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Entire Symmetric Functions on the Space of Essentially Bounded Integrable Functions on the Union of Lebesgue-Rohlin Spaces

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Abstract: The class of measure spaces which can be represented as unions of Lebesgue-Rohlin spaces with continuous measures contains a lot of important examples, such as $\mathbb{R}^n$ for any $n \in \mathbb{N}$ with the Lebesgue measure. In this work we consider symmetric functions on Banach spaces of all complex-valued integrable essentially bounded functions on such unions. We construct countable algebraic bases of algebras of continuous symmetric polynomials on these Banach spaces. The completions of such algebras of polynomials are Fréchet algebras of all complex-valued entire symmetric functions of bounded type on the abovementioned Banach spaces. We show that each such Fréchet algebra is isomorphic to the Fréchet algebra of all complex-valued entire symmetric functions of bounded type on the complex Banach space of all complex-valued essentially bounded functions on $[0,1]$.

Keywords: symmetric polynomial on a Banach space; continuous polynomial on a Banach space; algebraic basis; Lebesgue-Rohlin space

MSC: 46G25; 47H60; 46G20

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1. Introduction

The study of symmetric polynomials on infinite dimensional spaces started with the work [1] (for classical results in the finite dimensional case, see, e.g., [2–4]). In [1], the authors considered symmetric continuous polynomials on real Banach spaces ℓ_p and $L_p[0,1]$, where $p \in [1,+\infty)$. In particular, in [1] the authors constructed algebraic bases of algebras of the abovementioned polynomials. In [5], the authors considered symmetric continuous polynomials on separable sequence real Banach spaces with a symmetric basis (see [6] (Def. 3.a.1, p. 113)) and on a separable rearrangement invariant function the real Banach spaces (see [7] (Definition 2.a.1, p. 117)). Topological algebras of symmetric holomorphic functions on ℓ_p were studied first in [8]. Symmetric polynomials and symmetric holomorphic functions of bounded type on sequence Banach spaces were studied in [9–34] (see also the survey [35]). Symmetric holomorphic functions of unbounded type on sequence Banach spaces were studied in [36–39]. Symmetric polynomials and symmetric holomorphic functions on Banach spaces of Lebesgue measurable functions and on Cartesian powers of such spaces were studied in [40–49]. In [50–54], the authors used the most general approach to the study of symmetric functions.

In [41], the authors constructed an algebraic basis of the algebra of symmetric continuous complex-valued polynomials on the complex Banach space $L_\infty[0,1]$ of complex-valued Lebesgue measurable essentially bounded functions on $[0,1]$ and described the spectrum of the Fréchet algebra $H_{bs}(L_\infty[0,1])$ of symmetric analytic entire functions, which are bounded on bounded sets, on $L_\infty[0,1]$. In [42], the authors showed that the algebra $H_{bs}(L_\infty[0,1])$

is isomorphic to the algebra of all analytic functions on the strong dual of the topological vector space of entire functions on the complex plane $\mathbb{C}$. In addition in [42], it was shown that the algebra $H_{bs}(L_\infty[0,1])$ is a test algebra for the famous Michael problem (see [55]). In [49], the authors showed that the algebra $H_{bs}(L_\infty[0,1])$ is isomorphic to the algebra of symmetric entire functions on the complex Banach space of complex-valued Lebesgue integrable essentially bounded functions on the semi-axis.

In this work, we generalize the results of the work [49], replacing the semi-axis with the arbitrary union of Lebesgue-Rohlin spaces (which are also known as standard probability spaces) with continuous measures. Note that there are a lot of important measure spaces which can be represented as the abovementioned union. For example, $\mathbb{R}^n$ for any $n \in \mathbb{N}$ with the Lebesgue measure is one such space. We consider symmetric functions on Banach spaces of all complex-valued integrable essentially bounded functions on the unions of Lebesgue-Rohlin spaces with continuous measures. We construct countable algebraic bases of algebras of continuous symmetric polynomials on these Banach spaces. The completions of such algebras of polynomials are Fréchet algebras of all complex-valued entire symmetric functions of bounded type on the abovementioned Banach spaces. We show that every such Fréchet algebra is isomorphic to the Fréchet algebra $H_{bs}(L_\infty[0,1])$.

2. Preliminaries

Let us denote by $\mathbb{N}$ and $\mathbb{Z}_+$ the set of all positive integers and the set of all nonnegative integers, respectively.

2.1. Polynomials

Let X be a complex Banach space.

Let $N \in \mathbb{N}$. A mapping $P : X \to \mathbb{C}$, which is the restriction to the diagonal of some N-linear mapping $A_P : X^N \to \mathbb{C}$, i.e.,

$$P(x) = A_P(\underbrace{x, \ldots, x}_{N})$$

for every $x \in X$, is called an N-homogeneous polynomial.

A mapping $P : X \to \mathbb{C}$, which can be represented in the form

$$P = P_0 + P_1 + \ldots + P_N,$$

where $N \in \mathbb{N}$, P_0 is a constant mapping, and $P_n : X \to \mathbb{C}$ is an n-homogeneous polynomial for every $n \in \{1, \ldots, N\}$, is called a polynomial of a degree at most N.

It is known that a polynomial $P : X \to \mathbb{C}$ is continuous if and only if its norm

$$\|P\| = \sup_{\|x\| \le 1} |P(x)|$$

is finite. Consequently, for every continuous N-homogeneous polynomial $P : X \to \mathbb{C}$ and for every $x \in X$ we have the following inequality:

$$|P(x)| \le \|P\| \|x\|^N. \tag{1}$$

2.2. Holomorphic Functions

Definition 1. ([56] (Def. 2.1, p. 53)) *A subset U of a vector space E is said to be* finitely open *if $U \cap F$ is an open subset of the Euclidean space F for each finite dimensional subspace F of E.*

(See [56] (p. 53)). The finitely open subsets of E define a translation invariant topology τ_f. The balanced τ_f-neighborhoods of zero form a basis for the τ_f-neighborhoods of zero. On a topological vector space (E, τ), the topology τ_f is finer than τ, i.e., $\tau_f \ge \tau$.

Definition 2. (See [56] (Def. 2.2, p. 54)) *The complex-valued function f, defined on a finitely open subset U of a complex vector space E is said to be* G-holomorphic *if for each $a \in U, b \in E$ the complex-valued function of one complex variable*

$$\lambda \mapsto f(a + \lambda b)$$

is holomorphic in some neighborhood of zero. We let $H_G(U)$ denote the set of all G-holomorphic mappings from U into $\mathbb{C}$.

The following proposition is a partial result of [56] (Prop. 2.4, p. 55).

Proposition 1. *If U is a finitely open subset of a complex vector space E and $f \in H_G(U)$, then for each $a \in U$ there exists a unique sequence of homogeneous polynomials from E into $\mathbb{C}$, $\{f_m^{(a)}\}_{m=0}^{\infty}$, such that*

$$f(a+y) = \sum_{m=0}^{\infty} f_m^{(a)}(y)$$

for all y in some τ_f-neighborhood of zero. This series is called the Taylor series of f at a.

Definition 3. (See [56] (Def. 2.6, p. 57)) *Let (E, τ) be a complex locally convex space, and let U be a finitely open subset of E. A function $f : U \to \mathbb{C}$ is called* holomorphic *or* analytic *if it is G-holomorphic and for each $a \in U$ the function*

$$y \mapsto \sum_{m=0}^{\infty} f_m^{(a)}(y)$$

converges and defines a continuous function on some τ-neighborhood of zero. We let $H(U)$ denote the algebra of all holomorphic functions from U into $\mathbb{C}$ endowed with the compact open topology (the topology of uniform convergence on the compact subsets of U). A function, which is holomorphic on E, is called entire.

The following proposition is a partial result of [56] (Lemma 2.8, p. 58).

Proposition 2. *If U is an open subset of a complex locally convex space E and $f : U \to \mathbb{C}$ is G-holomorphic, then $f \in H(U)$ if and only if f is locally bounded.*

The following proposition is a partial result of [56] (Cor. 2.9, p. 59).

Proposition 3. *Let E be a complex locally convex space. Let U be an open subset of E, and suppose $f \in H(U)$. Then for every a in U and every $m \in \mathbb{N}$, the m-homogeneous polynomial $f_m^{(a)}$ is continuous.*

(See [56] (p. 166)). Let U be an open subset of a complex locally convex space E, and let B be a balanced closed subset of E. We let

$$d_B(a, U) = \sup\{|\lambda| : \ \lambda \in \mathbb{C}, a + \lambda B \subset U\}$$

for every $a \in U$. If E is a complex normed linear space and B is the unit ball of E, then $d_B(a, U)$ is the usual distance of a to the complement of U in E.

