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# Remarks on Radial Solutions of a Parabolic Gelfand-Type Equation

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**Abstract:** We consider an equation with exponential nonlinearity under the Dirichlet boundary condition. For a one- or two-dimensional domain, a global solution has been obtained. In this paper, to extend the result to a higher dimensional case, we concentrate on the radial solutions in an annulus. First, we construct a time-local solution with an abstract theory of differential equations. Next, we show that decreasing energy exists in this problem. Finally, we establish a global solution for the sufficiently small initial value and parameter by Sobolev embedding and Poincaré inequalities together with some technical estimates. Moreover, when we take the smaller parameter, we prove that the global solution tends to zero as time goes to infinity.

**Keywords:** exponential nonlinearity; Lyapunov function; global solution

**MSC:** 35K58; 35B45

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**Citation:** Miyasita, T. Remarks on Radial Solutions of a Parabolic Gelfand-Type Equation. *Axioms* **2022**, *11*, 429. https://doi.org/10.3390/axioms11090429

Academic Editor: Palle E.T. Jorgensen

Received: 29 July 2022
Accepted: 23 August 2022
Published: 25 August 2022



## 1. Introduction

In [1,2], we considered the following parabolic equation:

$$\begin{cases} u_t = \Delta u + \lambda(e^u - 1) & x \in \Omega,\ t \in (0, T_{u_0}), \\ u(x,t) = 0 & x \in \partial\Omega,\ t \in (0, T_{u_0}), \\ u(x,0) = u_0(x) & x \in \Omega \end{cases} \tag{1}$$

and the corresponding elliptic equation:

$$\begin{cases} \Delta v + \lambda(e^v - 1) = 0 & x \in \Omega, \\ v(x) = 0 & x \in \partial\Omega, \end{cases} \tag{2}$$

where $\lambda > 0$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ for $n \in \mathbb{N}$, and $T_{u_0}$ denotes the maximal existing time of the local solution for an initial function $u_0$. In [2], the author established a unique global solution for a sufficiently small $\lambda > 0$ and $u_0 \in H_0^1(\Omega)$, with $n = 1, 2$. In fact, we have following theorems:

**Theorem 1** (Theorem 2 in [2])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary $\partial\Omega$. For any $\lambda > 0$ and $u_0 \in H_0^1(\Omega)$ satisfying*

$$\left(C_{TM}^2 + \frac{2|\Omega|}{\mu}\right)\lambda^2 + \|u_0\|_{H_0^1}^2 < 4\pi(\log 4\pi - 1), \tag{3}$$

*there exists a unique global solution for (1) satisfying*

$$u \in C\big([0,+\infty); H_0^1(\Omega)\big) \cap C^1\big((0,+\infty); L^2(\Omega)\big),$$

*where $\mu$ is the first eigenvalue of $-\Delta$ in $\Omega$ with the Dirichlet boundary condition, $|\Omega|$ is the measure of $\Omega$ in $\mathbb{R}^n$, and $C_{TM} > 0$ is a constant which depends only on $\Omega$ coming from the Trudinger–Moser inequality. Moreover, there is some $\lambda_1 > 0$ such that for any $\lambda < \lambda_1$, we have $\|u(\cdot,t)\|_{H_0^1} \to 0$ as $t \to +\infty$.*

**Theorem 2** (Theorem 3 in [2])**.** *Let $\Omega = (0,1)$. If we replace (3) with*

$$2\left(e^{2eC_S^2} + \frac{1}{\pi^2}\right)\lambda^2 + \|u_0\|^2_{H_0^1} < e \log 2,$$

*then the conclusion of Theorem 1 is still true, where $C_S > 0$ is an embedding constant which depends only on $\Omega$ coming from $H_0^1(\Omega) \subset C(\overline{\Omega})$.*

