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Geometry of Indefinite Kenmotsu Manifolds as $*\eta$-Ricci-Yamabe Solitons

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Abstract: In this paper, we study the properties of ϵ-Kenmotsu manifolds if its metrics are $*\eta$-Ricci-Yamabe solitons. It is proven that an ϵ-Kenmotsu manifold endowed with a $*\eta$-Ricci-Yamabe soliton is η-Einstein. The necessary conditions for an ϵ-Kenmotsu manifold, whose metric is a $*\eta$-Ricci-Yamabe soliton, to be an Einstein manifold are derived. Finally, we model an indefinite Kenmotsu manifold example of dimension 5 to examine the existence $*\eta$-Ricci-Yamabe solitons.

Keywords: indefinite Kenmotsu manifolds; Ricci solitons; Yamabe solitons; η-Einstein manifolds; cyclic parallel Ricci tensor; Codazzi-type Ricci tensor

MSC: 53C21; 53C25; 5350; 53E20

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1. Introduction

In 1969, Takahashi [1] introduced an almost contact manifold equipped with an associated indefinite metric and explored some geometrical properties of almost contact manifolds (particularly, Sasakian manifolds) with indefinite metrics [2]. Later on, in 1972, Kenmotsu established a new class of almost contact manifold known as Kenmotsu manifolds [3]. A Kenmotsu manifold admitting an indefinite metric is termed as an ϵ-Kenmotsu manifold, which was proposed by De and Sarkar [4] and its geometrical properties were studied by several researchers, for instance [5–9]. Since the index of a metric generates variety of vector fields such as space-like, time-like and light-like vector fields, therefore the study of indefinite structures on manifolds becomes very interesting and of great importance, which attracts the researchers from different research areas.

In response to his own work on Ricci flow, Hamilton [10] defined Yamabe flow on a Riemannian manifold M as:

$$r\,g + \frac{\partial}{\partial t}g = 0,$$

where $g(0) = g_0$; g, r and t denote the Riemannian metric, the scalar curvature of g and the time, respectively. Notice that the Yamabe flow coincides with the Ricci flow ($\frac{\partial}{\partial t}g + 2S = 0$) for $dim\, M = 2$, where S is the Ricci tensor of M, but in case of $dim M > 2$ they differ. Extending the notion of Ricci flow to a nonlinear PDE which involves the Riemann curvature tensor R, the Riemann flow ($\frac{\partial}{\partial t}G + 2R = 0$, where $G = \frac{1}{2}g \odot g$, for $\odot$ the Kulkarni–Nomizu product) has very similar properties to that of the Ricci flow [11]. If $n \geq 3$, then the Riemann flow of the type $\frac{\partial G}{\partial t} = \alpha R + \beta \frac{\partial \mathcal{G}}{\partial t}G$, where $\mathcal{G}$ be the determinant of the metric g, with $\alpha = 2(n-2)$ and $\beta = \frac{1}{n-1}$, determines a standard Ricci flow $\frac{\partial}{\partial t}g + 2S = 0$.

The Ricci, Yamabe and Riemann solitons correspond to self-similar solutions of the Ricci, Yamabe and Riemann flows and are given respectively by [12,13]

$$\pounds_V g + 2\lambda g + 2S = 0,$$

$$\pounds_V g = 2(r - \lambda)g,$$

and

$$\pounds_V g \odot g + 2R = 2\lambda G,$$

where $\pounds_V$ represents the Lie derivative operator along the smooth vector field V (called soliton vector field) on M, $\lambda \in \mathbb{R}$ (called soliton constant of M) and $\mathbb{R}$ represents the set of real numbers. Recently, Blaga [14] studied almost-Riemann solitons (V, λ) in a Riemannian manifold and stablished their relation to almost-Ricci solitons. For a solenoidal vector field V, the Riemann soliton on an n-dimensional Riemannian manifold $(M, g), n \geq 3$, the soliton $(\bar{V}, \bar{\lambda})$, where $\bar{V} = (n-2)V$ and $\bar{\lambda} = (n-1)\lambda$ defines a Ricci soliton.

As a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [15]. This notion has also been studied in [16] for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is a tuple (g, V, λ, μ) satisfying the equation

$$(\pounds_V g)(F_1, F_2) + 2S(F_1, F_2) + 2\lambda g(F_1, F_2) + 2\mu\eta(F_1) \otimes \eta(F_2) = 0,$$

$\forall\ F_1, F_2 \in \mathfrak{X}(M)$, where $\lambda, \mu \in \mathbb{R}$. Here $\mathfrak{X}(M)$ refers to the set of all smooth vector fields of M.

In [17], authors defined the notion of Ricci–Yamabe flow of type (α, β) on M as:

$$\frac{\partial}{\partial t} g(t) + \beta g(t) r(t) + 2\alpha S(g(t)) = 0, \quad g(0) = g_0,$$

for some scalars α and β on M.

A solution to the Ricci–Yamabe flow is called Ricci–Yamabe soliton in case it depends only on one parameter group of diffeomorphism and scaling. A Riemannian manifold M is said to have a Ricci–Yamabe soliton [18] if g satisfies

$$(\pounds_V g)(F_1, F_2) + 2\alpha S(F_1, F_2) - (\beta r - 2\lambda) g(F_1, F_2) = 0,$$

where $\alpha, \beta, \lambda \in \mathbb{R}$.

The Riemannian manifold M is said to have an η-Ricci–Yamabe soliton [19] if g satisfies

$$(\pounds_V g)(F_1, F_2) + 2\alpha S(F_1, F_2) - (\beta r - 2\lambda) g(F_1, F_2) + 2\mu\eta(F_1)\eta(F_2) = 0,$$

where $\pounds_V, \alpha, \beta, \lambda, \mu, r$ are defined earlier.

The above equation with $\mu = 0$ infers that M has a Ricci–Yamabe soliton of type (α, β). Note that Ricci–Yamabe solitons of type $(1, 0)$, $(\alpha, 0)$, $(0, 1)$ and $(0, \beta)$ are Ricci solitons, α-Ricci solitons, Yamabe solitons and β-Yamabe solitons, respectively.

In 1959, Tachibana [20], proposed the concept of $*$-Ricci tensor on almost-Hermitian manifolds, and this concept gained wide importance in the fields of mathematics and physics. Further, in the non-flat complex space forms, Hamada [21] defined and studied the $*$-Ricci tensor of real hypersurfaces, while in contact metric manifolds $*$-Ricci tensor was defined by Blair [22].

