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Article

Approximation for the Ratios of the Confluent Hypergeometric Function $\Phi_D^{(N)}$ by the Branched Continued Fractions

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Abstract: The paper deals with the problem of expansion of the ratios of the confluent hypergeometric function of N variables $\Phi_D^{(N)}(a,\bar{b};c;\bar{z})$ into the branched continued fractions (BCF) of the general form with N branches of branching and investigates the convergence of these BCF. The algorithms of construction for BCF expansions of confluent hypergeometric function $\Phi_D^{(N)}$ ratios are based on some given recurrence relations for this function. The case of nonnegative parameters $a, b_1, \ldots, b_{N-1}$ and positive c is considered. Some convergence criteria for obtained BCF with elements in $\mathbb{R}^N$ and $\mathbb{C}^N$ are established. It is proven that these BCF converge to the functions which are an analytic continuation of the above-mentioned ratios of function $\Phi_D^{(N)}(a,\bar{b};c;\bar{z})$ in some domain of $\mathbb{C}^N$.

Keywords: confluent hypergeometric function of several variables; recurrence relations; branched continued fraction; approximant; uniform convergence

MSC: 33C65; 11J70; 30B70; 40A15

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1. Introduction

In the course of the last three centuries the necessity of solving the problems arising in the fields of hydrodynamics, control theory, classical and quantum mechanics stimulated the development of the theory of special functions of one and several variables [1–5]. Functions of hypergeometric type constitute an important class of special functions.

For hypergeometric functions of one variables there exists a well-developed theory with numerous applications. All advanced computer algebra systems support calculations involving hypergeometric functions. In the multivariate case there exist several approaches to the notion of a hypergeometric functions. Such a function can be defined as a sum of a power series of a certain kind (the so-called Γ-series), as a solution to a system of partial differential equations, as the Euler-type integral or as the Mellin–Barnes integral [1,3].

It is known that continued fractions have numerous applications in the theory of approximation of hypergeometric functions of one variable [6–9]. Multidimensional generalizations of continued fractions can be considered as a tool of rational approximation of functions of several variables [10–20]. In particular, branched continued fractions (BCF) of the form

$$d_0(\bar{z}) + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{c_{i(k)}(\bar{z})}{d_{i(k)}(\bar{z})} = d_0(\bar{z}) + \sum_{i_1=1}^{N} \cfrac{c_{i(1)}(\bar{z})}{d_{i(1)}(\bar{z}) + \sum\limits_{i_2=1}^{N} \cfrac{c_{i(2)}(\bar{z})}{d_{i(2)}(\bar{z}) + \sum\limits_{i_3=1}^{N} \cfrac{c_{i(3)}(\bar{z})}{d_{i(3)}(\bar{z}) + \ddots}}}, \qquad (1)$$

where $N \in \mathbb{N}$, $i(k) = (i_1, i_2, \ldots, i_k)$ be a multi-index,

$$\mathcal{I} = \{i(k) : \ 1 \le i_r \le N, \ 1 \le r \le k, \ k \ge 1\}$$

be a set of multi-indices, the $d_0(\bar{z})$ and the elements $c_{i(k)}(\bar{z})$ and $d_{i(k)}(\bar{z})$, $i(k) \in \mathcal{I}$ are certain polynomials, $\bar{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$ are used to approximate the ratios of some hypergeometric functions of one or several variables [21–29]. Note that the symbol D, proposed by I. Sleshynsky in 1888 [30], is used here to denote BCF.

In this paper, we construct the branched continued fraction expansions for confluent hypergeometric functions of N variables $\Phi_D^{(N)}$ ratios and investigate their convergence. The confluent hypergeometric function $\Phi_D^{(N)}$ is defined by the multiply power series [3]

$$\Phi_D^{(N)}(a, \bar{b}; c; \bar{z}) = \sum_{k_1, k_2, \ldots, k_N = 0}^{\infty} \frac{(a)_{k_1+k_2+\ldots+k_N} (b_1)_{k_1} (b_2)_{k_2} \ldots (b_{N-1})_{k_{N-1}}}{(c)_{k_1+k_2+\ldots+k_N}} \frac{z_1^{k_1}}{k_1!} \frac{z_2^{k_2}}{k_2!} \ldots \frac{z_N^{k_N}}{k_N!}, \quad (2)$$

where $a, b_1, \ldots, b_{N-1}, c$ are complex constants (parameters of function), $c \ne 0, -1, -2, \ldots$, $\bar{b} = (b_1, \ldots, b_{N-1})$, $(\alpha)_k$ is the Pochhammer symbol: $(\alpha)_0 = 1$, $(\alpha)_k = \alpha(\alpha+1)_{k-1}$, $k \ge 1$. Series (2) converges for $|z_i| < 1$, $1 \le i \le N-1$, $z_N \in \mathbb{C}$. Function $\Phi_D^{(N)}$ was originated by H. Exton and H. Srivastava. This function is a generalization of the Humbert function $\Phi_D^{(2)} = \Phi_1$. At $z_N = 0$ value of the function, $\Phi_D^{(N)}$ coincides with the value of the Lauricella function $F_D^{(N-1)}$.

The algorithms of construction for branched continued fraction expansions of confluent hypergeometric function $\Phi_D^{(N)}$ ratios are based on some recurrence relations for this function (Section 2). We stated and proved some convergence properties for the obtained BCF (Section 3).

Let us recall some basic concepts and notations (we refer the reader to the books [31,32] to learn more). The finite BCF

$$f_n(\bar{z}) = d_0(\bar{z}) + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{N} \frac{c_{i(k)}(\bar{z})}{d_{i(k)}(\bar{z})}$$

is called the nth approximant of the BCF (1). Note that for each $n \in \mathbb{N}$ the approximant $f_n(\bar{z})$ can also be written as

$$f_n(\bar{z}) = d_0(\bar{z}) + \sum_{i_1=1}^{N} \frac{c_{i(1)}(\bar{z})}{Q_{i(1)}^{(n)}(\bar{z})},$$

where the tails, $Q_{i(k)}^{(n)}(\bar{z})$, $i(k) \in \mathcal{I}$, $1 \le k \le n$, are defined as follows

$$Q_{i(n)}^{(n)}(\bar{z}) = d_{i(n)}(\bar{z}), \quad n \ge 1, \quad (3)$$
$$Q_{i(k)}^{(n)}(\bar{z}) = d_{i(k)}(\bar{z}) + \mathop{\mathrm{D}}_{r=1}^{n-k} \sum_{i_{k+r}=1}^{N} \frac{c_{i(k+r)}(\bar{z})}{d_{i(k+r)}(\bar{z})}, \quad i(k) \in \mathcal{I}, \ 1 \le k \le n-1, \ n \ge 2.$$

It is clear that the following recurrence relations hold

$$Q_{i(k)}^{(n)}(\bar{z}) = d_{i(k)}(\bar{z}) + \sum_{i_{k+1}=1}^{N} \frac{c_{i(k+1)}(\bar{z})}{Q_{i(k+1)}^{(n)}(\bar{z})}, \quad i(k) \in \mathcal{I}, \ 1 \le k \le n-1, \ n \ge 2. \quad (4)$$

Definition 1. *The BCF (1), whose elements are functions of N variables, is said to converge uniformly in a certain domain D, $D \subset \mathbb{C}^N$, if for each $\bar{z} \in D$ at most its approximants $f_n(\bar{z})$ have sense and are finite and for a given $\epsilon > 0$ there exists n_ϵ such that for all $m, n \geq n_\epsilon$ and for each $\bar{z} \in D$ the following inequality $|f_m(\bar{z}) - f_n(\bar{z})| < \epsilon$ is valid.*

Definition 2. *The BCF (1), whose elements are functions of N variables in a domain D, $D \subset \mathbb{C}^N$, is said to converge uniformly on a compact subset K of D if there exists $n(K)$ such that $f_n(\bar{z})$ is holomorphic in some domain containing K for all $n \geq n(K)$ and for a given $\epsilon > 0$ there exists $n_\epsilon > n(K)$ such that $\sup_{\bar{z} \in K} |f_m(\bar{z}) - f_n(\bar{z})| < \epsilon$ for $m, n \geq n_\epsilon$.*

If $Q^{(n)}_{i(k)}(\bar{z}) \not\equiv 0$ for all $i(k) \in \mathcal{I}$, $1 \leq k \leq n$, $n \geq 1$, the following formula of difference for two approximants of BCF of the form (1) is valid (see [31], p. 28)

$$f_m(\bar{z}) - f_n(\bar{z}) = (-1)^n \sum_{i_1=1}^{N} \dots \sum_{i_{n+1}=1}^{N} \frac{\prod_{k=1}^{n+1} c_{i(k)}(\bar{z})}{\prod_{k=1}^{n+1} Q^{(m)}_{i(k)}(\bar{z}) \prod_{p=1}^{n} Q^{(n)}_{i(k)}(\bar{z})}, \quad m > n, \ n \geq 1. \tag{5}$$

Note that this formula is used to study the properties of a sequence $\{f_n(\bar{z})\}$.

