, g)$ is a monotone operator. Indeed, consider $(f, g), (h, k) \in$

$C(I,\mathbb{R}) \times C(I,\mathbb{R})$ such that $(f,g) \precsim (h,k)$; that is, $f(p) \leqslant h(p)$, $g(p) \leqslant k(p)$, for every $p \in I$. For every $p \in I$, if $(f,g) \to q_i(\gamma,p,f,g)$ and $f \to r_i(\gamma,p,f)$, $i = 1,2$, are non-decreasing operators, then

$$\begin{aligned} q_i(\gamma,p,f(p),g(p)) \leqslant q_i(\gamma,p,h(p),k(p)), &\quad \text{for every } p \in I,\ i = 1,2, \\ r_1(\gamma,p,f(p)) \leqslant r_1(\gamma,p,h(p)), &\quad \text{for every } p \in I, \\ r_2(\gamma,p,g(p)) \leqslant r_2(\gamma,p,k(p)), &\quad \text{for every } p \in I, \end{aligned}$$

which implies that

$$Q_i(\gamma,f,g)(p) \leqslant Q_i(\gamma,h,k)(p), \quad \text{for every } p \in I,\ i = 1,2.$$

Therefore $Q(\gamma,f,g) \precsim Q(\gamma,h,k)$. In a similar way, we can conclude that if the functions $(f,g) \to q_i(\gamma,p,f,g)$ and $f \to r_i(\gamma,p,f)$ are non-decreasing functions, then for every $p \in I$ and $i = 1,2$, we have $Q(\gamma,h,k) \precsim Q(\gamma,f,g)$. Next, we have to show that Q is a contraction operator. At this step, we consider condition (H5) and assume that $\gamma \in \Gamma$ and $(f,g),(h,k) \in C(I,\mathbb{R}) \times C(I,\mathbb{R})$ such that $(f,g) \precsim (h,k)$. We have to show

$$M^V(Q(\gamma,f,g),Q(\gamma,h,k),\boldsymbol{t}) \succeq M^V((f,g),(h,k),\varphi_\gamma(\boldsymbol{t})).$$

Again, consider $\gamma \in \Gamma$ fixed. Let $(f,g),(h,k) \in C(I,\mathbb{R}) \times C(I,\mathbb{R})$ be such that $(f,g) \precsim (h,k)$, $p \in I$, then

$$\begin{aligned} &N^V(Q_1(\gamma,f,g)(p) - Q_1(\gamma,h,k)(p),\boldsymbol{t}) \\ &= N^V\bigg(\int_0^1 T(p,s)[q_1(\gamma,s,f(s),g(s)) - q_1(\gamma,s,h(s),k(s))]ds \\ &+ [r_{1,f}(\gamma,p) + r_{1,h}(\gamma,p)],\boldsymbol{t}\bigg) \\ &\succeq N^V\bigg(\int_0^1 T(p,s)[q_1(\gamma,s,f(s),g(s)) - q_1(\gamma,s,h(s),k(s))]ds, \frac{1}{2}\boldsymbol{t}\bigg) \\ &\star N^V\bigg(r_{1,f}(\gamma,p) + r_{1,h}(\gamma,p)], \frac{1}{2}\boldsymbol{t}\bigg) \\ &= N^V\bigg(\lim_{\|\Delta s\| \to 0} \sum_{i=1}^k T(p,\eta_i)[q_1(\gamma,\eta_i,f(\eta_i),g(\eta_i)) - q_1(\gamma,\eta_i,h(\eta_i),k(\eta_i))]\Delta s_i, \frac{1}{2}\boldsymbol{t}\bigg) \\ &\star N^V\bigg(r_{1,f}(\gamma,p) - r_{1,h}(\gamma,p), \frac{1}{2}\boldsymbol{t}\bigg), \end{aligned}$$

where $0 = \eta_1 < \eta_2 < \ldots < \eta_k = 1$, $\Delta s_i = \eta_i - \eta_{i-1} = \frac{1}{k}$, $i = 1,2,\ldots,k$ and $\|\Delta s\| = \max_{1 \leqslant i \leqslant k}(\Delta s_i)$.