To prove the results, first of all, we derive the energy inequality from the Lyapunov function. Next, we apply the Sobolev embedding theorem for $n = 1$ and the Trudinger–Moser inequality for $n = 2$. Thus, it is not easy to extend this result for $n \geq 3$. In this paper, we assume that the domain is an annulus

$$A_a \equiv \left\{x \in \mathbb{R}^n \mid a < |x| < \frac{1}{a} \text{ for } 0 < a < 1 \text{ and } n \geq 2\right\}$$

and concentrate on the radial solutions $u(r) = u(|x|)$ for $r = |x|$. Then, problems (1) and (2) are reduced to

$$\begin{cases} u_t = u_{rr} + \frac{n-1}{r}u_r + \lambda(e^u - 1) & r \in (a, a^{-1}),\ t \in (0, T_{u_0}), \\ u(a,t) = u(a^{-1},t) = 0 & t \in (0, T_{u_0}), \\ u(r,0) = u_0(r) & r \in (a, a^{-1}) \end{cases} \tag{4}$$

and

$$\begin{cases} v_{rr} + \frac{n-1}{r}v_r + \lambda(e^v - 1) = 0 & r \in (a, a^{-1}), \\ v(a) = v(a^{-1}) = 0, \end{cases}$$

respectively. In [3], the author considered the radial solutions of the Keller–Segel model in an annulus. First, they derived an inequality similar to Lemma 6 in this paper. The difference is the boundary condition. They imposed the Neumann boundary condition in [3]. Next, they established a global solution by the Lyapunov function and the Sobolev embedding theorem.

Note that any interval $\alpha < s < \beta$ for $0 < \alpha < \beta$ is transformed into $a < r < a^{-1}$ through the relations $r = (\alpha\beta)^{-1/2}s$ and $a = \alpha^{1/2}\beta^{-1/2}$. Hence, the problem on any interval is equivalent to that on $(a, a^{-1})$. Henceforward, we denote $I \equiv (a, a^{-1})$ and $|I| = a^{-1} - a$. We denote the $H_0^1$ space in relation to $r$ with $\mathcal{H} = H_0^1(a, a^{-1})$ that is equipped with

$$\|u\|_{\mathcal{H}} = \left(\int_I |u_r|^2\, dr\right)^{\frac{1}{2}}.$$

Nowadays, it seems that there are not enough studies that concern (1) and (2). If $\Omega$ is a unit ball, the authors of [1] studied the bifurcation diagram of the positive solution of (2) and computed the bound for the Morse index globally, not locally, around a bifurcation point. If the solution was positive and radially symmetric, they established the existence of a singular solution, the multiple existence of the regular solution, and the bound for its Morse index. In [2], they dealt with the bifurcation diagram of the solution for (2), which was not always positive, for $n = 1$, and proved that nontrivial solutions bifurcate from trivial solutions and compute the Morse index locally around each bifurcation point. They found blow-up criteria and proved the existence of a global solution for (1) for a sufficiently small initial value and parameter. The aim of this paper is therefore to make a few remarks regarding the solution for (1) for a higher dimensional case. We introduce the main theorem on the global existence of the solution for (4) for a small initial value and parameter. We also construct the global solution with the Lyapunov function. We present similar statements to Theorems 1 and 2.

**Theorem 3.** *Let $n \in \mathbb{N}$ and $n \geq 2$. For any $\lambda > 0$ and $u_0 \in \mathcal{H}$ satisfying*

$$\frac{1}{n^2}\left\{\left(\frac{1}{a}\right)^n - a^n\right\}^2 \left(e^{2|I|e} + |I|\right)\lambda^2 + \|u_0\|_{\mathcal{H}}^2 < 2ea^{2(n-1)} \log 2, \tag{5}$$

*there exists a unique global solution for (4) that satisfies*

$$u \in C\big((0,+\infty); \mathcal{H} \cap H^2(I)\big) \cap C\big([0,+\infty); \mathcal{H}\big) \cap C^1\big((0,+\infty); L^2(I)\big).$$

*Moreover, there is some $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1)$ and $u_0 \in \mathcal{H}$ satisfying (5), we have*

$$\|u(\cdot,t)\|_{\mathcal{H}} \to 0$$

*as $t \to +\infty$.*

We would now like to remark on the corresponding nonlocal problems. In [4,5], the nonlocal problems corresponding to (1) and (2) were formulated as

$$\begin{cases} u_t = \Delta u + \frac{\lambda}{2}\left(\frac{e^u}{\int_J e^u\,dx} - 1\right) & x \in J \equiv (0,1),\ t > 0, \\ u_x(0,t) = u_x(1,t) = 0 & t > 0, \\ u(x,0) = u_0(x) & x \in J, \\ \int_J u(x,t)\,dx = 0 & t > 0 \end{cases} \tag{6}$$

