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On Special Fuzzy Differential Subordinations Obtained for Riemann–Liouville Fractional Integral of Ruscheweyh and Sălăgean Operators

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Abstract: New results concerning fuzzy differential subordination theory are obtained in this paper using the operator denoted by $D_z^{-\lambda}L_\alpha^n$, previously introduced by applying the Riemann–Liouville fractional integral to the convex combination of well-known Ruscheweyh and Sălăgean differential operators. A new fuzzy subclass $DL_n^{\mathcal{F}}(\delta,\alpha,\lambda)$ is defined and studied involving the operator $D_z^{-\lambda}L_\alpha^n$. Fuzzy differential subordinations are obtained considering functions from class $DL_n^{\mathcal{F}}(\delta,\alpha,\lambda)$ and the fuzzy best dominants are also given. Using particular functions interesting corollaries are obtained and an example shows how the obtained results can be applied.

Keywords: differential operator; fuzzy differential subordination; fuzzy best dominant; fractional integral

MSC: 30C45; 30A10; 33C05

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1. Introduction

The concept of fuzzy set, introduced by Lotfi A. Zadeh in 1965 [1], has opened the way for a new theory called fuzzy set theory. It has developed intensely, nowadays having applications in many branches of science and technology.

The fuzzy set concept was applied for developing new directions of study in many mathematical theories. In geometric function theory, it was used for introducing the new concepts of fuzzy subordination [2] and fuzzy differential subordinations [3] as generalizations of the classical notion of differential subordination due to Miller and Mocanu [4,5]. The main aspects regarding the theory of differential subordination can be found in [6]. Steps in the evolution of the theory of fuzzy differential subordination can be followed in [7].

The general context of the study presented in this paper contains notions familiar to geometric function theory merged with fuzzy set theory. We first present the main classes of analytic functions involved and the definitions regarding fuzzy differential subordination theory.

$U = \{z \in \mathbb{C} : |z| < 1\}$ represents the unit disc of the complex plane and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Consider $\mathcal{A} = \{f \in \mathcal{H}(U) : f(z) = z + a_2 z^2 + \dots,\ z \in U\}$, and $\mathcal{H}[a,m] = \{f \in \mathcal{H}(U) : f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \dots,\ z \in U\}$, for $a \in \mathbb{C}$ and $m \in \mathbb{N}$.

We remember the usual definitions needed for fuzzy differential subordination:

Definition 1 ([8]). *A fuzzy subset of X is a pair (M, F_A), with $M = \{x \in X : 0 < F_M(x) \le 1\}$ the support of the fuzzy set and $F_M : X \to [0,1]$ the membership function of the fuzzy set. It is denoted $M = \operatorname{supp}(M, F_M)$.*

Remark 1. *When $M \subset X$, we have* $F_M(x) = \begin{cases} 1, \text{ if } x \in M, \\ 0, \text{ if } x \notin M. \end{cases}$

Evidently $F_{\emptyset}(x) = 0$, $x \in X$, *and* $F_X(x) = 1$, $x \in X$.

Definition 2 ([2]). *Let* $D \subset \mathbb{C}$ *and let* $z_0 \in D$ *be a fixed point. We take the functions* $f, g \in \mathcal{H}(D)$. *The function* f *is said to be fuzzy subordinate to* g *and we write* $f \prec_{\mathcal{F}} g$, *if there exists a function* $F : \mathbb{C} \to [0,1]$ *such that* $f(z_0) = g(z_0)$ *and* $F_{f(D)} f(z) \le F_{g(D)} g(z)$, $z \in D$.

Remark 2. *(1) If* g *is univalent, then* $f \prec_{\mathcal{F}} g$ *if and only if* $f(z_0) = g(z_0)$ *and* $f(D) \subset g(D)$.

(2) Such a function $F : \mathbb{C} \to [0,1]$ *can be consider* $F(z) = \frac{|z|}{1+|z|}$, $F(z) = \frac{1}{1+|z|}$.

(3) If $D = U$ *the conditions become* $f(0) = g(0)$ *and* $f(U) \subset g(U)$, *which is equivalent to the classical definition of subordination.*

Definition 3 ([3]). *Consider* h *an univalent function in* U *and* $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$, *such that* $h(0) = \psi(a, 0; 0) = a$. *When the fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3 \times U)} \psi(p(z), zp'(z), z^2 p''(z); z) \le F_{h(U)} h(z), \quad z \in U, \tag{1}$$

is satisfied for an analytic function p *in* U, *such that* $p(0) = a$, *then* p *is called a fuzzy solution of the fuzzy differential subordination. A fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination is an univalent function* q *for which* $F_{p(U)} p(z) \le F_{q(U)} q(z)$, $z \in U$, *for all* p *satisfying (1). The fuzzy best dominant of (1) is a fuzzy dominant* $\widetilde{q}$, *such that* $F_{\widetilde{q}(U)} \widetilde{q}(z) \le F_{q(U)} q(z)$, $z \in U$, *for all fuzzy dominants* q *of (1).*

Lemma 1 ([6]). *Consider* $h \in \mathcal{A}$. *If* $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$, $z \in U$, *then* $\frac{1}{z}\int_0^z h(t)dt$ *is a convex function,* $z \in U$.

Lemma 2 ([9]). *Consider a convex function* h *with* $h(0) = a$, *and* $\gamma \in \mathbb{C}^*$ *such that* $Re\, \gamma \ge 0$. *When* $p \in \mathcal{H}[a, m]$, $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = \frac{1}{\gamma} zp'(z) + p(z)$ *is an analytic function in* U *and*

$$F_{\psi(\mathbb{C}^2 \times U)}\left(\frac{1}{\gamma} zp'(z) + p(z)\right) \le F_{h(U)} h(z), \ \ z \in U,$$

then

$$F_{p(U)} p(z) \le F_{g(U)} g(z) \le F_{h(U)} h(z), \quad z \in U,$$

with the convex function $g(z) = \frac{\gamma}{m z^{\frac{\gamma}{m}}} \int_0^z h(t) t^{\frac{\gamma}{m} - 1} dt$, $z \in U$ *as the fuzzy best dominant.*

Lemma 3 ([9]). *Consider a convex function* g *in* U *and define* $h(z) = m\alpha z g'(z) + g(z)$, $z \in U$, *with* $m \in \mathbb{N}$ *and* $\alpha > 0$.

