axioms

Article

Stability for a Class of Differential Set-Valued Inverse Variational Inequalities in Finite Dimensional Spaces

Xinyue Zhu [1], Wei Li [1,*] and Xueping Luo [2]

1 Geomathematics Key Laboratory of Sichuan Province, College of Mathematics and Physics, Chengdu University of Technology, Chengdu 610059, China
2 Department of Mathematics, Southwest Minzu University, Chengdu 610041, China
* Correspondence: lovelylw@126.com

Abstract: In this paper, we introduce and study a new class of differential set-valued inverse variational inequalities in finite dimensional spaces. By applying a result on differential inclusions involving an upper semicontinuous set-valued mapping with closed convex values, we first prove the existence of Carathéodory weak solutions for differential set-valued inverse variational inequalities. Then, by the existence result, we establish the stability for the differential set-valued inverse variational inequality problem when the constraint set and the mapping are perturbed by two different parameters. The closedness and continuity of Carathéodory weak solutions with respect to the two different parameters are obtained.

Keywords: differential set-valued inverse variational inequality; stability; Carathéodory weak solution

MSC: 49J40; 35B35; 49J53

Citation: Zhu, X.; Li, W.; Luo, X. Stability for a Class of Differential Set-Valued Inverse Variational Inequalities in Finite Dimensional Spaces. *Axioms* **2022**, *11*, 475. https://doi.org/10.3390/axioms11090475

Academic Editor: Simeon Reich

Received: 17 August 2022
Accepted: 13 September 2022
Published: 16 September 2022

1. Introduction

Let K be a nonempty closed convex set of R^n and $F : R^n \to 2^{R^n}$ be a set-valued mapping. A set-valued inverse variational inequality, denoted by SIVI(K, F), is to find $u \in R^n$ and $u^* \in F(u) \cap K$ such that

$$\langle y - u^*, u\rangle \geq 0, \quad \forall y \in K. \tag{1}$$

The solution to this problem is denoted by SOL(K, F). We write $\dot{x} := \frac{dx}{dt}$ for the time-derivative of function $x(t)$. In this paper, we study the following initial-value differential set-valued inverse variational inequality (denoted by DSIVI):

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + B(t, x(t))u(t) \\ u(t) \in SOL(K, G(t, x(t)) + F(\cdot)) \\ x(0) = x_0, \end{cases} \tag{2}$$

where $\Omega := [0, T] \times R^m$, $(f, B, G) := \Omega \to R^m \times R^{m\times n} \times R^n$, $F := R^n \to 2^{R^n}$. Time-dependent functions $x(t)$ and $u(t)$ satisfy (2) in the weak sense of Carathéodory for $t \in [0, T]$ means that x is an absolutely continuous function on $[0, T]$, $x(t)$ satisfies the differential equation for almost all $t \in [0, T]$ and the initial-value condition. Moreover, u is an integrable function on $[0, T]$ and $u(t) \in SOL(K, G(t, x(t)) + F)$ for almost all $t \in [0, T]$.

Differential variational inequalities (DVIs) arise in some applied problems such as, for example, differential Nash games, operations research, physical sciences, and structural dynamics [1,2]. DVIs were first systematically studied in finite dimensional spaces by Pang and Stewart [1] in 2008 and gained much more attention to theoretical results, numerical algorithms, and applications. Stewart [3] investigated the uniqueness for a class of index-one DVIs in finite dimensional spaces. Li et al. [4,5] researched differential mixed variational inequalities and impulsive differential variational inequalities in finite dimensional spaces and obtained some existence results and numerical methods by using some results on differential inclusions and discrete Euler time-dependent procedures. Li et al. [6] proved the existence of the Carathéodory weak solutions for differential inverse variational inequalities in finite dimensional spaces and gave an application on the time-dependent price equilibrium problem. In [7], Liu et al. first explored partial differential variational inequalities in Banach spaces and proved the nonemptiness and compactness of the solution set. For more related work about DVIs, see [8–12].

The inverse variational inequality, like the variational inequality, has broad applications in optimization, engineering, economics, mechanics, and transportation [13–20]. Very recently, Luo [21] studied the stability for the set-valued inverse variational inequality (1) on Banach spaces. If F is single-valued, the set-valued inverse variational inequality (1) can be reduced to the singe-valued inverse variational inequality in [13]. Furthermore, if F is single-valued and inverse, the set-valued inverse variational inequality (1) can be transformed into the classical variational inequality. However, the above transforms both failed if F is set-valued.

The stability analysis of a DVI with perturbed data is very helpful in identifying sensitive parameters, predicting the coming changes of the equilibria as a result of the changes in the governing system, and providing helpful information for designing different equilibrium systems. Gwinner [22] researched stability of the solution set for a DVI and obtained a novel upper set convergence result with respect to perturbations in the data. When the mapping and the constraint set are perturbed by different parameters, Wang et al. [23] studied the stability for a class differential mixed variational inequality in finite dimensional spaces. To the best of our knowledge, there are some results about the existence of solutions for differential variational and inverse variational inequalities in finite dimensional spaces. However, there are very few results about the existence of solutions for differential set-valued inverse variational inequalities and the stability for differential single-valued or set-valued inverse variational inequalities in finite dimensional spaces. Motivated by the aforementioned work, in this paper we are devoted to stability analysis for the DSIVI (2) in finite dimensional spaces.

The goal of this paper is to study the existence of the Carathéodory weak solutions and the stability for DSIVI (2) in finite dimensional spaces with the constraint set K and the set-valued mapping F being perturbed by two different parameters. Our results about the existence of the Carathéodory weak solutions for DSIVI (2) generalize the corresponding results in [6]. Our stability results about the differential set-valued inverse variational inequality are very new. We also give an example of a time-dependent price equilibrium control problem influenced by the seasons to show that the realistic problem can be transformed into the stability for the differential inverse variational inequality.

The paper is organized as follows. Section 2 contains some useful definitions and lemmas. In Section 3, the existence and uniqueness results of Carathéodory solutions for DSIVI (2) are considered. Furthermore, the closedness and continuity of Carathéodory solution set with respect to the perturbed data in the constraint set K and the set-valued mapping F are obtained.

2. Preliminaries

In this section, we will introduce some basic notations and preliminary results.

Definition 1 ([24])**.** *Let X and Y be two metric spaces; Y^* is the dual space of Y. We say a set-valued mapping $F : X \to 2^Y$ is*

(i) Upper semicontinuous at $x \in X$ if and only if for any neighborhood U of $F(x)$, there exists the neighborhood $B(x, \eta)$ of x with $\eta > 0$ such that

$$\forall x' \in B(x, \eta), \quad F(x') \subset U;$$

(ii) *Lower semicontinuous at $x \in X$ if and only if for any $y \in F(x)$ and for any sequence of elements $x_n \in X$ converging to x, there exists a sequence of elements $y_n \in F(x_n)$ converging to y;*

(iii) *Upper hemicontinuous at $x \in X$ if and only if for any $r \in Y^*$, the function $x \mapsto \sup\limits_{y \in F(x)} \langle r, y \rangle$ is upper semicontinuous at x.*

Definition 2 ([23,25])**.** *The set-valued mapping $F : R^n \to 2^{R^n}$ is said to be*