M is said to have a $*\eta$-Ricci–Yamabe soliton $(g, V, \lambda, \mu, \alpha, \beta)$ if the following equation holds [23]:

$$(\pounds_V g)(F_1, F_2) + 2\alpha S^*(F_1, F_2) + (2\lambda - \beta r) g(F_1, F_2) + 2\mu\eta(F_1)\eta(F_2) = 0, \tag{1}$$

where

$$S^*(F_1, F_2) = g(Q^* F_1, F_2) = Trace\{\phi \circ R(F_1, \phi F_2)\}.$$

Here ϕ is a tensor field of type $(1,1)$, S^* is the $*$-Ricci tensor, Q^* is the $*$-Ricci operator. $(g, V, \lambda, \alpha, \beta)$ is expanding, steady or shrinking if $\lambda > 0$, =0 or < 0, respectively. For more details (c.f., [24–36]).

In 1970, Pokhariyal and Mishra [37] first defined the W_2-curvature tensor and they studied its physical and geometrical properties. The W_2-curvature tensor possesses properties almost similar to the Weyl projective curvature tensor. Thus we can very well use W_2-curvature tensor in various physical and geometrical spheres in place of the Weyl projective curvature tensor. The W_2-curvature tensor has also been studied by various authors in different structures such as Mallick and De [38], Pokhariyal [39,40], Shaikh, Matsuyama and Jana [41], Zengin [42] and many others.

As a weaker notion of locally symmetric manifolds, Takahashi [43] introduced the notion of locally ϕ-Symmetric Sasakian manifolds. In 2008, De [44] studied ϕ-Symmetric Kenmotsu manifolds and obtained some interesting results of this manifold. Recently, the notion of ϕ-Ricci Symmetry was studied by Shukla and Shukla [45] in the context of Kenmotsu manifolds.

In this paper, we have studied the properties of ϵ-Kenmotsu manifolds with $*\eta$-Ricci–Yamabe solitons. Throughout the manuscript, we denote an n-dimensional ϵ-Kenmotsu manifold by $M^n(\epsilon)$. We arrange our work as follows: In Section 2, we have given some preliminary results and basic definitions of $M^n(\epsilon)$. Section 3 is concerned with the study $*\eta$-Ricci–Yamabe solitons on $M^n(\epsilon)$, and derives some interesting results of $M^n(\epsilon)$. Also, the study of $*\eta$-Ricci–Yamabe solitons in $M^n(\epsilon)$ admitting Codazzi-type and cyclic parallel Ricci tensors is illustrated in Section 3. Section 4 is concerned with the study of $\mathcal{W}_2$-curvature tensor satisfying certain conditions on $M^n(\epsilon)$ admitting $*\eta$-Ricci–Yamabe solitons. The geometrical properties of ϕ-Ricci symmetric ϵ-Kenmotsu manifolds admitting $*\eta$-Ricci–Yamabe solitons are studied in Section 5. Finally, we construct a non-trivial example of five-dimensional ϵ-Kenmotsu manifold to prove some of our results.

2. Preliminaries

An odd-dimensional manifold M of class C^∞ is termed as an ϵ-almost contact metric manifold [2] if there exist ϕ, ξ, η and g on M, respectively known as a tensor field of type $(1,1)$, $(1,0)$-type vector field, 1-form and an indefinite metric g, satisfying

$$\eta(\xi) = 1, \quad \phi^2 F_1 = -F_1 + \eta(F_1)\xi, \quad g(\xi, \xi) = \epsilon, \quad \eta(F_1) = \epsilon g(F_1, \xi), \tag{2}$$

$$g(\phi F_1, \phi F_2) = g(F_1, F_2) - \epsilon\eta(F_1)\eta(F_2), \ \forall F_1, F_2 \in \mathfrak{X}(M). \tag{3}$$

If the structure vector field ξ is timelike or spacelike, then $\epsilon = -1$ or $\epsilon = 1$, respectively. If the exterior derivative operator d of g satisfies $d\eta(F_1, F_2) = g(F_1, \phi F_2)$, then M becomes an ϵ-contact metric manifold, then (2) implies that

$$\phi\xi = 0, \ rank(\phi) = n-1, \ \eta(\phi F_1) = 0, \tag{4}$$

where $n = dimM$. Let

$$(\nabla_{F_1}\phi)(F_2) = -g(F_1, \phi F_2)\xi - \epsilon\eta(F_2)\phi F_1, \tag{5}$$

where ∇ refers to the Levi-Civita connection. Then the manifold M satisfying (5) is named as ϵ-Kenmotsu manifold of dimension n [44]. From the last equation, we infer that

$$\nabla_{F_1}\xi = \epsilon(F_1 - \eta(F_1)\xi). \tag{6}$$

From the above equations, we can easily conclude that $M^n(\epsilon)$ satisfies the following:

$$(\nabla_{F_1}\eta)F_2 = g(F_1, F_2) - \epsilon\eta(F_1)\eta(F_2), \tag{7}$$

$$R(F_1, F_2)\xi = \eta(F_1)F_2 - \eta(F_2)F_1, \tag{8}$$

$$R(\xi, F_1)F_2 = \eta(F_2)F_1 - \epsilon g(F_1, F_2)\xi, \tag{9}$$

$$R(F_1, \xi)\xi + F_1 - \eta(F_1)\xi = 0, \tag{10}$$

which after contraction gives

$$S(F_1, \xi) + (n-1)\eta(F_1) = 0, \tag{11}$$

where R, S and Q represent the curvature tensor, the Ricci tensor and the Ricci operator, respectively. For $\epsilon = 1$, $M^n(\epsilon)$ reduces to a usual Kenmotsu manifold. Throughout the manuscript, we denote an n-dimensional ϵ-Kenmotsu manifold endowed with a $*\eta$-Ricci–Yamabe soliton $(g, V, \lambda, \mu, \alpha, \beta)$ by $(M^n(\epsilon), g, V, \lambda, \mu, \alpha, \beta)$.

From (6) and the definition of Lie derivative, we get

$$(\pounds_\xi g)(F_1, F_2) = 2\epsilon\{g(F_1, F_2) - \epsilon\eta(F_1)\eta(F_2)\} \tag{12}$$

for any F_1, F_2 on $M^n(\epsilon)$.

Definition 1. *If the Ricci operator Q of $M^n(\epsilon)$ is non-zero and satisfies*

$$g(QF_1, F_2) = lg(F_1, F_2) + m\eta(F_1)\eta(F_2),$$

where l and m are smooth functions on $M^n(\epsilon)$, then $M^n(\epsilon)$ is termed as an η-Einstein manifold. If $m = 0$, then $M^n(\epsilon)$ becomes Einstein manifold.