If $p(z) = g(0) + p_m z^m + p_{m+1} z^{m+1} + \ldots$, $z \in U$, *is a holomorphic function in* U *and*

$$F_{p(U)}\big(p(z) + \alpha z p'(z)\big) \le F_{h(U)} h(z), \quad z \in U,$$

then we obtain the sharp result

$$F_{p(U)} p(z) \le F_{g(U)} g(z), z \in U.$$

The original results exposed in this paper are obtained using the well-known Ruscheweyh and Sălăgean differential operators combined with Riemann–Liouville fractional integral. The resulting operator was introduced in [10], where it was used for obtaining results involving classical differential subordination theory. The necessary definitions are reminded:

Definition 4 (Ruscheweyh [11]). *The Ruscheweyh operator R^n is introduced by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z), \\ R^1 f(z) &= z f'(z), \\ &\ldots \\ (n+1) R^{n+1} f(z) &= n R^n f(z) + z (R^n f(z))', \end{aligned}$$

for $f \in \mathcal{A}$, $n \in \mathbb{N}$, $z \in U$.

Remark 3. *For a function $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$, the Ruscheweyh operator can be written using the following form $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)} a_j z^j$, $z \in U$, where Γ denotes the gamma function.*

Definition 5 (Sălăgean [12]). *The Sălăgean operator S^n is introduced by $S^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} S^0 f(z) &= f(z), \\ S^1 f(z) &= z f'(z), \\ &\ldots \\ S^{n+1} f(z) &= z (S^n f(z))', \end{aligned}$$

for $f \in \mathcal{A}$, $n \in \mathbb{N}$, $z \in U$.

Remark 4. *For a function $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$, the Sălăgean operator can be written using the following form $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, $z \in U$.*

Definition 6 ([13]). *Define the linear operator $L_\alpha^n : \mathcal{A} \to \mathcal{A}$, given by*

$$L_\alpha^n f(z) = \alpha S^n f(z) + (1-\alpha) R^n f(z), \quad z \in U,$$

where $\alpha \geq 0$, $n \in \mathbb{N}$.

Remark 5. *For a function $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$, the defined operator can be written using the following form $L_\alpha^n f(z) = z + \sum_{j=2}^{\infty} \left[\alpha j^n + (1-\alpha) \frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)} \right] a_j z^j$, $z \in U$.*

We also remind the definition of Riemann–Liouville fractional integral:

Definition 7 ([14]). *The Riemann–Liouville fractional integral of order λ applied to an analytic function f is defined by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt,$$

with $\lambda > 0$.

In [10] we defined the Riemann–Liouville fractional integral applied to the operator L_α^n as follows:

Definition 8 ([10]). *The Riemann–Liouville fractional integral applied to the differential operator $L_\alpha^n f$ is introduced by*

$$D_z^{-\lambda} L_\alpha^n f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{L_\alpha^n f(t)}{(z-t)^{1-\lambda}} dt =$$

$$\frac{1}{\Gamma(\lambda)} \int_0^z \frac{t}{(z-t)^{1-\lambda}} dt + \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha) \frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)} \right) a_j \int_0^z \frac{t^j}{(z-t)^{1-\lambda}} dt,$$

where $\alpha \geq 0$, $\lambda > 0$ *and* $n \in \mathbb{N}$.

Remark 6. *For a function* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$, *the Riemann–Liouville fractional integral of* $L_\alpha^n f$ *has the following form*

$$D_z^{-\lambda} L_\alpha^n f(z) = \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{j=2}^{\infty} \left[\frac{\alpha j^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} + \frac{(1-\alpha) j \Gamma(m+j)}{\Gamma(m+1)\Gamma(j+\lambda+1)} \right] a_j z^{j+\lambda},$$

and $D_z^{-\lambda} L_\alpha^n f(z) \in \mathcal{H}[0, \lambda+1]$.

The results exposed in this paper follow a line of research concerned with fuzzy differential subordinations which is popular nowadays, namely introducing new operators and using them for defining and studying new fuzzy classes of functions.

Fuzzy differential subordinations involving Ruscheweyh and Sălăgean differential operators were obtained in many studies, such as [15]. New operators introduced using fractional integral and applied in fuzzy differential subordination theory were studied in [16] where Riemann–Liouville fractional integral is applied for Gaussian hypergeometric function and in [17] where Riemann–Liouville fractional integral is combined with confluent hypergeometric function.

Motivated by the nice results obtained in fuzzy differential subordination theory using Ruscheweyh and Sălăgean differential operators and fractional integral applied to different known operators, the study presented in this paper uses the previously defined operator $D_z^{-\lambda} L_\alpha^n$ given in Definition 8 applied for obtaining new fuzzy differential subordinations. In the next section, a new fuzzy class will be defined and studied in order to obtain fuzzy differential subordinations inspired by recently published studies concerned with the same topic seen in [18–20].

The main results contained in Section 2 of the paper, begin with the definition of a new fuzzy class $DL_n^{\mathcal{F}}(\delta, \alpha, \lambda)$ for which the operator $D_z^{-\lambda} L_\alpha^n$ given in Definition 8 is used. The property of this class to be convex is proved and certain fuzzy differential subordinations involving functions from the class and the operator $D_z^{-\lambda} L_\alpha^n$ are obtained. The fuzzy best dominants are given for the considered fuzzy differential subordinations in theorems which generate interesting corollaries when specific functions with remarkable geometric properties are used as fuzzy best dominants. An example is also shown in order to prove the applicability of the new results.