(i) *Strictly monotone on set $L \subset R^n$ iff for any $x, y \in L$, $x \neq y$, $x^* \in F(x)$, $y^* \in F(y)$, we have*

$$\langle x^* - y^*, x - y \rangle > 0;$$

(ii) *Strongly monotone with modulus $\mu > 0$, if for any $x, y \in R^n$ and $x^* \in F(x)$, $y^* \in F(y)$, we have*

$$\langle x^* - y^*, x - y \rangle \geq \mu \|x - y\|^2.$$

Definition 3. *A mapping $f : \Omega \to R^m$ (respectively, $B : \Omega \to R^{m \times n}$) is said to be Lipschitz continuous if there exists a constant $L_f > 0$ (respectively, $L_B > 0$) such that, for any $(t_1, x), (t_2, y) \in \Omega$, we have*

$$\|f(t_1, x) - f(t_2, y)\| \leq L_f(|t_1 - t_2| + \|x - y\|),$$

$$(respectively, \|B(t_1, x) - B(t_2, y)\| \leq L_B(|t_1 - t_2| + \|x - y\|)).$$

Lemma 1 ([26], Lemma 1)**.** *Let X and Y be metric spaces. If a set-valued mapping $F : X \to P_f(Y) := \{D \subset Y : D \text{ is nonempty, closed}\}$ is upper semicontinuous, then F is closed.*

Lemma 2 ([27], Theorem 5.1)**.** *Let $\mathbb{F} : \Omega \to 2^{R^m}$ be an upper semicontinuous set-valued mapping with nonempty closed convex values. Suppose that there exists a scalar $\rho^{\mathbb{F}} > 0$ satisfying*

$$\sup\{\|y\| : y \in \mathbb{F}(t, x)\} \leq \rho^{\mathbb{F}}(1 + \|x\|), \quad \forall (t, x) \in \Omega. \tag{3}$$

Then for every $x_0 \in R^m$, the DI:

$$\dot{x} \in \mathbb{F}(t, x), \quad x(0) = x_0$$

has a weak solution in the sense of Carathéodory.

Lemma 3 ([1], Lemma 6.3)**.** *Let $h : \Omega \times R^n \to R^m$ be a continuous function and $U : \Omega \to 2^{R^n}$ be a closed set-valued map such that for some constant $\eta_U > 0$,*

$$\sup_{u \in U(t,x)} \|u\| \leq \eta_U(1 + \|x\|), \quad \forall (t, x) \in \Omega.$$

Let $v : [0, T] \to R^m$ be a measurable function and $x : [0, T] \to R^m$ be a continuous function satisfying $v(t) \in h(t, x(t), U(t, x(t)))$ for almost all $t \in [0, T]$. There exists a measurable function $u : [0, T] \to R^n$ such that $u(t) \in U(t, x(t))$ and $v(t) = h(t, x(t), U(t))$ for almost all $t \in [0, T]$.

Throughout the rest of this paper, let $K \subset R^n$ be a nonempty, closed, and convex subset. The symbols " $\rightharpoonup$" and " $\to$" are used to denote the weak convergence and strong convergence. Let the barrier cone of K be denoted by

$$barr(K) := \{y \in R^n : \sup_{x \in K} \langle y, x \rangle < \infty\}.$$

The recession cone of K, denoted by K_∞, is defined by

$$K_\infty := \{d \in R^n : \exists t_n \to 0, \exists x_n \in K, t_n x_n \rightharpoonup d\}.$$

The negative polar cone of the nonempty set $D \subset R^n$, denoted by D^-, is defined by

$$D^- := \{y \in R^n : \langle y, x \rangle \le 0, \forall x \in D\}.$$

Lemma 4 ([21], Theorem 4.2)**.** *Let $L : Z_1 \to 2^{R^n}$ be a continuous set-valued mapping; $p_0 \in Z_1$, $\lambda_0 \in Z_2$ are given points; $F : R^n \times Z_2 \to 2^{R^n}$ is a set-valued mapping and lower semicontinuous on Z_2. Suppose that there exists a neighborhood of $P \times \Lambda$ of (p_0, λ_0), such that $L(p)$ has nonempty, closed, and convex values for any $p \in P$, and $F(x, \lambda)$ has nonempty closed values for every $x \in R^n$ and $\lambda \in \Lambda$. Moreover, for each $\lambda \in \Lambda$ and $q \in G(\Omega)$, the mapping $x \mapsto q + F(x, \lambda)$ is upper hemicontinuous and monotone. If*

$$(L(p_0))_\infty \cap \{x \in R^n : q + F(x, \lambda_0) \cap L(p_0) \neq \emptyset\}^- = \{0\},$$

then there exists a neighborhood $P' \times \Lambda'$ of (p_0, λ_0) with $P' \times \Lambda' \subset P \times \Lambda$, such that for every $(p, \lambda) \in P' \times \Lambda'$, the set $SOL(L(p), q + F(\cdot, \lambda))$ is nonempty and bounded.

In the rest of this paper, we assume (A) and (B) hold.

(A) f, B and G are Lipschitz continuous functions on Ω with Lipschitz constants $L_f > 0$, $L_B > 0$ and $L_G > 0$, respectively;

(B) B is bounded on Ω with ffi$_B := \sup_{(t,x)\in\Omega} \| B(t, x) \|$.

Remark 1. *If $f : \Omega \to R^m$ is a Lipschitz continuous function on Ω, we obtain that there exists a constant $\rho_f > 0$, for any $(t, x) \in \Omega$, such that*

$$\begin{aligned} \|f(t,x)\| &= \|f(t,x) - f(t_0,0) + f(t_0,0)\| \\ &\le \|f(t,x) - f(t_0,0)\| + \|f(t_0,0)\| \\ &\le L_f(|t - t_0| + \|x\|) + \|f(t_0,0)\| \\ &\le L_f(2T + \|x\|) + \|f(t_0,0)\| \\ &\le \rho_f(1 + \|x\|), \end{aligned}$$

where $t_0 \in [0, T], \rho_f = max\{L_f, 2L_f T + \|f(t_0, 0)\|\}$. Similarly, $G : \Omega \to R^n$ is a Lipschitz continuous function on Ω, so there exists a constant $\rho_G > 0$ such that $\|G(t, x)\| \le \rho_G(1 + \|x\|)$ for any $(t, x) \in \Omega$.

3. Existence and Uniqueness of Solutions for DSIVI (2)

In this section, we will show the existence and uniqueness of Carathéodory weak solutions for DSIVI (2) by applying Lemmas 2 and 3. For this purpose, we define a set-valued mapping $\mathbb{F} : \Omega \to 2^{R^n}$ as follows:

$$\mathbb{F}(t, x) := \{f(t, x) + B(t, x)u : u \in SOL(K, G(t, x) + F(\cdot))\}. \quad (4)$$

The following lemma presents some properties of the set-valued mapping $\mathbb{F}$ defined by (4) under the hypotheses (A) and (B).

In the following, $L^2([0, T]; R^n)$ is the set of all measurable functions $u : [0, T] \to R^n$, that satisfies $\int_0^T \|u(t)\|^2 dt < +\infty$. The norm of $\|u\|$ is defined by

$$\|u\| := (\int_0^T \|u(t)\|^2 dt)^{\frac{1}{2}}.$$

Lemma 5. *Let* (f, G, B) *satisfy conditions* (A) *and* (B). *Let* $K \subset R^n$ *be a nonempty, bounded, closed, and convex set and* $F : R^n \to 2^{R^n}$ *be an upper semicontinuous set-valued mapping with nonempty closed convex values. Suppose that there exists a constant* $\rho > 0$ *such that, for all* $q \in G(\Omega)$,

$$\sup\{\|u\| : u \in SOL(K, q + F(\cdot))\} \le \rho(1 + \|q\|). \tag{5}$$

Then, there exists a constant $\rho^{\mathbb{F}} > 0$ *such that* (3) *holds for the mapping* $\mathbb{F}$. *Hence,* $\mathbb{F}$ *is an upper semicontinuous closed-valued mapping on* Ω.