Lemma 1 ([5])**.** *S^* in $M^n(\epsilon)$ satisfies*

$$S^*(F_1, F_2) = S(F_1, F_2) + \eta(F_1)\eta(F_2) + \epsilon(n-2)g(F_1, F_2), \tag{13}$$

for any F_1, F_2 on $M^n(\epsilon)$.

3. ϵ-Kenmotsu Manifolds Admitting $*\eta$-Ricci–Yamabe Solitons

Let the metric g of $M^n(\epsilon)$ be a $*\eta$-Ricci–Yamabe soliton $(g, \xi, \lambda, \mu, \alpha, \beta)$. Then (1) and (12) lead to

$$\alpha S^*(F_1, F_2) = (\epsilon + \lambda - \frac{\beta r}{2})g(F_1, F_2) - (\mu - 1)\eta(F_1)\eta(F_2), \tag{14}$$

provided $\alpha \neq 0$.

By employing (13), (14) leads to

$$S(F_1, F_2) = A_1 g(F_1, F_2) + A_2\eta(F_1)\eta(F_2), \tag{15}$$

where $A_1 = -[\epsilon(n-2) + \frac{1}{\alpha}(\epsilon + \lambda - \frac{\beta r}{2})]$ and $A_2 = -[1 - \frac{1}{\alpha}(1-\mu)]$.

Putting $F_2 = \xi$ in (15) and using (2) and (3), we find

$$S(F_1, \xi) = A_3\eta(F_1), \tag{16}$$

where $A_3 = \epsilon A_1 + A_2 = -n + 1 - \frac{\epsilon}{\alpha}(\lambda + \epsilon\mu - \frac{\beta r}{2})$ and $\alpha \neq 0$.

From (15) we also have

$$QF_1 = A_1 F_1 + \epsilon A_2\eta(F_1)\xi \implies Q\xi = \epsilon A_3\xi. \tag{17}$$

In view of (11) and (16), it follows that

$$\lambda + \epsilon\mu = \frac{\beta r}{2}. \tag{18}$$

On contracting (15) and using the values of A_1, A_2, we obtain

$$r = -\epsilon(n-1)\big[n-1+\frac{1-\mu}{\alpha}\big], \tag{19}$$

where μ and $\alpha(\neq 0)$ are constants. Thus, by virtue of (15), (18) and (19), we conclude:

Theorem 1. *$(M^n(\epsilon), g, \xi, \lambda, \mu, \alpha, \beta)$ is an η-Einstein manifold. Furthermore, the scalar curvature of $M^n(\epsilon)$ is constant and $\lambda + \epsilon\mu = \frac{\beta r}{2}$.*

Particularly, if we take $\mu = 0$ and $\alpha(\neq 0) \in \mathbb{R}$, then the $*\eta$-Ricci–Yamabe soliton reduces to the $*$-Ricci–Yamabe soliton. From Equations (15), (18) and (19), we find that $S + [\epsilon(n-2) + \frac{\epsilon}{\alpha}]g + (1-\frac{1}{\alpha})\eta \otimes \eta = 0$ and $\lambda = \frac{\beta r}{2} = -\frac{\epsilon\beta(n-1)[1+\alpha(n-1)]}{2\alpha}$. Thus, we have

Corollary 1. *An $M^n(\epsilon)$ admitting a soliton $(g, \xi, \lambda, \alpha, \beta)$ is an η-Einstein manifold and the soliton $(g, \xi, \lambda, \alpha, \beta)$ on $M^n(\epsilon)$ is concluded as follows:*

Values of ϵ	*Values of α*	*Conditions for soliton to be expanding, shrinking or steady*
$\epsilon = 1$	*(i)* $\alpha > 0$	*(i) soliton is expanding, shrinking or steady if $\beta <, >$ or $= 0$, respectively.*
	(ii) $\alpha < 0$	*(ii) soliton is expanding, shrinking or steady if $\beta >, <$ or $= 0$, respectively*
$\epsilon = -1$	*(i)* $\alpha > 0$	*(i) soliton is expanding, shrinking or steady if $\beta >, <$ or $= 0$, respectively.*
	(ii) $\alpha < 0$	*(ii) soliton is expanding, shrinking or steady if $\beta <, >$ or $= 0$, respectively*

For $\beta = 0$, Corollary 1 shows that the soliton $(g, \xi, \lambda, \alpha, \beta)$ becomes $*$-Ricci soliton $(g, \xi, \lambda, \alpha)$ of type $\alpha(\neq 0)$ and $M^n(\epsilon)$ is η-Einstein. Moreover, $\lambda = 0$. Thus, we conclude our result as:

Corollary 2. *Let the metric of $M^n(\epsilon)$ be a soliton $(g, \xi, \lambda, \alpha)$. Then $M^n(\epsilon)$ is an η-Einstein manifold and the soliton $(g, \xi, \lambda, \alpha)$ is steady.*

Next, we consider that an $M^n(\epsilon)$ admits a soliton $(g, V, \lambda, \mu, \alpha, \beta)$. If $V = k\xi$ for some function k, then (1) gives

$$kg(\nabla_{F_1}\xi, F_2) + \epsilon(F_1 k)\eta(F_2) + kg(F_1, \nabla_{F_2}\xi) + \epsilon(F_2 k)\eta(F_1)$$
$$+2\alpha S^*(F_1, F_2) + (2\lambda - \beta r)g(F_1, F_2) + 2\mu\eta(F_1)\eta(F_2) = 0,$$

which in view of (6) and (13) takes the form

$$2\alpha S(F_1, F_2) + \{2\epsilon k + 2\epsilon\alpha(n-2) + 2\lambda - \beta r\}g(F_1, F_2) + \epsilon(F_1 k)\eta(F_2) \tag{20}$$
$$+\epsilon(F_2 k)\eta(F_1) + \{2\alpha + 2\mu - 2k\}\eta(F_1)\eta(F_2) = 0.$$

Taking $F_2 = \xi$ in (20), and then using (2) and (3) it follows that

$$(F_1 k) + \{(\xi k) + (2-\alpha)(\lambda + \epsilon\mu - \frac{\beta r}{2})\}\eta(F_1) = 0. \tag{21}$$

Again putting $F_1 = \xi$ in (21) and using (2), we get

$$\xi k = -(1-\frac{\alpha}{2})(\lambda + \epsilon\mu - \frac{\beta r}{2}). \tag{22}$$

On combining (21) and (22), we get

$$dk = (1 - \frac{\alpha}{2})(\lambda + \epsilon\mu - \frac{\beta r}{2})\eta. \quad (23)$$

Now, by operating d on (23) and using the facts $d^2 = 0$ and $d\eta = 0$, it follows that either $\alpha = 2$, or $\beta = 0$ or $r =$ constant.