Let $f \in H(U)$. The B-radius of boundedness of f at $a \in U$, is defined as

$$r_f(a, B) = \sup\left\{|\lambda| : \ \lambda \in \mathbb{C}, a + \lambda B \subset U, \sup_{y \in a + \lambda B} |f(y)| < \infty\right\}.$$

The B-radius of uniform convergence of f at $a \in U$ is defined as

$$R_f(a,B) = \sup\Big\{|\lambda| : \ \lambda \in \mathbb{C}, a + \lambda B \subset U, \text{and the Taylor series of } f \text{ at } a$$
$$\text{converges to } f \text{ uniformly on } a + \lambda B\Big\}.$$

The following proposition is a partial result of [56] (Prop. 4.7, p. 166).

Proposition 4. *Let U be an open subset of a complex locally convex space E. Suppose $f \in H(U)$. If $a \in U$, B is a closed balanced subset of E and $r_f(a,B) > 0$, then*

$$r_f(a,B) = R_f(a,B) = \min\left\{ d_B(a,U), \left(\limsup_{n\to\infty} \sup_{y\in B} \left| f_n^{(a)}(y) \right|^{1/n} \right)^{-1} \right\}.$$

Let E be a complex normed space. An entire function $f : X \to \mathbb{C}$, for which $r_f(0,B) = +\infty$, where B is a closed unit ball in E, is called a function of *bounded type*. In other words, f is called a function of bounded type if it is bounded on every bounded subset of E. By Proposition 4, for every such a function f, its Taylor series at zero, $\sum_{m=0}^{\infty} f_m$, converges uniformly to f on every bounded subset of E (we denote $f_m^{(0)}$ by f_m).

By [57] (Cor. 7.3, p. 47),

$$f_m(y) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi y)}{\xi^{m+1}}\, d\xi, \qquad (2)$$

where $m \in \mathbb{Z}_+$, $y \in E$ and $r > 0$. Equation (2) is called the Cauchy Integral Formula.

Let E be a complex Banach space. Let $H_b(E)$ be the Fréchet algebra of all entire functions of bounded type $f : E \to \mathbb{C}$ endowed with the topology of the uniform convergence on bounded subsets. Let

$$\|f\|_r = \sup_{\|x\|\le r} |f(x)|$$

for $f \in H_b(E)$ and $r \in (0,+\infty)$. The topology of the Fréchet algebra $H_b(E)$ is generated by any set of norms

$$\{\|\cdot\|_r : \ r \in I\},$$

where I is an arbitrary unbounded subset of $(0,+\infty)$.

For details on holomorphic functions on Banach spaces, we refer the reader to [57] or [56,58] .

2.3. Measure Spaces

A measure space is a triple $(\Omega, \mathcal{F}, \nu)$, where Ω is a set, $\mathcal{F}$ is a σ algebra of its subsets, and $\nu : \mathcal{F} \to [0,+\infty]$ is a measure. In addition, we assume ν to be a complete measure, i.e., every subset of a measurable set with null measure (so called null set) is measurable too. An isomorphism between two measure spaces $(\Omega_1, \mathcal{F}_1, \nu_1)$ and $(\Omega_2, \mathcal{F}_2, \nu_2)$ is an invertible map $f : \Omega_1 \to \Omega_2$ such that f and f^{-1} are both measurable and measure-preserving maps. In the case $(\Omega_1, \mathcal{F}_1, \nu_1) = (\Omega_2, \mathcal{F}_2, \nu_2)$, the mapping f is called a measurable automorphism. Two measure spaces $(\Omega_1, \mathcal{F}_1, \nu_1)$ and $(\Omega_2, \mathcal{F}_2, \nu_2)$ are called isomorphic modulo zero if there exist null sets $M \subset \Omega_1$ and $N \subset \Omega_2$ such that measure spaces $\Omega_1 \setminus M$ and $\Omega_2 \setminus N$ are isomorphic [59] (§1, No. 5).

Let a measure space $(\Omega, \mathcal{F}, \nu)$ be such that $\nu(\Omega) = 1$. The measure space $(\Omega, \mathcal{F}, \nu)$ is called separable ([59] (§2, No. 1)), if there exists a countable system $\mathcal{G}$ of measurable sets having the following two properties:

1. For every measurable set $A \subset \Omega$, there exists a set B such that $A \subset B \subset \Omega$, B is identical with A modulo zero, and B is an element of the σ algebra generated by $\mathcal{G}$.
2. For every pair of points $x, y \in \Omega$, there exists a set $G \subset \mathcal{G}$ such that either $x \in G, y \notin G$, or $x \notin G, y \in G$.

Every countable system $\mathcal{G}$ of measurable sets satisfying conditions (1) and (2) is called a basis of the space $(\Omega, \mathcal{F}, \nu)$.

Let $(\Omega, \mathcal{F}, \nu)$ be a separable measure space, and let $B = \{B_n\}_{n=1}^{\infty}$ be an arbitrary basis in $(\Omega, \mathcal{F}, \nu)$. If all intersections of the form $\cap_{n=1}^{\infty} A_n$, where A_n is one of the two sets B_n and $\Omega \setminus B_n$, are nonempty, then the space $(\Omega, \mathcal{F}, \nu)$ is called complete with respect to the basis B. By [59] (§2, No. 2), if the space $(\Omega, \mathcal{F}, \nu)$ is complete modulo zero (i.e., isomorphic modulo zero to some complete measure space) with respect to some basis, then it is complete modulo zero with respect to every other basis. Separable measure spaces which are complete modulo zero with respect to their bases are called Lebesgue-Rohlin spaces or standard probability spaces. By [59] (§2, No. 4), every Lebesgue-Rohlin space with continuous measure (i.e., there are no points of positive measure) is isomorphic modulo zero to $[0,1]$ with Lebesgue measure. The following simple lemma shows that every such space is isomorphic to $[0,1]$ with Lebesgue measure.

Lemma 1. *Every Lebesgue-Rohlin measure space with continuous measure is isomorphic to* $[0,1]$ *with Lebesgue measure.*

Proof. Let $(\Omega, \mathcal{F}, \nu)$ be a Lebesgue-Rohlin measure space with continuous measure. By [59] (§2, No. 4), $(\Omega, \mathcal{F}, \nu)$ is isomorphic modulo zero to $[0,1]$ with Lebesgue measure, i.e., there exist null sets $M \subset \Omega$ and $N \subset [0,1]$ such that $\Omega \setminus M$ is isomorphic to $[0,1] \setminus N$. Let $f : \Omega \setminus M \to [0,1] \setminus N$ be the isomorphism. Let K be an arbitrary null subset of $[0,1] \setminus N$ with the cardinality of the continuum. Then $f^{-1}(K)$ is a null subset of $\Omega \setminus M$ with the cardinality of the continuum. Consequently, both sets $C_1 = M \cup f^{-1}(K)$ and $C_2 = N \cup K$ are null sets of the cardinality of the continuum. Let $h : C_1 \to C_2$ be a bijection. Let $g : \Omega \to [0,1]$ be defined by

$$g(t) = \begin{cases} h(t), & \text{if } t \in C_1, \\ f(t), & \text{if } t \in [0,1] \setminus C_1. \end{cases}$$

Evidently, g is an isomorphism between $(\Omega, \mathcal{F}, \nu)$ and $[0,1]$ with Lebesgue measure. □

2.4. Symmetric Functions

In general, symmetric functions are defined in the following way.

Definition 4. *Let A be an arbitrary nonempty set, and let S be a nonempty set of mappings acting from A to itself. A function f, defined on A, is called symmetric with respect to the set S if $f(s(a)) = f(a)$ for every $s \in S$ and $a \in A$.*

Let us describe the partial case of Definition 4, which we will use in this work. The set of all measurable automorphisms of some measure space $(\Omega, \mathcal{F}, \nu)$ we will denote by Ξ_Ω. A complex Banach space X of measurable functions $x : \Omega \to \mathbb{C}$ such that $x \circ \sigma$ belongs to X for every $x \in X$ and $\sigma \in \Xi_\Omega$ will be in the role of the set A from Definition 4. The set of operators

$$\{x \in X \mapsto x \circ \sigma \in X : \ \sigma \in \Xi_\Omega\}$$

will be in the role of the set S from Definition 4. So, a function f, defined on X, is called *symmetric* if

$$f(x \circ \sigma) = f(x)$$

for every $x \in X$ and $\sigma \in \Xi_\Omega$.