Proof. We first prove that there is a constant $\rho^{\mathbb{F}} > 0$ such that (3) holds for the mapping $\mathbb{F}$. For any $y \in \mathbb{F}(t, x)$, from the definition of $\mathbb{F}$, we know there exists $u \in SOL(K, G(t, x) + F(\cdot))$ such that $y = f(t, x) + B(t, x)u$. From conditions (A) and (B), it is easy to see that there exists positive constants ρ_f and ρ_G such that

$$\|f(t, x)\| \le \rho_f(1 + \|x\|), \tag{6}$$

and

$$\|G(t, x)\| \le \rho_G(1 + \|x\|). \tag{7}$$

Applying (5), we obtain

$$\|y\| \le \|f(t, x)\| + \|B(t, x)u\| \le (\rho_f + \delta_B\rho + \delta_B\rho\rho_G)(1 + \|x\|).$$

If we let $\rho^{\mathbb{F}} = \rho_f + \delta_B\rho + \delta_B\rho\rho_G$, then (3) holds.

Next, we prove that $\mathbb{F}$ is upper semicontinuous. We note that under the linear growth condition (3), the upper semicontinuity of F holds if $\mathbb{F}$ is closed. Therefore, we need to prove $\mathbb{F}$ is closed on Ω. Let $\{(t_n, x_n)\} \subset \Omega$ be a sequence converging to some vector $(t_0, x_0) \in \Omega$ and $\{f(t_n, x_n) + B(t_n, x_n)u_n\} \subset \mathbb{F}(t_n, x_n)$ converging to z_0, where $u_n \in SOL(K, G(t_n, x_n) + F(\cdot))$. It follows that sequence $\{u_n\}$ is bounded by (5). Therefore, $\{u_n\}$ has a convergent subsequence, denoted again by $\{u_n\}$, with a limit point $u_0 \in R^n$. According to $u_n \in SOL(K, G(t_n, x_n) + F(\cdot))$, it is easy to see that there exists $u_n^* \in F(u_n)$ and $G(t_n, x_n) + u_n^* \in K$ such that

$$\langle y - G(t_n, x_n) - u_n^*, u_n \rangle \ge 0, \quad \forall y \in K.$$

By the boundedness of K, we get $\{u_n^*\}$ is bounded and has a convergent subsequence with a limit u_0^*, as the set-valued mapping F is upper semicontinuous with nonempty closed convex values. By Lemma 1, we obtain that F is closed, which means $Graph(F) := \{(x, y) \in R^n \times R^n : y \in F(x)\}$ is closed. We know $(u_n, u_n^*) \in Graph(F)$ since $u_n^* \in F(u_n)$. Therefore, $u_0^* \in F(u_0)$. Since G is Lipschitz continuous, it follows $\{G(t_n, x_n)\}$ converges to $G(t_0, x_0)$. Since K is closed, it follows that $G(t_0, x_0) + u_0^* \in (G(t_0, x_0) + F(u_0)) \cap K$ and

$$\langle y - G(t_0, x_0) - u_0^*, u_0 \rangle \ge 0, \quad \forall y \in K.$$

That means $u_0 \in SOL(K, G(t_0, x_0) + F(\cdot))$ and so

$$f(t_n, x_n) + B(t_n, x_n)u_n \to z_0 = f(t_0, x_0) + B(t_0, x_0)u_0 \in \mathbb{F}(t_0, x_0).$$

Therefore, $\mathbb{F}$ is closed. This completes the proof. □

Remark 2. *We would like to point out that Lemma 5 extends Lemma 2.5 in [6].*

Theorem 1. *Let* (f, G, B) *satisfy conditions* (A) *and* (B). *Let* $K \subset R^n$ *be a nonempty, bounded, closed, and convex set and* $F : R^n \to 2^{R^n}$ *be an upper semicontinuous set-valued mapping with nonempty closed convex values. Suppose for any* $q \in G(\Omega)$, *there exists a constant* $\rho > 0$ *such* *that* (5) *holds and the set* $SOL(K, q + F(\cdot))$ *is nonempty, closed, and convex. Then, DSIVI* (2) *has a Carathéodory weak solution.*

Proof. By Lemma 5, we obtain that there exists a constant $\rho^{\mathbb{F}} > 0$ such that (3) holds for the mapping $\mathbb{F}$ defined by (4) and $\mathbb{F}$ is an upper semicontinuous closed-valued mapping on Ω. Next, for any $(t, x) \in \Omega$, we prove $\mathbb{F}(t, x)$ is convex. Since for any $q \in G(\Omega)$, $SOL(K, q + F(\cdot))$ is nonempty, it is easy to see that $\mathbb{F}(t, x)$ is nonempty. However, for any $f(t, x) + B(t, x)u_1, f(t, x) + B(t, x)u_2 \in \mathbb{F}(t, x)$, where $u_1, u_2 \in SOL(K, G(t, x) + F(\cdot))$, by the convex of $SOL(K, G(t, x) + F(\cdot))$, we know that there exists a constant $\eta \in (0, 1)$ such that

$$\begin{aligned} & \eta(f(t,x) + B(t,x)u_1) + (1-\eta)(f(t,x) + B(t,x)u_2) \\ = \; & f(t,x) + B(t,x)(\eta u_1 + (1-\eta)u_2) \in \mathbb{F}(t,x), \end{aligned}$$

where $\eta u_1 + (1 - \eta)u_2 \in SOL(K, G(t, x) + F(\cdot))$. This means $\mathbb{F}(t, x)$ is convex.

Because $\mathbb{F}$ is an upper semicontinuous set-valued mapping with nonempty closed convex values and there exists a constant $\rho^{\mathbb{F}} > 0$ such that (3) holds for the mapping $\mathbb{F}$, by Lemma 2, we obtain that the following differential inclusion $\dot{x} \in \mathbb{F}(t, x), x(0) = x_0$ has a Carathéodory weak solution $x(t)$. Thus, we have for any $t \in [0, T]$,

$$\int_0^t \dot{x}(s)ds \quad = \quad \int_0^t [f(s, x(s)) + B(s, x(s))u(s)]ds,$$

and

$$\|x(t)\| \quad \leq \quad \|x_0\| + \int_0^t \rho^{\mathbb{F}}(1 + \|x(s)\|)ds.$$

Then, by the Gronwall inequality, we obtain

$$\|x(t)\| \leq (\|x_0\| + \rho^{\mathbb{F}}T)e^{\rho^{\mathbb{F}}T}. \tag{8}$$

Therefore, from the above two inequalities we can obtain that $x(t)$ is absolutely continuous on $[0, T]$. Let $U(t, x) := SOL(K, G(t, x) + F(\cdot))$ and $h(t, x, u) := f(t, x) + B(t, x)u$. We conclude by Lemma 3 that there exists a measurable function $u(t)$ such that $u(t) \in SOL(K, G(t, x(t)) + F(\cdot))$ and $\dot{x}(t) = f(t, x) + B(t, x)u(t)$ for almost all t. By Lemma 6, it follows that for almost all $t \in [0, T]$, there exists $\rho > 0$ such that

$$\|u(t)\| \leq \rho(1 + \|G(t, x(t))\|),$$

where $u(t) \in SOL(K, G(t, x(t)) + F(\cdot))$. From (6) and (8), it follows from the above inequality that for almost all $t \in [0, T]$,

$$\|u(t)\| \leq \rho(1 + \rho_G(1 + (\|x_0\| + \rho^{\mathbb{F}}T)e^{\rho^{\mathbb{F}}T})).$$

Therefore, $u(t)$ is integrable on $[0, T]$. This completes the proof. □

Lemma 6. *Let* (f, G, B) *satisfy conditions* (A) *and* (B)*. Let* $K \subset R^n$ *be a nonempty, bounded, closed, and convex set. Suppose the following statements hold:*

(i) $F : R^n \to 2^{R^n}$ *is strictly monotone and upper hemicontinuous on* R^n*;*
(ii) *For any* $q \in G(\Omega)$, $K_\infty \cap \{x \in R^n : q + F(x) \cap K \neq \emptyset\}^- = \{0\}$*;*
(iii) *The interior of* $barr(K)$ *is nonempty.*

Then, $SOL(K, q + F(\cdot))$ *is a singleton for any* $q \in G(\Omega)$*. Moreover, there exists a constant* $\rho > 0$ *such that* (5) *holds for any* $q \in G(\Omega)$*.*