2. Recurrence Relations for Function $\Phi_D^{(N)}$: Expansions for the Ratios of Function $\Phi_D^{(N)}$ into the Branched Continued Fractions

To construct the expansion of the ratio of hypergeometric series of one or several variables, the recurrence relations between these series are used. Here we give some recurrence relations for multiply power series (2).

We denote $e_i = (\delta_i^1, \delta_i^2, \dots, \delta_i^{N-1})$, where δ_i^j is the Kronecker delta: $\delta_i^j = 1$, if $i = j$, and $\delta_i^j = 0$, if $i \neq j$.

The recurrence relations for function $\Phi_D^{(N)}$ are valid

$$\begin{aligned} \Phi_D^{(N)}(a, \bar{b}; c; \bar{z}) &= \Phi_D^{(N)}(a+1, \bar{b}; c; \bar{z}) - \sum_{i=1}^{N-1} \frac{b_i z_i}{c} \Phi_D^{(N)}(a+1, \bar{b}+e_i; c+1; \bar{z}) \\ &\quad - \frac{z_N}{c} \Phi_D^{(N)}(a+1, \bar{b}; c+1; \bar{z}), \end{aligned} \tag{6}$$

$$\begin{aligned} \Phi_D^{(N)}(a, \bar{b}; c; \bar{z}) &= \Phi_D^{(N)}(a, \bar{b}; c+1; \bar{z}) + \sum_{i=1}^{N-1} \frac{a b_i z_i}{c(c+1)} \Phi_D^{(N)}(a+1, \bar{b}+e_i; c+2; \bar{z}) \\ &\quad + \frac{a z_N}{c(c+1)} \Phi_D^{(N)}(a+1, \bar{b}; c+2; \bar{z}), \end{aligned} \tag{7}$$

$$\begin{aligned} \Phi_D^{(N)}(a, \bar{b}; c; \bar{z}) &= \Phi_D^{(N)}(a, \bar{b}+e_i; c; \bar{z}) \\ &\quad - \frac{a z_i}{c} \Phi_D^{(N)}(a+1, \bar{b}+e_i; c+1; \bar{z}), \quad 1 \leq i \leq N-1. \end{aligned} \tag{8}$$

These formal identities can be derived from (2) by comparing the coefficients of $z_1^{k_1} z_2^{k_2} \dots z_N^{k_N}$ on both sides of the identities.

From (6)–(8) it follows that

$$\begin{aligned}
\Phi_D^{(N)}(a,\bar{b};c;\bar{z}) &= \Phi_D^{(N)}(a+1,\bar{b};c;\bar{z}) - \sum_{j=1}^{N-1}\frac{b_j z_j}{c}\Phi_D^{(N)}(a+1,\bar{b}+e_j;c+1;\bar{z}) \\
&\quad - \frac{z_N}{c}\Phi_D^{(N)}(a+1,\bar{b};c+1;\bar{z}) \\
&= \Phi_D^{(N)}(a+1,\bar{b};c;\bar{z}) - \frac{z_N}{c}\Phi_D^{(N)}(a+1,\bar{b};c+1;\bar{z}) \\
&\quad - \sum_{j=1}^{N-1}\frac{b_j z_j}{c}\left(\Phi_D^{(N)}(a+1,\bar{b};c+1;\bar{z}) + \frac{a+1}{c+1}z_j\Phi_D^{(N)}(a+2,\bar{b}+e_j;c+2;\bar{z})\right) \\
&= \Phi_D^{(N)}(a+1,\bar{b};c+1;\bar{z}) - \frac{z_N}{c}\Phi_D^{(N)}(a+1,\bar{b};c+1;\bar{z}) \\
&\quad + \sum_{j=1}^{N-1}\frac{(a+1)b_j z_j}{c(c+1)}\Phi_D^{(N)}(a+2,\bar{b}+e_j;c+2;\bar{z}) \\
&\quad + \frac{(a+1)z_N}{c(c+1)}\Phi_D^{(N)}(a+2,\bar{b};c+2;\bar{z}) \\
&\quad - \sum_{j=1}^{N-1}\frac{b_j}{c}z_j\left(\Phi_D^{(N)}(a+1,\bar{b};c+1;\bar{z}) + \frac{a+1}{c+1}z_j\Phi_D^{(N)}(a+2,\bar{b}+e_j;c+2;\bar{z})\right).
\end{aligned}$$

So,

$$\begin{aligned}
\Phi_D^{(N)}(a,\bar{b};c;\bar{z}) = \Phi_D^{(N)}(a+1,\bar{b};c+1;\bar{z})&\left(1 - \frac{z_N}{c} - \sum_{j=1}^{N-1}\frac{b_j}{c}z_j\right) \\
&+ \sum_{j=1}^{N-1}\frac{(a+1)b_j}{c(c+1)}z_j(1-z_j)\Phi_D^{(N)}(a+2,\bar{b}+e_j;c+2;\bar{z}) \\
&+ \frac{a+1}{c(c+1)}z_N\Phi_D^{(N)}(a+2,\bar{b};c+2;\bar{z}).
\end{aligned} \tag{9}$$

Using the recurrence relations (8), (9) the expansions of the ratios

$$X_i(a,\bar{b};c;\bar{z}) = \frac{\Phi_D^{(N)}(a,\bar{b};c;\bar{z})}{\Phi_D^{(N)}(a+1,\bar{b}+e_i;c+1;\bar{z})}, \quad 1 \le i \le N-1,$$
$$X_N(a,\bar{b};c;\bar{z}) = \frac{\Phi_D^{(N)}(a,\bar{b};c;\bar{z})}{\Phi_D^{(N)}(a+1,\bar{b};c+1;\bar{z})},$$

into the branched continued fraction (BCF) of the general form with N branches of branching can be constructed. Indeed, performing the termwise division of the identity (9) by $\Phi_D^{(N)}(a+1,\bar{b};c+1;\bar{z})$, we obtain

$$\begin{aligned}
X_N(a,\bar{b};c;\bar{z}) = 1 - \frac{z_N}{c} - \sum_{j=1}^{N-1}\frac{b_j}{c}z_j + \sum_{j=1}^{N-1}\frac{(a+1)b_j}{c(c+1)}\frac{z_j(1-z_j)}{X_j(a+1,\bar{b};c+1;\bar{z})} \\
+ \frac{a+1}{c(c+1)}\frac{z_N}{X_N(a+1,\bar{b};c+1;\bar{z})}.
\end{aligned} \tag{10}$$

Moreover, from (8) it follows that

$$X_i(a,\bar{b};c;\bar{z}) = X_N(a,\bar{b}+e_i;c;\bar{z}) - \frac{az_i}{c}, \quad 1 \le i \le N-1. \tag{11}$$