2.5. Algebraic Combinations

A mapping

$$t \in T \mapsto Q(f_1(t), \ldots, f_k(t)) \in \mathbb{C},$$

where T is a nonempty set, $k \in \mathbb{N}$, $f_1, \ldots, f_k$ are mappings acting from T to $\mathbb{C}$ and Q is a polynomial acting from $\mathbb{C}^k$ to $\mathbb{C}$, is called an algebraic combination of mappings $f_1, \ldots, f_k$.

Let $\mathcal{A}$ be some algebra of complex-valued mappings. Let $\mathcal{B} \subset \mathcal{A}$ be such that every element of $\mathcal{A}$ can be uniquely represented as an algebraic combination of some elements of $\mathcal{B}$. Then $\mathcal{B}$ is called an *algebraic basis* of $\mathcal{A}$.

2.6. Entire Symmetric Functions on $L_\infty[0,1]$

Let $L_\infty[0,1]$ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0,1]$ with norm

$$\|x\|_\infty = \operatorname{ess\,sup}_{t \in [0,1]} |x(t)|.$$

For every $n \in \mathbb{N}$, let $R_n : L_\infty[0,1] \to \mathbb{C}$ be defined by

$$R_n(x) = \int_{[0,1]} (x(t))^n \, dt.$$

Note that R_n is a symmetric continuous n-homogeneous polynomial such that $\|R_n\| = 1$ for every $n \in \mathbb{N}$.

Theorem 1. ([41] (Theorem 4.3)) *Every symmetric continuous n-homogeneous polynomial $P : L_\infty[0,1] \to \mathbb{C}$ can be uniquely represented as*

$$P(x) = \sum_{k_1 + 2k_2 + \ldots + nk_n = n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x) \cdots R_n^{k_n}(x),$$

where $k_1, \ldots, k_n \in \mathbb{Z}_+$ and $\alpha_{k_1,\ldots,k_n} \in \mathbb{C}$. In other words, $\{R_n\}$ forms an algebraic basis in the algebra of symmetric continuous polynomials on $L_\infty[0,1]$.

Theorem 2. ([41] (Theorem 3.1)) *For every sequence $\xi = \{\xi_n\}_{n=1}^\infty \subset \mathbb{C}$ such that the sequence $\{\sqrt[n]{|\xi_n|}\}_{n=1}^\infty$ is bounded, there exists $x_\xi \in L_\infty[0,1]$ such that $R_n(x_\xi) = \xi_n$ for every $n \in \mathbb{N}$ and*

$$\|x_\xi\|_\infty \le \frac{2}{M} \sup_{n \in \mathbb{N}} \sqrt[n]{|\xi_n|},$$

where

$$M = \prod_{k=1}^\infty \cos\left(\frac{\pi}{2} \cdot \frac{1}{k+1}\right). \tag{3}$$

Let $H_{bs}(L_\infty[0,1])$ be the subalgebra of the Fréchet algebra $H_b(L_\infty[0,1])$, which consists of all symmetric elements of $H_b(L_\infty[0,1])$. It can be checked that $H_{bs}(L_\infty[0,1])$ is closed in $H_b(L_\infty[0,1])$.

For every function $f \in H_{bs}(L_\infty[0,1])$, its Taylor series converges uniformly to f on every bounded set. The nth term, where $n \in \mathbb{N}$, of the Taylor series is a continuous n-homogeneous polynomial, which is symmetric by the symmetry of f and by the Cauchy Integral Equation (2). Therefore, by Theorem 1, every $f \in H_{bs}(L_\infty[0,1])$ can be represented as

$$f(x) = \alpha_0 + \sum_{n=1}^\infty \sum_{\substack{k_1 + 2k_2 + \ldots + nk_n = n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x) \cdots R_n^{k_n}(x) \tag{4}$$

where $\alpha_{k_1,\ldots,k_n} \in \mathbb{C}$, $x \in L_\infty[0,1]$, and the series converges uniformly on every bounded subset of $L_\infty[0,1]$.

2.7. Entire Symmetric Functions on $(L_1 \cap L_\infty)[0,+\infty)$

Let $L_1[0,+\infty)$ be the complex Banach space of all Lebesgue integrable functions $x : [0,+\infty) \to \mathbb{C}$ with norm $\|x\|_1 = \int_{[0,+\infty)} |x(t)|\,dt$. Let $L_\infty[0,+\infty)$ be the complex Banach space of all Lebesgue measurable essentially bounded functions $x : [0,+\infty) \to \mathbb{C}$ with norm

$$\|x\|_\infty = \operatorname{ess\,sup}_{t\in[0,+\infty)}|x(t)|.$$

Let us consider the space $(L_1 \cap L_\infty)[0,+\infty) := L_1[0,+\infty) \cap L_\infty[0,+\infty)$ with norm $\|x\| = \max\{\|x\|_1, \|x\|_\infty\}$. By [60] (p. 97, Thm. 1.3), this space is complete. For $n \in \mathbb{N}$, let us define $\hat{R}_n : (L_1 \cap L_\infty)[0,+\infty) \to \mathbb{C}$ by

$$\hat{R}_n(x) = \int_{[0,+\infty)} (x(t))^n\,dt.$$

For every $n \in \mathbb{N}$, $\hat{R}_n$ is a symmetric n-homogeneous polynomial and $\|\hat{R}_n\| = 1$.

Theorem 3. ([48] (Theorem 2)) *Every symmetric continuous n-homogeneous polynomial $P : (L_1 \cap L_\infty)[0,+\infty) \to \mathbb{C}$ can be uniquely represented as*

$$P(x) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n\in\mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} \hat{R}_1^{k_1}(x)\cdots\hat{R}_n^{k_n}(x),$$

where $\alpha_{k_1,\ldots,k_n} \in \mathbb{C}$.

By [49] (Thm. 2), Fréchet algebras $H_{bs}((L_1 \cap L_\infty)[0,+\infty))$ and $H_{bs}(L_\infty[0,1])$ are isomorphic.

3. The Main Result

Let $(\Omega_\gamma, \mathcal{F}_\gamma, \nu_\gamma)$ be a Lebesgue-Rohlin measure space with continuous measure for every $\gamma \in \Gamma$, where Γ is an arbitrary index set. Let $(\Omega, \mathcal{F}, \nu)$ be the disjoint union of all the spaces belonging to the set $\{(\Omega_\gamma, \mathcal{F}_\gamma, \nu_\gamma) : \gamma \in \Gamma\}$, i.e.,

$$\Omega = \bigsqcup_{\gamma\in\Gamma} \Omega_\gamma,$$

$$\mathcal{F} = \Big\{A \in \Omega : \quad A \cap \Omega_\gamma \in \mathcal{F}_\gamma \text{ for every } \gamma \in \Gamma\Big\}$$

and

$$\nu(A) = \sum_{\gamma\in\Gamma} \nu_\gamma(A \cap \Omega_\gamma)$$

for $A \in \mathcal{F}$. By Lemma 1, for every $\gamma \in \Gamma$ there exists an isomorphism w_γ between $(\Omega_\gamma, \mathcal{F}_\gamma, \nu_\gamma)$ and $[0,1]$ with Lebesgue measure. Therefore, for every $\gamma \in \Gamma$, the mapping $W_\gamma : L_\infty[0,1] \to L_\infty(\Omega_\gamma)$, defined by

$$W_\gamma(x) = x \circ w_\gamma \tag{5}$$

for $x \in L_\infty[0,1]$, is a linear isometrical bijection, where $L_\infty(\Omega_\gamma)$ is the complex Banach space of all complex-valued measurable essentially bounded functions on $(\Omega_\gamma, \mathcal{F}_\gamma, \nu_\gamma)$.

Let $(L_1 \cap L_\infty)(\Omega)$ be the complex Banach space of all measurable integrable essentially bounded functions $x : \Omega \to \mathbb{C}$ with norm

$$\|x\| = \max\{\|x\|_1, \|x\|_\infty\},$$

where

$$\|x\|_1 = \int_\Omega |x(t)|\,dt$$

and

$$\|x\|_\infty = \operatorname{ess\,sup}_{t\in\Omega}|x(t)|.$$

Lemma 2. *Let $\gamma \in \Gamma$. The mapping $J_\gamma : L_\infty(\Omega_\gamma) \to (L_1 \cap L_\infty)(\Omega)$, defined by*

$$J_\gamma(x)(t) = \begin{cases} x(t), & \text{if } t \in \Omega_\gamma, \\ 0, & \text{if } t \in \Omega \setminus \Omega_\gamma \end{cases} \tag{6}$$

for $x \in L_\infty(\Omega_\gamma)$, is a linear isometrical injective mapping. Consequently, $L_\infty(\Omega_\gamma)$ can be considered as a subspace of $(L_1 \cap L_\infty)(\Omega)$.