For any given $p \in I$ and $\boldsymbol{t} \gg \bar{0}$, then we have

$$
\begin{aligned}
& N^V(Q_1(\gamma,f,g)(p)-Q_1(\gamma,h,k)(p),\boldsymbol{t}) \\
& \succeq \lim_{\|\Delta s\|\to 0} N^V\left(\sum_{i=1}^{k} T(p,\eta_i)[q_1(\gamma,\eta_i,f(\eta_i),g(\eta_i))-q_1(\gamma,\eta_i,h(\eta_i),k(\eta_i))]\Delta s_i,\frac{1}{2}\boldsymbol{t}\right) \\
& \star N^V\left(r_{1,f}(\gamma,p)-r_{1,h}(\gamma,p),\frac{1}{2}\boldsymbol{t}\right) \\
& \succeq \lim_{\|\Delta s\|\to 0}\left[N^V\left(T(p,\eta_1)[q_1(\gamma,\eta_1,f(\eta_1),g(\eta_1))-q_1(\gamma,\eta_1,h(\eta_1),k(\eta_1))],\frac{1}{2|\Delta s_1|}\boldsymbol{t}\right)\right. \\
& \left.\star\cdots\star N^V\left(T(p,\eta_k)[q_1(\gamma,\eta_k,f(\eta_k),g(\eta_k))-q_1(\gamma,\eta_k,h(\eta_k),k(\eta_k)),\frac{1}{2|\Delta s_k|}\boldsymbol{t}\right)\right] \\
& \star N^V\left(r_{1,f}(\gamma,p)-r_{1,h}(\gamma,p),\frac{1}{2}\boldsymbol{t}\right) \\
& \succeq \lim_{\|\Delta s\|\to 0}\left[N^V\left([q_1(\gamma,\eta_1,f(\eta_1),g(\eta_1))-q_1(\gamma,\eta_1,h(\eta_1),k(\eta_1))],\frac{1}{2|\Delta s_1||T(p,\eta_1)|}\boldsymbol{t}\right)\right. \\
& \left.\star\cdots\star N^V\left([q_1(\gamma,\eta_k,f(\eta_k),g(\eta_k))-q_1(\gamma,\eta_k,h(\eta_k),k(\eta_k)],\frac{1}{2|\Delta s_k||T(p,\eta_k)|}\boldsymbol{t}\right)\right] \\
& \star N^V\left(r_{1,f}(\gamma,p)-r_{1,h}(\gamma,p),\frac{1}{2}\boldsymbol{t}\right) \\
& \succeq \lim_{\|\Delta s\|\to 0}\min N^V\left(q_1(\gamma,\eta_i,f(\eta_i),g(\eta_i))-q_1(\gamma,\eta_i,h(\eta_i),k(\eta_i)),\frac{1}{2}\boldsymbol{t}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \star N^V\left(r_{1,f}(\gamma,p)-r_{1,h}(\gamma,p),\frac{1}{2}\boldsymbol{t}\right) \\
& \succeq \inf_{s\in I} N^V\left(q_1(\gamma,s,f(s),g(s))-q_1(\gamma,s,h(s),k(s)),\frac{1}{2}\boldsymbol{t}\right) \\
& \star N^V\left(r_{1,f}(\gamma,p)-r_{1,h}(\gamma,p),\frac{1}{2}\boldsymbol{t}\right) \\
& \succeq \left(\inf_{s\in I} N\left((f-h)(s),\phi_{\gamma,i}\left(\frac{1}{2}\boldsymbol{t}\right)\right),\inf_{s\in I} N\left((g-k)(s),\phi_{\gamma,i}\left(\frac{1}{2}\boldsymbol{t}\right)\right)\right) \\
& \star N^V\left(r_{1,f}(\gamma,p)-r_{1,h}(\gamma,p),\frac{1}{2}\boldsymbol{t}\right) \\
& \succeq \left(\inf_{s\in I} N\left((f-h)(s),\varphi_{\gamma,i}\left(\frac{1}{2}\boldsymbol{t}\right)\right),\inf_{s\in I} N\left((g-k)(s),\varphi_{\gamma,i}\left(\frac{1}{2}\boldsymbol{t}\right)\right)\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \inf_{s\in I} N^V(Q_1(\gamma,f,g)(s)-Q_1(\gamma,h,k)(s),\boldsymbol{t}) \\
\succeq\ & \left(\inf_{s\in I} N((f-h)(s),\varphi_{\gamma,1}(\boldsymbol{t})),\inf_{s\in I} N((g-k)(s),\varphi_{\gamma,1}(\boldsymbol{t}))\right),
\end{aligned}
$$

by (H8). By analogous reasoning, one has

$$
\begin{aligned}
& \inf_{s\in I} N^V(Q_2(\gamma,f,g)(s)-Q_2(\gamma,h,k)(s),\boldsymbol{t}) \\
\succeq\ & \left(\inf_{s\in I} N((f-h)(s),\varphi_{\gamma,2}(\boldsymbol{t})),\inf_{s\in I} N((g-k)(s),\varphi_{\gamma,2}(\boldsymbol{t}))\right),
\end{aligned}
$$

then

$$
M^V(Q(\gamma,f,g),Q(\gamma,h,k),\boldsymbol{t}) \succeq M^V((f,g),(h,k),\varphi_\gamma(\boldsymbol{t})).
$$