2. Main Results

The usage of the operator $D_z^{-\lambda} L_\alpha^n$ seen in Definition 8 defines a new fuzzy subclass of analytic functions as follows:

Definition 9. *The class* $DL_n^{\mathcal{F}}(\delta, \alpha, \lambda)$ *is composed of all functions* $f \in \mathcal{A}$ *with the property*

$$F_{(D_z^{-\lambda} L_\alpha^n f)'(U)} \left(D_z^{-\lambda} L_\alpha^n f(z) \right)' > \delta, \quad z \in U,$$

where $n \in \mathbb{N}$, $\delta \in [0,1)$, $\alpha \geq 0$, $\lambda > 0$.

We begin studying this subclass of functions:

Theorem 1. $DL_n^{\mathcal{F}}(\delta, \alpha, \lambda)$ *is a convex set.*

Proof. Taking the functions

$$f_k(z) = z + \sum_{j=2}^{\infty} a_{jk} z^j, \quad k = 1, 2, \quad z \in U,$$

belonging to the class $DL_n^{\mathcal{F}}(\delta,\alpha,\lambda)$, we have to prove that the function

$$h(z)=\gamma_1 f_1(z)+\gamma_2 f_2(z)$$

belongs to the class $DL_n^{\mathcal{F}}(\delta,\alpha,\lambda)$ with $\gamma_1,\gamma_2\geq 0$, $\gamma_1+\gamma_2=1$.

We have $h'(z)=(\gamma_1 f_1+\gamma_2 f_2)'(z)=\gamma_1 f_1'(z)+\gamma_2 f_2'(z)$, $z\in U$, and $\left(D_z^{-\lambda}L_\alpha^n h(z)\right)'=\left(D_z^{-\lambda}L_\alpha^n(\gamma_1 f_1+\gamma_2 f_2)(z)\right)'=\gamma_1\left(D_z^{-\lambda}L_\alpha^n f_1(z)\right)'+\gamma_2\left(D_z^{-\lambda}L_\alpha^n f_2(z)\right)'$.

and we can write

$F_{\left(D_z^{-\lambda}L_\alpha^n h\right)'(U)}\left(D_z^{-\lambda}L_\alpha^n h(z)\right)'=F_{\left(D_z^{-\lambda}L_\alpha^n(\gamma_1 f_1+\gamma_2 f_2)\right)'(U)}\left(D_z^{-\lambda}L_\alpha^n(\gamma_1 f_1+\gamma_2 f_2)(z)\right)'=$
$F_{\left(D_z^{-\lambda}L_\alpha^n(\gamma_1 f_1+\gamma_2 f_2)\right)'(U)}\left(\gamma_1\left(D_z^{-\lambda}L_\alpha^n f_1(z)\right)'+\gamma_2\left(D_z^{-\lambda}L_\alpha^n f_2(z)\right)'\right)=$
$\frac{F_{\left(\gamma_1 D_z^{-\lambda}L_\alpha^n f_1\right)'(U)}\left(\gamma_1\left(D_z^{-\lambda}L_\alpha^n f_1(z)\right)'\right)+F_{\left(\gamma_2 D_z^{-\lambda}L_\alpha^n f_2\right)'(U)}\left(\gamma_2\left(D_z^{-\lambda}L_\alpha^n f_2(z)\right)'\right)}{2}=$
$\frac{F_{\left(D_z^{-\lambda}L_\alpha^n f_1\right)'(U)}\left(D_z^{-\lambda}L_\alpha^n f_1(z)\right)'+F_{\left(D_z^{-\lambda}L_\alpha^n f_2\right)'(U)}\left(D_z^{-\lambda}L_\alpha^n f_2(z)\right)'}{2}$.

Having $f_1,f_2\in DL_n^{\mathcal{F}}(\delta,\alpha,\lambda)$ we get $\delta<F_{\left(D_z^{-\lambda}L_\alpha^n f_1\right)'(U)}\left(D_z^{-\lambda}L_\alpha^n f_1(z)\right)'\leq 1$ and $\delta<F_{\left(D_z^{-\lambda}L_\alpha^n f_2\right)'(U)}\left(D_z^{-\lambda}L_\alpha^n f_2(z)\right)'\leq 1$, $z\in U$.

In these conditions $\delta<\frac{F_{\left(D_z^{-\lambda}L_\alpha^n f_1\right)'(U)}\left(D_z^{-\lambda}L_\alpha^n f_1(z)\right)'+F_{\left(D_z^{-\lambda}L_\alpha^n f_2\right)'(U)}\left(D_z^{-\lambda}L_\alpha^n f_2(z)\right)'}{2}\leq 1$ and we get $\delta<F_{\left(D_z^{-\lambda}L_\alpha^n h\right)'(U)}\left(D_z^{-\lambda}L_\alpha^n h(z)\right)'\leq 1$, equivalently with $h\in DL_n^{\mathcal{F}}(\delta,\alpha,\lambda)$ and $DL_n^{\mathcal{F}}(\delta,\alpha,\lambda)$ is a convex set. □

We give fuzzy differential subordinations obtained for the operator $D_z^{-\lambda}L_a^n$.