Proof. Using conditions (i)–(iii) and according to Theorem 3.2 in [21], we can obtain that $SOL(K, q + F(\cdot)) \neq \emptyset$ for any $q \in G(\Omega)$. Next, we show $SOL(K, q + F(\cdot))$ is a singleton for any $q \in G(\Omega)$. We assume $u_1, u_2 \in SOL(K, q + F(\cdot))$ and $u_1 \neq u_2$, and we have

$$q + u_1^* \in (q + F(u_1)) \cap K, \quad \langle y - q - u_1^*, u_1 \rangle \geq 0, \quad \forall y \in K, \tag{9}$$

and

$$q + u_2^* \in (q + F(u_2)) \cap K, \quad \langle y - q - u_2^*, u_2 \rangle \geq 0, \quad \forall y \in K. \tag{10}$$

Letting $y = q + u_2^*$ in (9), we have

$$\langle u_2^* - u_1^*, u_1 \rangle \geq 0.$$

Letting $y = q + u_1^*$ in (10), we have

$$\langle u_1^* - u_2^*, u_2 \rangle \geq 0.$$

It follows from the above two inequalities that

$$\langle u_2^* - u_1^*, u_1 - u_2 \rangle \geq 0. \tag{11}$$

Since F is strictly monotone, $u_1 \neq u_2$, $u_1^* \in F(u_1)$, $u_2^* \in F(u_2)$, we obtain

$$\langle u_2^* - u_1^*, u_2 - u_1 \rangle > 0,$$

which contradicts (11). That means $SOL(K, q + F(\cdot))$ is a singleton for any $q \in G(\Omega)$ and so there exists a constant $\rho > 0$ such that (5) holds for any $q \in G(\Omega)$. This completes the proof. □

Theorem 2. *Let $K \subset R^n$ be a nonempty, bounded, closed, and convex set. Let (f, G, B) satisfy conditions (A) and (B). Suppose the following statements hold:*

(i) $F : R^n \to 2^{R^n}$ is strictly monotone and upper hemicontinuous on R^n;

(ii) $F : R^n \to 2^{R^n}$ is an upper semicontinuous set-valued map with nonempty closed convex values;

(iii) For any $q \in G(\Omega)$, $K_\infty \cap \{x \in R^n : q + F(x) \cap K \neq \emptyset\}^- = \{0\}$;

(iv) The interior of $barr(K)$ is nonempty.

Then, DSIVI (2) has a Carathéodory weak solution.

Proof. It follows from conditions (i), (iii), (iv), and Lemma 6 that (5) holds. By condition (ii) and Lemma 5, we know there exists a constant $\rho^{\mathbb{F}} > 0$ such that (3) holds, where $\mathbb{F}$ is defined by (4). Applying Theorem 1, we get DSIVI (2) has a Carathéodory weak solution. This completes the proof. □

Remark 3. *From the above proof, it is easy to see that $u \in L^2([0, T]; R^n)$.*

Theorem 3. *Assume conditions (ii)–(iv) in Theorem 2 hold and $F : R^n \to 2^{R^n}$ is strongly monotone and upper hemicontinuous on R^n. Then, DSIVI (2) has a unique Carathéodory weak solution $(x, u) \in C([0, T]; R^m) \times L^2([0, T]; R^n)$.*

Proof. By Theorem 2, we know DSIVI (2) has Carathéodory weak solutions. Now, we only need to prove the uniqueness of the Carathéodory weak solution for DSIVI (2). For this purpose, we let (x_1, u_1) and (x_2, u_2) be the Carathéodory weak solutions for DSIVI (2). Therefore,

$$\begin{cases} x_1(t) = x_0 + \int_0^t f(\tau, x_1(\tau)) + B(\tau, x_1(\tau))u_1(\tau)d\tau, & \text{for any } t \in [0,T] \\ u_1(t) \in SOL(K, G(t, x_1(t)) + F(\cdot)), & \text{for almost all } t \in [0,T] \\ x_1(0) = x_0, \end{cases} \tag{12}$$

and

$$\begin{cases} x_2(t) = x_0 + \int_0^t f(\tau, x_2(\tau)) + B(\tau, x_2(\tau))u_2(\tau)d\tau, & \text{for any } t \in [0,T] \\ u_2(t) \in SOL(K, G(t, x_2(t)) + F(\cdot)), & \text{for almost all } t \in [0,T] \\ x_2(0) = x_0. \end{cases} \tag{13}$$

For almost all $t \in [0,T]$, it is easy to see $u_1(t) \in SOL(K, G(t, x_1(t)) + F(\cdot))$ and $u_2(t) \in SOL(K, G(t, x_2(t)) + F(\cdot))$. Therefore, there exists a measurable E on $[0,T]$ with $mE = 0$ (mE denotes the Lebesgue measure of the set E on $[0,T]$) such that for any $t \in [0,T] \setminus E$, there exists $u_1^*(t) \in F(u_1(t))$ and $G(t, x_1(t)) + u_1^*(t) \in K$ such that

$$\langle y - G(t, x_1(t)) - u_1^*(t), u_1(t) \rangle \geq 0, \quad \forall y \in K \tag{14}$$

and there exists $u_2^*(t) \in F(u_2(t))$ and $G(t, x_2(t)) + u_2^*(t) \in K$ such that

$$\langle y - G(t, x_2(t)) - u_2^*(t), u_2(t) \rangle \geq 0, \quad \forall y \in K. \tag{15}$$

For $t \in [0,T] \setminus E$, letting $y = G(t, x_2(t)) + u_2^*(t)$ in (14), we get

$$\langle G(t, x_2(t)) + u_2^*(t) - G(t, x_1(t)) - u_1^*(t), u_1(t) \rangle \geq 0.$$

For $t \in [0,T] \setminus E$, letting $y = G(t, x_1(t)) + u_1^*(t)$ in (15), we get

$$\langle G(t, x_1(t)) + u_1^*(t) - G(t, x_2(t)) - u_2^*(t), u_2(t) \rangle \geq 0.$$

Therefore, for $t \in [0,T] \setminus E$, one has

$$\langle G(t, x_1(t)) + u_1^*(t) - G(t, x_2(t)) - u_2^*(t), u_2(t) - u_1(t) \rangle \geq 0,$$

and

$$\langle G(t, x_1(t)) - G(t, x_2(t)), u_2(t) - u_1(t) \rangle \geq \langle u_1^*(t) - u_2^*(t), u_1(t) - u_2(t) \rangle.$$

Since F is strongly monotone on R^n, it yields for almost all $t \in [0,T]$,

$$\langle u_1^*(t) - u_2^*(t), u_1(t) - u_2(t) \rangle \geq \mu \|u_1(t) - u_2(t)\|^2. \tag{16}$$

From the Cauchy–Schwarz inequality, we know that

$$\begin{aligned} & \langle G(t, x_1(t)) - G(t, x_2(t)), u_2(t) - u_1(t) \rangle \\ \leq \quad & \|G(t, x_1(t)) - G(t, x_2(t))\| \|u_1(t) - u_2(t)\|. \end{aligned} \tag{17}$$

Therefore, combining (16) and (17), we get for almost all $t \in [0,T]$,

$$\begin{aligned} \mu \|u_1(t) - u_2(t)\| &\leq \|G(t, x_1(t)) - G(t, x_2(t))\| \\ &\leq L_G \|x_1(t) - x_2(t)\| \end{aligned} \tag{18}$$

and

$$\|u_1(t) - u_2(t)\| \leq \frac{L_G}{\mu} \|x_1(t) - x_2(t)\|. \tag{19}$$

Furthermore, from (12), (13), and (18), we infer that for any $t \in [0, T]$,

$$\begin{aligned} & \|x_1(t) - x_2(t)\| \\ = & \int_0^t \|f(\tau, x_1(\tau)) + B(\tau, x_1(\tau))u_1(\tau) - f(\tau, x_2(\tau)) + B(\tau, x_2(\tau))u_2(\tau)\|d\tau \\ \le & \int_0^t \|f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))\|d\tau + \delta_B \int_0^t \|u_1(\tau) - u_2(\tau)\|d\tau \\ \le & L_f \int_0^t \|x_1(\tau) - x_2(\tau)\|d\tau + \delta_B \frac{L_G}{\mu} \int_0^t \|x_1(\tau) - x_2(\tau)\|d\tau \\ \le & (L_f + \delta_B \frac{L_G}{\mu}) \int_0^t \|x_1(\tau) - x_2(\tau)\|d\tau. \end{aligned}$$