Taking into account (11), we rewrite formula (10) as follows

$$X_N(a,\bar{b};c;\bar{z}) = 1 - \frac{z_N}{c} - \sum_{j=1}^{N-1} \frac{b_j}{c} z_j + \sum_{j=1}^{N-1} \frac{(a+1)b_j}{c(c+1)} \frac{z_j(1-z_j)}{X_N(a+1,\bar{b}+e_j;c+1;\bar{z}) - \frac{(a+1)z_j}{c+1}} + \frac{a+1}{c(c+1)} \frac{z_N}{X_N(a+1,\bar{b};c+1;\bar{z})}$$

or

$$X_N(a,\bar{b};c;\bar{z}) = 1 - \frac{z_N}{c} - \sum_{j=1}^{N-1} \frac{b_j}{c} z_j + \sum_{i_1=1}^{N} \frac{\frac{(a+1)((1-\delta_{i_1}^N)b_{i_1} + \delta_{i_1}^N)}{c(c+1)} z_{i_1}(1-(1-\delta_{i_1}^N)z_{i_1})}{X_N(a+1,\bar{b}+e_{i_1};c+1;\bar{z}) - (1-\delta_{i_1}^N)\frac{(a+1)z_{i_1}}{c+1}}. \tag{12}$$

Then

$$X_N(a,\bar{b};c;\bar{z}) = 1 - \frac{z_N}{c} - \sum_{j=1}^{N-1} \frac{b_j}{c} z_j + \sum_{i_1=1}^{N} \frac{(a+1)((1-\delta_{i_1}^N)b_{i_1} + \delta_{i_1}^N)}{c(c+1)} z_{i_1}(1-(1-\delta_{i_1}^N)z_{i_1}) \times \left(1 - \frac{z_N}{c+1} - \sum_{j=1}^{N-1} \frac{b_j + \delta_{i_1}^j}{c+1} z_j - (1-\delta_{i_1}^N)\frac{a+1}{c+1} z_{i_1} + \sum_{i_2=1}^{N} \frac{\frac{(a+2)((1-\delta_{i_2}^N)(b_{i_2} + \delta_{i_2}^{i_1}) + \delta_{i_2}^N)}{(c+1)(c+2)} z_{i_2} - (1-(1-\delta_{i_2}^N)z_{i_2})}{X_N(a+2,\bar{b}+e_{i_1}+e_{i_2};c+2;\bar{z}) - (1-\delta_{i_2}^N)\frac{a+2}{c+2} z_{i_2}}\right)^{-1}.$$

Substituting expressions for X_N with corresponding parameters into formula (12), after n steps we obtain the expansion for the ratio $X_N(a,\bar{b};c;\bar{z})$ into the finite BCF of the general form with N branches:

$$X_N(a,\bar{b};c;\bar{z}) = 1 - \frac{z_N}{c} - \sum_{j=1}^{N-1} \frac{b_j}{c} z_j + \sum_{i_1=1}^{N} \frac{c_{i(1)}(\bar{z})|}{|d_{i(1)}(\bar{z})} + \sum_{i_2=1}^{N} \frac{c_{i(2)}(\bar{z})|}{|d_{i(2)}(\bar{z})} + \ldots + \sum_{i_n=1}^{N} \frac{c_{i(n)}(\bar{z})|}{|X_N(a+n,\bar{b}+\sum_{p=1}^{n} e_{i_p};c+n;\bar{z}) - (1-\delta_{i_n}^N)\frac{a+n}{c+n} z_{i_n}}, \tag{13}$$

where for $i(k) \in \mathcal{I}, 1 \le k \le n$,

$$c_{i(k)}(\bar{z}) = \begin{cases} \frac{(a+k)(b_{i_k} + \sum_{p=1}^{k-1} \delta_{i_k}^{i_p})}{(c+k-1)(c+k)} z_{i_k}(1-z_{i_k}), & \text{if} \quad 1 \le i_k \le N-1, \\ \frac{a+k}{(c+k-1)(c+k)} z_{i_k}, & \text{if} \quad i_k = N, \end{cases} \tag{14}$$

and for $i(k) \in \mathcal{I}, 1 \le k \le n-1$,

$$d_{i(k)}(\bar{z}) = \begin{cases} 1 - \dfrac{z_N}{c+k} - \dfrac{a+k}{c+k} z_{i_k} - \displaystyle\sum_{j=1}^{N-1} \dfrac{b_j + \sum_{p=1}^{k} \delta_j^{i_p}}{c+k} z_j, & \text{if} \quad 1 \le i_k \le N-1, \\ 1 - \dfrac{z_N}{c+k} - \displaystyle\sum_{j=1}^{N-1} \dfrac{b_j + \sum_{p=1}^{k} \delta_j^{i_p}}{c+k} z_j, & \text{if} \quad i_k = N. \end{cases} \tag{15}$$

It is easy to prove, by induction, that expansion (13)–(15) is true.

Passing n to ∞, we obtain the formal expansion of $X_n(a, \bar{b}; c; \bar{z})$ into infinite BCF of the form

$$1 - \frac{z_N}{c} - \sum_{j=1}^{N-1} \frac{b_j}{c} z_j + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{c_{i(k)}(\bar{z})}{d_{i(k)}(\bar{z})}. \tag{16}$$

Elements of BCF (16) are defined by Formulas (14) and (15) under $i(k) \in \mathcal{I}, k \ge 1$.

Taking into account Formula (11), we obtain the formal expansion of the ratio $X_{i_0}(a, \bar{b}; c; \bar{z})$, $i_0 \in \{1, \ldots, N-1\}$, into such BCF

$$1 - \frac{z_N}{c} - \frac{(a+1) z_{i_0}}{c} - \sum_{j=1}^{N-1} \frac{b_j}{c} z_j + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{l_{i(k)}(\bar{z})}{q_{i(k)}(\bar{z})}, \tag{17}$$

where for $i(k) \in \mathcal{I}, k \ge 1$,

$$l_{i(k)}(\bar{z}) = \begin{cases} \dfrac{(a+k)(b_{i_k} + \sum_{p=0}^{k-1} \delta_{i_k}^{i_p})}{(c+k-1)(c+k)} z_{i_k}(1 - z_{i_k}), & \text{if} \quad 1 \le i_k \le N-1, \\ \dfrac{a+k}{(c+k-1)(c+k)} z_{i_k}, & \text{if} \quad i_k = N, \end{cases} \tag{18}$$

$$q_{i(k)}(\bar{z}) = \begin{cases} 1 - \dfrac{z_N}{c+k} - \dfrac{a+k}{c+k} z_{i_k} - \displaystyle\sum_{j=1}^{N-1} \dfrac{b_j + \sum_{p=0}^{k} \delta_j^{i_p}}{c+k} z_j, & \text{if} \quad 1 \le i_k \le N-1, \\ 1 - \dfrac{z_N}{c+k} - \displaystyle\sum_{j=1}^{N-1} \dfrac{b_j + \sum_{p=0}^{k} \delta_j^{i_p}}{c+k} z_j, & \text{if} \quad i_k = N. \end{cases} \tag{19}$$

If $z_N = 0$, then the formal expansion of $X_1(a, \bar{b}; c; z_1, \ldots, z_{N-1}, 0)$ coincides with the expansion of the ratio of the Lauricella function $F_D^{(N-1)}$

$$\frac{F_D^{(N-1)}(a, \bar{b}; c; z_1, \ldots, z_{N-1})}{F_D^{(N-1)}(a+1, \bar{b}+e_1; c+1; z_1, \ldots, z_{N-1})}$$

into the $(N-1)$-dimensional analogue of Nörlund's continued fraction [23]. If $z_1 = z_2 = \ldots = z_{N-1} = 0$, then the formal expansion of $X_N(a, \bar{b}; c; 0, \ldots, 0, z_N)$ coincides with the continued fraction expansion of the ratio of Kummer's confluent function

$$\frac{\Phi(a; c; z_N)}{\Phi(a+1; c+1; z_N)}.$$

3. Convergence of the Branched Continued Fraction Expansions of the Confluent Hypergeometric Function $\Phi_D^{(N)}$ Ratios

Theorem 1. *Let parameters $a, b_1, \ldots, b_{N-1}, c$ of the confluent hypergeometric function $\Phi_D^{(N)}$ be real numbers such that*

$$a, b_1, \ldots, b_{N-1} \ge 0, \quad 2c > a + b_1 + \ldots + b_{N-1} > 0. \tag{20}$$

Then the BCF (16) *with elements* $c_{i(k)}, d_{i(k)}, i(k) \in \mathcal{I}$, *defined by* (14), (15), *under* $k \geq 1$, *converges uniformly in the domain*

$$G_\epsilon = \left\{ \bar{z} \in \mathbb{R}^N : \ 0 < z_i < \frac{1}{2} - \epsilon, \ 1 \leq i \leq N-1, \ 0 < z_N < \frac{2c - a - \sum_{j=1}^{N-1} b_j}{2} \right\},$$

where $0 < \epsilon < 1/2$, *to the function* $X_N(a, \bar{b}; c; \bar{z})$.