Next, we have to show that (H6) holds. We prove using (H10)

$$0 \leqslant Q_1(\gamma, \times, 0, 0) \quad \text{and} \quad 0 \leqslant Q_2(\gamma, \times, 0, 0), \quad \text{for every } \gamma \in \Gamma,$$

or

$$0 \geqslant Q_1(\gamma, \times, 0, 0) \quad \text{and} \quad 0 \geqslant Q_2(\gamma, \times, 0, 0), \quad \text{for every } \gamma \in \Gamma,$$

that is, $Q(\gamma, 0, 0) \succeq \bar{0}$ for every $\gamma \in \Gamma$ or $Q(\gamma, 0, 0) \preceq \bar{0}$ for every $\gamma \in \Gamma$. Then, for null random variables $\bar{0} : \Gamma \to C^2(I, \mathbb{R})$, which are defined as follows

$$\bar{0}(\gamma) = \bar{0},$$

one of the following conditions is true

$$Q(\gamma, \bar{0}(\gamma)) \succeq \bar{0}(\gamma), \quad \text{for every } \gamma \in \Gamma,$$

or

$$Q(\gamma, \bar{0}(\gamma)) \preceq \bar{0}(\gamma), \quad \text{for every } \gamma \in \Gamma.$$

Considering that the condition of uniqueness also exists, then all the conditions of Theorem 4 are satisfied and we conclude that there is a unique FP for Q, which results directly from Theorem 4.

□

Here we have a theorem that gives us a unique random solution (URS) of (2). This theorem is proposed for a specific selection of FC functions $r_1, r_2 : \Gamma \times I \times \mathbb{R} \to \mathbb{R}$. Therefore, we consider

- $r_i(\gamma, p, h(p)) = \psi_i\left(\int_0^1 h(\gamma)(s)ds\right)p$, where $\psi_i \in C(\mathbb{R}, \mathbb{R})$, for $i = 1, 2$;
- the random integral operator

$$\widetilde{Q}(\gamma, f, g)(p) = \left(\widetilde{Q}_1(\gamma, f, g)(p), \widetilde{Q}_2(\gamma, f, g)(p)\right), \quad f, g \in C(I, \mathbb{R}), \ p \in I,$$

with

$$\widetilde{Q}_1(\gamma, f, g)(p) = \int_0^1 T(p,s)q_1(\gamma, s, f(s), g(s))ds + \psi_1\left(\int_0^1 f(\gamma)(s)ds\right)p,$$
$$\widetilde{Q}_2(\gamma, f, g)(p) = \int_0^1 T(p,s)f_2(\gamma, s, x(s), y(s))ds + \psi_2\left(\int_0^1 g(\gamma)(s)ds\right)p,$$

where $T : I \times I \to \mathbb{R}$ is given by (4).

Theorem 6. *We assume that conditions* (H7)–(H10) *are fulfilled. Then* (2) *has a URS.*

Proof. Here we point out that every RBF of $\widetilde{Q}$ is a solution of (2) and also every solution of (2) is a RBF of $\widetilde{Q}$. That is, considering the random variables $(f, g) : \Gamma \to C(I, \mathbb{R}) \times C(I, \mathbb{R})$, we have

$$\widetilde{Q}(\gamma, f(\gamma), g(\gamma)) = (f(\gamma), g(\gamma)), \quad \text{for every } \gamma \in \Gamma,$$

is equivalent to

$$f(\gamma)(p) = \int_0^1 T(p,s)q_1(\gamma, s, f(\gamma)(s), g(\gamma)(s))ds + \psi_1\left(\int_0^1 f(\gamma)(s)ds\right)p, \quad 0 < p < 1,$$
$$g(\gamma)(p) = \int_0^1 T(p,s)q_2(\gamma, s, f(\gamma)(s), g(\gamma)(s))ds + \psi_2\left(\int_0^1 g(\gamma)(s)ds\right)p, \quad 0 < p < 1,$$

Such that $f(\gamma, p) = f(\gamma)(p)$, $g(\gamma, p) = g(\gamma)(p)$ for $p \in I$ and $\gamma \in \Gamma$ is a solution of (2). Then, by Theorem 5, there exists a unique random solution of the Problem (2). $\square$

6. Conclusions

In this paper, a general system of fuzzy random differential equations with boundary conditions has been investigated. In addition, the existence theorem that yields a unique random solution was proved by a random fixed point theorem in fuzzy metric spaces. The obtained results would be used for modelling dynamical systems in environments, especially in air pollution problems. The applied procedure can also be useful in the future for some other types of fuzzy equations.

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