Apparently, there exists a constant $C = L_f + \delta_B \frac{L_G}{\mu} > 0$ such that

$$\|x_1(t) - x_2(t)\| \le C \int_0^t \|x_1(\tau) - x_2(\tau)\|d\tau.$$

According to the Gronwall inequality, we get $x_1(t) = x_2(t)$ for all $t \in [0, T]$, so $x_1 = x_2$ in $C([0, T], R^m)$. From (18), we have $u_1(t) = u_2(t)$ in R^n for almost all $t \in [0, T]$. This means $u_1 = u_2$ in $L^2([0, T], R^n)$. This completes the proof. □

4. Stability for DSIVI (2)

In this section, we aim to study the stability for DSIVI (2) in finite dimensional spaces when both the mapping and the constraint set are perturbed by two different parameters. For this purpose, we consider the parametric DSIVI, denoted by DSIVI ($L(p)$, $G(t, x(t)) + F(\cdot, \lambda)$), as follows:

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + B(t, x(t))u(t) \\ u(t) \in SOL(L(p), G(t, x(t)) + F(\cdot, \lambda)) \\ x(0) = x_0, \end{cases} \tag{20}$$

where (Z_1, d_1) and (Z_2, d_2) are two metric spaces. The nonempty closed convex subset K of R^n in DSIVI (2) is perturbed by a parameter p, which varies over (Z_1, d_1). Therefore, K is a perturbed set. That means $L : Z_1 \to 2^{R^n}$ is a set-valued mapping with nonempty closed convex values. The mapping $F : R^n \to 2^{R^n}$ is a set-valued mapping that is perturbed by a parameter λ, and λ varies over (Z_2, d_2). That is to say, $F : R^n \times Z_2 \to 2^{R^n}$. In what follows, to simplify notation, we let $S(p, \lambda)$ denote the Carathéodory weak solution for DSIVI (20). Next, we will establish the closedness and continuity of the mapping $(p, \lambda) \to S(p, \lambda)$.

Theorem 4. *Let (f, G, B) satisfy conditions (A) and (B), $p_0 \in Z_1$, $\lambda_0 \in Z_2$ be two given points. Assume the following conditions hold.*

(i) *$L : Z_1 \to 2^{R^n}$ is a continuous set-valued mapping with nonempty bounded closed convex values and $\bigcup_{p \in U(p_0)} L(p)$ is compact, where $U(p_0)$ is a neighborhood of p_0;*
(ii) *$F : R^n \times Z_2 \to 2^{R^n}$ is an upper semicontinuous set-valued mapping with nonempty closed convex values on $R^n \times Z_2$ and lower semicontinuous on Z_2;*
(iii) *There exists a neighborhood Λ of λ_0, for each $\lambda \in \Lambda$, the mapping $x \mapsto q + F(x, \lambda)$ is upper hemicontinuous and monotone for any $q \in G(\Omega)$;*
(iv) *The set $SOL(L(p_0), q + F(\cdot, \lambda_0))$ is nonempty and bounded for any $q \in G(\Omega)$;*
(v) *$F : R^n \times Z_2 \to 2^{R^n}$ is strictly monotone and upper hemicontinuous on R^n .*

Then, $S(p, \lambda)$ is closed at $(p_0, \lambda_0) \in Z_1 \times Z_2$.

Proof. From Theorem 3.2 in [21], we know that condition (iv) is equivalent to conditions (iii) and (iv) in Theorem 2. By conditions (i)–(iv), it follows from Lemma 4 that there exists a neighborhood $P' \times \Lambda'$ of (p_0, λ_0), $P' \times \Lambda' \subset P \times \Lambda$, such that for each $(p, \lambda) \in P' \times \Lambda'$, the set $SOL(L(p), q + F(\cdot, \lambda))$ is nonempty and bounded. It follows from Lemma 6 that there exists a constant $\rho > 0$ such that (5) holds for any $q \in G(\Omega)$. It is obvious that DSIVI (20) has solutions by Theorem 2.

Now, we prove $S(p, \lambda)$ is closed at (p_0, λ_0). Let $\{(p_n, \lambda_n)\} \subset P \times \Lambda$ be a given sequence with $(p_n, \lambda_n) \to (p_0, \lambda_0)$ and $(x_n, u_n) \in S(p_n, \lambda_n)$ with $(x_n, u_n) \to (x_0, u_0)$ in $C([0, T]; R^m) \times L^2([0, T], R^n)$. Therefore,

(a) For any $0 \leq s \leq t \leq T$,

$$x_n(t) - x_n(s) = \int_s^t f(\tau, x_n(\tau)) + B(\tau, x_n(\tau))u_n(\tau)d\tau;$$

(b) For almost all $t \in [0, T]$, there exists $u_n^*(t) \in F(u_n(t), \lambda_n)$ and $G(t, x_n(t)) + u_n^*(t) \in L(p_n)$, for any $y_n \in L(p_n)$, such that

$$\langle y_n - G(t, x_n(t)) - u_n^*(t), u_n(t)\rangle \geq 0;$$

(c) The initial condition

$$x_n(0) = x_0.$$

From the convergence u_n converges to u_0 in $L^2([0, T], R^n)$, we obtain

$$\int_{[0,T]} \|u_n(t) - u_0(t)\|^2 dt < \infty$$

and

$$\begin{aligned} \lim_{n\to\infty} \|u_n - u_0\|_{L^2} &= \lim_{n\to\infty} \Big(\int_{[0,T]} \|u_n(t) - u_0(t)\|^2 dt\Big)^{\frac{1}{2}} \\ &= 0. \end{aligned}$$

Moreover, applying the Holder inequality, we know

$$\int_{[0,T]} \|u_n(t) - u_0(t)\| dt \leq \Big(\int_{[0,T]} \|u_n(t) - u_0(t)\|^2 dt\Big)^{\frac{1}{2}} \Big(\int_{[0,T]} 1^2 dt\Big)^{\frac{1}{2}}$$

and

$$\lim_{n\to\infty} \int_{[0,T]} \|u_n(t) - u_0(t)\| dt \leq \lim_{n\to\infty} \Big(\int_{[0,T]} \|u_n(t) - u_0(t)\|^2 dt\Big)^{\frac{1}{2}} = 0.$$

This means u_n converges to u_0 in $L^1([0, T], R^n)$, which is equivalent to $\|u_n - u_0\|_{L^1} \to 0$. By Theorem 4.9 in [28], there exists a sequence $u_n(t)$ and a function $h \in L^1$ such that

$$u_n(t) \to u_0(t), \text{ for almost all } t \in [0, T] \tag{21}$$

and

$$\|u_n(t)\| \leq h(t), \text{ for almost all } t \in [0, T]. \tag{22}$$

Combining (21) and (22), by the Lebesgue control convergence theorem, we know

$$\lim_{n\to\infty} \int_{[0,T]} u_n(t) dt = \int_{[0,T]} u_0(t) dt.$$

However, from (b), it is easy to see that $G(t, x_n(t)) + u_n^*(t) \in L(p_n)$ for almost all $t \in [0, T]$. By condition (i), there exists a neighborhood $U(p_0)$ of p_0 such that $\bigcup_{p \in U(p_0)} L(p)$

is compact. Therefore, $\{u_n^*(t)\}$ has a subsequence, denoted again by $\{u_n^*(t)\}$, such that $u_n^*(t) \to u_0^*(t)$. Since $(u_n, \lambda_n) \to (u_0, \lambda_0)$, it follows from Lemma 1 and condition (ii) that $u_0^*(t) \in F(u_0(t), \lambda_0)$. Moreover, the lower semicontinuity of L implies that, for any $y \in L(p_0)$, there exists a sequence $\{y_n\}$ with $y_n \in L(p_n)$ such that $y_n \to y$.