Proof. It is obvious that partial numerators $c_{i(k)}(\bar{z})$, $i(k) \in \mathcal{I}$, $k \geq 1$, for all $\bar{z} \in G_\epsilon$ are positive under conditions (20).

We will find lower bound of the denominators $d_{i(k)}(\bar{z})$, $i(k) \in \mathcal{I}$, $k \geq 1$, for $\bar{z} \in G_\epsilon$. If $1 \leq i_k \leq N-1$, then we have

$$\begin{aligned}
d_{i(k)}(\bar{z}) &= 1 - \frac{z_N}{c+k} - \frac{a+k}{c+k} z_{i_k} - \sum_{j=1}^{N-1} \frac{b_j + \sum_{p=1}^{k} \delta_j^{i_p}}{c+k} z_j \\
&> 1 - \frac{2c - a - \sum_{j=1}^{N-1} b_j}{2(c+k)} - \frac{a+k}{c+k}\left(\frac{1}{2} - \epsilon\right) \\
&\quad - \sum_{j=1}^{N-1} \frac{b_j}{c+k}\left(\frac{1}{2} - \epsilon\right) - \sum_{j=1}^{N-1} \frac{\sum_{p=1}^{k} \delta_j^{i_p}}{c+k}\left(\frac{1}{2} - \epsilon\right) \\
&= \frac{k}{2(c+k)} + \epsilon \frac{a + k + \sum_{j=1}^{N-1} b_j}{c+k} - \left(\frac{1}{2} - \epsilon\right) \frac{\sum_{p=1}^{k} \sum_{j=1}^{N-1} \delta_j^{i_p}}{c+k} \\
&\geq \epsilon \frac{a + 2k + \sum_{j=1}^{N-1} b_j}{c+k}.
\end{aligned}$$

If $i_k = N$, then

$$\begin{aligned}
d_{i(k)}(\bar{z}) &= 1 - \frac{z_N}{c+k} - \sum_{j=1}^{N-1} \frac{b_j + \sum_{p=1}^{k} \delta_j^{i_p}}{c+k} z_j \\
&> 1 - \frac{z_N}{c+k} - \sum_{j=1}^{N-1} \frac{a + k + b_j + \sum_{p=1}^{k} \delta_j^{i_p}}{c+k} z_j \\
&> \epsilon \frac{a + 2k + \sum_{j=1}^{N-1} b_j}{c+k}.
\end{aligned}$$

So,

$$Q_{i(k)}^{(n)}(\bar{z}) > d_{i(k)}(\bar{z}) > \epsilon \frac{a + 2k + \sum_{j=1}^{N-1} b_j}{c+k}, \quad i(k) \in \mathcal{I}, \ k \geq 1. \tag{21}$$

We will show that for an arbitrary $\bar{z} \in G_\epsilon$ following inequality

$$|f_m(\bar{z}) - f_n(\bar{z})| < M \left(\frac{\eta}{\eta + 1}\right)^n, \quad m > n, \tag{22}$$

where

$$M = \left(\frac{1}{4\epsilon} - \epsilon\right)\frac{a+1}{c} + \frac{2c - a - \sum_{j=1}^{N-1} b_j}{2c\epsilon}, \quad \eta = \left(\frac{1}{4\epsilon^2} - 1\right) + \frac{2c - a - \sum_{j=1}^{N-1} b_j}{2\epsilon^2 (a + \sum_{j=1}^{N-1} b_j)},$$

is valid. Formula (5) can be rewritten as follows

$$f_m(\bar{z}) - f_n(\bar{z}) = (-1)^n \sum_{i_1=1}^{N} \dots \sum_{i_{n+1}=1}^{N} \frac{c_{i(1)}(\bar{z})}{Q^{(q)}_{i(1)}(\bar{z})} \times \prod_{j=1}^{[(n+1)/2]} \frac{c_{i(2j)}(\bar{z})}{Q^{(r)}_{i(2j-1)}(\bar{z}) Q^{(r)}_{i(2j)}(\bar{z})} \prod_{j=1}^{[n/2]} \frac{c_{i(2j+1)}(\bar{z})}{Q^{(q)}_{i(2j)}(\bar{z}) Q^{(q)}_{i(2j+1)}(\bar{z})}, \quad (23)$$

where $q = m$, $r = n$, if $n = 2p$, and $q = n$, $r = m$, if $n = 2p - 1$, $p \geq 1$.

We note, that

$$\sum_{i_{k+1}=1}^{N} \frac{c_{i(k+1)}(\bar{z})}{Q^{(r)}_{i(k)}(\bar{z}) Q^{(r)}_{i(k+1)}(\bar{z})} = \frac{\sum_{i_{k+1}=1}^{N} \frac{c_{i(k+1)}(\bar{z})}{Q^{(r)}_{i(k+1)}(\bar{z})}}{d_{i(k)}(\bar{z}) + \sum_{i_{k+1}=1}^{N} \frac{c_{i(k+1)}(\bar{z})}{Q^{(r)}_{i(k+1)}(\bar{z})}} \leq \frac{\sum_{i_{k+1}=1}^{N} \frac{c_{i(k+1)}(\bar{z})}{d_{i(k)}(\bar{z}) d_{i(k+1)}(\bar{z})}}{1 + \sum_{i_{k+1}=1}^{N} \frac{c_{i(k+1)}(\bar{z})}{d_{i(k)}(\bar{z}) d_{i(k+1)}(\bar{z})}}.$$

Taking into account the inequality (21), we obtain

$$\begin{aligned}
&\sum_{i_{k+1}=1}^{N} \frac{c_{i(k+1)}(\bar{z})}{d_{i(k)}(\bar{z}) d_{i(k+1)}(\bar{z})} \\
&< \frac{(c+k)(c+k+1)}{\epsilon^2 (a + \sum_{j=1}^{N-1} b_j + 2k)(a + \sum_{j=1}^{N-1} b_j + 2k + 2)} \sum_{i_{k+1}=1}^{N} c_{i(k+1)}(\bar{z}) \\
&< \sum_{i_{k+1}=1}^{N-1} \frac{(a+k+1)(b_{i_{k+1}} + \sum_{p=1}^{k} \delta^{i_p}_{i_{k+1}}) z_{i_{k+1}} (1 - z_{i_{k+1}})}{\epsilon^2 (a + \sum_{j=1}^{N-1} b_j + 2k)(a + \sum_{j=1}^{N-1} b_j + 2k + 2)} \\
&\quad + \frac{(a+k+1) z_N}{\epsilon^2 (a + \sum_{j=1}^{N-1} b_j + 2k)(a + \sum_{j=1}^{N-1} b_j + 2k + 2)} \\
&< \left(\frac{1}{4\epsilon^2} - 1 \right) \frac{\sum_{i_{k+1}}^{N-1} (b_{i_{k+1}} + \sum_{p=1}^{k} \delta^{i_p}_{i_{k+1}})}{(a + \sum_{j=1}^{N-1} b_j + 2k)} + \frac{2c - a - \sum_{j=1}^{N-1} b_j}{2\epsilon^2 (a + \sum_{j=1}^{N-1} b_j + k)} \\
&< \left(\frac{1}{4\epsilon} - \epsilon \right) + \frac{2c - a - \sum_{j=1}^{N-1} b_j}{2\epsilon^2 (a + \sum_{j=1}^{N-1} b_j)}.
\end{aligned}$$