and

$$\begin{cases} \Delta v + \frac{\lambda}{2}\left(\frac{e^v}{\int_J e^v\,dx} - 1\right) = 0 \quad x \in J, \\ v_x(0) = v_x(1) = 0, \\ \int_J v(x)\,dx = 0, \end{cases} \tag{7}$$

respectively. To introduce the result, we thus define

$$X = \left\{u \in H^1(J) \mid \int_J u\,dx = 0\right\}$$

and

$$H_N^2(J) = \left\{u \in H^2(J) \mid u_x(0) = u_x(1) = 0\right\},$$

respectively. In a one-dimensional case, the situation in (6) seems to be different from that in (1). Owing to a nonlocal term, and with the use of the Lyapunov function and the Sobolev embedding theorem, we can derive a uniform estimate of the norm of $u$ in $H^1(J)$ independently for $\lambda > 0$. It is an open problem to obtain an a priori estimate for $n \geq 2$. Then, we can prove that (6) admits a unique global solution in $X$.

**Theorem 4** (Theorem 3 in [4])**.** *For $u_0 \in X$, (6) admits a unique global solution $u = u(x,t)$, such that*

$$u \in C([0,+\infty); X), \qquad u_t \in L^2\Big((0,+\infty); L^2(J)\Big).$$

*For any $T > 0$, we have*

$$u \in L^2\Big((0,T); H_N^2(J)\Big).$$

Finally, we would like to remark on a bifurcation diagram of the solution set of (2) and (7). First, we note that $v(x) = 0$ is always a solution for all $\lambda > 0$. Then, we argue for the bifurcation problem with the use of the abstract theory in [6]. However, it is complicated to investigate an eigenvalue problem for a general domain $\Omega$. Hence, in [1,2,4,5], the authors obtained the results of the elliptic properties such as the structure of the solution set and the monotonicity of the Morse index for a one-dimensional or radial case. In fact, there exists $\lambda_m$ with

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots \uparrow +\infty$$

such that two continua $\mathcal{S}_m^{\pm}$ of a solution for (2) and (7) bifurcate from $(\lambda, v) = (\lambda_m, 0)$. Moreover, we can compute the Morse index on a trivial solution $(\lambda, v) = (\lambda, 0)$ for any $\lambda > 0$ and $(\lambda, v) \in \mathcal{S}_m^{\pm}$ that are sufficiently close to $(\lambda_m, 0)$. For details, see Proposition 2 in [2] and Theorem 2 in [4], respectively.

This paper is composed of 3 sections. In Section 2, we recall the definitions, notations, and known results. In Section 3, we first transform (4) in order to construct a local solution. Next, using the Lyapunov function and the Sobolev embedding theorem, we obtain the necessary estimate of the result of the proof of the global existence for Theorem 3.

## 2. Preliminary

We recall the definitions, notations, and known results. By defining the norm of $\mathcal{H}$ with $\|u\|_{\mathcal{H}} = \|u_r\|_2$ for $u \in \mathcal{H}$, where $\|\cdot\|_p$ is the standard $L^p$ norm over $I$ defined by

$$\|u\|_p = \left( \int_I |u(r)|^p \, dr \right)^{\frac{1}{p}}$$

for $p \in [1, +\infty)$ and

$$\|u\|_\infty = \operatorname*{ess\,sup}_{r \in I} |u(r)|,$$

we then introduce the Sobolev embedding and Poincaré inequalities.

**Lemma 1.** *We have the following inequalities*:

$$\|u\|_\infty \le \sqrt{\frac{|I|}{2}} \|u\|_{\mathcal{H}} \qquad \text{for } u \in \mathcal{H},$$
$$\|u\|_2 \le \frac{|I|}{\sqrt{2}} \|u\|_{\mathcal{H}} \qquad \text{for } u \in \mathcal{H}.$$

**Lemma 2.** *We have the following inequality*:

$$\int_I u_r^2 r^{n-1} \, dr \le \frac{|I|^2}{2a^{n-1}} \int_I \left\{ \left( u_r r^{n-1} \right)_r \right\}^2 dr \qquad \text{for } u \in \mathcal{H} \cap H^2(I).$$