Now, by (a), (b), and (c), we have

(a') For any $0 \le s \le t \le T$,

$$x_0(t) - x_0(s) = \int_s^t f(\tau, x_0(\tau)) + B(\tau, x_0(\tau))u_0(\tau)d\tau;$$

(b') For almost all $t \in [0, T]$, there exists $u_0^*(t) \in F(u_0(t), \lambda_0)$ and $G(t, x_0(t)) + u_0^*(t) \in L(p_0)$, for any $y \in L(p_0)$, such that

$$\langle y - G(t, x_0(t)) - u_0^*(t), u_0(t)\rangle \ge 0;$$

(c') The initial condition

$$x_0(0) = x_0.$$

Therefore, it deduces that $(x_0, u_0) \in S(p_0, \lambda_0)$. This completes the proof. □

Theorem 5. *Let (f, G, B) satisfy conditions (A) and (B); $p_0 \in Z_1$, $\lambda_0 \in Z_2$ are given points. Assume the following conditions hold.*

(i) $L : Z_1 \to 2^{R^n}$ is a continuous set-valued mapping with nonempty bounded closed convex values, and there exists a neighborhood $U(p_0)$ of p_0 such that $\bigcup_{p\in U(p_0)} L(p)$ is compact;

(ii) $F : R^n \times Z_2 \to 2^{R^n}$ is a upper semicontinuous set-valued mapping with nonempty closed convex values on $R^n \times Z_2$ and lower semicontinuous on Z_2;

(iii) For each $\lambda \in \Lambda$ and $q \in G(\Omega)$, the mapping $x \mapsto q + F(x, \lambda)$ is upper hemicontinuous and monotone, where Λ is a neighborhood of λ_0;

(iv) There exists a neighborhood $U(p_0, \lambda_0)$ of (p_0, λ_0) such that

$$\bigcup_{(p,\lambda)\in U(p_0,\lambda_0)} SOL(L(p), q + F(\cdot, \lambda))$$

is bounded for any $q \in G(\Omega)$;

(v) $F : R^n \times Z_2 \to 2^{R^n}$ is strongly monotone and upper hemicontinuous on R^n.

Then, $S(p, \lambda)$ is continuous at $(p_0, \lambda_0) \in Z_1 \times Z_2$.

Proof. From Theorem 3.2 in [21], we know that condition (iv) is equivalent to conditions (iii) and (iv) in Theorem 2. It follows from Theorem 3 that $S(p, \lambda)$ is a singleton by conditions (i), (ii), (iv), and (v). Let $S(p_n, \lambda_n) = (x_n, u_n)$ with $(p_n, \lambda_n) \to (p_0, \lambda_0)$. Next, we need to prove sequence $\{x_n\}$ and $\{u_n\}$ have convergent subsequences, respectively.

Step 1. $\{x_n\}$ is uniformly bounded.

It is known that $(x_n, u_n) \in S(p_n, \lambda_n)$. Therefore, for almost all $t \in [0, T]$,

$$\dot{x}_n(t) = f(t, x_n(t)) + B(t, x_n(t))u_n(t), \quad n = 1, 2, \cdots. \tag{23}$$

Since $\mathbb{F}(t, x_n)$ defined by (4) satisfies (3), for any $t \in [0, T]$, we have

$$\|x_n(t)\| \le \|x_0\| + \int_0^t \rho^{\mathbb{F}}(1 + \|x_n(s)\|)ds.$$

Applying the Gronwall inequality, we know

$$\|x_n(t)\| \le (\|x_0\| + \rho^{\mathbb{F}}T)e^{\rho^{\mathbb{F}}T}.$$

Clearly, $\{x_n\}$ is uniformly bounded with $\|x\| = \sup_{t\in[0,T]} \|x(t)\|$.

Step 2. $\{x_n\}$ is an equicontinuous family of functions. Since $(x_n, u_n) \in S(p_n, \lambda_n)$, for almost all $t \in [0, T]$, $u_n(t) \in SOL(L(p_n), G(t, x_n(t)) + F(\cdot, \lambda_n))$. By condition (iv), for almost all $t \in [0, T]$ and $n = 1, 2, \cdots$, there exists a constant C such that $\|u_n(t)\| \le C$.

In reality, (23) means for all $0 \le s \le t \le T$,

$$x_n(t) - x_n(s) = \int_s^t f(\tau, x_n(\tau)) + B(\tau, x_n(\tau))u_n(\tau)d\tau, \quad n = 1, 2, \cdots.$$

We note that f is a Lipschitz continuous function on Ω, so for all $(t, x) \in \Omega$, there exists a constant $\rho_f > 0$ such that

$$\|f(t, x)\| \le \rho_f(1 + \|x\|). \tag{24}$$

Since B is bounded on Ω and $\{u_n(t)\}$ is bounded for almost all $t \in [0, T]$, by (24), we have

$$\begin{aligned}
& \|x_n(t) - x_n(s)\| \\
= \ & \|\int_s^t f(\tau, x_n(\tau)) + B(\tau, x_n(\tau))u_n(\tau)d\tau\| \\
\le \ & \int_s^t \|f(\tau, x_n(\tau))\|d\tau + \int_s^t \|B(\tau, x_n(\tau))u_n(\tau)\|d\tau \\
\le \ & \int_s^t \|f(\tau, x_n(\tau))\|d\tau + \int_s^t \|B(\tau, x_n(\tau))\|\|u_n(\tau)\|d\tau \\
\le \ & \int_s^t \rho_f(1 + \|x_n(\tau)\|)d\tau + \delta_B C|t - s| \\
\le \ & \rho_f|t - s| + \rho_f(\|x_0\| + \rho^{\mathbb{F}} T)e^{\rho^{\mathbb{F}} T}|t - s| + \delta_B C|t - s| \\
\le \ & (\rho_f(1 + (\|x_0\| + \rho^{\mathbb{F}} T)e^{\rho^{\mathbb{F}} T}) + \delta_B C)|t - s|.
\end{aligned}$$

Let $M = \rho_f(1 + (\|x_0\| + \rho^{\mathbb{F}} T)e^{\rho^{\mathbb{F}} T}) + \delta_B C$. Therefore, there exists a constant M such that, for any $n = 1, 2, \cdots$,

$$\|x_n(t) - x_n(s)\| \le M|t - s|.$$

Then, sequence $\{x_n\}$ is equicontinuous. We can apply the Arzelà–Ascoli theorem to deduce that $\{x_n\}$ has a subsequence, denoted again by $\{x_n\}$, which converges to x_0.