Theorem 2. *Considering a convex function g in U and defining $h(z)=g(z)+\frac{1}{c+2}zg'(z)$, with $c>0$, $z\in U$, when $f\in DL_n^{\mathcal{F}}(\delta,\alpha,\lambda)$ and $G(z)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t)dt$, $z\in U$, then*

$$F_{\left(D_z^{-\lambda}L_\alpha^n f\right)'(U)}\left(D_z^{-\lambda}L_\alpha^n f(z)\right)'\leq F_{h(U)}h(z),\quad z\in U, \tag{2}$$

implies the sharp result

$$F_{\left(D_z^{-\lambda}L_\alpha^n G\right)'(U)}\left(D_z^{-\lambda}L_\alpha^n G(z)\right)'\leq F_{g(U)}g(z),\quad z\in U.$$

Proof. Differentiating relation

$$z^{c+1}G(z)=(c+2)\int_0^z t^c f(t)dt,$$

considering z as variable, we get $(c+1)G(z)+zG'(z)=(c+2)f(z)$ and

$$(c+1)D_z^{-\lambda}L_\alpha^n G(z)+z\left(D_z^{-\lambda}L_\alpha^n G(z)\right)'=(c+2)D_z^{-\lambda}L_\alpha^n f(z),\quad z\in U,$$

and differentiating it again with respect to z, we obtain

$$\left(D_z^{-\lambda}L_\alpha^n G(z)\right)'+\frac{1}{c+2}z\left(D_z^{-\lambda}L_\alpha^n G(z)\right)''=\left(D_z^{-\lambda}L_\alpha^n f(z)\right)',\quad z\in U.$$

and the inequality (2) representing the fuzzy differential subordination can be written

$$F_{D_z^{-\lambda}L_\alpha^n G(U)}\left(\frac{1}{c+2}z\left(D_z^{-\lambda}L_\alpha^n G(z)\right)''+\left(D_z^{-\lambda}L_\alpha^n G(z)\right)'\right)\leq F_{g(U)}\left(\frac{1}{c+2}zg'(z)+g(z)\right).$$

Denoted

$$p(z) = \left(D_z^{-\lambda} L_\alpha^n G(z)\right)', \ z \in U,$$

where $p \in \mathcal{H}[1,n]$, we obtain

$$F_{p(U)}\left(\frac{1}{c+2} z p'(z) + p(z)\right) \leq F_{g(U)}\left(\frac{1}{c+2} z g'(z) + g(z)\right), \ z \in U.$$

Applying Lemma 3, we get

$$F_{(D_z^{-\lambda} L_\alpha^n G)'(U)}\left(D_z^{-\lambda} L_\alpha^n G(z)\right)' \leq F_{g(U)} g(z), \ z \in U,$$

and g is the best dominant. □

We give an inclusion result for the class $DL_n^{\mathcal{F}}(\delta, \alpha, \lambda)$:

Theorem 3. *Taking* $h(z) = \frac{1+(2\delta-1)z}{1+z}$ *and* $G(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t) dt$, $z \in U$, *with* $\delta \in [0,1)$, $c > 0$, $n \in \mathbb{N}$, $\alpha \geq 0$, $\lambda > 0$, *then*

$$G\left[DL_n^{\mathcal{F}}(\delta, \alpha, \lambda)\right] \subset DL_n^{\mathcal{F}}(\delta^*, \alpha, \lambda), \tag{3}$$

where $\delta^* = 2\delta - 1 + 2(2+c)(1-\delta) \int_0^1 \frac{t^{c+1}}{t+1} dt$.

Proof. Making the same steps such as in the proof of Theorem 2, taking account the hypothesis of Theorem 3 and that $h(z) = \frac{1+(2\delta-1)z}{1+z}$ is a convex function, we obtain

$$F_{p(U)}\left(\frac{1}{c+2} z p'(z) + p(z)\right) \leq f_{h(U)} h(z),$$

with $p(z) = \left(D_z^{-\lambda} L_\alpha^n G(z)\right)'$, $z \in U$.

Applying Lemma 2, we get

$$F_{(D_z^{-\lambda} L_\alpha^n G)'(U)}\left(D_z^{-\lambda} L_\alpha^n G(z)\right)' \leq F_{g(U)} g(z) \leq F_{h(U)} h(z),$$

where

$$g(z) = \frac{2+c}{n z^{\frac{2+c}{n}}} \int_0^z t^{\frac{2+c}{n}-1} \frac{1+(2\delta-1)t}{1+t} dt = (2\delta - 1) + \frac{2(c+2)(1-\delta)}{n z^{\frac{c+2}{n}}} \int_0^z \frac{t^{\frac{2+c}{n}-1}}{t+1} dt.$$

Since the function g is convex and $g(U)$ is symmetric with respect to the real axis, we can write

$$F_{D_z^{-\lambda} L_\alpha^n G(U)}\left(D_z^{-\lambda} L_\alpha^n G(z)\right)' \geq \min_{|z|=1} F_{g(U)} g(z) = F_{g(U)} g(1) \tag{4}$$

and $\delta^* = g(1) = 2\delta - 1 + \frac{2(2+c)(1-\delta)}{n} \int_0^1 \frac{t^{\frac{2+c}{n}-1}}{t+1} dt$, that give the inclusion (3). □

Theorem 4. *Taking a convex function* g *with the property* $g(0) = 0$, *define* $h(z) = g(z) + z g'(z)$, $z \in U$. *When* $f \in \mathcal{A}$, $n \in \mathbb{N}$, $\alpha \geq 0$, $\lambda > 0$, *and the fuzzy differential subordination holds*

$$F_{(D_z^{-\lambda} L_\alpha^n f)'(U)}\left(D_z^{-\lambda} L_\alpha^n f(z)\right)' \leq F_{h(U)} h(z), \quad z \in U, \tag{5}$$

then we get the sharp result

$$F_{D_z^{-\lambda} L_\alpha^n f(U)} \frac{D_z^{-\lambda} L_\alpha^n f(z)}{z} \leq F_{g(U)} g(z), \quad z \in U.$$

Proof. Considering $p(z) = \frac{D_z^{-\lambda} L_\alpha^n f(z)}{z} \in \mathcal{H}[0,\lambda]$, we can write $zp(z) = D_z^{-\lambda} L_\alpha^n f(z)$, $z \in U$, and differentiating it we get $zp'(z) + p(z) = \left(D_z^{-\lambda} L_\alpha^n f(z)\right)'$, $z \in U$.