**Proof.** For a constant function, the conclusion is obvious. Assume that $u$ is not a constant function. By Lemma 1, we have

$$\begin{aligned} \int_I u_r^2 r^{n-1} \, dr &= -\int_I u \left( u_r r^{n-1} \right)_r dr \\ &\le \sqrt{\int_I u^2 \, dr} \sqrt{\int_I \{(u_r r^{n-1})_r\}^2 \, dr} \\ &\le \frac{|I|}{\sqrt{2}} \sqrt{\int_I u_r^2 \, dr} \sqrt{\int_I \{(u_r r^{n-1})_r\}^2 \, dr} \\ &\le \frac{|I|}{\sqrt{2a^{n-1}}} \sqrt{\int_I u_r^2 r^{n-1} \, dr} \sqrt{\int_I \{(u_r r^{n-1})_r\}^2 \, dr}, \end{aligned}$$

which completes the proof of the lemma. □

Next, we introduce a technical lemma that plays an important role in deriving the necessary estimate for the global existence of the solution.

**Lemma 3** (Lemma 6 in [2]). *Let $A, B, \alpha > 0$. Assume that $A\alpha e < 1$ holds. We define*

$$f(t) = Ae^{\alpha t} + B - t$$

*for $t \geq 0$. If*

$$B < -\frac{1}{\alpha} \log A\alpha e \tag{8}$$

*holds, then there exist $t_1 \in (0, t_0)$ and $t_2 \in (t_0, +\infty)$ such that $f(t) \geq 0$ is equivalent to $0 \leq t \leq t_1$ or $t \geq t_2$, where $t_0$, $t_1$ and $t_2$ satisfy $t_0 = -(1/\alpha) \log A\alpha$ and $f'(t_0) = f(t_1) = f(t_2) = 0$.*

## 3. Global Solution

First of all, we transform (4) through the relation

$$w(r,t) = r^{\frac{1}{2}(n-1)} u(r,t)$$

and obtain

$$\begin{cases} w_t = w_{rr} - \rho(r)w + \lambda\sigma(r)\left(e^{\sigma^{-1}(r)w} - 1\right) & r \in I,\ t \in (0, T_{u_0}), \\ w(a,t) = w(a^{-1},t) = 0 & t \in (0, T_{u_0}), \\ w(r,0) = w_0(r) \equiv \sigma(r)u_0(r) & r \in I, \end{cases} \tag{9}$$

where

$$\rho = \rho(r) = \frac{(n-1)(n-3)}{4r^2} \qquad \text{and} \qquad \sigma = \sigma(r) = r^{\frac{1}{2}(n-1)}.$$

For $w_0 \in \mathcal{H}$, we transform (9) into the integral equation

$$w(t) = e^{-At}w_0 + \int_0^t e^{-A(t-s)}\left\{ -\rho(r)w(s) + \lambda\sigma(r)\left(e^{\sigma^{-1}(r)w(s)} - 1\right)\right\} ds,$$

where we extend $A \equiv -d^2/dr^2$ to be a self-adjoint positive operator in $L^2(I)$ with the domain $\mathcal{D}(A) = \mathcal{H} \cap H^2(I)$ and write the semi-group generated by $A$ as $e^{-At}$. We prove the local existence and uniqueness of the solution by the theories of an abstract evolution equation according to Theorem 4.1 in [7]. Thus, by omitting the details, we can establish a time-local solution $w = w(r,t)$. Hence, we have the following time-local solution:

$$u \in C\big((0, T_{u_0}); \mathcal{H} \cap H^2(I)\big) \cap C\big([0, T_{u_0}); \mathcal{H}\big) \cap C^1\big((0, T_{u_0}); L^2(I)\big).$$

We then introduce a decreasing energy, which plays an important role in deriving the necessary estimate for the global solution.

