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The Asymptotic Behavior for Generalized Jiřina Process

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Abstract: As the classic branching process, the Galton-Watson process has obtained intensive attentions in the past decades. However, this model has two idealized assumptions–discrete states and time-homogeneity. In the present paper, we consider a branching process with continuous states, and for any given $n \in \mathbb{N}$, the branching law of every particle in generation n is determined by the population size of generation n. We consider the case that the process is extinct with Probability 1 since in this case the process will be substantially different from the size-dependent branching process with discrete states. We give the extinction rate in the sense of L^2 and almost surely by the form of harmonic moments, that is to say, we show how fast $\{Z_n^{-1}\}$ grows under a group of sufficient conditions. From the result of the present paper, we observe that the extinction rate will be determined by an asymptotic behavior of the mean of the branching law. The results obtained in this paper have the more superiority than the counterpart from the existing literature.

Keywords: size-dependent Jiřina process; L^2-convergence; extinction

MSC: 60J80

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1. Introduction and Preliminaries

Branching process is an important class of Markov processes, which describes the survival and extinction of a particle system. The most classical branching process is called the Galton-Watson process (see [1]). For a chosen family, Galton and Watson [1] used this process to record the number of males in each generation. For a Galton-Watson process $\{Z_n\}$,, we usually set $Z_0 = 1$, which means that there is a male ancestor in the family. The relationship between Z_{n+1} and Z_n is written by

$$Z_{n+1} = \mathbf{1}_{Z_n \geq 1} \sum_{i=1}^{Z_n} \eta_{n,i},$$

where $\eta_{n,i}$ presents the number of boys whose father (in generation n) is indexed by i. In a Galton-Watson process, the random array $\{\eta_{n,i}\}_{n,i \in \mathbb{N}}$ is set to be i.i.d. Hence, Galton-Watson process is a time homogeneous Markov chains with discrete state. There are two idealized assumptions in this model: one is the discrete state space, the other is the property of time homogeneous. In other words, there are two directions to extend this model.

Jiřina process (see [2–5]) is the continuous version of the Galton-Watson process, which stresses that the role of $\eta_{n,i}$ can take value in $\mathbb{R}^+$ ($\mathbb{R}^+ := [0, +\infty)$) instead of $\mathbb{N}$. Since the state space of this process is a subset of $\mathbb{R}^+$, we use the Laplace transform to describe the relationship between the number of particles in generation n and $n+1$, which is described by

$$\mathbb{E}(e^{-sZ_{n+1}} | Z_n = x) = e^{-xF(s)}, \ \ x \in \mathbb{R}^+,$$

where $F(s)$ is a cumulate generate function of a certain infinitely divisible distribution G. G can be observed as the common branching mechanism (i.e., the law of $\eta_{1,1}$) of each particle.

It should be noted that in the above equality, $F(s)$ is independent of n, thus, we see that the Jiřina process is still time-homogeneous.

To break the feature of time homogeneous, several time-inhomogeneous branching processes have been studied over the past decades. There are different motivations to construct the time-inhomogeneous property for a branching process, one of which assumes that the common law of $\eta_{n,1}, \eta_{n,2}, \ldots$ is depending on Z_n, and $\eta_{n,i}$ takes value in $\mathbb{N}$ for every n, i. We call this a time-inhomogeneous branching process as the size-dependent branching process (with discrete time and discrete state). This assumption (the law of $\eta_{n,1}$ depends on Z_n) has a strong practical background; for example, when a country is overpopulated, the government may promote family planning, while if a country faces the problem of population scarcity, the government will encourage childbearing. This model has been investigated in [6–8] and some other papers.

In the present paper, the model we consider is the continuous version of the size-dependent branching process, which is also called the generalized Jiřina process (for short, GJP). This model was introduced in [9], where the model is defined by the Laplace transform as

$$\mathbb{E}(\mathrm{e}^{-Z_{n+1}\tau}|Z_n = x) = \mathrm{e}^{-xF(x,s)}, \quad x \in \mathbb{R}^+, \tag{1}$$

where $F(x,s)$ is called a reproduction cumulative function (for short, r.c.f.) and it has the following representation:

$$F(x,s) = r(x)\tau + \int_{0^+}^{+\infty} (1 - \mathrm{e}^{-us}) v(x, \mathrm{d}u). \tag{2}$$