We also obtain

$$\begin{aligned}
\sum_{i_1=1}^{N} \frac{c_{i(1)}(\bar{z})}{Q^{(q)}_{i(1)}(\bar{z})} &\leq \frac{c+1}{\epsilon (a + \sum_{j=1}^{N-1} b_j + 2)} \sum_{i_1=1}^{N-1} \frac{(a+1)b}{c(c+1)} \left(\frac{1}{4} - \epsilon^2 \right) \\
&\quad + \frac{c+1}{\epsilon (a + \sum_{j=1}^{N-1} b_j + 2)} \frac{(a+1)(2c - a - \sum_{j=1}^{N-1} b_j)}{2c(c+1)} \\
&< \left(\frac{1}{4\epsilon} - \epsilon \right) \frac{a+1}{c} + \frac{2c - a - \sum_{j=1}^{N-1} b_j}{2c\epsilon}.
\end{aligned}$$

Substituting the above estimates in Formula (23) we obtain inequality (22).

We will consider the difference $X_N(a,\bar{b};c;\bar{z}) - f_n(\bar{z})$. Let

$$\tilde{Q}^{(s)}_{i(s)}(\bar{z}) = X_N\left(a+s, \bar{b} + \sum_{j=1}^{s} e_{i_j}; c+s; \bar{z}\right) - (1-\delta^N_{i_s})\frac{a+s}{c+s} z_{i_s},$$

$$\tilde{Q}^{(p)}_{i(k)}(\bar{z}) = d_{i(k)}(\bar{z}) + \sum_{i_{k+1}=1}^{N} \frac{c_{i(k+1)}(\bar{z})|}{|d_{i(k+1)}(\bar{z})} + \sum_{i_{k+2}=1}^{N} \frac{c_{i(k+2)}(\bar{z})|}{|d_{i(k+2)}(\bar{z})}$$
$$+ \ldots + \sum_{i_n=1}^{N} \frac{c_{i(n)}(\bar{z})|}{|X_N(a+n, \bar{b} + \sum_{p=1}^{n} e_{i_p}; c+n; \bar{z}) - (1-\delta^N_{i_n})\frac{a+n}{c+n} z_{i_n}},$$

where $s \geq 1$, $p \geq 2$, $1 \leq k \leq p-1$. It is clear that the following recurrence relations hold

$$\tilde{Q}^{(p)}_{i(k)}(\bar{z}) = d_{i(k)}(\bar{z}) + \sum_{i_{k+1}=1}^{N} \frac{c_{i(k+1)}(\bar{z})}{\tilde{Q}^{(p)}_{i(k+1)}(\bar{z})}, \quad s \geq 1,\ p \geq 2,\ 1 \leq k \leq p-1.$$

Applying the method suggested in [31], p. 28, for $n \geq 1$ on the first step we obtain

$$X_N(a,\bar{b};c;\bar{z}) - f_n(\bar{z}) = 1 - \frac{z_N}{c} - \sum_{j=1}^{N-1} \frac{b_j}{c} z_j + \sum_{i_1=1}^{N} \frac{c_{i(1)}(\bar{z})}{\tilde{Q}^{(n+1)}_{i(1)}(\bar{z})}$$
$$- \left(1 - \frac{z_N}{c} - \sum_{j=1}^{N-1} \frac{b_j}{c} z_j + \sum_{i_1=1}^{N} \frac{c_{i(1)}(\bar{z})}{Q^{(n)}_{i(1)}(\bar{z})}\right)$$
$$= -\sum_{i_1=1}^{N} \frac{c_{i(1)}(\bar{z})}{\tilde{Q}^{(n+1)}_{i(1)}(\bar{z}) Q^{(n)}_{i(1)}(\bar{z})} \left(\tilde{Q}^{(n+1)}_{i(1)}(\bar{z}) - Q^{(n)}_{i(1)}(\bar{z})\right).$$

Let k be an arbitrary natural number and $i(k)$ be an arbitrary multi-index from $\mathcal{I}$; moreover $1 \leq k \leq n-1$, $n \geq 2$. Then we have

$$\tilde{Q}^{(n+1)}_{i(k)}(\bar{z}) - Q^{(n)}_{i(k)}(\bar{z}) = d_{i(k)}(\bar{z}) + \sum_{i_{k+1}=1}^{N} \frac{c_{i(k+1)}(\bar{z})}{\tilde{Q}^{(n+1)}_{i(k+1)}(\bar{z})} - \left(d_{i(k)}(\bar{z}) + \sum_{i_{k+1}=1}^{N} \frac{c_{i(k+1)}(\bar{z})}{Q^{(n)}_{i(k+1)}(\bar{z})}\right)$$
$$= -\sum_{i_{k+1}=1}^{N} \frac{c_{i(k+1)}(\bar{z})}{\tilde{Q}^{(n+1)}_{i(k+1)}(\bar{z}) Q^{(n)}_{i(k+1)}(\bar{z})} \left(\tilde{Q}^{(n+1)}_{i(k+1)}(\bar{z}) - Q^{(n)}_{i(k+1)}(\bar{z})\right). \tag{24}$$

Applying recurrence relation (24) and taking into account that

$$\tilde{Q}^{(n+1)}_{i(n)}(\bar{z}) - Q^{(n)}_{i(n)}(\bar{z}) = \sum_{i_{n+1}=1}^{N} \frac{c_{i(n+1)}(\bar{z})}{\tilde{Q}^{(n+1)}_{i(n+1)}(\bar{z})},$$

after nth step we obtain

$$X_N(a,\bar{b};c;\bar{z}) - f_n(\bar{z}) = (-1)^n \sum_{i_1=1}^{N} \ldots \sum_{i_{n+1}=1}^{N} \frac{\prod_{p=1}^{n+1} c_{i(p)}(\bar{z})}{\prod_{p=1}^{n+1} \tilde{Q}^{(n+1)}_{i(p)}(\bar{z}) \prod_{p=1}^{n} Q^{(n)}_{i(p)}(\bar{z})}. \tag{25}$$

From (25) it follows that

$$f_{2m}(\bar{z}) < X_N(\bar{z})(a,\bar{b};c;\bar{z}) < f_{2m-1}(\bar{z}).$$

Since

$$\lim_{m\to\infty} f_{2m}(\bar{z}) = \lim_{m\to\infty} f_{2m-1}(\bar{z}) = f(\bar{z}),$$

then $X_N(\bar{z})(a, \bar{b}; c; \bar{z}) = f(\bar{z})$. □

Theorem 2. *Let parameters $a, b_1, \ldots, b_{N-1}, c$ of the confluent hypergeometric function $\Phi_D^{(N)}$ satisfy conditions (20). Then:*

(A) the BCF (16) with elements $c_{i(k)}$, $d_{i(k)}$, $i(k) \in \mathcal{I}$, defined by (14), (15), $i(k) \in \mathcal{I}$, $k \geq 1$, converges uniformly on every compact subset of the domain

$$G = \left\{ \bar{z} \in \mathbb{C}^N : \operatorname{Re} z_i < \frac{1}{2}, \; i = \overline{1, N-1}, \quad |z_N| < \frac{2c - a - \sum_{j=1}^{N-1} b_j}{2} \right\}$$

to a function $f(\bar{z})$ holomorphic in G;

(B) $f(\bar{z})$ is the analytic continuation of the function $X_N(a, \bar{b}; c; \bar{z})$ which is holomorphic in some neighborhood of the origin in the domain G.

We will use the following auxiliary lemmas.