**Lemma 4.** *For $u_0 \in \mathcal{H}$ and $t \in [0, T_{u_0})$,*

$$L_\lambda(u(t)) \equiv \frac{1}{2}\int_I (u_r)^2 r^{n-1}\, dr - \lambda \int_I (e^u - u) r^{n-1}\, dr$$

*is a decreasing energy in $t \in [0, T_{u_0})$ for (4).*

**Proof.** The conclusion follows from

$$\frac{d}{dt} L_\lambda(u(t)) = -\int_I (u_t)^2 r^{n-1}\, dr \leq 0.$$

□

**Proof of Theorem 3.** By Lemma 4, we have

$$L_\lambda(u(t)) \le L_\lambda(u_0)$$

for all $t \in [0, T_{u_0})$, which yields

$$\frac{1}{2}\int_I (u_r)^2 r^{n-1}\,dr \le \lambda \int_I e^u r^{n-1}\,dr - \lambda \int_I u r^{n-1}\,dr + L_\lambda(u_0). \tag{10}$$

We then estimate each term of (10) in the following way:

**Lemma 5** (Estimate of (10))**.** *We have*

$$a^{n-1}\|u\|_{\mathcal{H}}^2 \le \int_I (u_r)^2 r^{n-1}\,dr.$$

**Lemma 6** (Estimate of (10))**.** *Let*

$$k = \sqrt{2|I|e} \quad \textit{and} \quad l = \left(\frac{1}{a}\right)^n - a^n.$$

*We have*

$$\lambda \int_I e^u r^{n-1}\,dr \le \frac{1}{2}a^{n-1}e^{\frac{|I|}{2k^2}\|u\|_{\mathcal{H}}^2} + \frac{l^2 e^{k^2}}{2a^{n-1}n^2}\lambda^2.$$

**Proof.** We have

$$\begin{aligned}
\lambda \int_I e^u r^{n-1}\,dr &\le \lambda e^{k\cdot\frac{1}{k}\|u\|_\infty}\int_I r^{n-1}\,dr \\
&\le \frac{l}{n}\lambda e^{\frac{1}{2}k^2 + \frac{1}{2k^2}\|u\|_\infty^2} \\
&\le \frac{l}{n}\lambda e^{\frac{1}{2}k^2} e^{\frac{|I|}{4k^2}\|u\|_{\mathcal{H}}^2} \\
&= \sqrt{a^{n-1}}e^{\frac{|I|}{4k^2}\|u\|_{\mathcal{H}}^2}\cdot\frac{1}{\sqrt{a^{n-1}}}\lambda e^{\frac{1}{2}k^2}\frac{l}{n} \\
&\le \frac{1}{2}a^{n-1}e^{\frac{|I|}{2k^2}\|u\|_{\mathcal{H}}^2} + \frac{l^2 e^{k^2}}{2a^{n-1}n^2}\lambda^2
\end{aligned}$$

according to Young's inequality and Lemma 1. □

**Lemma 7** (Estimate of (10))**.** *We have*

$$\left|\lambda \int_I u r^{n-1}\,dr\right| \le \frac{1}{4}a^{n-1}\|u\|_{\mathcal{H}}^2 + \frac{l^2|I|}{2a^{n-1}n^2}\lambda^2.$$

**Proof.** We have

$$\begin{aligned}\left|\lambda\int_I ur^{n-1}\,dr\right| &\leq \lambda\|u\|_\infty\frac{l}{n}\\ &\leq \lambda\sqrt{\frac{|I|}{2}}\|u\|_{\mathcal{H}}\frac{l}{n}\\ &= \sqrt{\frac{1}{2}a^{n-1}}\|u\|_{\mathcal{H}}\cdot\sqrt{\frac{|I|}{a^{n-1}}}\frac{l}{n}\lambda\\ &\leq \frac{1}{4}a^{n-1}\|u\|_{\mathcal{H}}^2+\frac{l^2|I|}{2a^{n-1}n^2}\lambda^2\end{aligned}$$

according to Young's inequality and Lemma 1. □

**Lemma 8** (Estimate of (10))**.** *We have*

$$L_\lambda(u_0)\leq\frac{1}{2a^{n-1}}\|u_0\|_{\mathcal{H}}^2.$$

**Proof.** The lemma follows from $r\leq a^{-1}$ and $e^p-p>0$ for $p\in\mathbb{R}$. □

We now go back to the proof of the theorem. Substituting the results of Lemmas 5, 6, 7, and 8 into (10), we have

$$\|u\|_{\mathcal{H}}^2 \leq 2e^{\frac{|I|}{2k^2}\|u\|_{\mathcal{H}}^2}+\frac{2l^2}{a^{2(n-1)}n^2}\left(e^{k^2}+|I|\right)\lambda^2+\frac{2}{a^{2(n-1)}}\|u_0\|_{\mathcal{H}}^2.$$