Proof. Clearly, J_γ is linear and injective. Let us show that J_γ is isometrical. Let $x \in L_\infty(\Omega_\gamma)$. Note that $\|J_\gamma(x)\|_\infty = \|x\|_\infty$. Since $\nu_\gamma(\Omega_\gamma) = 1$, it follows that

$$\|J_\gamma(x)\|_1 = \int_\Omega |J_\gamma(x)(t)|\,dt = \int_{\Omega_\gamma} |x(t)|\,dt \le \|x\|_\infty.$$

Therefore,

$$\|J_\gamma(x)\| = \max\{\|J_\gamma(x)\|_1, \|J_\gamma(x)\|_\infty\} = \|x\|_\infty.$$

Hence, J_γ is an isometrical mapping. □

For every $E \subset \Omega$, let

$$1_E(t) = \begin{cases} 1, & \text{if } t \in E, \\ 0, & \text{if } t \in \Omega \setminus E. \end{cases}$$

For $n \in \mathbb{N}$, let the polynomial $\tilde{R}_n : (L_1 \cap L_\infty)(\Omega) \to \mathbb{C}$ be defined by

$$\tilde{R}_n(x) = \int_\Omega (x(t))^n\,dt.$$

The symmetry and the n-homogeneity of the polynomial $\tilde{R}_n$, for every $n \in \mathbb{N}$, can be easily verified. Let us prove the continuity of $\tilde{R}_n$.

Lemma 3. *For every $n \in \mathbb{N}$,*

$$\|\tilde{R}_n\| = 1$$

and, consequently, $\tilde{R}_n$ is continuous.

Proof. Let us show that $\|\tilde{R}_n\| = 1$. Let $x \in (L_1 \cap L_\infty)(\Omega)$ be such that $\|x\| \le 1$. Then $\|x\|_1 \le 1$ and $\|x\|_\infty \le 1$. Since $\|x\|_\infty \le 1$, it follows that $|x(t)| \le 1$ for almost all $t \in \Omega$. Consequently, $|x(t)|^n \le |x(t)|$ for almost all $t \in \Omega$. Therefore,

$$|\tilde{R}_n(x)| \le \int_\Omega |x(t)|^n\,dt \le \int_\Omega |x(t)|\,dt = \|x\|_1 \le 1.$$

Hence,

$$\|\tilde{R}_n\| = \sup_{\|x\|\le 1} |\tilde{R}_n(x)| \le 1.$$

On the other hand, for an arbitrary fixed $\gamma \in \Gamma$, we have $\|1_{\Omega_\gamma}\| = 1$ and $\tilde{R}_n(1_{\Omega_\gamma}) = 1$. Therefore, $\|\tilde{R}_n\| = 1$. Consequently, $\tilde{R}_n$ is continuous. □

Theorem 4. *Every symmetric continuous n-homogeneous polynomial $P : (L_1 \cap L_\infty)(\Omega) \to \mathbb{C}$ can be uniquely represented as*

$$P(x) = \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} \tilde{R}_1^{k_1}(x) \cdots \tilde{R}_n^{k_n}(x),$$

where $k_1, \ldots, k_n \in \mathbb{Z}_+$ and $\alpha_{k_1,\ldots,k_n} \in \mathbb{C}$.

Proof. For $\gamma \in \Gamma$, let $Q_\gamma : L_\infty[0,1] \to \mathbb{C}$ be defined by

$$Q_\gamma = P \circ J_\gamma \circ W_\gamma, \tag{7}$$

where W_γ and J_γ are defined by (5) and (6), respectively. We have the following diagram:

$$L_\infty[0,1] \xrightarrow{W_\gamma} L_\infty(\Omega_\gamma) \xrightarrow{J_\gamma} (L_1 \cap L_\infty)(\Omega) \xrightarrow{P} \mathbb{C}.$$

Since W_γ and J_γ are linear continuous mappings and P is a continuous n-homogeneous polynomial, it follows that $P \circ J_\gamma$ and Q_γ are continuous n-homogeneous polynomials.

Let us prove that $P \circ J_\gamma$ is a symmetric polynomial on $L_\infty(\Omega_\gamma)$. Let $x \in L_\infty(\Omega_\gamma)$ and $\sigma \in \Xi_{\Omega_\gamma}$. Let us show that $(P \circ J_\gamma)(x \circ \sigma) = (P \circ J_\gamma)(x)$. Note that

$$J_\gamma(x \circ \sigma) = J_\gamma(x) \circ \sigma',$$

where $\sigma' : \Omega \to \Omega$ is defined by

$$\sigma'(t) = \begin{cases} \sigma(t), & \text{if } t \in \Omega_\gamma, \\ t, & \text{if } t \in \Omega \setminus \Omega_\gamma. \end{cases}$$

It can be easily checked that $\sigma' \in \Xi_\Omega$. Since P is symmetric, it follows that

$$(P \circ J_\gamma)(x \circ \sigma) = P(J_\gamma(x) \circ \sigma') = P(J_\gamma(x)) = (P \circ J_\gamma)(x).$$

Thus, $P \circ J_\gamma$ is symmetric.

Let us prove that Q_γ is symmetric. Let $x \in L_\infty[0,1]$ and $\tau \in \Xi_{[0,1]}$. By (7),

$$Q_\gamma(x \circ \tau) = (P \circ J_\gamma)(W_\gamma(x \circ \tau)).$$

By (5), $W_\gamma(x \circ \tau) = x \circ \tau \circ w_\gamma$. Therefore

$$Q_\gamma(x \circ \tau) = (P \circ J_\gamma)(x \circ \tau \circ w_\gamma).$$

Note that $x \circ \tau \circ w_\gamma = x \circ w_\gamma \circ w_\gamma^{-1} \circ \tau \circ w_\gamma$. We have the following diagram:

$$\Omega_\gamma \xrightarrow{w_\gamma} [0,1] \xrightarrow{\tau} [0,1] \xrightarrow{w_\gamma^{-1}} \Omega_\gamma \xrightarrow{w_\gamma} [0,1] \xrightarrow{x} \mathbb{C}.$$

Since w_γ and τ are isomorphisms, it follows that $w_\gamma^{-1} \circ \tau \circ w_\gamma \in \Xi_{\Omega_\gamma}$. Since $P \circ J_\gamma$ is symmetric, it follows that

$$(P \circ J_\gamma)(x \circ \tau \circ w_\gamma) = (P \circ J_\gamma)\big(x \circ w_\gamma \circ (w_\gamma^{-1} \circ \tau \circ w_\gamma)\big) = (P \circ J_\gamma)(x \circ w_\gamma).$$

By (5) and (7), $(P \circ J_\gamma)(x \circ w_\gamma) = Q_\gamma(x)$. Therefore $Q_\gamma(x \circ \tau) = Q_\gamma(x)$. Hence, Q_γ is symmetric.