Step 3. $S(p_n, \lambda_n) \to S(p_0, \lambda_0)$ in $C([0, T]; R^m) \times L^2([0, T]; R^n)$. We know that $u_n(t) \in SOL(L(p_n), G(t, x_n(t)) + F(\cdot, \lambda_n))$ for almost all $t \in [0, T]$. Then, there exists a measure E with $mE = 0$ such that $u_n(t) \in SOL(L(p_n), G(t, x_n(t)) + F(\cdot, \lambda_n))$ for any $t \in [0, T] \setminus E$. That is, for any $t \in [0, T] \setminus E$, there exists $u_n^*(t) \in F(u_n(t), \lambda_n)$ and $G(t, x_n(t)) + u_n^*(t) \in L(p_n)$ such that

$$\langle y - G(t, x_n(t)) - u_n^*(t), u_n(t)\rangle \ge 0, \quad \forall y \in L(p_n). \tag{25}$$

Take any small h such that $t + h \in [0, T] \setminus E$ and $u_n(t + h) \in SOL(L(p_n), G(t + h, x_n(t + h)) + F(\cdot, \lambda_n))$. Then, there exists $u_n^*(t + h) \in F(u_n(t + h), \lambda_n)$ and $G(t + h, x_n(t + h)) + u_n^*(t + h) \in L(p_n)$ such that, for any $t + h \in [0, T] \setminus E$,

$$\langle y - G(t + h, x_n(t + h)) - u_n^*(t + h), u_n(t + h)\rangle \ge 0, \quad \forall y \in L(p_n). \tag{26}$$

For any $t + h \in [0, T] \setminus E$, letting $y = G(t + h, x_n(t + h)) + u_n^*(t + h)$ in (25), we have

$$\langle G(t + h, x_n(t + h)) + u_n^*(t + h) - G(t, x_n(t)) - u_n^*(t), u_n(t)\rangle \ge 0.$$

For any $t \in [0, T] \setminus E$, letting $y = G(t, x_n(t)) + u_n^*(t)$ in (26), we have

$$\langle G(t, x_n(t)) + u_n^*(t) - G(t + h, x_n(t + h)) - u_n^*(t + h), u_n(t + h)\rangle \ge 0.$$

Therefore,

$$\begin{aligned} & \langle G(t+h, x_n(t+h)) - G(t, x_n(t)), u_n(t) - u_n(t+h)\rangle && (27)\\ \geq\ & \langle u_n^*(t) - u_n^*(t+h), u_n(t) - u_n(t+h)\rangle. && (28)\end{aligned}$$

By the monotonicity of F,

$$\langle u_n^*(t) - u_n^*(t+h), u_n(t) - u_n(t+h)\rangle \geq \mu \|u_n(t) - u_n(t+h)\|^2.$$

Applying the Cauchy–Schwarz inequality, we get

$$\begin{aligned} & \langle G(t+h, x_n(t+h)) - G(t, x_n(t)), u_n(t) - u_n(t+h)\rangle \\ \leq\ & \|G(t+h, x_n(t+h)) - G(t, x_n(t))\| \|u_n(t) - u_n(t+h)\|.\end{aligned}$$

Thus, combining the above two inequalities and applying the Lipschitz continuity of G, we obtain

$$\begin{aligned} \mu\|u_n(t) - u_n(t+h)\| &\leq \|G(t+h, x_n(t+h)) - G(t, x_n(t))\| \\ &\leq L_G(|h| + \|x_n(t+h) - x_n(t)\|),\end{aligned}$$

which means

$$\begin{aligned} \|u_n(t) - u_n(t+h)\| &\leq \frac{L_G}{\mu}(|h| + \|x_n(t+h) - x_n(t)\|) \\ &\leq \frac{L_G}{\mu}(|h| + M|h|) \\ &\leq \frac{L_G}{\mu}(M+1)|h|. && (29)\end{aligned}$$

Let $l = \frac{L_G}{\mu}(M+1)$. Since $\{x_n\}$ is equicontinuous, it follows from (29) that for any $\epsilon > 0$, there exists $\delta = min\{T, \frac{\epsilon}{l\sqrt{2T}}\}$ such that, for all $n = 1, 2, \ldots$ and all $|h| \leq \delta$,

$$\begin{aligned} \int_0^{T-h} \|u_n(t+h) - u_n(h)\|^2 dt &\leq \int_0^{T-h} l^2 h^2 dt \\ &\leq l^2 h^2 (T-h) \\ &\leq l^2 \delta^2 (T+\delta) \\ &< \epsilon^2. && (30)\end{aligned}$$

It is known that $\|u_n\|_{L^2} = (\int_{[0,T]} \|u_n(t)\|^2 dt)^{\frac{1}{2}} < \infty$, which means $\{u_n\}$ is bounded in $L^2[0,T]$. Applying inequality (30) and the boundedness of $\{u_n\}$, by Corollary 1.34 in [29], we get that the sequence $\{u_n\}$ is relatively compact in $L^2[0,T]$. We can obtain the closure of $\{u_n\}$ is compact. Therefore, $\{u_n\}$ exists a convergent subsequence, denoted again by $\{u_n\}$, which converges to u_0. Up to now, we get subsequence $(x_n, u_n) = S(p_n, \lambda_n)$ with $x_n \to x_0$ and $u_n \to u_0$. From Theorem 4, $S(p, \lambda)$ is closed at (p_0, λ_0). This means $(x_n, u_n) \to (x_0, u_0) = S(p_0, \lambda_0)$ and so $S(p, \lambda)$ is continuous at (p_0, λ_0). This completes the proof. □

5. An Example of a Time-Dependent Spatial Price Equilibrium Control Problem

In this section, we will give an example of the differential inverse variational inequality to the time-dependent spatial price equilibrium control problem. As discussed by Scrimali [15], assume that a single commodity is produced at m supply market, with typical supply market denoted by i, and is consumed at n demand markets, with typical demand market denoted by j, during the time interval $[0, T]$ with $T > 0$. Let (i, j) denote the typical pair of producers and consumers for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. Let $S_i(t)$ be the supply of the commodity produced at supply market i at time $t \in [0, T]$ and group the supplies into a column vector

$$S(t) = (S_1(t), \cdots S_m(t)) \in R^m.$$

Let $D_j(t)$ be the demand of the commodity associated with demand market j at time $t \in [0, T]$ and group the demands into a column vector

$$D(t) = (D_1(t), \cdots D_n(t)) \in R^n.$$

Let $x_{ij}(t)$ be the commodity shipment from supply market i to demand market j at time $t \in [0, T]$ and group the commodity shipments into a column vector $x(t) \in R^{mn}$.

Li et al. [6] studied the time-dependent spatial price equilibrium control problem by establishing the relation between the problem and a differential inverse variational inequality. We restate it here with a concise version.

Assume that, for any $t \in [0, T]$,

$$S_i(t) = \sum_{j=1}^{n} x_{ij}(t), \qquad D_j(t) = \sum_{i=1}^{m} x_{ij}(t)$$

and resource exploitations $S(x(t), u(t))$ at supply market and consumption $D(x(t), u(t))$ at demands market can be controlled by adjusting the tax $u(t)$. Let

$$W(t, x(t), u(t)) = (S(x(t), u(t)), D(x(t), u(t))), \quad \forall t \in [0, T],$$

which can be written as

$$W(t, x(t), u(t)) = G(t, x(t)) + F(u(t)),$$

where $G(t, x)$ is a Carathéodory function with $\gamma(t) \in L^2[0, T]$ such that

$$\|G(t, x)\| \leq \gamma(t) + \|x\|.$$

and F is a continuous mapping. Let

$$L = \{w \in L^2([0, T], R^{m+n}) : \underline{w}(t) \leq w(t) \leq \overline{w}(t) \text{ for almost all } t \in [0, T]\}$$

be the set of a feasible state influenced by the adjusted taxes $u(t)$, where $\underline{w}(t) = (\underline{S}(t), \underline{D}(t))$ and $\overline{w}(t) = (\overline{S}(t), \overline{D}(t))$ denote the lower and upper capacity constraints, respectively. Under some appropriate assumptions, finding the solution of a time-dependent optimal control equilibrium problem is equivalent to finding the Carathéodory solution $(x(t), u(t))$ for the following differential inverse variational inequality:

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + B(t, x(t))u(t) \\ u(t) \in SOL(-L, -G(t, x(t)) - F(\cdot)) \\ x(0) = x_0, \end{cases}$$

For more details, we refer the reader to [6].