The inequality (5) can be written as following

$$F_{p(U)}\left(zp'(z) + p(z)\right) \leq F_{h(U)} h(z) = F_{g(U)}\left(zg'(z) + g(z)\right), \quad z \in U,$$

and applying Lemma 3, we get the sharp result

$$F_{\left(D_z^{-\lambda} L_\alpha^n f\right)'(U)} \frac{D_z^{-\lambda} L_\alpha^n f(z)}{z} \leq F_{g(U)} g(z), \quad z \in U.$$

□

Example 1. *Consider*

$$g(z) = \frac{-2z}{1+z}$$

a convex function in U and we obtain that $g(0) = 0$, $g'(z) = \frac{-2}{(1+z)^2}$. *Define*

$$h(z) = g(z) + zg'(z) = \frac{-2z}{1+z} - \frac{2z}{(1+z)^2} = \frac{-2z^2 - 4z}{(1+z)^2}.$$

Take $\alpha = 2$, $n = 1$, $f(z) = z + z^2$, $z \in U$, *and after a short computation we obtain*

$$L_2^1 f(z) = z + 2z^2$$

and

$$D_z^{-\lambda} L_2^1 f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{L_2^1 f(t)}{(z-t)^{1-\lambda}} dt = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{t + 2t^2}{(z-t)^{1-\lambda}} dt$$
$$= \frac{1}{\Gamma(\lambda+2)} z^{1+\lambda} + \frac{4}{\Gamma(\lambda+3)} z^{2+\lambda}$$

and differentiating it

$$\left(D_z^{-\lambda} L_2^1 f(z)\right)' = \frac{1}{\Gamma(\lambda+1)} z^{\lambda} + \frac{4}{\Gamma(\lambda+2)} z^{\lambda+1}.$$

Applying Theorem 4 we get the following fuzzy differential subordination

$$\frac{1}{\Gamma(1+\lambda)} z^{\lambda} + \frac{4}{\Gamma(2+\lambda)} z^{1+\lambda} \prec_{\mathcal{F}} \frac{-2z^2 - 4z}{(1+z)^2}, \ z \in U,$$

induce the following fuzzy differential subordination

$$\frac{1}{\Gamma(2+\lambda)} z^{\lambda} + \frac{4}{\Gamma(3+\lambda)} z^{1+\lambda} \prec_{\mathcal{F}} \frac{-2z}{1+z}, \ z \in U.$$

Theorem 5. *Taking a holomorphic function h, such that* $h(0) = 0$ *and* $\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, *when* $f \in \mathcal{A}$, $n \in \mathbb{N}$, $\alpha \geq 0$, $\lambda > 0$, *and the fuzzy differential subordination holds*

$$F_{\left(D_z^{-\lambda} L_\alpha^n f\right)'(U)} \left(D_z^{-\lambda} L_\alpha^n f(z)\right)' \leq F_{h(U)} h(z), \quad z \in U, \tag{6}$$

then

$$F_{D_z^{-\lambda} L_\alpha^n f(U)} \frac{D_z^{-\lambda} L_\alpha^n f(z)}{z} \leq F_{q(U)} q(z), \quad z \in U,$$

where the fuzzy best dominant $q(z) = \frac{1}{z} \int_0^z h(t) dt$ *is convex.*

Proof. Considering Re $\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, and using Lemma 1, we deduce that $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is a convex function and it is a solution of the differential equation defining the fuzzy differential subordination (6) $zq'(z) + q(z) = h(z)$, therefore it is the fuzzy best dominant.

Differentiating $zp(z) = D_z^{-\lambda} L_\alpha^n f(z)$, we get $\left(D_z^{-\lambda} L_\alpha^n f(z)\right)' = zp'(z) + p(z)$, $z \in U$, and (6) can be written

$$F_{p(U)}\left(zp'(z) + p(z)\right) \leq F_{h(U)}h(z), \quad z \in U.$$

Applying Lemma 3, we get

$$F_{D_z^{-\lambda} L_\alpha^n f(U)} \frac{D_z^{-\lambda} L_\alpha^n f(z)}{z} \leq F_{q(U)}q(z), \ z \in U.$$

□

Corollary 1. *Taking the convex function in* U*,* $h(z) = \frac{1+(2\delta-1)z}{1+z}$*, with* $\delta \in [0,1)$*, when* $f \in \mathcal{A}$ *and the fuzzy differential subordination holds*

$$F_{\left(D_z^{-\lambda} L_\alpha^n f\right)'(U)}\left(D_z^{-\lambda} L_\alpha^n f(z)\right)' \leq F_{h(U)}h(z), z \in U, \quad (7)$$

then

$$F_{D_z^{-\lambda} L_\alpha^n f(U)} \frac{D_z^{-\lambda} L_\alpha^n f(z)}{z} \leq F_{q(U)}q(z), z \in U,$$

where the fuzzy best dominant $q(z) = 2\delta - 1 + 2(1-\delta)\frac{\ln(z+1)}{z}$*,* $z \in U$*, is convex.*

Proof. Taking $h(z) = \frac{1+(2\delta-1)z}{1+z}$, we obtain $h(0) = 1$, $h'(z) = \frac{-2(1-\delta)}{(1+z)^2}$ and $h''(z) = \frac{4(1-\delta)}{(1+z)^3}$, therefore $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) = Re\left(\frac{1-z}{1+z}\right) = Re\left(\frac{1-\rho\cos\theta - i\rho\sin\theta}{1+\rho\cos\theta + i\rho\sin\theta}\right) = \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} > 0 > -\frac{1}{2}$.