If $\alpha = 2$, then (23) reveals that $k =$ constant and the soliton vector field V of $(g, V, \lambda, \mu, \alpha, \beta)$ is a constant multiple of ξ. Moreover, from (20) we infer that

$$S = ag + b\eta \otimes \eta,$$

where $a = \frac{\beta r - 2\epsilon k - 4\epsilon(n-2) - \lambda}{4}$ and $b = \frac{k - \mu - 2}{2}$. This shows that $M^n(\epsilon)$ is an η-Einstein manifold.

If $\beta = 0$, then metric of $M^n(\epsilon)$ forces to be a $*\eta$-Ricci soliton, therefore (23) gives

$$dk = (1 - \frac{\alpha}{2})(\lambda + \epsilon\mu)\eta \Longleftrightarrow grad\, k = (1 - \frac{\alpha}{2})(\lambda + \epsilon\mu)\xi. \quad (24)$$

This shows that the gradient of k is a constant multiple of the Reeb vector field of $M^n(\epsilon)$. In view of (20) and (24), we find

$$\alpha S + \{\epsilon k + \epsilon\alpha(n-2) + \lambda\}g + [\epsilon(1 - \frac{\alpha}{2})(\lambda + \epsilon\mu) + \alpha + \mu - k]\eta \otimes \eta = 0,$$

which shows that the manifold $M^n(\epsilon)$ under consideration is an η-Einstein manifold, provided $\alpha \neq 0$. Similarly, we can prove that if $r =$ constant then $M^n(\epsilon)$ is said to be an η-Einstein manifold. Thus, we have

Theorem 2. *Let the soliton vector field V on $(M^n(\epsilon), g, V, \lambda, \mu, \alpha, \beta)$ be pointwise collinear with ξ. Then $M^n(\epsilon)$ is η-Einstein.*

Theorem 3. *Let $V = k\xi$ for some smooth function k on $(M^n(\epsilon), g, V, \lambda, \mu, \alpha, \beta)$. Then either*

(i) *the soliton vector field V is a constant multiple of ξ, or*

(ii) *the metric of $M^n(\epsilon)$ forces to be a $*\eta$-Ricci soliton and gradient of k is a constant multiple of ξ, or*

(iii) *scalar curvature of $M^n(\epsilon)$ is constant.*

Codazzi-type and cyclic parallel Ricci tensors are special types of Ricci tensors introduced and extensively studied by Gray [46]. Now, we explore the properties of $*\eta$-Ricci–Yamabe solitons on $M^n(\epsilon)$ if the Ricci tensors of $M^n(\epsilon)$ are of Codazzi and cyclic parallel types.

Definition 2. *An $M^n(\epsilon)$ possesses a Codazzi-type Ricci tensor $S(\neq 0)$ if*

$$(\nabla_{F_1} S)(F_2, F_3) = (\nabla_{F_2} S)(F_1, F_3) \quad (25)$$

for all F_1, F_2, F_3 on $M^n(\epsilon)$.

Let an $M^n(\epsilon)$ admitting a soliton $(g, V = \xi, \lambda, \mu, \alpha, \beta)$ have Codazzi-type Ricci tensor, then (25) holds. In view of (15), the expression $(\nabla_{F_1} S)(F_2, F_3) = F_1 S(F_2, F_3) - S(\nabla_{F_1} F_2, F_3) - S(F_2, \nabla_{F_1} F_3)$ gives

$$(\nabla_{F_1} S)(F_2, F_3) = A_2[(\nabla_{F_1}\eta)(F_2)\eta(F_3) + (\nabla_{F_1}\eta)(F_3)\eta(F_2)],$$

which by using (7) takes the form

$$(\nabla_{F_1} S)(F_2, F_3) = A_2[g(F_1, F_2)\eta(F_3) + g(F_1, F_3)\eta(F_2) - 2\epsilon\eta(F_1)\eta(F_2)\eta(F_3)]. \quad (26)$$

By virtue of (26), (25) leads to

$$A_2[g(F_1,F_3)\eta(F_2)-g(F_2,F_3)\eta(F_1)]=0. \tag{27}$$

By putting $F_2=\xi$ in (27) and using (2), we have

$$A_2 g(\phi F_1,\phi F_3)=0,$$

from which it follows that $A_2=0$ (as $g(\phi F_1,\phi F_3)\neq 0$). This implies that $\mu=1-\alpha$, and hence (18) infers that $\lambda=\frac{\beta r}{2}-\epsilon(1-\alpha)$. Now, we state our results as:

Theorem 4. *Let an $M^n(\epsilon)$ admitting a soliton $(g,\xi,\lambda,\mu,\alpha,\beta)$. If the Ricci tensor of $M^n(\epsilon)$ is of Codazzi type, then $\alpha+\mu=1$ and $\lambda=\frac{\beta r}{2}-\epsilon(1-\alpha)$.*

Next, by using the values $\lambda=-\epsilon(1-\alpha)+\frac{\beta r}{2}$ and $\mu=1-\alpha$ in (15), we obtain $S(F_1,F_2)=-\epsilon(n-1)g(F_1,F_2)$. Conversely, if $(M^n(\epsilon),g,\xi,\lambda,\mu,\alpha,\beta)$ is Einstein, then we can easily verify that S is of Codazzi type. Thus, we have

Theorem 5. *Let the metric of an $M^n(\epsilon)$ be a soliton $(g,\xi,\lambda,\mu,\alpha,\beta)$. Then the Ricci tensor of $M^n(\epsilon)$ is of Codazzi type if and only if $M^n(\epsilon)$ is an Einstein manifold.*

Let the Ricci tensor of $(M^n(\epsilon),g,\xi,\lambda,\mu,\alpha,\beta)$ be of Codazzi type, then from Theorem 4 $A_2=0$, and hence from (26) we have $\nabla S=0$. A Riemannian (semi-Riemannian) manifold is said to be Ricci symmetric if $\nabla S=0$. This definition with the above results lead to

Corollary 3. *$(M^n(\epsilon),g,\xi,\lambda,\mu,\alpha,\beta)$ with Codazzi-type Ricci tensor is Ricci symmetric.*