Hence, we have

$$\|u\|_{\mathcal{H}}^2\leq Ae^{\alpha\|u\|_{\mathcal{H}}^2}+B,$$

where

$$A=2,\qquad \alpha=\frac{|I|}{2k^2}=\frac{1}{4e}$$

and

$$B=\frac{2l^2}{a^{2(n-1)}n^2}\left(e^{k^2}+|I|\right)\lambda^2+\frac{2}{a^{2(n-1)}}\|u_0\|_{\mathcal{H}}^2.$$

We note that

$$A\alpha e=\frac{1}{2}<1$$

and that (5) is equivalent to (8). Hence, we can apply Lemma 3 to $f(\|u\|_{\mathcal{H}}^2)\geq 0$. Since

$$\|u_0\|_{\mathcal{H}}^2<\frac{a^{2(n-1)}}{2}B<-\frac{1}{2\alpha}\log A\alpha e=\frac{1}{2}\left(t_0-\frac{1}{\alpha}\right)<t_0$$

holds, we have

$$\|u(t)\|_{\mathcal{H}}^2\leq t_1$$

as long as the local solution exists. Hence, a global solution exists in $\mathcal{H}$.

Finally, we deal with the convergence problem. According to Lemmas 1 and 2, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I u_r^2 r^{n-1}\,dr \\
=&\ -\int_I u_t\left(r^{n-1}u_r\right)_r dr \\
=&\ -\int_I\left\{\left(r^{n-1}u_r\right)_r+\lambda r^{n-1}(e^u-1)\right\}\frac{1}{r^{n-1}}\left(r^{n-1}u_r\right)_r dr \\
=&\ -\int_I\left\{\left(r^{n-1}u_r\right)_r\right\}^2\frac{1}{r^{n-1}}\,dr-\lambda\int_I\left\{\left(r^{n-1}u_r\right)_r(e^u-1)\right\}dr \\
\leq&\ -a^{n-1}\int_I\left\{\left(r^{n-1}u_r\right)_r\right\}^2 dr+\lambda\int_I r^{n-1}e^u u_r^2\,dr \\
\leq&\ \left(\lambda e^{\sqrt{\frac{|I|}{2}}\|u\|_{\mathcal{H}}}-\frac{2a^{2n-2}}{|I|^2}\right)\int_I u_r^2 r^{n-1}\,dr \\
\leq&\ \left(\lambda e^{\sqrt{\frac{|I|}{2}t_1}}-\frac{2a^{2n-2}}{|I|^2}\right)\int_I u_r^2 r^{n-1}\,dr
\end{aligned}$$

owing to $\|u\|_{\mathcal{H}} \leq \sqrt{t_1}$. Thus, let

$$\lambda_1 \equiv \frac{2a^{2n-2}}{|I|^2}e^{-\sqrt{\frac{|I|}{2}t_1}}.$$

If $\lambda < \lambda_1$, we have

$$\int_I u_r^2 r^{n-1}\,dr \quad \leq \quad \exp\left\{-2e^{\sqrt{\frac{|I|}{2}t_1}}(\lambda_1-\lambda)t\right\}\int_I\{(u_0)_r\}^2 r^{n-1}\,dr \to 0$$

as $t \to +\infty$, which completes the proof of the theorem. □

**Remark 1.** *In [8], the authors extended the Trudinger–Moser inequality to the case of radially symmetric functions. Similarly in [3], the Trudinger–Moser inequality was formulated for radially symmetric functions in an annulus. The advantage of these is that the inequality holds for any constant because an interval does not contain* 0*. In our paper, we proved Lemma 6 in a manner similar to the inequality in [3]. As a result, the global existence of radially symmetric solutions was proved. The method may be applied to the global existence of radially symmetric solutions for* (6) *with $n \geq 2$.*

**Funding:** This research received no external funding.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The author would like to express their deepest gratitude to the anonymous reviewers for their careful reading of the manuscript.

**Conflicts of Interest:** The author declares no conflicts of interest.

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