We can refer to [9] on how to obtain (2). On the other hand, ref. [9] also explains that $r(x)$ is a non-negative Borel function, and $(1 \wedge u)v(x, \mathrm{d}u)$ is a bounded kernel from $\mathbb{R}^+$ to $(0, +\infty)$. That is to say,

$$\forall x \geq 0, \quad \int_{0^+}^{+\infty} (1 \wedge u) v(x, \mathrm{d}u) < +\infty.$$

Hence, we see that the r.c.f. $F(x,\tau)$ is determined by $r(x)$ and $v(x, \mathrm{d}u)$. Obviously, if there exist a constant r and a measure v on $(0, +\infty)$ such that

$$r(x) \equiv r, \qquad v(x, \mathrm{d}u) \equiv v(\mathrm{d}u),$$

then GJP will degenerate to the Jiřina process. Moreover, from (1) one can see

$$\mathbb{E}(Z_{n+1}|Z_n = x) = -\frac{\partial \mathrm{e}^{-xF(x,s)}}{\partial s}\Big|_{s=0^+}.$$

Note that

$$\lim_{s \to 0^+} F(x,s) = 0$$

and

$$\lim_{s \to 0^+} \frac{\partial F(x,s)}{\partial s} = r(x) + \int_{0^+}^{+\infty} u v(x, \mathrm{d}u).$$

Actually, we have set that $(1 \wedge u)v(x, \mathrm{d}u)$ is a bounded kernel. Denote

$$m(x) := r(x) + \int_{0^+}^{+\infty} u v(x, \mathrm{d}u),$$

then we have

$$\mathbb{E}(Z_{n+1}|Z_n = x) = x m(x),$$

which means that $m(x)$ presents the expectation of the children reproduced by unit parent when the generation of the parent contains x particle(s). The above equality is equivalent to

$$\mathbb{E}(Z_{n+1}|Z_n) = Z_n m(Z_n).$$

Denote

$$\sigma^2(x) := \int_{0^+}^{+\infty} u^2 v(x, \mathrm{d}u) = \frac{\partial^2 \mathrm{e}^{-xF(x,s)}}{\partial^2 s}\Big|_{s=0^+}.$$

By a direct calculation we obtain

$$\mathbb{E}(Z_{n+1}^2|Z_n) = \sigma^2(Z_n)Z_n + Z_n^2 m^2(Z_n).$$

For a branching process $\{Z_n\}$, a very important topic, which is usually considered first, is the limit behavior of Z_n and the distribution of the limit (if it exists). For example, the celebrated Kesten-Stigum theorem (see [1], Chapter 1) for the Galton-Watson process and various generalized Kesten-Stigum theorem for different types of branching processes (see [3,7,10]). In summary, the Kesten-Stigum theorem and its various of generalized versions demonstrate that $\{Z_n\}$ converges to 0 with Probability p_0 and to $+\infty$ with Probability $1 - p_0$ and p_0 depends on the branching mechanism (reproduction law) of the branching process. Ref. [11] showed that the asymptotic behavior of GJP also behaves as

$$\mathbb{P}\Big(\lim_{n\to\infty} Z_n \in \{0, +\infty\}\Big) = 1$$

and $p_0 := \mathbb{P}(Z_n \to 0)$ is depending on some properties of $F(x,s)$. The author of [11] also pointed out that it is as similar as the asymptotic behavior of size-dependent branching process for the case $\{Z_n \to +\infty\}$. The most interesting and worth investigating is the case that $\{Z_n \to 0\}$, since when the state space is $\mathbb{N}$, then $Z_n \to 0$ means that there exists a finite generation n such that $Z_n = 0$ but Z_n can always be positive even though $Z_n \to 0$ when the state space is $\mathbb{R}^+$. Under some mild assumptions, ref. [10] gave the extinction rate of Z_n in the sense of almost surely when $\mathbb{P}(Z_n \to 0) = 1$. The idea to deal with the extinction rate is to consider the growth rate of Z_n^{-1}, then, the method to show the growth rate of the size-dependent branching process $\{X_n\}$ when $\mathbb{P}(X_n \to +\infty) = 1$ can be referred. Ref. [12] gave a sufficient condition to ensure that the extinction rate in the sense of it almost surely is also the extinction rate in the sense of L^2. In the present paper, we obtain a new extinction rate, which is easier to understanding by the definition of the mean function $m(\cdot)$ (see Section 3 for detail). Combining with the result in [12], we can observe that an extinct GJP may have different extinction rates under different conditions.

In this paper, we consider the rate of Z_n in the sense of almost surely and L^2 when the GJP behaves as $\mathbb{P}(Z_n \to 0) = 1$. We will give another group of sufficient conditions to ensure that there exists a constant sequence $\{c_n\}$ such that $\{c_n/Z_n\}$ has a limit in the sense of almost surely and L^2. Compared with the previous results, our results have more values for applications.