However, the total amount of supply for a commodity and the relevant tax adjustments policy on the markets always vary with the sales season and the off-season [21]. In real life, any minute change in the proportion of each strategy seen will lead to a change in strategy. Let 0 denote the off-season and 1 denote the sales season. During the off-season, policy-makers will motivate manufacturers to develop resources by lowering the taxes they need to bear. During the sales season, policy-makers resist more development resources by increasing taxes on manufacturers. That means the set L of a feasible state is influenced by a parameter p, where $p \in \{0, 1\}$. Because the supply and demand of the commodity are also influenced by the seasons, we assume the mapping F is influenced by a parameter λ, where $\lambda \in \{0, 1\}$. Now, the time-dependent spatial price equilibrium control problem can be transformed into the following differential inverse variational inequality including parameters:

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + B(t, x(t))u(t) \\ u(t) \in SOL(-L(p), -G(t, x(t)) - F(\cdot, \lambda)) \\ x(0) = x_0, \end{cases}$$

Therefore, the time-dependent spatial price equilibrium control problem influenced by seasons will lead to a stability problem for a class of differential inverse variational inequalities.

6. Conclusions

The paper is concerned with the stability analysis of differential set-valued inverse variational inequalities in finite dimensional spaces. First, we proved an important result about a set-valued mapping, Lemma 5, which extends Lemma 2.5 in [6] and plays an important role in proving the existence of Carathéodory weak solutions for DSIVI (2). Then we obtained the existence of Carathéodory weak solutions for DSIVI (2). Second, we established closedness and continuity for the differential set-valued inverse variational inequality problem when the constraint set and the mapping are perturbed by two different parameters. Finally, we gave an example of a time-dependent spatial price equilibrium control problem, which can be transformed into a differential inverse variational inequality in finite dimensional spaces.

For further research, we can note the following directions: First, to adapt the main methods to study the existence of Carathéodory weak solutions and stability for differential set-valued inverse mixed variational inequalities in finite dimensional spaces; second, to use the theory of semigroups, set-valued mappings, and variational inequality to study the partial differential set-valued inverse variational inequalities in Banach spaces.

Author Contributions: X.Z. and W.L. analyzed the existence and stability for the differential set-valued inverse variational inequality; X.Z. and X.L. analyzed the applications for the differential inverse variational inequality; X.Z. is the major contributor in writing the manuscript text. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by Opening Fund of Geomathematics Key Laboratory of Sichuan Province (scsxdz2021yb06) and National Natural Science Foundation of China (Grant 11701480).

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the anonymous reviewers for their valuable comments and suggestions that helped us to improve the content of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Pang, J.S.; Stewart, D.E. Differential variational inequaliities. *Math. Program.* **2008**, *113*, 345–424. [CrossRef]
2. Chen, X.J.; Wang, Z.Y. Differential variational inequalitiy approach to dynamic games with shared constraints. *Math. Program.* **2014**, *146*, 379–408. [CrossRef]
3. Stewart, D.E. Uniqueness for index-one differential variational inequalities. *Nonlinear Anal. Hybrid Syst.* **2008**, *2*, 812–818. [CrossRef]
4. Li, X.S.; Huang, N.J.; O'Regan, D. Differential mixed variational inequalities in finite dimensional spaces. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 3875–3886. [CrossRef]
5. Li, X.S.; Huang, N.J. A class of impulsive differential variational inequalities in finite demensional space. *J. Frankl. Inst.* **2016**, *353*, 3151–3175. [CrossRef]
6. Li, W.; Wang, X.; Huang, N.J. Differential inverse variational inequalities in finite dimensional spaces. *Acta Math. Sci.* **2015**, *35*, 407–422. [CrossRef]
7. Liu, Z.H.; Zeng, S.D.; Motreanu, D. Evolutionary problems driven by variational inequalities. *J. Differ. Equ.* **2016**, *260*, 6787–6799. [CrossRef]
8. Liu, Z.H.; Sofonea, M. Differential quasivariational inequalities in contact machanics. *Acta Math. Sci.* **2017**, *37*, 26–32. [CrossRef]

9. Liu, Z.H.; Migrski, S.; Zeng, S.D. Partial differential variational inequalities involving nonlocal boundary conditions in Banach spaces. *J. Differ. Equ.* **2017**, *263*, 3989–4006. [CrossRef]
10. Chen, T.; Huang, N.J.; Li, X.S.; Zhou, Y.Z. A new class of differential ninliner system involving parabolic variational and history-dependent hemi-variational inequalities arising in contact mechanics. *Common. Nonliner Sci. Number. Simul.* **2021**, *101*, 24.
11. Li, W.; Wang, X.F. Stability analysis of partial differential set-valued variational inequalities on Banach spaces. *J. Convex Anal.* **2020**, *27*, 423–442.
12. Tang, G.J.; Cen, J.X.; Nguyen, V.T.; Zeng, S.D. Differential variational-hemivariational inequalities: Existence, uniqueness, stability, and convergence. *J. Fixed Point Theory Appl.* **2020**, *22*, 83. [CrossRef]
13. He, B.S.; Liu, X.H. Inverse variational inequalities in the economic field: Applications and algorithms. *Science Paper Online* 2006.
14. Yang, J.F. Dynamic power price problem: An inverse variational inequality approach. *J. Ind. Manag. Optim.* **2008**, *4*, 673–684. [CrossRef]
15. Scrimali, L. An inverse variational inequality approach to the evolutionary spatial price equilibrium problem. *Optim. Eng.* **2012**, *13*, 375–387. [CrossRef]
16. Xu, H.K.; Dey, S.; Vetrivel, V. Notes on a neural network approach to inverse variational inequalities. *Optimization* **2020**, *70*, 901–910. [CrossRef]
17. Jiang, Y.; Cai, X.G.; Han, D. Solving policy design problems: Alternating direction method of multipliers-based methods for structured inverse variational inequalities. *Eur. J. Oper. Res.* **2020**, *280*, 417–427. [CrossRef]
18. Yang, H.; Bell, M.G.H. Traffic restraint, road pricing and network equilibrium. *Transp. Res. Part B Methodol.* **1997**, *31*, 303–314. [CrossRef]
19. He, B.S.; Liu, H.X.; Li, M.; He, X.Z. PPA-based methods for monotone inverse variational inequalities. *Sci. Pap. Online* **2006**.
20. He, B.S.; He, X.Z.; Liu, H.X. Solving a class of constrained 'black-box' inverse variational inequalities. *Eur. J. Oper. Res.* **2010**, *204*, 391–401. [CrossRef]
21. Luo, X.P. Stability analysis of set-valued inverse variational inequalities in reflexive Banach spaces. *J. Fixed Point Theory Appl.* **2021**, *23*, 41. [CrossRef]
22. Gwinner, J. On a new class of differential variational inequalities and a stability result. *Math. Program.* **2013**, *139*, 205–221. [CrossRef]
23. Wang, X.; Li, W.; Li, X.S.; Huang, N.J. Stability for differential mixed variational inequalities. *Optim. Lett.* **2014**, *8*, 1873–1887. [CrossRef]
24. Aubin, J.P.; Frankowska, H. *Set-Valued Analysis*; Birkhauser: Boston, MA, USA, 1990.
25. Fan, J.H.; Wang, X.G. Gap functions and global error bounds for set-valued variational inequalities. *J. Comput. Appl. Math.* **2010**, *233*, 2956–2965. [CrossRef]
26. Guo, F.M.; Li, W.; Xiao, Y.B.; Migórski, S. Stability analysis of partial differential variational inequalities in Banach spaces. *Nonlinear Anal. Model. Control.* **2020**, *25*, 69–83. [CrossRef]
27. Deimling, K. *Multivalued Differential Equations*; Walter de Gruyter: Berlin, Germany, 1992.
28. Haim, B. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*; Springer: New York, NY, USA, 2010; pp. 181–199. [CrossRef]
29. Hu, S.C.; Papageorgiou, N.S. *Handbook of Multivalued Analysis: Volume II: Applications*; Mathematics and Its Applications; Springer: New York, NY, USA, 1997.