Let us prove that Q_γ does not depend on γ. Let $\gamma_1, \gamma_2 \in \Gamma$ be such that $\gamma_1 \neq \gamma_2$. Let us show that $Q_{\gamma_1} \equiv Q_{\gamma_2}$. Let $x \in L_\infty[0,1]$. By (5) and (7),

$$Q_{\gamma_1}(x) = P(J_{\gamma_1}(x \circ w_{\gamma_1})). \tag{8}$$

Let $\sigma_{\gamma_1\gamma_2} : \Omega \to \Omega$ be defined by

$$\sigma_{\gamma_1\gamma_2}(t) = \begin{cases} (w_{\gamma_2}^{-1} \circ w_{\gamma_1})(t), & \text{if } t \in \Omega_{\gamma_1}, \\ (w_{\gamma_1}^{-1} \circ w_{\gamma_2})(t), & \text{if } t \in \Omega_{\gamma_2}, \\ t, & \text{if } t \in \Omega \setminus (\Omega_{\gamma_1} \cap \Omega_{\gamma_2}). \end{cases}$$

Since w_{γ_1} and w_{γ_2} are isomorphisms, it follows that $\sigma_{\gamma_1\gamma_2} \in \Xi_\Omega$. Since P is symmetric, it follows that

$$P(J_{\gamma_1}(x \circ w_{\gamma_1})) = P(J_{\gamma_1}(x \circ w_{\gamma_1}) \circ \sigma_{\gamma_1\gamma_2}). \tag{9}$$

Let us show that

$$J_{\gamma_1}(x \circ w_{\gamma_1}) \circ \sigma_{\gamma_1\gamma_2} = J_{\gamma_2}(x \circ w_{\gamma_2}).$$

If $t \in \Omega_{\gamma_2}$, then $\sigma_{\gamma_1\gamma_2}(t) = (w_{\gamma_1}^{-1} \circ w_{\gamma_2})(t)$. In this case, $\sigma_{\gamma_1\gamma_2}(t) \in \Omega_{\gamma_1}$,; therefore, by (6),

$$\big(J_{\gamma_1}(x \circ w_{\gamma_1})\big)\big(\sigma_{\gamma_1\gamma_2}(t)\big) = \big(x \circ w_{\gamma_1}\big)\big((w_{\gamma_1}^{-1} \circ w_{\gamma_2})(t)\big) = (x \circ w_{\gamma_2})(t).$$

If $t \in \Omega \setminus \Omega_{\gamma_2}$, then $\sigma_{\gamma_1\gamma_2}(t) \in \Omega \setminus \Omega_{\gamma_1}$,; therefore, by (6),

$$\big(J_{\gamma_1}(x \circ w_{\gamma_1})\big)\big(\sigma_{\gamma_1\gamma_2}(t)\big) = 0.$$

Thus,

$$\big(J_{\gamma_1}(x \circ w_{\gamma_1})\big)\big(\sigma_{\gamma_1\gamma_2}(t)\big) = \begin{cases} (x \circ w_{\gamma_2})(t), & \text{if } t \in \Omega_{\gamma_2}, \\ 0, & \text{if } t \in \Omega \setminus \Omega_{\gamma_2}, \end{cases}$$

that is,

$$\big(J_{\gamma_1}(x \circ w_{\gamma_1})\big)\big(\sigma_{\gamma_1\gamma_2}(t)\big) = (J_{\gamma_2}(x \circ w_{\gamma_2}))(t)$$

for every $t \in \Omega$. Therefore,

$$J_{\gamma_1}(x \circ w_{\gamma_1}) \circ \sigma_{\gamma_1\gamma_2} = J_{\gamma_2}(x \circ w_{\gamma_2}). \tag{10}$$

Consequently, by (8)–(10),

$$Q_{\gamma_1}(x) = P(J_{\gamma_2}(x \circ w_{\gamma_2})) = Q_{\gamma_2}(x).$$

Therefore, $Q_{\gamma_1} \equiv Q_{\gamma_2}$.

Since Q_γ is a continuous n-homogeneous symmetric polynomial on $L_\infty[0,1]$, by Theorem 1, Q_γ can be uniquely represented as

$$Q(x) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x) \cdots R_n^{k_n}(x), \tag{11}$$

where $x \in L_\infty[0,1]$ and $\alpha_{k_1,\ldots,k_n} \in \mathbb{C}$.

Recall that for every index $\gamma \in \Gamma$, the mapping w_γ is an isomorphism between $(\Omega_\gamma, \mathcal{F}_\gamma, \nu_\gamma)$ and $[0,1]$ with Lebesgue measure μ. For every index $\gamma \in \Gamma$, let us construct the isomorphism w'_γ between $(\Omega_\gamma, \mathcal{F}_\gamma, \nu_\gamma)$ and $[0,1)$ with Lebesgue measure. Choose a countable set $M \subset \Omega_\gamma$ such that $w_\gamma^{-1}(1) \in M$. Let $N = w_\gamma(M) \setminus \{1\}$. Since the mapping w_γ is a bijection, the set N is countable. Since measures ν_γ and μ are continuous, the sets M and N are null sets. Let $h : M \to N$ be an arbitrary bijection. Let us define the mapping $w'_\gamma : \Omega_\gamma \to [0,1)$ by

$$w'_\gamma(t) = \begin{cases} w_\gamma(t), & \text{if } t \in \Omega_\gamma \setminus M, \\ h(t), & \text{if } t \in M. \end{cases}$$

It can be checked that the mapping w'_γ is an isomorphism between $(\Omega_\gamma, \mathcal{F}_\gamma, \nu_\gamma)$ and $[0,1)$ with Lebesgue measure.

Let $\{\gamma_n\}_{n=1}^\infty \in \Gamma$ be an arbitrary sequence of pairwise distinct indexes. Let us define the mapping $v_{\{\gamma_n\}_{n=1}^\infty} : \bigsqcup_{n=1}^{\infty} \Omega_{\gamma_n} \to [0,+\infty)$ by

$$v_{\{\gamma_n\}_{n=1}^\infty}(t) = w'_k(t) + k - 1$$

for $t \in \Omega_{\gamma_k}$, $k \in \mathbb{N}$. Note that the mapping $v_{\{\gamma_n\}_{n=1}^\infty}$ is an isomorphism between $\bigsqcup_{n=1}^{\infty} \Omega_{\gamma_n}$ and $[0, +\infty)$.

Let us define the mapping $V_{\{\gamma_n\}_{n=1}^\infty} : (L_1 \cap L_\infty)[0, +\infty) \to (L_1 \cap L_\infty)\left(\bigsqcup_{n=1}^{\infty} \Omega_{\gamma_n}\right)$ by

$$V_{\{\gamma_n\}_{n=1}^\infty}(x) = x \circ v_{\{\gamma_n\}_{n=1}^\infty},$$

where $x \in (L_1 \cap L_\infty)[0, +\infty)$. Since the mapping $v_{\{\gamma_n\}_{n=1}^\infty}$ is an isomorphism, it follows that the mapping $V_{\{\gamma_n\}_{n=1}^\infty}$ is a linear isometric bijection.

Let us define the mapping $I_{\{\gamma_n\}_{n=1}^\infty} : (L_1 \cap L_\infty)\left(\bigsqcup_{n=1}^{\infty} \Omega_{\gamma_n}\right) \to (L_1 \cap L_\infty)(\Omega)$ by

$$I_{\{\gamma_n\}_{n=1}^\infty}(x)(t) = \begin{cases} x(t), & \text{if } t \in \bigsqcup_{n=1}^{\infty} \Omega_{\gamma_n}, \\ 0, & \text{if } t \in \Omega \setminus \bigsqcup_{n=1}^{\infty} \Omega_{\gamma_n}, \end{cases}$$

where $x \in (L_1 \cap L_\infty)\left(\bigsqcup_{n=1}^{\infty} \Omega_{\gamma_n}\right)$. It can be checked that the mapping $I_{\{\gamma_n\}_{n=1}^\infty}$ is linear, isometric and injective.

Since mappings $V_{\{\gamma_n\}_{n=1}^\infty}$ and $I_{\{\gamma_n\}_{n=1}^\infty}$ are linear and continuous, and the mapping P is a continuous n-homogeneous polynomial, it follows that the mapping $P \circ I_{\{\gamma_n\}_{n=1}^\infty} \circ V_{\{\gamma_n\}_{n=1}^\infty}$ is a continuous n-homogeneous polynomial. It can be checked that $P \circ I_{\{\gamma_n\}_{n=1}^\infty} \circ V_{\{\gamma_n\}_{n=1}^\infty}$ is symmetric. Therefore, by Theorem 3, $P \circ I_{\{\gamma_n\}_{n=1}^\infty} \circ V_{\{\gamma_n\}_{n=1}^\infty}$ can be uniquely represented in the form

$$\left(P \circ I_{\{\gamma_n\}_{n=1}^\infty} \circ V_{\{\gamma_n\}_{n=1}^\infty}\right)(x) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \beta_{k_1,\ldots,k_n} \hat{R}_1^{k_1}(x) \cdots \hat{R}_n^{k_n}(x),$$

where $x \in (L_1 \cap L_\infty)[0, +\infty)$ and $\beta_{k_1,\ldots,k_n} \in \mathbb{C}$. Since the mapping $V_{\{\gamma_n\}_{n=1}^\infty}$ is an isomorphism, it follows that