Following the same steps like in the proof of Theorem 5 with $p(z) = \frac{D_z^{-\lambda} L_\alpha^n f(z)}{z}$, the fuzzy differential subordination (7) can be written

$$F_{D_z^{-\lambda} L_\alpha^n f(U)}\left(zp'(z) + p(z)\right) \leq F_{h(U)}h(z), \quad z \in U.$$

Applying Lemma 2 for $m = 1$ and $\gamma = 1$, we obtain

$$F_{D_z^{-\lambda} L_\alpha^n f(U)} \frac{D_z^{-\lambda} L_\alpha^n f(z)}{z} \leq F_{q(U)}q(z),$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt = \frac{1}{z}\int_0^z \frac{1+(2\delta-1)t}{t+1}dt =$$

$$2\delta - 1 + \frac{2(1-\delta)}{z}\int_0^z \frac{1}{t+1}dt = 2\delta - 1 + 2(1-\delta)\frac{\ln(z+1)}{z}, \ z \in U.$$

□

Example 2. *Consider*

$$h(z) = \frac{-2z}{1+z}$$

and we obtain that $h(0) = 0$*,* $h'(z) = \frac{-2}{(1+z)^2}$ *and* $h''(z) = \frac{4}{(1+z)^3}$*.*

Taking account that

$$Re\left(1 + \frac{zh''(z)}{h'(z)}\right) = Re\left(\frac{1-z}{1+z}\right) = Re\left(\frac{1-\rho\cos\theta - i\rho\sin\theta}{1+\rho\cos\theta + i\rho\sin\theta}\right)$$

$$= \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} > 0 > -\frac{1}{2},$$

h is a convex function in U.

Taking $\alpha = 2$, $n = 1$, $f(z) = z + z^2$, $z \in U$, as in Example 1, we have

$$L_2^1 f(z) = z + 2z^2$$

and

$$D_z^{-\lambda} L_2^1 f(z) = \frac{1}{\Gamma(\lambda+2)} z^{1+\lambda} + \frac{4}{\Gamma(\lambda+3)} z^{2+\lambda}$$

and differentiating it

$$\left(D_z^{-\lambda} L_2^1 f(z)\right)' = \frac{1}{\Gamma(\lambda+1)} z^{\lambda} + \frac{4}{\Gamma(\lambda+2)} z^{\lambda+1}.$$

Additionally, we get

$$q(z) = \frac{1}{z}\int_0^z \frac{-2t}{1+t} dt = \frac{2\ln(1+z)}{z} - 2.$$

Applying Theorem 5 we get the following fuzzy differential subordination

$$\frac{1}{\Gamma(1+\lambda)} z^{\lambda} + \frac{4}{\Gamma(2+\lambda)} z^{1+\lambda} \prec_{\mathcal{F}} \frac{2z}{1+z},\ z \in U,$$

induce the following fuzzy differential subordination

$$\frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \frac{4}{\Gamma(3+\lambda)} z^{2+\lambda} \prec_{\mathcal{F}} \frac{2\ln(1+z)}{z} - 2,\ z \in U.$$

Theorem 6. *Taking a convex function g with the property $g(0) = 0$ and defining $h(z) = zg'(z) + g(z)$, $z \in U$, when $f \in \mathcal{A}$, $n \in \mathbb{N}$, $\alpha \geq 0$, $\lambda > 0$, and the fuzzy differential subordination*

$$F_{D_z^{-\lambda} L_\alpha^n f(U)} \left(\frac{z D_z^{-\lambda} L_\alpha^{n+1} f(z)}{D_z^{-\lambda} L_\alpha^n f(z)} \right)' \leq F_{h(U)} h(z), \quad z \in U, \tag{8}$$

holds, then we obtain the sharp result

$$F_{D_z^{-\lambda} L_\alpha^n f(U)} \frac{z D_z^{-\lambda} L_\alpha^{n+1} f(z)}{D_z^{-\lambda} L_\alpha^n f(z)} \leq F_{g(U)} g(z), \quad z \in U.$$

Proof. Considering $p(z) = \frac{D_z^{-\lambda} L_\alpha^{n+1} f(z)}{D_z^{-\lambda} L_\alpha^n f(z)}$ and differentiating it we get $zp'(z) + p(z) = \left(\frac{z L_\alpha^{n+1} f(z)}{L_\alpha^n f(z)}\right)'$. With this notation, inequality (8) can be written as

$$F_{p(U)}\left(zp'(z) + p(z)\right) \leq F_{h(U)} h(z) = F_{g(U)}\left(zg'(z) + g(z)\right),\ z \in U.$$

Applying Lemma 3, we get

$$F_{D_z^{-\lambda} L_\alpha^n f(U)} \frac{D_z^{-\lambda} L_\alpha^{n+1} f(z)}{D_z^{-\lambda} L_\alpha^n f(z)} \leq F_{g(U)} g(z), \quad z \in U.$$

□

Example 3. *Consider*

$$g(z) = \frac{-2z}{1+z}$$

and

$$h(z) = g(z) + zg'(z) = \frac{-2z^2 - 4z}{(1+z)^2}$$

as given in Example 1.

Taking $\alpha = 2$, $n = 1$, $f(z) = z + z^2$, $z \in U$, as in Example 1, we get

$$L_2^1 f(z) = z + 2z^2$$

and

$$L_2^2 f(z) = z + 2z^2$$

and applying Riemann–Liouville fractional integral of order λ we have

$$D_z^{-\lambda} L_2^1 f(z) = \frac{1}{\Gamma(\lambda+2)} z^{1+\lambda} + \frac{4}{\Gamma(\lambda+3)} z^{2+\lambda} = D_z^{-\lambda} L_2^2 f(z).$$