In particular, if we take $\alpha=1$, then the above values of μ and λ reduces to 0 and $\frac{\beta r}{2}$, respectively. Moreover, from (19) we find $r=-\epsilon n(n-1)$. Thus, we have

Corollary 4. *Let the Ricci tensor of $M^n(\epsilon)$ be of Codazzi type. Then the $*\eta$-Ricci–Yamabe soliton $(g,\xi,\lambda,\mu,1,\beta)$ on $M^n(\epsilon)$ forces to be $*$-Ricci–Yamabe soliton $(g,\xi,\lambda,1,\beta)$.*

Corollary 5. *Let $(g,\xi,\lambda,1,\beta)$ be a soliton on $M^n(\epsilon)$. If the Ricci tensor of $M^n(\epsilon)$ is of Codazzi type, then $(g,\xi,\lambda,1,\beta)$ is concluded as follows:*

(i) if $\epsilon=1$ (i.e., ξ is space-like), then the soliton is expanding, steady or shrinking according to $\beta<0,=0$ or $\beta>0$, respectively, and

(ii) if $\epsilon=-1$ (i.e., ξ is time-like), then the soliton is expanding, steady or shrinking according to $\beta>0,=0$ or $\beta<0$, respectively.

Definition 3. *If the Ricci tensor $S(\neq 0)$ of $M^n(\epsilon)$ satisfies*

$$(\nabla_{F_1}S)(F_2,F_3)+(\nabla_{F_2}S)(F_3,F_1)+(\nabla_{F_3}S)(F_1,F_2)=0, \tag{28}$$

for all F_1,F_2,F_3 on $M^n(\epsilon)$, then $M^n(\epsilon)$ possesses a cyclic parallel Ricci tensor.

Suppose that an $M^n(\epsilon)$ admits $(g,V=\xi,\lambda,\mu,\alpha,\beta)$. If the Ricci tensor of $M^n(\epsilon)$ is cyclic parallel, then (28) holds. By virtue of (26), we can write the following equations:

$$(\nabla_{F_2}S)(F_3,F_1)=A_2[g(F_2,F_3)\eta(F_1)+g(F_1,F_2)\eta(F_3)-2\epsilon\eta(F_1)\eta(F_2)\eta(F_3)], \tag{29}$$

$$(\nabla_{F_3}S)(F_1,F_2)=A_2[g(F_3,F_1)\eta(F_2)+g(F_2,F_3)\eta(F_1)-2\epsilon\eta(F_1)\eta(F_2)\eta(F_3)]. \tag{30}$$

By making use of (26), (29) and (30) in (28), we have

$$A_2[g(F_1,F_2)\eta(F_3)+g(F_2,F_3)\eta(F_1)+g(F_1,F_3)\eta(F_2)-3\epsilon\eta(F_1)\eta(F_2)\eta(F_3)]=0,$$

which by putting $F_3 = \xi$ and using (2) leads to $A_2 g(\phi F_1, \phi F_2) = 0 \implies A_2 = 0$, as $g(\phi F_1, \phi F_2) \neq 0$. Consequently, (15) reduces to an equation of Einstein manifold. Conversely, we can easily prove that the Ricci tensor of the Einstein manifold is cyclic parallel. Thus we can write:

Theorem 6. *Let the metric of an $M^n(\epsilon)$ be a soliton $(g, \xi, \lambda, \mu, \alpha, \beta)$. Then $M^n(\epsilon)$ is an Einstein manifold if and only if the Ricci tensor of $M^n(\epsilon)$ is cyclic parallel.*

Remark 1. *The conditions for the soliton $(g, \xi, \lambda, \mu, \alpha, \beta)$ to be expanding, shrinking or steady on an $M^n(\epsilon)$ admitting cyclic parallel Ricci tensor can be discussed as in Corollary 5.*

4. $*\eta$-Ricci–Yamabe Solitons on ϵ-Kenmotsu Manifolds Satisfying the Conditions $\mathcal{W}_2(\xi, F_1) \cdot S = 0$ and $S(\xi, F_1) \cdot \mathcal{W}_2 = 0$

In this section, we characterize an $M^n(\epsilon)$ admitting $(g, \xi, \lambda, \mu, \alpha, \beta)$, satisfying certain conditions on $\mathcal{W}_2$-curvature tensor. The $\mathcal{W}_2$-curvature tensor on an n-dimensional Riemannian manifold M is defined as [37]

$$\mathcal{W}_2(F_1, F_2)F_3 = R(F_1, F_2)F_3 - \frac{1}{n-1}[g(F_2, F_3)QF_1 - g(F_1, F_3)QF_2], \tag{31}$$

for all F_1, F_2, F_3 on $M^n(\epsilon)$.

First, let us consider that an $M^n(\epsilon)$ admitting $(g, \xi, \lambda, \mu, \alpha, \beta)$ satisfies the condition

$$\mathcal{W}_2(\xi, F_1) \cdot S = 0. \tag{32}$$

The condition (32) implies that

$$S(\mathcal{W}_2(\xi, F_1)F_2, F_3) + S(F_2, \mathcal{W}_2(\xi, F_1)F_3) = 0. \tag{33}$$

From (2), (9), (16) and (31), we find

$$\mathcal{W}_2(\xi, F_1)F_2 = \frac{\epsilon}{n-1}\eta(F_2)QF_1 + \eta(F_2)F_1, \tag{34}$$

$$\eta(\mathcal{W}_2(\xi, F_1)F_2) = (1 + \frac{A_3}{n-1})\eta(F_1)\eta(F_2). \tag{35}$$

Thus, in view of (15), (33) turns to

$$\begin{aligned} A_1\{\eta(F_2)g(F_1, F_3) + \eta(F_3)g(F_1, F_2) + \frac{\epsilon}{n-1}\eta(F_2)S(F_1, F_3)\} \\ + \frac{\epsilon}{n-1}\eta(F_3)S(F_1, F_2) + 2A_2(1 + \frac{A_3}{n-1})\eta(F_1)\eta(F_2)\eta(F_3) = 0, \end{aligned} \tag{36}$$

where (34) and (35) being used. Now, putting $F_2 = \xi$ in (36), then using (2), (3) and (16), we obtain

$$S(F_1, F_3) = -\epsilon(n-1)g(F_1, F_3) + \frac{1}{\alpha}(\epsilon + \frac{2A_2}{A_1})(\lambda + \epsilon\mu - \frac{\beta r}{2})\eta(F_1)\eta(F_3). \tag{37}$$