$$\begin{aligned} \left(P \circ I_{\{\gamma_n\}_{n=1}^\infty}\right)(y) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \beta_{k_1,\ldots,k_n} \times \\ \left(\left(\hat{R}_1 \circ V^{-1}_{\{\gamma_n\}_{n=1}^\infty}\right)(y)\right)^{k_1} \cdots \left(\left(\hat{R}_n \circ V^{-1}_{\{\gamma_n\}_{n=1}^\infty}\right)(y)\right)^{k_n} \end{aligned} \tag{12}$$

for every $y \in (L_1 \cap L_\infty)\left(\bigsqcup_{n=1}^{\infty} \Omega_{\gamma_n}\right)$. Let us show that coefficients $\beta_{k_1,\ldots,k_n}$ coincide with respective coefficients $\alpha_{k_1,\ldots,k_n}$, obtained in (11). Let us define the mapping $T : L_\infty(\Omega_{\gamma_1}) \to (L_1 \cap L_\infty)\left(\bigsqcup_{n=1}^{\infty} \Omega_{\gamma_n}\right)$ by

$$T(x)(t) = \begin{cases} x(t), & \text{if } t \in \Omega_{\gamma_1} \\ 0, & \text{if } t \in \bigsqcup_{n=2}^{\infty} \Omega_{\gamma_n}, \end{cases}$$

where $x \in L_\infty(\Omega_{\gamma_1})$. It can be verified that the mapping T is linear, isometric and injective. We have the following diagram:

$$L_\infty[0,1] \xrightarrow{W_{\gamma_1}} L_\infty(\Omega_{\gamma_1}) \xrightarrow{T} (L_1 \cap L_\infty)\left(\bigsqcup_{n=1}^{\infty} \Omega_{\gamma_n}\right) \xrightarrow{I_{\{\gamma_n\}_{n=1}^\infty}}$$

$$\xrightarrow{I_{\{\gamma_n\}_{n=1}^\infty}} (L_1 \cap L_\infty)(\Omega) \xrightarrow{P} \mathbb{C}.$$

By (12),

$$\left(P \circ I_{\{\gamma_n\}_{n=1}^\infty} \circ T \circ W_{\gamma_1}\right)(x) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \beta_{k_1,\ldots,k_n} \times \\ \left(\left(\hat{R}_1 \circ V^{-1}_{\{\gamma_n\}_{n=1}^\infty} \circ T \circ W_{\gamma_1}\right)(x)\right)^{k_1} \cdots \left(\left(\hat{R}_n \circ V^{-1}_{\{\gamma_n\}_{n=1}^\infty} \circ T \circ W_{\gamma_1}\right)(x)\right)^{k_n} \tag{13}$$

for every $x \in L_\infty[0,1]$. Taking into account that $P \circ I_{\{\gamma_n\}_{n=1}^\infty} \circ T \circ W_{\gamma_1} = Q_{\gamma_1}$ and $\tilde{R}_j \circ V^{-1}_{\{\gamma_n\}_{n=1}^\infty} \circ T \circ W_{\gamma_1} = R_j$ for every $j \in \mathbb{N}$, by (13),

$$Q_\gamma(x) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \beta_{k_1,\ldots,k_n} R_1^{k_1}(x) \cdots R_n^{k_n}(x)$$

for every $x \in L_\infty[0,1]$. By the uniqueness of the representation (11), we obtain the equality $\beta_{k_1,\ldots,k_n} = \alpha_{k_1,\ldots,k_n}$ for every $k_1,\ldots,k_n \in \mathbb{Z}_+$ such that $k_1 + 2k_2 + \ldots + nk_n = n$. Therefore, by (12),

$$\left(P \circ I_{\{\gamma_n\}_{n=1}^\infty}\right)(y) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} \times \\ \left(\left(\hat{R}_1 \circ V^{-1}_{\{\gamma_n\}_{n=1}^\infty}\right)(y)\right)^{k_1} \cdots \left(\left(\hat{R}_n \circ V^{-1}_{\{\gamma_n\}_{n=1}^\infty}\right)(y)\right)^{k_n},$$

for every $y \in (L_1 \cap L_\infty)\left(\bigsqcup_{n=1}^\infty \Omega_{\gamma_n}\right)$. Consequently, for every $z \in (L_1 \cap L_\infty)(\Omega)$, which belongs to $I_{\{\gamma_n\}_{n=1}^\infty}\left((L_1 \cap L_\infty)\left(\bigsqcup_{n=1}^\infty \Omega_{\gamma_n}\right)\right)$,

$$P(z) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} \times \\ \left(\left(\hat{R}_1 \circ V^{-1}_{\{\gamma_n\}_{n=1}^\infty} \circ I^{-1}_{\{\gamma_n\}_{n=1}^\infty}\right)(z)\right)^{k_1} \cdots \left(\left(\hat{R}_n \circ V^{-1}_{\{\gamma_n\}_{n=1}^\infty} \circ I^{-1}_{\{\gamma_n\}_{n=1}^\infty}\right)(z)\right)^{k_n}.$$

Taking into account that

$$\left(\hat{R}_j \circ V^{-1}_{\{\gamma_n\}_{n=1}^\infty} \circ I^{-1}_{\{\gamma_n\}_{n=1}^\infty}\right)(z) = \tilde{R}_j(z)$$

for every $j \in \mathbb{N}$,

$$P(z) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} \left(\tilde{R}_1(z)\right)^{k_1} \cdots \left(\tilde{R}_n(z)\right)^{k_1} \tag{14}$$

for every $z \in (L_1 \cap L_\infty)(\Omega)$, which belongs to $I_{\{\gamma_n\}_{n=1}^\infty}\left((L_1 \cap L_\infty)\left(\bigsqcup_{n=1}^\infty \Omega_{\gamma_n}\right)\right)$. As we can see, coefficients in this equality do not depend on the choice of the sequence of indexes $\{\gamma_n\}_{n=1}^\infty$.

Let us show that the equality (14) holds for every $z \in (L_1 \cap L_\infty)(\Omega)$. Let z be an arbitrary element of the space $(L_1 \cap L_\infty)(\Omega)$. Since

$$\|z\|_1 = \int_\Omega |z(t)|\, dt = \sum_{\gamma \in \Gamma} \int_{\Omega_\gamma} |z(t)|\, dt$$

is finite, there exists not more than a countable set of indexes $\gamma \in \Gamma$ such that $\int_{\Omega_\gamma} |z(t)|\, dt > 0$. So, there exists a sequence of pairwise distinct indexes $\{\gamma_n\}_{n=1}^\infty \subset \Gamma$ such that $z = 0$ a. e. on the set Ω_γ for every index $\gamma \in \Gamma \setminus \{\gamma_n\}_{n=1}^\infty$. Therefore, $z \in I_{\{\gamma_n\}_{n=1}^\infty}\left((L_1 \cap L_\infty)\left(\bigsqcup_{n=1}^\infty \Omega_{\gamma_n}\right)\right)$. Consequently, for the element z the equality (14) holds. This completes the proof. □

Theorem 4 and the Cauchy Integral Equation (2) imply the following corollary.

Corollary 1. *Every function $f \in H_{bs}((L_1 \cap L_\infty)(\Omega))$ can be uniquely represented in the form*

$$f = \alpha_0 + \sum_{n=1}^{\infty} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} \tilde{R}_1^{k_1} \cdots \tilde{R}_n^{k_n},$$

where $\alpha_{k_1 k_2 \ldots k_n} \in \mathbb{C}$, and the series converges uniformly on bounded sets.

Lemma 4. *For every $y \in (L_1 \cap L_\infty)(\Omega)$, there exists $x_y \in L_\infty[0,1]$ such that $\tilde{R}_n(y) = R_n(x_y)$ for every $n \in \mathbb{N}$ and the following estimate holds:*

$$\|x_y\|_\infty \le \frac{2}{M}\|y\|, \tag{15}$$

where M is defined by (3).