Applying Theorem 6 we get the following fuzzy differential subordination

$$1 \prec_{\mathcal{F}} \frac{-2z^2 - 4z}{(1+z)^2},\ z \in U,$$

induce the following fuzzy differential subordination

$$z \prec_{\mathcal{F}} \frac{-2z}{1+z},\ z \in U.$$

Theorem 7. *Taking a convex function g with the property $g(0) = 0$ and defining $h(z) = \lambda z g'(z) + g(z)$, $z \in U$, $\alpha \geq 0$, $\lambda, \delta > 0$, when $f \in \mathcal{A}$ and the fuzzy differential subordination*

$$F_{D_z^{-\lambda} L_\alpha^n f(U)}\left(\left(\frac{D_z^{-\lambda} L_\alpha^n f(z)}{z}\right)^{\delta-1} \left(D_z^{-\lambda} L_\alpha^n f(z)\right)'\right) \leq F_{h(U)} h(z)\,, \quad z \in U, \tag{9}$$

holds, then we obtain the sharp result

$$F_{D_z^{-\lambda} L_\alpha^n f(U)}\left(\frac{D_z^{-\lambda} L_\alpha^n f(z)}{z}\right)^{\delta} \leq F_{g(U)} g(z),\ \ z \in U.$$

Proof. Considering $p(z) = \left(\frac{D_z^{-\lambda} L_\alpha^n f(z)}{z}\right)^{\delta} \in \mathcal{H}[0, \lambda\delta]$, and differentiating it we obtain

$$zp'(z) = \delta\left(\frac{D_z^{-\lambda} L_\alpha^n f(z)}{z}\right)^{\delta-1}\left(D_z^{-\lambda} L_\alpha^n f(z)\right)' - \delta\left(\frac{D_z^{-\lambda} L_\alpha^n f(z)}{z}\right)^{\delta}$$

$$= \delta\left(\frac{D_z^{-\lambda} L_\alpha^n f(z)}{z}\right)^{\delta-1}\left(D_z^{-\lambda} L_\alpha^n f(z)\right)' - \delta p(z),$$

and $\frac{1}{\delta} z p'(z) + p(z) = \left(\frac{D_z^{-\lambda} L_\alpha^n f(z)}{z}\right)^{\delta-1} \left(D_z^{-\lambda} L_\alpha^n f(z)\right)'$, $z \in U$.

Inequality (9) can be written

$$F_{p(U)}\left(\frac{1}{\delta} zp'(z) + p(z)\right) \leq F_{h(U)} h(z) = F_{g(U)}\left(\lambda z g'(z) + g(z)\right),\ \ z \in U.$$

Applying Lemma 3 for $\alpha = \frac{1}{\delta}$ and $m = \lambda\delta$, we get

$$F_{D_z^{-\lambda} L_\alpha^n f(U)}\left(\frac{D_z^{-\lambda} L_\alpha^n f(z)}{z}\right)^{\delta} \leq F_{g(U)} g(z), \quad z \in U.$$

□

Example 4. *Consider*

$$g(z) = \frac{-2z}{1+z}$$

and

$$h(z) = g(z) + zg'(z) = \frac{-2z^2 - 4z}{(1+z)^2}$$

as given in Example 1.

Taking $\alpha = 2$, $n = 1$, $f(z) = z + z^2$, $z \in U$, *as in Example 1, we obtain*

$$L_2^1 f(z) = z + 2z^2$$

and

$$D_z^{-\lambda} L_2^1 f(z) = \frac{1}{\Gamma(\lambda+2)} z^{1+\lambda} + \frac{4}{\Gamma(\lambda+3)} z^{2+\lambda}$$

and differentiating it

$$\left(D_z^{-\lambda} L_2^1 f(z)\right)' = \frac{1}{\Gamma(\lambda+1)} z^{\lambda} + \frac{4}{\Gamma(\lambda+2)} z^{\lambda+1}.$$

Applying Theorem 7 we get the following fuzzy differential subordination

$$\left(\frac{1}{\Gamma(\lambda+2)} z^{\lambda} + \frac{4}{\Gamma(\lambda+3)} z^{1+\lambda}\right)^{\delta-1} \left(\frac{1}{\Gamma(1+\lambda)} z^{\lambda} + \frac{4}{\Gamma(2+\lambda)} z^{1+\lambda}\right) \prec_{\mathcal{F}} \frac{-2z^2-4z}{(1+z)^2},\ z \in U,$$

induce the following fuzzy differential subordination

$$\left(\frac{1}{\Gamma(\lambda+2)} z^{\lambda} + \frac{4}{\Gamma(\lambda+3)} z^{1+\lambda}\right)^{\delta} \prec_{\mathcal{F}} \frac{-2z}{1+z},\ z \in U.$$

Theorem 8. *Considering a holomorphic function h, such that* $h(0) = 0$ *and* $\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, *when* $f \in \mathcal{A}$, $\alpha \geq 0$, $\lambda, \delta > 0$, *and the fuzzy differential subordination*

$$F_{D_z^{-\lambda} L_\alpha^n f(U)}\left(\left(\frac{D_z^{-\lambda} L_\alpha^n f(z)}{z}\right)^{\delta-1} \left(D_z^{-\lambda} L_\alpha^n f(z)\right)'\right) \leq F_{h(U)} h(z), \quad z \in U, \tag{10}$$

holds, then

$$F_{D_z^{-\lambda} L_\alpha^n f(U)}\left(\frac{D_z^{-\lambda} L_\alpha^n f(z)}{z}\right)^{\delta} \leq F_{q(U)} q(z),\ \ z \in U,$$

where the fuzzy best dominant $q(z) = \frac{1}{z}\int_0^z h(t)dt$ *is convex.*

Proof. Considering $p(z) = \left(\frac{D_z^{-\lambda} L_\alpha^n f(z)}{z}\right)^{\delta} \in \mathcal{H}[0, \lambda\delta]$, after differentiating it and making an easy computation, we get

$$\frac{1}{\delta} zp'(z) + p(z) = \left(\frac{D_z^{-\lambda} L_\alpha^n (z)}{z}\right)^{\delta-1} \left(D_z^{-\lambda} L_\alpha^n f(z)\right)',\ z \in U,$$

and inequality (10) can be written

$$F_{p(U)}\left(\frac{1}{\delta} zp'(z) + p(z)\right) \leq F_{h(U)} h(z),\ z \in U.$$