Lemma 1 ([23])**.** *Let elements of the BCF (1) be the functions defined in some domain D, $D \subset \mathbb{C}^N$, and the following conditions for each $\bar{z} \in D$ and for all possible values of multi-indices $i(k) \in \mathcal{I}$ are valid:*

(A) $\operatorname{Re} d_{i(k)}(\bar{z}) > 0$;

(B) there exist such functions $g_{i(k)}(\bar{z})$ given in the domain D that $0 < g_{i(k)}(\bar{z}) \leq \operatorname{Re} d_{i(k)}(\bar{z})$ and

$$\sum_{i_{k+1}=1}^{2} \frac{|c_{i(k+1)}(\bar{z})| - \operatorname{Re} c_{i(k+1)}(\bar{z})}{g_{i(k+1)}(\bar{z})} \leq 2(\operatorname{Re} d_{i(k)}(\bar{z}) - g_{i(k)}(\bar{z})). \tag{26}$$

Then, for each $n \geq 1$,

$$\operatorname{Re}(Q_{i(k)}^{(n)}(\bar{z})) \geq g_{i(k)}(\bar{z}) \quad \text{for all} \quad i(k) \in \mathcal{I}, \; 1 \leq k \leq n, \quad \text{and} \quad \bar{z} \in D, \tag{27}$$

where $Q_{i(k)}^{(n)}(\bar{z})$, $i(k) \in \mathcal{I}$, $1 \leq k \leq n$, $n \geq 1$, defined by (3) and (4).

Lemma 2 ([23])**.** *Let w be a complex number. Then*

$$|w(1-w)| - \operatorname{Re}(w(1-w)) \leq 2\left(\frac{1}{2} - \operatorname{Re} w\right)^2,$$

and equality is achieved only when $\operatorname{Re} w = 1/2$.

In addition, we will use the convergence continuation Theorem 2.17 [31] (see also ([9], Theorem 24.2).

Theorem 3. *Let $\{f_n(\mathbf{z})\}$ be a sequence of functions, holomorphic in the domain D, $D \subset \mathbb{C}^N$, which is uniformly bounded on every compact subset of D. Let this sequence converge at each point of the set E, $E \subset D$, which is the N-dimensional real neighborhood of the point $\bar{z}^0$, $\bar{z}^0 \in D$. Then $\{f_n(\mathbf{z})\}$ converges uniformly on every compact subset of the domain D to a function holomorphic in D.*

Proof of Theorem 2. We will use the proof scheme from [23]. Let for $k \geq 1$

$$g_{i(k)}(\bar{z}) = \begin{cases} \dfrac{a+k}{c+k}\left(\dfrac{1}{2} - \operatorname{Re} z_{i_k}\right), & \text{if} \quad 1 \leq i_k \leq N-1, \\ \dfrac{a+k}{2(c+k)}, & \text{if} \quad i_k = N. \end{cases} \tag{28}$$

It is obvious that functions $g_{i(k)}(\bar{z})$ are positive. Next we have
(a) for $i_k = N$

$$\begin{aligned}
\operatorname{Re} d_{i(k)}(\bar{z}) - g_{i(k)}(\bar{z}) &= 1 - \frac{\operatorname{Re} z_N}{c+k} - \sum_{j=1}^{N-1} \frac{b_j + \sum_{p=1}^{k} \delta_j^{i_p}}{c+k} \operatorname{Re} z_j - \frac{a+k}{2(c+k)} \\
&= \frac{2c-a+k}{2(c+k)} - \frac{\operatorname{Re} z_N}{c+k} - \sum_{j=1}^{N-1} \frac{b_j + \sum_{p=1}^{k} \delta_j^{i_p}}{c+k} \operatorname{Re} z_j \\
&> \frac{2c-a+k}{2(c+k)} - \frac{2c-a-\sum_{j=1}^{N-1} b_j}{2(c+k)} - \sum_{j=1}^{N-1} \frac{b_j + \sum_{p=1}^{k} \delta_j^{i_p}}{2(c+k)} \\
&= \frac{1}{2(c+k)} \left(k - \sum_{j=1}^{N-1} \sum_{p=1}^{k} \delta_j^{i_p} \right) \\
&\geq \frac{1}{2(c+k)};
\end{aligned}$$

(b) for arbitrary $1 \leq i_k \leq N-1$

$$\begin{aligned}
\operatorname{Re} d_{i(k)}(\bar{z}) - g_{i(k)}(\bar{z}) &= 1 - \frac{\operatorname{Re} z_N}{c+k} - \frac{a+k}{c+k} \operatorname{Re} z_{i_k} \\
&\quad - \sum_{j=1}^{N-1} \frac{b_j + \sum_{p=1}^{k} \delta_j^{i_p}}{c+k} \operatorname{Re} z_j - \frac{a+k}{c+k} \left(\frac{1}{2} - \operatorname{Re} z_{i_k} \right) \\
&= \frac{2c-a+k}{2(c+k)} - \frac{\operatorname{Re} z_N}{c+k} - \sum_{j=1}^{N-1} \frac{b_j + \sum_{p=1}^{k} \delta_j^{i_p}}{c+k} \operatorname{Re} z_j \\
&> \frac{1}{2(c+k)} \left(k - \sum_{j=1}^{N-1} \sum_{p=1}^{k} \delta_j^{i_p} \right) \geq 0.
\end{aligned}$$

Thus, $\operatorname{Re} d_{i(k)}(\bar{z}) \geq g_{i(k)}(\bar{z})$.
On the other hand, taking into account Lemma 2, we obtain

$$\begin{aligned}
&\sum_{i_{k+1}=1}^{N} \frac{|c_{i(k+1)}(\bar{z})| - \operatorname{Re} c_{i(k+1)}(\bar{z})}{g_{i(k+1)}(\bar{z})} \\
&\quad = \sum_{i_{k+1}=1}^{N-1} \frac{b_{i_{k+1}} + \sum_{p=1}^{N-1} \delta_{i_{k+1}}^{i_p}}{c+k} \frac{|z_{i_{k+1}}(1-z_{i_{k+1}})| - \operatorname{Re} z_{i_{k+1}}(1-z_{i_{k+1}})}{(1/2 - \operatorname{Re} z_{i_{k+1}})} \\
&\quad + 2\frac{|z_N| - \operatorname{Re} z_N}{c+k} \\
&\quad \leq \sum_{i_{k+1}=1}^{N-1} \frac{b_{i_{k+1}} + \sum_{p=1}^{N-1} \delta_{i_{k+1}}^{i_p}}{c+k} - 2 \sum_{i_{k+1}=1}^{N-1} \frac{b_{i_{k+1}} + \sum_{p=1}^{N-1} \delta_{i_{k+1}}^{i_p}}{(c+k)} \operatorname{Re} z_{i_{k+1}} \\
&\quad + \frac{2c-a-\sum_{j=1}^{N-1} b_j}{c+k} - \frac{2\operatorname{Re} z_N}{c+k}
\end{aligned}$$

and

$$2(\operatorname{Re} d_{i(k)}(\bar{z}) - g_{i(k)}(\bar{z})) - \sum_{i_{k+1}=1}^{N} \frac{|c_{i(k+1)}(\bar{z})| - \operatorname{Re} c_{i(k+1)}(\bar{z})}{g_{i(k+1)}(\bar{z})}$$

$$\geq \frac{2c-a+k}{c+k} - \sum_{i_{k+1}=1}^{N-1} \frac{b_{i_{k+1}} + \sum_{p=1}^{N-1} \delta_{i_{k+1}}^{i_p}}{(c+k)} - \frac{2c-a-\sum_{j=1}^{N-1} b_j}{c+k}$$

$$= \frac{1}{2(c+k)} \left(k - \sum_{j=1}^{N-1} \sum_{p=1}^{k} \delta_j^{i_p} \right) \geq 0.$$

Therefore, the conditions (26) of Lemma 1 are satisfied and inequality (27) is valid, where $g_{i(k)}(\bar{z})$ is defined by (28). Thus, $\{f_n(\bar{z})\}$, $n \geq 1$, is a sequence of functions holomorphic in domain G.