Proof. Consider the sequence $c = \{c_n\}_{n=1}^\infty$, where $c_n = \tilde{R}_n(y)$ for $n \in \mathbb{N}$. Since $\tilde{R}_n$ is an n-homogeneous polynomial and $\|\tilde{R}_n\| = 1$, by (1),

$$|\tilde{R}_n(y)| \le \|y\|^n$$

for every $n \in \mathbb{N}$. Consequently,

$$\sup_{n\in\mathbb{N}} \sqrt[n]{|c_n|} \le \|y\| < \infty.$$

Therefore, by Theorem 2, there exists $x_c \in L_\infty[0,1]$ such that $R_n(x_c) = c_n$ for every $n \in \mathbb{N}$ and

$$\|x_c\|_\infty \le \frac{2}{M} \sup_{n\in\mathbb{N}} \sqrt[n]{|c_n|} \le \frac{2}{M}\|y\|,$$

where M is defined by (3). We set $x_y := x_c$. This completes the proof. □

Let us define the mapping $J : H_{bs}(L_\infty[0,1]) \to H_{bs}((L_1 \cap L_\infty)(\Omega))$ in the following way. Let $f \in H_{bs}(L_\infty[0,1])$. Then f can be uniquely represented in the form (4), that is,

$$f = \alpha_0 + \sum_{n=1}^{\infty} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} R_1^{k_1} \cdots R_n^{k_n}. \tag{16}$$

Let

$$J(f) = \alpha_0 + \sum_{n=1}^{\infty} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} \tilde{R}_1^{k_1} \cdots \tilde{R}_n^{k_n}. \tag{17}$$

Let us show that $J(f) \in H_{bs}((L_1 \cap L_\infty)(\Omega))$.

Proposition 5. *$J(f) \in H_{bs}((L_1 \cap L_\infty)(\Omega))$ for every $f \in H_{bs}(L_\infty[0,1])$ and*

$$\|J(f)\|_r \le \|f\|_{\frac{2}{M}r} \tag{18}$$

for every $r > 0$, where M is defined by (3).

Proof. By Lemma 4, for every $y \in (L_1 \cap L_\infty)(\Omega)$ there exists $x_y \in L_\infty[0,1]$ such that

$$\tilde{R}_n(y) = R_n(x_y) \tag{19}$$

for every $n \in \mathbb{N}$ and the inequality (15) holds. By (16), (17) and (19),

$$J(f)(y) = f(x_y) \tag{20}$$

for every $f \in H_{bs}(L_\infty[0,1])$ and $y \in (L_1 \cap L_\infty)(\Omega)$. By (15) and (20),

$$\begin{aligned} \|J(f)\|_r &= \sup\{|J(f)(y)| : \ y \in (L_1 \cap L_\infty)(\Omega) \text{ such that } \|y\| \le r\} \\ &= \sup\{|f(x_y)| : \ y \in (L_1 \cap L_\infty)(\Omega) \text{ such that } \|y\| \le r\} \\ &\le \sup\Big\{|f(x)| : \ x \in L_\infty[0,1] \text{ such that } \|x\|_\infty \le \frac{2}{M} r\Big\} \\ &\le \|f\|_{\frac{2}{M} r} \end{aligned} \tag{21}$$

for every $f \in H_{bs}(L_\infty[0,1])$ and $r > 0$. Thus, we have proved (18).

Let $f \in H_{bs}(L_\infty[0,1])$. Let us show that $J(f) \in H_{bs}((L_1 \cap L_\infty)(\Omega))$. The inequality (21) and the fact that f is the function of bounded type imply the fact that $J(f)$ is the function of bounded type. By (17) and by the symmetry of $\tilde{R}_n$, the function $J(f)$ is symmetric. Let us show that $J(f)$ is entire. By Proposition 4,

$$\limsup_{n\to\infty} \|P_n\|_1^{1/n} = 0, \tag{22}$$

where $P_0 = \alpha_0$ and

$$P_n = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} R_1^{k_1} \cdots R_n^{k_n}$$

for $n \in \mathbb{N}$. Consider the series

$$\sum_{n=0}^{\infty} \tilde{P}_n, \tag{23}$$

where $\tilde{P}_0 = \alpha_0$ and

$$\tilde{P}_n = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} \tilde{R}_1^{k_1} \cdots \tilde{R}_n^{k_n}$$

for $n \in \mathbb{N}$. Note that $\tilde{P}_n = J(P_n)$; therefore, by (21),

$$\|\tilde{P}_n\|_1 \le \|P_n\|_{\frac{2}{M}}$$

for every $n \in \mathbb{N}$. By the n-homogeneity of the polynomial P_n,

$$\|P_n\|_{\frac{2}{M}} = \sup_{\|x\| \le \frac{2}{M}} |P_n(x)| = \sup_{\|x\| \le 1} \left| P_n\Big(\frac{2}{M} x\Big) \right| = \Big(\frac{2}{M}\Big)^n \sup_{\|x\| \le 1} |P_n(x)| = \Big(\frac{2}{M}\Big)^n \|P_n\|_1.$$

Therefore,

$$\|\tilde{P}_n\|_1 \le \Big(\frac{2}{M}\Big)^n \|P_n\|_1. \tag{24}$$

By (22) and (24),

$$0 \le \limsup_{n\to\infty} \|\tilde{P}_n\|_1^{1/n} \le \frac{2}{M} \limsup_{n\to\infty} \|P_n\|_1^{1/n} = 0.$$

Therefore,

$$\limsup_{n\to\infty} \|\tilde{P}_n\|_1^{1/n} = 0$$

and, consequently, by Proposition 4, the series (23) converges to some entire function on the space $(L_1 \cap L_\infty)(\Omega)$ with the infinite radius of boundedness. By (17), this function is $J(f)$. Consequently, $J(f)$ is an entire function of bounded type. Thus, $J(f) \in H_{bs}((L_1 \cap L_\infty)(\Omega))$. $\square$

Theorem 5. *The mapping J, defined by (17), is an isomorphism of Fréchet algebras $H_{bs}(L_\infty[0,1])$ and $H_{bs}((L_1 \cap L_\infty)(\Omega))$.*

Proof. Let us show that J is linear. Let $f, g \in H_{bs}(L_\infty[0,1])$. Then functions f and g can be uniquely represented as

$$f = \alpha_0 + \sum_{n=1}^{\infty} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n\in\mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} R_1^{k_1} \cdots R_n^{k_n},$$

$$g = \beta_0 + \sum_{n=1}^{\infty} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n\in\mathbb{Z}_+}} \beta_{k_1,\ldots,k_n} R_1^{k_1} \cdots R_n^{k_n}$$

respectively. Let $\lambda \in \mathbb{C}$. Note that

$$\lambda f = \lambda\alpha_0 + \sum_{n=1}^{\infty} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n\in\mathbb{Z}_+}} \lambda\alpha_{k_1,\ldots,k_n} R_1^{k_1} \cdots R_n^{k_n}$$

and

$$f + g = \alpha_0 + \beta_0 + \sum_{n=1}^{\infty} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n\in\mathbb{Z}_+}} (\alpha_{k_1,\ldots,k_n} + \beta_{k_1,\ldots,k_n}) R_1^{k_1} \cdots R_n^{k_n}.$$

Therefore,

$$J(\lambda f) = \lambda\alpha_0 + \sum_{n=1}^{\infty} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n\in\mathbb{Z}_+}} \lambda\alpha_{k_1,\ldots,k_n} \tilde{R}_1^{k_1} \cdots \tilde{R}_n^{k_n} = \lambda J(f)$$

and

$$J(f+g) = \alpha_0 + \beta_0 + \sum_{n=1}^{\infty} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n\in\mathbb{Z}_+}} (\alpha_{k_1,\ldots,k_n} + \beta_{k_1,\ldots,k_n}) \tilde{R}_1^{k_1} \cdots \tilde{R}_n^{k_n} = J(f) + J(g).$$

Thus, J is linear.

Let us show that J is continuous. Since J is a linear mapping between Fréchet algebras, it follows that for J the continuity and the boundedness are equivalent. In turn, the boundedness of J follows from (18). Thus, J is continuous.

Let us show that J is multiplicative. By (17),

$$J(R_1^{k_1} \cdots R_n^{k_n}) = \tilde{R}_1^{k_1} \cdots \tilde{R}_n^{k_n} \tag{25}$$

for every $n \in \mathbb{N}$ and $k_1, \ldots, k_n \in \mathbb{Z}_+$. As a consequence of Theorem 1, every symmetric continuous polynomial $P : L_\infty[0,1] \to \mathbb{C}$ can be uniquely represented as

$$P = \alpha_0 + \sum_{n=1}^{N} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1} \cdots R_n^{k_n},$$

where $N \in \mathbb{N}$, $k_1, \ldots, k_n \in \mathbb{Z}_+$ and $\alpha_{k_1,\ldots,k_n} \in \mathbb{C}$. Therefore, since J is linear, taking into account (25),

$$J(P) = \alpha_0 + \sum_{n=1}^{N} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} \tilde{R}_1^{k_1} \cdots \tilde{R}_n^{k_n}. \tag{26}$$

By using (26), it can be verified the equality

$$J(P_1P_2) = J(P_1)J(P_2) \tag{27}$$

for arbitrary symmetric continuous polynomials $P_1, P_2 : L_\infty[0,1] \to \mathbb{C}$. Let $f, g \in H_{bs}(L_\infty[0,1])$. Let us show that $J(fg) = J(f)J(g)$. Let $f = \sum_{n=0}^{\infty} f_n$ and $g = \sum_{n=0}^{\infty} g_n$ be the Taylor series expansions of f and g respectively. Then

$$fg = \sum_{k=0}^{\infty} \sum_{s=0}^{k} f_s g_{k-s}.$$

Consequently, since J is linear and continuous, taking into account (26),

$$J(fg) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} J(f_s g_{k-s}) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} J(f_s)J(g_{k-s}) = \left(\sum_{n=0}^{\infty} J(f_n) \right) \left(\sum_{n=0}^{\infty} J(g_n) \right) = J(f)J(g).$$

Thus, J is multiplicative.