Applying Lemma 2, we obtain

$$F_{D_z^{-\lambda}L_\alpha^n f(U)}\left(\frac{D_z^{-\lambda}L_\alpha^n f(z)}{z}\right)^\delta \le F_{q(U)}q(z),\ \ z\in U.$$

Taking into account that Re $\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, applying Lemma 1 we obtain that $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is a convex function and it is a solution of the differential equation of the fuzzy differential subordination (10) $zq'(z) + q(z) = h(z)$, thus it is the fuzzy best dominant. □

Example 5. *Considering*

$$h(z) = \frac{-2z}{1+z},$$

as in Example 2, a convex function which satisfy conditions from Theorem 8, and taking $\alpha = 2$, $n = 1$, $f(z) = z + z^2$, $z \in U$, *we obtain*

$$L_2^1 f(z) = z + 2z^2$$

and

$$D_z^{-\lambda}L_2^1 f(z) = \frac{1}{\Gamma(\lambda+2)}z^{1+\lambda} + \frac{4}{\Gamma(\lambda+3)}z^{2+\lambda}$$

and differentiating it

$$\left(D_z^{-\lambda}L_2^1 f(z)\right)' = \frac{1}{\Gamma(\lambda+1)}z^{\lambda} + \frac{4}{\Gamma(\lambda+2)}z^{\lambda+1}.$$

Additionally, we get

$$q(z) = \frac{1}{z}\int_0^z \frac{-2t}{1+t}dt = \frac{2\ln(1+z)}{z} - 2.$$

Applying Theorem 8 we get the following fuzzy differential subordination

$$\left(\frac{1}{\Gamma(\lambda+2)}z^{\lambda} + \frac{4}{\Gamma(\lambda+3)}z^{1+\lambda}\right)^{\delta-1}\left(\frac{1}{\Gamma(1+\lambda)}z^{\lambda} + \frac{4}{\Gamma(2+\lambda)}z^{1+\lambda}\right) \prec_{\mathcal{F}} \frac{2z}{1+z},\ z\in U,$$

induce the following fuzzy differential subordination

$$\left(\frac{1}{\Gamma(\lambda+2)}z^{\lambda} + \frac{4}{\Gamma(\lambda+3)}z^{1+\lambda}\right)^{\delta} \prec_{\mathcal{F}} \frac{2\ln(1+z)}{z} - 2,\ z\in U.$$

Theorem 9. *Considering a convex function g with the property* $g(0) = \frac{1}{\lambda+1}$ *and defining* $h(z) = zg'(z) + g(z)$, $z \in U$, $\lambda > 0$, $\alpha \ge 0$, $n \in \mathbb{N}$, *when* $f \in \mathcal{A}$ *and the fuzzy differential subordination*

$$F_{D_z^{-\lambda}L_\alpha^n f(U)}\left(1 - \frac{D_z^{-\lambda}L_\alpha^n f(z)\left(D_z^{-\lambda}L_\alpha^n f(z)\right)''}{\left[\left(D_z^{-\lambda}L_\alpha^n f(z)\right)'\right]^2}\right) \le F_{h(U)}h(z),\quad z\in U,$$

holds, then we obtain the sharp result

$$F_{D_z^{-\lambda}L_\alpha^n f(U)}\left(\frac{D_z^{-\lambda}L_\alpha^n f(z)}{z\left(D_z^{-\lambda}L_\alpha^n f(z)\right)'}\right) \le F_{g(U)}g(z),\ \ z\in U.$$

Proof. Differentiating $p(z) = \frac{D_z^{-\lambda}L_\alpha^n f(z)}{z\left(D_z^{-\lambda}L_\alpha^n f(z)\right)'}$ we obtain $zp'(z) + p(z) = 1 - \frac{D_z^{-\lambda}L_\alpha^n f(z)\left(D_z^{-\lambda}L_\alpha^n f(z)\right)''}{\left[\left(D_z^{-\lambda}L_\alpha^n f(z)\right)'\right]^2}, z \in U.$

Using this notation, the fuzzy differential subordination can be written

$$F_{p(U)}\left(zp'(z) + p(z)\right) \leq F_{h(U)}h(z) = F_{g(U)}\left(zg'(z) + g(z)\right), \ z \in U,$$

and applying Lemma 3, we obtain the sharp result

$$F_{D_z^{-\lambda}L_\alpha^n f(U)}\left(\frac{D_z^{-\lambda}L_\alpha^n f(z)}{z\left(D_z^{-\lambda}L_\alpha^n f(z)\right)'}\right) \leq F_{g(U)}g(z), \quad z \in U.$$

□

3. Conclusions

Applying the theory of fuzzy differential subordination, we studied a subclass of analytic function $DL_n^{\mathcal{F}}(\delta, \alpha, \lambda)$ newly introduced regarding the operator $D_z^{-\lambda}L_\alpha^n$. Several interesting properties are obtained for the defining subclass $DL_n^{\mathcal{F}}(\delta, \alpha, \lambda)$. New fuzzy differential subordinations are obtained for $D_z^{-\lambda}L_\alpha^n$. To show how the results would be applied it is give an example. The operator $D_z^{-\lambda}L_\alpha^n$ introduced in Definition 8 and the subclass $DL_n^{\mathcal{F}}(\delta, \alpha, \lambda)$ introduced in Definition 9 can be objects in other future studies. Other subclasses of analytic functions can be introduced regarding this operator and some properties for these subclasses can be investigated regarding coefficient estimates, closure theorems, distortion theorems, neighborhoods, and the radii of starlikeness, convexity, or close-to-convexity.

The dual theory of fuzzy differential superordination introduced in [21] could be used for obtaining similar results involving the operator $D_z^{-\lambda}L_\alpha^n$ and the class $L_n^{\mathcal{F}}(\delta, \alpha, \lambda)$ which could be combined with the results presented here for sandwich-type theorems, as seen in [17].

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