Let K be an arbitrary compact subset of G. Then,

$$|f_n(\bar{z})| \leq 1 + \frac{|z_N|}{c} + \sum_{j=1}^{N-1} \frac{b_j}{c}|z_j| + \sum_{i_1=1}^{N} \frac{|c_{i(1)}(\bar{z})|}{g_{i(1)}(\bar{z})}$$

$$\leq 1 + \frac{|z_N|}{c} + \sum_{j=1}^{N-1} \frac{b_j|z_j|}{c} + \sum_{j=1}^{N-1} \frac{b_j|z_j(1-z_j)|}{c(1/2 - \operatorname{Re} z_j)} + \frac{2c - \sum_{j=1}^{N-1} b_j - a}{c}$$

$$\leq 1 + \sup_{\bar{z}\in K} \left(\frac{|z_N|}{c} + \sum_{j=1}^{N-1} \frac{b_j|z_j|}{c} + \sum_{j=1}^{N-1} \frac{b_j|z_j(1-z_j)|}{c(1/2 - \operatorname{Re} z_j)} + \frac{2c - \sum_{j=1}^{N-1} b_j - a}{c} \right)$$

$$= M(K),$$

where constant $M(K)$ depends only on K. Moreover, $G_\epsilon \subset G$. So, sequence of approximants $\{f_n(\bar{z})\}$ of the BCF (16) satisfies the conditions of Theorem 3 and it means that Statement (A) of Theorem 2 is proven.

The series (2) converges for each $\bar{z}$ from domain $\{\bar{z} \in \mathbb{C}^N : |z_i| < 1, 1 \leq i \leq N-1\}$ and $X_N(a, \bar{b}; c; \bar{z})|_{z_1=\ldots=z_N=0} = 1$. Therefore, there is such $\delta > 0$ that function $X_N(a, \bar{b}; c; \bar{z})$ is holomorphic in domain $G_\delta = \{\bar{z} \in \mathbb{C}^N : |z_i| < \delta, 1 \leq i \leq N\}$, $G_\delta \subset G$. Since investigated BCF converges uniformly in G_ϵ to $X_N(a, \bar{b}; c; \bar{z})$, then by the principle of analytic continuation ([33], p. 53), Statement (B) follows. □

Let us note that $X_N(0, \bar{b}; c; \bar{z}) = 1/\Phi_D^{(N)}(1, \bar{b}; c+1; \bar{z})$. We assume that $a = 0$ and

$$Q_0^{(0)}(\bar{z}) = 1 - \frac{z_N}{c} - \sum_{j=1}^{N-1} \frac{b_j z_j}{c}, \quad Q_0^{(n)}(\bar{z}) = 1 - \frac{z_N}{c} - \sum_{j=1}^{N-1} \frac{b_j z_j}{c} + \sum_{i_1=1}^{N} \frac{c_{i(1)}(\bar{z})}{Q_{i(1)}^{(n)}(\bar{z})}, \quad n \geq 1.$$

In the proof of the Theorem 2 it is shown that inequality (27) is valid. It can be similarly shown that

$$\operatorname{Re} Q_0^{(n)}(\bar{z}) > g_0(\bar{z}) = 1 - \frac{1}{2c} \sum_{j=1}^{N-1} b_j - \frac{|z_N|}{c} > 0, \quad n \geq 0, \quad \bar{z} \in G. \tag{29}$$

Indeed, for each $\bar{z} \in G$

$$
\begin{aligned}
\operatorname{Re} Q_0^{(n)}(\bar{z}) - g_0(\bar{z}) &= 1 - \frac{\operatorname{Re} z_N}{c} - \sum_{j=1}^{N-1} \operatorname{Re} \frac{b_j z_j}{c} - \left(1 - \frac{1}{2c}\sum_{j=1}^{N-1} b_- \frac{|z_N|}{c}\right) \\
&\geq \sum_{j=1}^{N-1} \frac{b_j}{2c}(1 - 2\operatorname{Re} z_j) + \frac{|z_N| - \operatorname{Re} z_N}{c} > 0, \\
\sum_{i_1=1}^{N} \frac{|c_{i(1)}(\bar{z})| - \operatorname{Re} c_{i(1)}(\bar{z})}{g_{i(1)}(\bar{z})} &= \sum_{i_1=1}^{N-1} \frac{b_{i_1}}{c} \frac{|z_{i_1}(1 - z_{i_1})| - \operatorname{Re} z_{i_1}(1 - z_{i_1})}{(1/2 - \operatorname{Re} z_{i_1})} + 2\frac{|z_N| - \operatorname{Re} z_N}{c} \\
&\leq \sum_{i_1=1}^{N-1} \frac{b_{i_1}}{c} - 2\sum_{i_1=1}^{N-1} \frac{b_{i_1}}{c} \operatorname{Re} z_{i_1} + 2\frac{|z_N| - \operatorname{Re} z_N}{c},
\end{aligned}
$$

and

$$
2\left(1 - \frac{\operatorname{Re} z_N}{c} - \sum_{j=1}^{N-1} \operatorname{Re} \frac{b_j z_j}{c} - g_0(\bar{z})\right) - \sum_{i_1=1}^{N} \frac{|c_{i(1)}(\bar{z})| - \operatorname{Re} c_{i(1)}(\bar{z})}{g_{i(1)}(\bar{z})} \geq 0.
$$

From (29) it follows that $\{h_n(\bar{z})\}$, where $h_n(\bar{z}) = (f_n(\bar{z}))^{-1}$, $n \geq 0$, is a sequence of functions holomorphic in G.

Setting $a = 0$, replacing c by $c - 1$ in Theorem 2 and taking into account the above considerations we obtain the corollary.

Corollary 1. *Let parameters $b_1, b_2, \ldots, b_{N-1}, c$ of function $\Phi_D^{(N)}$ satisfy inequalities*

$$
b_1, \ldots, b_{N-1} \geq 0, \quad 2c > b_1 + \ldots + b_{N-1} + 2 > 2.
$$

Then:

(A) the BCF

$$
\left(1 - \frac{z_N}{c-1} - \sum_{j=1}^{N-1} \frac{b_j z_j}{c-1} + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N-1} \frac{c_{i(k)}(\bar{z})}{d_{i(k)}(\bar{z})}\right)^{-1} \tag{30}
$$

with elements $c_{i(k)}, d_{i(k)}, i(k) \in \mathcal{I}$, defined by

$$
c_{i(k)}(\bar{z}) = \begin{cases} \dfrac{k(b_{i_k} + \sum_{p=1}^{k-1} \delta_{i_k}^{i_p})}{(c+k-2)(c+k-1)} z_{i_k}(1 - z_{i_k}), & \text{if} \quad 1 \leq i_k \leq N-1, \\ \dfrac{k}{(c+k-2)(c+k-1)} z_{i_k}, & \text{if} \quad i_k = N, \end{cases} \tag{31}
$$

$$
d_{i(k)}(\bar{z}) = \begin{cases} 1 - \dfrac{z_N + k z_{i_k}}{c+k-1} - \displaystyle\sum_{j=1}^{N-1} \dfrac{b_j + \sum_{p=1}^{k} \delta_j^{i_p}}{c+k-1} z_j, & \text{if} \quad 1 \leq i_k \leq N-1, \\ 1 - \dfrac{z_N}{c+k-1} - \displaystyle\sum_{j=1}^{N-1} \dfrac{b_j + \sum_{p=1}^{k} \delta_j^{i_p}}{c+k-1} z_j, & \text{if} \quad i_k = N, \end{cases} \tag{32}
$$

converges uniformly on every compact subset of H to a function $h(\bar{z})$ holomorphic in H, where

$$
H = \left\{\bar{z} \in \mathbb{C}^N : \operatorname{Re} z_i < \frac{1}{2}, 1 \leq i \leq N-1, \quad |z_N| < c - 1 - \frac{1}{2}\sum_{j=1}^{N-1} b_j\right\};
$$

(B) $h(\bar{z})$ is an analytic continuation of function $\Phi_D^{(N)}(1, \bar{b}; c; \bar{z})$ in domain H.