Since $*\eta$-Ricci–Yamabe soliton on $M^n(\epsilon)$ satisfies (18), therefore (37) reduces to

$$S(F_1, F_3) = -\epsilon(n-1)g(F_1, F_3).$$

Thus, we have

Theorem 7. *Let an $M^n(\epsilon)$ admit a soliton $(g, \xi, \lambda, \mu, \alpha, \beta)$. If $M^n(\epsilon)$ satisfies the condition $\mathcal{W}_2(\xi, F_1) \cdot S = 0$, then it is an Einstein manifold.*

Next, suppose that an $M^n(\epsilon)$ admits a $*\eta$-Ricci–Yamabe soliton $(g,\xi,\lambda,\mu,\alpha,\beta)$ and $S(\xi,F_1)\cdot\mathcal{W}_2=0$, which infers that

$$\begin{aligned}&S(F_1,\mathcal{W}_2(F_2,F_3)F_4)\xi-S(\xi,\mathcal{W}_2(F_2,F_3)F_4)F_1+S(F_1,F_2)\mathcal{W}_2(\xi,F_3)F_4\\&-S(\xi,F_2)\mathcal{W}_2(F_1,F_3)F_4+S(F_1,F_3)\mathcal{W}_2(F_2,\xi)F_4-S(\xi,F_3)\mathcal{W}_2(F_2,F_1)F_4\\&+S(F_1,F_4)\mathcal{W}_2(F_2,F_3)\xi-S(\xi,F_4)\mathcal{W}_2(F_2,F_3)F_1=0,\end{aligned}\tag{38}$$

for any for all F_1,F_2,F_3 on $M^n(\epsilon)$. Taking the inner product of (38) with ξ and using (15) and (16), we have

$$\begin{aligned}&A_1g(F_1,\mathcal{W}_2(F_2,F_3)F_4)+A_2\eta(F_1)\eta(\mathcal{W}_2(F_2,F_3)F_4)-A_3\eta(\mathcal{W}_2(F_2,F_3)F_4)\eta(F_1)\\&-A_3\{\eta(F_4)\eta(\mathcal{W}_2(F_2,F_3)F_1)+\eta(F_3)\eta(\mathcal{W}_2(F_2,F_1)F_4)+\eta(F_2)\eta(\mathcal{W}_2(F_1,F_3)F_4)\}\\&+A_1g(F_1,F_2)\eta(\mathcal{W}_2(\xi,F_3)F_4)+A_2\eta(F_1)\eta(F_2)\eta(\mathcal{W}_2(\xi,F_3)F_4)\\&+A_1g(F_1,F_3)\eta(\mathcal{W}_2(F_2,\xi)F_4)+A_2\eta(F_1)\eta(F_3)\eta(\mathcal{W}_2(F_2,\xi)F_4)\\&+A_1g(F_1,F_4)\eta(\mathcal{W}_2(F_2,F_3)\xi)+A_2\eta(F_1)\eta(F_4)\eta(\mathcal{W}_2(F_2,F_3)\xi)=0.\end{aligned}\tag{39}$$

From (2), (8), (9), (16), (18) and (31), we find

$$\begin{aligned}g(F_1,\mathcal{W}_2(F_2,F_3)F_4)&=g(F_1,R(F_2,F_3)F_4)-\frac{1}{n-1}\{g(F_3,F_4)S(F_1,F_2)\\&\quad-g(F_2,F_4)S(F_1,F_3)\},\end{aligned}\tag{40}$$

$$\eta(\mathcal{W}_2(F_2,F_3)F_4)=\eta(R(F_2,F_3)F_4)-\frac{\epsilon A_3}{n-1}\{g(F_3,F_4)\eta(F_2)-g(F_2,F_4)\eta(F_3)\},\tag{41}$$

$$\eta(\mathcal{W}_2(\xi,F_3)F_4)=(1+\frac{A_3}{n-1})\eta(F_4)\eta(F_3),\tag{42}$$

$$\eta(\mathcal{W}_2(F_2,F_3)\xi)=0.\tag{43}$$

Now by making use of (40)–(43) in (39), we arrive at

$$\begin{aligned}&A_1g(F_1,\mathcal{W}_2(F_2,F_3)F_4)+A_2\eta(\mathcal{W}_2(F_2,F_3)F_4)\eta(F_1)\\&+A_1(1+\frac{A_3}{A_1})\{g(F_1,F_2)\eta(F_4)\eta(F_3)-g(F_1,F_3)\eta(F_4)\eta(F_2)\}\\&-A_3\{\eta(\mathcal{W}_2(F_2,F_3)F_4)\eta(F_1)+\eta(\mathcal{W}_2(F_1,F_3)F_4)\eta(F_2)\\&+\eta(\mathcal{W}_2(F_2,F_1)F_4)\eta(F_3)+\eta(\mathcal{W}_2(F_2,F_3)F_1)\eta(F_4)\}=0,\end{aligned}$$

which by substituting $F_4=\xi$ and using (43) reduces to

$$\begin{aligned}&A_1g(F_1,\mathcal{W}_2(F_2,F_3)\xi)-A_3\eta(\mathcal{W}_2(F_2,F_3)F_1)\eta(F_4)\\&+A_1(1+\frac{A_3}{A_1})\{g(F_1,F_2)\eta(F_3)-g(F_1,F_3)\eta(F_2)\}=0.\end{aligned}\tag{44}$$

After employing (40) and (42) in (44), we lead to

$$\begin{aligned}&A_1\{S(F_1,F_3)\eta(F_2)-S(F_1,F_2)\eta(F_3)\}\\&+A_3(\epsilon A_1-A_3-(n-1))\{g(F_1,F_2)\eta(F_3)-g(F_1,F_3)\eta(F_2)\}=0.\end{aligned}\tag{45}$$

Again putting $F_2=\xi$ in (45) and using (2), (3) and (16), we find

$$\begin{aligned}S(F_1,F_3)&=\frac{A_3}{A_1}\{\epsilon A_1-A_3-n+1\}g(F_1,F_3)\\&\quad+\frac{\epsilon A_3}{A_1}(A_3+n-1)\eta(F_1)\eta(F_3).\end{aligned}\tag{46}$$

Note that an $M^n(\epsilon)$ endowed with $(g, \xi, \lambda, \mu, \alpha, \beta)$ satisfies (18). Thus, $A_3 = -n+1$. Consequently, (46) reduces to

$$S(F_1, F_3) = \epsilon(n-1)g(F_1, F_3).$$

Thus, we have

Theorem 8. *Let the metric of an $M^n(\epsilon)$ be a soliton $(g, \xi, \lambda, \mu, \alpha, \beta)$ and $S(\xi, F_1) \cdot \mathcal{W}_2 = 0$. Then $M^n(\epsilon)$ is an Einstein manifold.*