Let us show that J is a bijection. Let γ_0 be an arbitrary element of Γ. Let $v : L_\infty[0,1] \to (L_1 \cap L_\infty)(\Omega)$ be defined by

$$v = J_{\gamma_0} \circ W_{\gamma_0}, \tag{28}$$

where W_{γ_0} is defined by (5), and J_{γ_0} is defined by (6). Since W_{γ_0} is a linear isometrical bijection and J_{γ_0} is a linear isometrical injective mapping (by Lemma 2), it follows that v is a linear isometrical injective mapping. Therefore, for every $r > 0$, the image of the closed ball with the center at 0 and the radius r of the space $L_\infty[0,1]$ under v is a subset of the closed ball with the center at 0 and the radius r of the space $(L_1 \cap L_\infty)(\Omega)$. Therefore,

$$\begin{aligned} &\sup\{|g(v(x))| : \ x \in L_\infty[0,1], \ \|x\|_\infty \le r\} \\ &\qquad \le \sup\{|g(y)| : \ y \in (L_1 \cap L_\infty)(\Omega), \ \|y\| \le r\}. \end{aligned} \tag{29}$$

for every function of bounded type $g : (L_1 \cap L_\infty)(\Omega) \to \mathbb{C}$ and for every $r > 0$. Let us prove the following auxiliary statement.

Lemma 5. *For every function $f \in H_{bs}((L_1 \cap L_\infty)(\Omega))$, the function $f \circ v$ belongs to the Fréchet algebra $H_{bs}(L_\infty[0,1])$.*

Proof of Lemma 5. Let $f \in H_{bs}((L_1 \cap L_\infty)(\Omega))$. Since f is a function of bounded type, it follows that the value $\|f\|_r$ is finite for every $r > 0$. Therefore, by (29), the value $\|f \circ v\|_r$ is finite for every $r > 0$. Thus, the function $f \circ v$ is of bounded type.

Let us show that $f \circ v$ is symmetric. For every $\sigma \in \Xi_{[0,1]}$, let us define the function $\hat{\sigma} : \Omega \to \Omega$ by

$$\hat{\sigma}(t) = \begin{cases} (w_{\gamma_0}^{-1} \circ \sigma \circ w_{\gamma_0})(t), & \text{if } t \in \Omega_{\gamma_0} \\ t, & \text{if } t \in \Omega \setminus \Omega_{\gamma_0}. \end{cases}$$

It can be checked that $\hat{\sigma} \in \Xi_\Omega$ and $v(x \circ \sigma) = v(x) \circ \hat{\sigma}$ for every $x \in L_\infty[0,1]$. Therefore, taking into account the symmetry of f,

$$(f \circ v)(x \circ \sigma) = f(v(x) \circ \hat{\sigma}) = f(v(x)) = (f \circ v)(x)$$

for every $\sigma \in \Xi_{[0,1]}$ and $x \in L_\infty[0,1]$. Thus, $f \circ v$ is symmetric.

Let us show that $f \circ v$ is an entire function. Since the function f is an entire function of bounded type, its Taylor series, terms of which we denote by $f_0, f_1, \ldots, f_n, \ldots$, is uniformly convergent to f on every bounded subset of the space $(L_1 \cap L_\infty)(\Omega)$. By Proposition 4,

$$\limsup_{n\to\infty} \|f_n\|_1^{1/n} = 0.$$

Consider the series

$$\sum_{n=0}^{\infty} f_n \circ v. \tag{30}$$

By (29), $\|f_n \circ v\|_1 \le \|f_n\|_1$ for every $n \in \mathbb{N}$. Consequently,

$$\limsup_{n\to\infty} \|f_n \circ v\|_1^{1/n} = 0,$$

that is, the series (30) converges uniformly to some entire function of bounded type on every bounded subset of the space $L_\infty[0,1]$. Let us show that this function is equal to $f \circ v$. Since $\sum_{n=0}^{\infty} f_n$ converges uniformly to f on every bounded subset of $(L_1 \cap L_\infty)(\Omega)$, it follows that for every $\varepsilon > 0$ and $r > 0$ there exists $N \in \mathbb{N}$ such that

$$\left\| f - \sum_{n=0}^{m} f_n \right\|_r < \varepsilon$$

for every $m > N$. Therefore, by (29),

$$\left\| f \circ v - \sum_{n=0}^{m} f_n \circ v \right\|_r \le \left\| f - \sum_{n=0}^{m} f_n \right\|_r < \varepsilon,$$

where $m > N$. Thus, the series (30) converges uniformly to $f \circ v$ on every bounded subset of the space $L_\infty[0,1]$. Consequently, the function $f \circ v$ is entire. This completes the proof of Lemma 5. □

We now continue with the proof of Theorem 5. Let us show that J is surjective. Let g be an arbitrary element of $H_{bs}((L_1 \cap L_\infty)(\Omega))$. Then g can be represented in the form

$$g = \alpha_0 + \sum_{n=1}^{\infty} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} \tilde{R}_1^{k_1} \cdots \tilde{R}_n^{k_n}. \tag{31}$$

Let $f = g \circ v$. By Lemma 5, $f \in H_{bs}(L_\infty[0,1])$. By (31),

$$f = \alpha_0 + \sum_{n=1}^{\infty} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} (\tilde{R}_1 \circ v)^{k_1} \cdots (\tilde{R}_n \circ v)^{k_n}.$$

Taking into account the equality $\tilde{R}_n \circ v = R_n$,

$$f = \alpha_0 + \sum_{n=1}^{\infty} \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_1,\ldots,k_n \in \mathbb{Z}_+}} \alpha_{k_1,\ldots,k_n} R_1^{k_1} \cdots R_n^{k_n}.$$

By (17), $J(f) = g$. Thus, the mapping J is surjective and

$$J(f) \circ v = f \tag{32}$$

for every $f \in H_{bs}(L_\infty[0,1])$.

Let us prove that J is injective. Recall that J is linear. For a linear mapping, the injectivity is equivalent to the fact that the image of every nonzero element is nonzero. Let

f be a nonzero element of $H_{bs}(L_\infty[0,1])$. Let us show that $J(f) \neq 0$. Suppose $J(f) = 0$. Then $J(f) \circ v = 0$. Therefore, by (32), $f = 0$., which is a contradiction. Thus, $J(f) \neq 0$. Consequently, J is injective. So, J is bijective.

By (18) and (29),

$$\|f\|_r \leq \|J(f)\|_r \leq \|f\|_{\frac{2}{M}r}$$

for every $f \in H_{bs}(L_\infty[0,1])$ and for every $r > 0$. This inequality implies the continuity of J and J^{-1}. This completes the proof of Theorem 5. □

4. Conclusions

This work is a significant generalization of the work [49]. We consider symmetric functions on Banach spaces of all complex-valued integrable essentially bounded functions on the unions of Lebesgue-Rohlin spaces with continuous measures. Note that there are a lot of important measure spaces which can be represented as the abovementioned union. For example, $\mathbb{R}^n$ for any $n \in \mathbb{N}$ with the Lebesgue measure is one such space. We investigate algebras of symmetric polynomials and entire symmetric functions on the abovementioned spaces. In particular, we show that Fréchet algebras of all complex-valued entire symmetric functions of bounded type on these Banach spaces are isomorphic to the Fréchet algebra of all complex-valued entire symmetric functions of bounded type on the complex Banach space $L_\infty[0,1]$.

The next step in this investigation is to consider the case of unions of arbitrary Lebesgue-Rohlin spaces.

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