Example 1. *We set $a = 0$, $b_1 = 0.5$, $b_2 = 1$, $c = 4$. The results of computation of the approximants $h_n(\bar{z})$, $0 \leq n \leq 12$, of BCF (30) with elements $c_{i(k)}, d_{i(k)}, i(k) \in \mathcal{I}$, defined* *by* (31), (32)*, and partial sums* $S_n(\bar{z})$, $0 \le n \le 12$, *of* $\Phi_D^{(3)}(1, 0.5, 1; 4; \bar{z})$ *for* $\bar{z} = (0.3, 0.4, 1)$ *and* $\bar{z} = (-0.7, -0.4, 1)$ *are given in Table* 1*.*

For given parameters and $\bar{z} = (0.3, 0.4, 1)$ *elements of BCF* (30) *are positive and*

$$h_{2m-1}(\bar{z}) < \Phi_D^{(3)}(1, 0.5, 1; 4; \bar{z}) < h_{2m}(\bar{z}), \quad 1 \le m \le 6.$$

If $\bar{z} = (-0.7, -0.4, 1)$*, then*

$$|h_m(\bar{z}) - h_{m-1}(\bar{z})| < |S_m(\bar{z}) - S_{m-1}(\bar{z})|, \quad 1 \le m \le 12.$$

Table 1. Values of $h_n(\bar{z})$, $S_n(\bar{z})$ for different values of $\bar{z} = (z_1, z_2, z_3)$.

n	$h_n(0.3, 0.4, 1)$	$S_n(0.3, 0.4, 1)$	$h_n(-0.7, -0.4, 1)$	$S_n(-0.7, -0.4, 1)$
0	2.0689655172413793	1.0000000000000000	1.0909090909090909	1.0000000000000000
1	1.4560459283938569	1.3875000000000000	1.0798919301578482	1.0625000000000000
2	1.6062420542029685	1.5178750000000000	1.0854460271288587	1.0858750000000000
3	1.5663393776978655	1.5581427083333333	1.0854992029539980	1.0846114583333333
4	1.5774800126642679	1.5700380133928571	1.0855766580493781	1.0858623586309523
5	1.5741237293361620	1.5734982670665922	1.0855849420453230	1.0855849420453230
6	1.5752175755666838	1.5745081593644076	1.0855871865334549	1.0856431331367290
7	1.5748338584710080	1.5748069122651405	1.0855876480094189	1.0855617160131383
8	1.5749774440398022	1.5748968805416772	1.0855877608401303	1.0856005483413065
9	1.5749206724246927	1.5749244851830382	1.0855877888742481	1.0855813521666234
10	1.5749441919671161	1.5749331078713755	1.0855877962018176	1.0855911502163538
11	1.5749340537588600	1.5749358459608639	1.0855877981816782	1.0855860154913730
12	1.5749385748468521	1.5749367284599484	1.0855877987333202	1.0855887673017868

Example 2. *We set* $a = 0$, $b_1 = 1$, $c = 4$*. The results of computation of the approximants* $h_n(\bar{z})$, $0 \le n \le 12$*, of BCF* (30) *with elements* $c_{i(k)}$, $d_{i(k)}$, $i(k) \in \mathcal{I}$*, defined by* (31), (32)*, for* $\bar{z} = (-1.2, 1)$ *and* $\bar{z} = (-1.2 + 0.2i, 1 + 0, 5i)$ *are given in Table* 2*. These values of* $\bar{z}$ *do not belong to a convergence domain of double power series for* $\Phi(1, 1; 4; \bar{z})$*.*

Table 2. Values of $h_n(0, 1; 4; \bar{z})$ for different values of $\bar{z} = (z_1, z_2)$.

$\bar{z}$	$(-1.2, 1)$	$(-1.2 + 0.2i, 1 + 0, 5i)$
$h_0(\bar{z})$	0.9375000000000000	$0.8946877912395153 + 0.1957129543336439i$
$h_1(\bar{z})$	0.9874608150470219	$0.9682330302329962 + 0.1636661528464738i$
$h_2(\bar{z})$	0.9999386478760991	$0.9783495727203259 + 0.1621180086394217i$
$h_3(\bar{z})$	1.0021612335538261	$0.9810777556363008 + 0.1611130234246828i$
$h_4(\bar{z})$	1.0027828150938215	$0.9816708481472565 + 0.1608142450994196i$
$h_5(\bar{z})$	1.0029538035362679	$0.9818431129623030 + 0.1607160062318091i$
$h_6(\bar{z})$	1.0030069414508122	$0.9818931929871372 + 0.1606803796414656i$
$h_7(\bar{z})$	1.0030242918372864	$0.9819087721653132 + 0.1606673323815619i$
$h_8(\bar{z})$	1.0030302600610872	$0.9819137968090410 + 0.1606623598862077i$
$h_9(\bar{z})$	1.0030323958017573	$0.9819154619391862 + 0.1606604197525936i$
$h_{10}(\bar{z})$	1.0030331862564592	$0.9819160242737400 + 0.1606596462158485i$
$h_{11}(\bar{z})$	1.0030334872518964	$0.9819162161393741 + 0.1606593322207231i$
$h_{12}(\bar{z})$	1.0030336047089570	$0.9819162817068529 + 0.1606592028011838i$

The following theorems can be proven in much the same way as Theorems 1 and 2.

Theorem 4. *Let parameters* $a, b_1, \ldots, b_{N-1}, c$ *of the confluent hypergeometric function* $\Phi_D^{(N)}$ *be real numbers such that*

$$a, b_1, \ldots, b_{N-1} \ge 0, \quad 2c > a + b_1 + \ldots + b_{N-1} + 1 > 1. \tag{33}$$

Then, the BCF (17) *with elements* $l_{i(k)}$, $q_{i(k)}$, $i(k) \in \mathcal{I}$, *defined by* (18), (19), *converges uniformly in the domain*

$$L_\epsilon = \left\{ \bar{z} \in \mathbb{R}^N : \ 0 < z_i < \frac{1}{2} - \epsilon, \ 1 \le i \le N-1, \ 0 < z_N < \frac{2c - a - \sum_{j=1}^{N-1} b_j - 1}{2} \right\},$$

where $0 < \epsilon < 1/2$, *to the function* $X_{i_0}(a, \bar{b}; c; \bar{z})$, $1 \le i_0 \le N-1$.

Theorem 5. *Let parameters* $a, b_1, \dots, b_{N-1}, c$ *of the confluent hypergeometric function* $\Phi_D^{(N)}$ *satisfy conditions* (33). *Then:*

(A) the BCF (17) *with elements* $l_{i(k)}$, $q_{i(k)}$, $i(k) \in \mathcal{I}$, *defined by* (18), (19), $i(k) \in \mathcal{I}$, $k \ge 1$, *converges uniformly on every compact subset of the domain*

$$L = \left\{ \bar{z} \in \mathbb{C}^N : \ \operatorname{Re} z_i < \frac{1}{2}, \ 1 \le i \le N-1, \ |z_N| < \frac{2c - a - \sum_{j=1}^{N-1} b_j - 1}{2} \right\}$$

to a function $f(\bar{z})$ *holomorphic in* L*;*

(B) $f(\bar{z})$ *is the analytical continuation of the function* $X_{i_0}(a, \bar{b}; c; \bar{z})$, $1 \le i_0 \le N-1$, *which is holomorphic in some neighborhood of the origin in the domain* L.

4. Conclusions

In the paper we have constructed and investigated the branched continued fraction expansions of the confluent hypergeometric function $\Phi_D^{(N)}$ ratios.

In particular, we have proven that the branched continued fraction expansions converges to the functions which are an analytic continuation of the above-mentioned ratios in some domains. The problem of studying wider convergence domains and establishing estimates of the rate of convergence of the above-mentioned expansions still remains open.

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