5. $*\eta$-Ricci–Yamabe Solitons on ϕ-Ricci Symmetric ϵ-Kenmotsu Manifolds

Let an $M^n(\epsilon)$ admitting $(g, \xi, \lambda, \mu, \alpha, \beta)$ be ϕ-Ricci-symmetric, i.e., $\phi^2(\nabla_{F_1} Q)F_2 = 0$. Then by virtue of (2), we have

$$(\nabla_{F_1} Q)F_2 - \eta((\nabla_{F_1} Q)F_2)\xi = 0. \tag{47}$$

The inner product of (47) with F_4 leads to

$$g((\nabla_{F_1} Q)F_2, F_4) - \epsilon\eta((\nabla_{F_1} Q)F_2)\eta(F_4) = 0,$$

which can be written as

$$g(\nabla_{F_1} QF_2, F_4) - S(\nabla_{F_1} F_2, F_4) - \epsilon\eta((\nabla_{F_1} Q)F_2)\eta(F_4) = 0. \tag{48}$$

By putting $F_2 = \xi$ in (48) and using (17), we arrive at

$$S(F_1, F_4) = \epsilon A_3 g(F_1, F_4) - \epsilon\eta((\nabla_{F_1} Q)\xi)\eta(F_4). \tag{49}$$

Replacing $F_1 = \phi F_1$, $F_4 = \phi F_4$ in (49) and using (15), we find $A_1 - \epsilon A_3 = 0$ (since $g(\phi F_1, \phi F_4) \neq 0$), from which we have $\mu = 1 - \alpha$ and thus (18) gives $\lambda = -\epsilon(1-\alpha) + \frac{\beta r}{2}$. Thus, we have

Theorem 9. *Let $(M^n(\epsilon), g, \xi, \lambda, \mu, \alpha, \beta)$ be ϕ-Ricci symmetric. Then $\lambda = \frac{\beta r}{2} - \epsilon(1-\alpha)$ and $\mu = 1 - \alpha$.*

In particular, for $\alpha = 1$ we have $\lambda = \frac{\beta r}{2}$ and $\mu = 0$. Moreover, from (15) and (19) we find $S(F_1, F_2) = -\epsilon(n-1)g(F_1, F_2)$ and $r = -\epsilon n(n-1)$, respectively. Thus, we have

Corollary 6. *An n-dimensional ϕ-Ricci symmetric ϵ-Kenmotsu manifold admitting a soliton $(g, \xi, \lambda, 1, \beta)$ is Einstein manifold. Furthermore, the soliton $(g, \xi, \lambda, 1, \beta)$ on $M^n(\epsilon)$ is concluded as follows:*

(i) *if $\epsilon = 1$ (i.e., ξ is space-like), then the soliton is expanding, steady or shrinking according to $\beta < 0, = 0$ or $\beta > 0$,*

(ii) *if $\epsilon = -1$ (i.e., ξ is time-like), then the soliton is expanding, steady or shrinking according to $\beta > 0, = 0$ or $\beta < 0$.*

6. Example

We consider the manifold $M = \{(x_1, x_2, y_1, y_2, z) \in \mathbb{R}^5, z \neq 0\}$ of dimension 5, where (x_1, x_2, y_1, y_2, z) are the standard coordinates in $\mathbb{R}^5$. Let the vector fields ς_1, ς_2, ς_3, ς_4 and ς_5 be defined on M as:

$$\varsigma_1 = e^{-\epsilon z}\frac{\partial}{\partial x_1}, \quad \varsigma_2 = e^{-\epsilon z}\frac{\partial}{\partial x_2}, \quad \varsigma_3 = e^{-\epsilon z}\frac{\partial}{\partial y_1}, \quad \varsigma_4 = e^{-\epsilon z}\frac{\partial}{\partial y_2}, \quad \varsigma_5 = e^{-\epsilon z}\frac{\partial}{\partial z} = \xi.$$

Then they form a basis on M. Let g be the metric defined by

$$g(\varsigma_i, \varsigma_j) = \begin{cases} \epsilon, & 1 \leq i = j \leq 5, \\ 0, & 1 \leq i \neq j \leq 5. \end{cases}$$

Define the 1-form η and the $(1,1)$-tensor ϕ on M as:

$$\eta(F_1) = \epsilon g(F_1, \varsigma_5) = \epsilon g(F_1, \xi) \ \ \forall F_1 \in \mathfrak{X}(M),$$

$$\phi\varsigma_1 = \epsilon\varsigma_2, \ \phi\varsigma_2 = \epsilon\varsigma_1, \ \phi\varsigma_3 = \epsilon\varsigma_4, \ \phi\varsigma_4 = \epsilon\varsigma_3, \ \phi\varsigma_5 = 0.$$

The linearity property of ϕ and g yields

$$\eta(\varsigma_5) = 1, \ \phi^2 F_1 = -F_1 + \eta(F_1)\xi, \ g(\phi F_1, \phi F_2) = g(F_1, F_2) - \epsilon\eta(F_1)\eta(F_2),$$

for all $F_1, F_2 \in \mathfrak{X}(M)$. Obvious that $(\phi, \xi, \eta, g, \epsilon)$ is an almost contact structure on M for $\varsigma_5 = \xi$. We list the components of Lie bracket as:

$$[\varsigma_1, \varsigma_2] = [\varsigma_1, \varsigma_3] = [\varsigma_1, \varsigma_4] = [\varsigma_2, \varsigma_3] = [\varsigma_2, \varsigma_4] = [\varsigma_3, \varsigma_4] = 0,$$

$$[\varsigma_1, \varsigma_5] = \epsilon\varsigma_1, [\varsigma_2, \varsigma_5] = \epsilon\varsigma_2, [\varsigma_3, \varsigma_5] = \epsilon\varsigma_3, [\varsigma_4, \varsigma_5] = \epsilon\varsigma_4.$$

From Koszul's formula, we can easily calculate

$$\nabla_{\varsigma_1}\varsigma_1 = -\epsilon\varsigma_5, \ \ \nabla_{\varsigma_1}\varsigma_2 = 0, \ \ \nabla_{\varsigma_1}\varsigma_3 = 0, \ \ \nabla_{\varsigma_1}\varsigma_4 = 0, \ \ \nabla_{\varsigma_1}\varsigma_5 = \epsilon\varsigma_1,$$

$$\nabla_{\varsigma_2}\varsigma_1 = 0, \ \ \nabla_{\varsigma_2}\varsigma_2 = -\epsilon\varsigma_5, \ \ \nabla_{\varsigma_2}\varsigma_3 = 0, \ \ \nabla_{\varsigma_2}\varsigma_4 = 0, \ \ \nabla_{\varsigma_2}\varsigma_5 = \epsilon\varsigma_2,$$
$$\nabla_{\varsigma_3}\varsigma_1 = 0, \ \ \nabla_{\varsigma_3}\varsigma_2 = 0, \ \ \nabla_{\varsigma_3}\varsigma_3 = -\epsilon\varsigma_5, \ \ \nabla_{\varsigma_3}\varsigma_4 = 0, \ \ \nabla_{\varsigma_3}\varsigma_5 = \epsilon\varsigma_3,$$
$$\nabla_{\varsigma_4}\varsigma_1 = 0, \ \ \nabla_{\varsigma_4}\varsigma_2 = 0, \ \ \nabla_{\varsigma_4}\varsigma_3 = 0, \ \ \nabla_{\varsigma_4}\varsigma_4 = -\epsilon\varsigma_5, \ \ \nabla_{\varsigma_4}\varsigma_5 = \epsilon\varsigma_4,$$
$$\nabla_{\varsigma_5}\varsigma_1 = 0, \ \ \nabla_{\varsigma_5}\varsigma_2 = 0, \ \ \nabla_{\varsigma_5}\varsigma_3 = 0, \ \ \nabla_{\varsigma_5}\varsigma_4 = 0, \ \ \nabla_{\varsigma_5}\varsigma_5 = 0,$$

which reflect that $\nabla_{\varsigma_i}\xi = \epsilon(\varsigma_i - \eta(\varsigma_i)\xi)$ for all ς_i, $i = 1,2,3,4,5$. Thus, M is a five-dimensional ϵ-Kenmotsu manifold $M^5(\epsilon)$. We can easily obtain the following components of R and S:

$$R(\varsigma_1, \varsigma_2)\varsigma_2 = R(\varsigma_1, \varsigma_3)\varsigma_3 = R(\varsigma_1, \varsigma_4)\varsigma_4 = R(\varsigma_1, \varsigma_5)\varsigma_5 = -\varsigma_1,$$

$$R(\varsigma_1, \varsigma_2)\varsigma_1 = \varsigma_2, \ R(\varsigma_1, \varsigma_3)\varsigma_1 = R(\varsigma_2, \varsigma_3)\varsigma_2 = R(\varsigma_5, \varsigma_3)\varsigma_5 = \varsigma_3,$$
$$R(\varsigma_2, \varsigma_3)\varsigma_3 = R(\varsigma_2, \varsigma_4)\varsigma_4 = R(\varsigma_2, \varsigma_5)\varsigma_5 = -\varsigma_2, R(\varsigma_3, \varsigma_4)\varsigma_4 = -\varsigma_3,$$
$$R(\varsigma_1, \varsigma_5)\varsigma_1 = R(\varsigma_2, \varsigma_5)\varsigma_2 = R(\varsigma_4, \varsigma_5)\varsigma_4 = R(\varsigma_3, \varsigma_5)\varsigma_3 = \varsigma_5,$$
$$R(\varsigma_1, \varsigma_4)\varsigma_1 = R(\varsigma_2, \varsigma_4)\varsigma_2 = R(\varsigma_3, \varsigma_4)\varsigma_3 = R(\varsigma_5, \varsigma_4)\varsigma_5 = \varsigma_4.$$

The components of the Ricci tensor can be easily obtained as follows:

$$S(\varsigma_1, \varsigma_1) = S(\varsigma_2, \varsigma_2) = S(\varsigma_3, \varsigma_3) = S(\varsigma_4, \varsigma_4) = S(\varsigma_5, \varsigma_5) = -4. \tag{50}$$

Obviously, the scalar curvature of $M^5(\epsilon)$ is -20.

Let F_1 and F_2 be the arbitrary vector fields of $M^5(\epsilon)$. Then we can write F_1 and F_2 as:

$$F_1 = F_1^1\varsigma_1 + F_1^2\varsigma_2 + F_1^3\varsigma_3 + F_1^4\varsigma_4 + F_1^5\xi \text{ and } F_2 = F_2^1\varsigma_1 + F_2^2\varsigma_2 + F_2^3\varsigma_3 + F_2^4\varsigma_4 + F_2^5\xi,$$

where $F_i^j, j = 1,2,3,4,5, i = 1,2$ denotes the scalar on $M^5(\epsilon)$. From straightforward calculations, we have

$$(\pounds_\xi g)(F_1, F_2) = g(\nabla_{F_1}\xi, F_2) + g(F_1, \nabla_{F_2}\xi) = 2(F_1^1F_2^1 + F_1^2F_2^2 + F_1^3F_2^3 + F_1^4F_2^4),$$

$$S(F_1, F_2) = -4(F_1^1 F_2^1 + F_1^2 F_2^2 + F_1^3 F_2^3 + F_1^4 F_2^4 + F_1^5 F_2^5),$$
$$g(F_1, F_2) = \epsilon(F_1^1 F_2^1 + F_1^2 F_2^2 + F_1^3 F_2^3 + F_1^4 F_2^4 + F_1^5 F_2^5),$$
$$\eta(F_1) = F_1^5$$

and

$$S^*(F_1, F_2) = -(F_1^1 F_2^1 + F_1^2 F_2^2 + F_1^3 F_2^3 + F_1^4 F_2^4).$$

Let us choose the set of values of $\lambda, \mu, \alpha, \beta$ to satisfy the relation $\lambda - \alpha + 10\beta + 1 = 0$. For instance, choose $\alpha = 2, \beta = 1, \lambda = -9$. It is obvious that the metric g of $M^5(\epsilon)$ satisfies the $*\eta$-Ricci–Yamabe soliton Equation (1), that is,

$$(\pounds_\xi g)(F_1, F_2) + 2\alpha S^*(F_1, F_2) + (2\lambda - \beta r)g(F_1, F_2) + 2\mu\eta(F_1)\eta(F_2) = 0,$$

and the relations $\lambda + \epsilon\mu = \frac{\beta r}{2}$ and $\alpha = 1 - \mu$, which verifies Theorem 1 and Theorem 4. Also, we notice that $\mathcal{W}_2 \cdot S = 0$ holds on $M^5(\epsilon)$ for all F_1 and F_2, and $M^5(\epsilon)$ is Einstein. Thus, the Theorem 7 is verified.

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