The GJP has a strong connection with reality. We can use GJP to model a number of chemical reactions and biological situations. For instance, it is proper to describe the trend of the concentration by GJP for some bacteria or virus whose reproduction depend on their concentration in the medium. For more examples, we recommend [7] and the references therein.

2. Main Results

For the sake of presenting our results, first of all, we give some basic assumptions as follows.

(A1) $r := \lim_{x\to 0^+} m(x)$, where $0 < r < 1$.

(A2) There exits a function $\bar{m}(x) \geq m(x)$ for all $x \geq 0$ which satisfies that $\inf_{x\geq 0} \bar{m}(x) \geq r$ and

$$p(x) := |\bar{m}(1/x) - r| (= \bar{m}(1/x) - r)$$

is non-increasing, $xp(x)$ is non-decreasing and concave, and

$$\int_1^{+\infty} \frac{p(x)}{x} \mathrm{d}x < +\infty.$$

(A3) For any $x \geq 0$, it satisfies $rx\int_0^{+\infty} \mathrm{e}^{-xF(x,s)} \mathrm{d}s \leq 1$.

(A4) $xp(\sqrt{x})$ is non-decreasing, concave and $xp^2(\sqrt{x})$ is concave.

(A5) For any $x \geq 0$, it satisfies $r^2x^2\int_0^{+\infty} s\mathrm{e}^{-xF(x,s)} \mathrm{d}s \leq 1$.

We remind that if $p''(x)$ exists on $(0, +\infty)$, then (A2) implies (A4). Denote

$$Y_n := \frac{1}{Z_n}, \quad q := \frac{1}{r}, \quad S_n := \frac{Y_n}{q^n} = \frac{r^n}{Z_n}.$$

First, we give some lemmas and results which will be used during, as we prove our main theorems.

Lemma 1. *Suppose that $h(x)$ is a positive and non-increasing function, then for any $t > 1, \epsilon > 0$, the following propositions are equivalent:*

(1) $\int_1^{+\infty} \frac{h(x)}{x} \mathrm{d}x < +\infty$;

(2) $\sum_{n=1}^{\infty} h(\epsilon t^n) < +\infty$.

Proof. See ([6], p. 42). □

Lemma 2. *Let $h(x)$ be a positive and non-increasing real function defined on $[0, +\infty)$. Assume that $xh(x)$ is non-decreasing and $\int_1^{+\infty} \frac{h(x)}{x} \mathrm{d}x < +\infty$. Let $\{c_n\}$ be a positive sequence and there exists a $t > 1$ such that for any n, it satisfies*

$$|c_{n+1} - c_n| \leq c_n h(c_n t^n),$$

then $\{c_n\}$ exists a finite non-negative limit. Moreover, there exists a constant $\tilde{c}$ depending on $h(x)$ and t such that, $\lim_{n\to\infty} c_n > 0$ only if the first term $c_0 > \tilde{c}$.

Proof. See ([6], p. 45). □

It is worth mentioning that the method from this paper is mainly concentrating on the martingale convergence theorems listed below.

Theorem 1. *(Martingale convergence theorem, (ref. [13], p. 270)) If $\{\xi_n\}$ is a sub-martingale and* $\sup_n \mathbb{E}|\xi_n| < +\infty$, *then there exists a random variable (denoted by $\xi_{+\infty}$) satisfying that*

$$\lim_{n\to\infty} \xi_n = \xi_{+\infty}, \text{ a.s.}, \quad \mathbb{P}(\xi_{+\infty} < +\infty) = 1, \quad \mathbb{E}|\xi_{+\infty}| < +\infty.$$

Theorem 2. *(Martingale L^p convergence theorem, (ref. [14], p. 60)) If $\{\xi_n\}$ is a sub-martingale and $\sup_n \mathbb{E}|\xi_n^p| < +\infty$ for some $p > 1$, then, there exists a random variable (denoted by $\xi_{+\infty}$) satisfying that*

$$\lim_{n\to\infty} \xi_n = \xi_{+\infty}, \text{ a.s., } L^p, \quad \mathbb{P}(\xi_{+\infty} < +\infty) = 1, \quad \mathbb{E}|\xi_{+\infty}^p| < +\infty.$$

Now, we give our main results as follows:

Theorem 3. *Let $\{Z_n\}$ be a GJP, if Assumptions (A1)–(A3) hold and $\mathbb{P}(Z_0 = z_0) = 1$, where z_0 is a positive constant, then there exist a constant $\gamma \in (0, +\infty)$ and a random variable S (both depending on z_0) such that*

$$\gamma = \lim_{n\to\infty} \mathbb{E}(S_n | Z_0 = z_0),$$

$$S = \lim_{n\to\infty} S_n, \text{ a.s.,}$$

and

$$\mathbb{E}S < +\infty.$$

Proof. Let $\mathcal{F}_n$ be the σ-algebra field, which is generated by $Z_0, Z_1, \ldots, Z_n$. Recalling that $Z_n := \frac{1}{Y_n}$, which means that

$$\begin{aligned} \mathbb{E}(Y_{n+1}|\mathcal{F}_n) &= \mathbb{E}\left(\frac{1}{Z_{n+1}}|\mathcal{F}_n\right) \\ &= \mathbb{E}\left(\int_0^{+\infty} \mathrm{e}^{-sZ_{n+1}}\mathrm{d}s \big| Z_n\right) \\ &= \int_0^{+\infty} \mathbb{E}(\mathrm{e}^{-sZ_{n+1}}|Z_n)\mathrm{d}s \\ &= \int_0^{+\infty} \mathrm{e}^{-Z_n F(Z_n, s)}\mathrm{d}s \\ &= \int_0^{+\infty} \mathrm{e}^{-\frac{1}{Y_n}F(\frac{1}{Y_n}, s)}\mathrm{d}s \\ &:= h(Y_n). \end{aligned} \tag{3}$$

The second equality above is due to $c\int_0^{+\infty} \mathrm{e}^{-cs}\mathrm{d}s = 1$, where $c > 0$ is a constant. By Taylor's expansion we can observe

$$F(x, s) \leq m(x)s. \tag{4}$$

Assumption (A3) and (4) imply that

$$\int_0^{+\infty} \mathrm{e}^{-x\bar{m}(x)s}\mathrm{d}s \leq \int_0^{+\infty} \mathrm{e}^{-xm(x)s}\mathrm{d}s \leq \int_0^{+\infty} \mathrm{e}^{-xF(x,s)}\mathrm{d}s \leq \frac{1}{rs} = \int_0^{+\infty} \mathrm{e}^{-xrs}\mathrm{d}s.$$

By the smoothing property of conditional expectation and (3), we obtain that

$$\begin{aligned} &|\mathbb{E}S_n - \mathbb{E}S_{n+1}| \\ =& \frac{1}{q^{n+1}}|\mathbb{E}(qY_n) - \mathbb{E}(h(Y_n))| \\ =& \frac{1}{q^{n+1}}\left|\mathbb{E}\int_0^{+\infty} \mathrm{e}^{-\frac{s}{qY_n}}\mathrm{d}s - \mathbb{E}\int_0^{+\infty} \mathrm{e}^{-\frac{1}{Y_n}F(\frac{1}{Y_n}, s)}\mathrm{d}s\right| \\ \leq& \frac{1}{q^{n+1}}\left|\mathbb{E}\int_0^{+\infty} (\mathrm{e}^{-rsZ_n} - \mathrm{e}^{-Z_n\bar{m}(Z_n)s})\mathrm{d}s\right|. \end{aligned}$$

According to the mean value theorem, there exists a constant $\vartheta \in [0,1]$ such that

$$\begin{aligned}
&\left|\mathbb{E}\int_0^{+\infty}(\mathrm{e}^{-rsZ_n}-\mathrm{e}^{-Z_n\bar{m}(Z_n)s})\mathrm{d}s\right| \\
&=\left|\mathbb{E}\int_0^{+\infty}(\mathrm{e}^{-(rsZ_n+\vartheta(Z_n\bar{m}(Z_n)s-rsZ_n))})(sZ_n\bar{m}(Z_n)-rsZ_n)\mathrm{d}s\right| \\
&\leq\left|\mathbb{E}\int_0^{+\infty}\mathrm{e}^{-rsZ_n}(\bar{m}(Z_n)sZ_n-rsZ_n)\mathrm{d}s\right| \\
&=\left|(\bar{m}(Z_n)-r)Z_n\mathbb{E}\int_0^{+\infty}\mathrm{e}^{-rsZ_n}s\mathrm{d}s\right|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&|\mathbb{E}S_n-\mathbb{E}S_{n+1}| \\
&\leq\frac{1}{q^{n+1}}\left|\mathbb{E}\left(\frac{(\bar{m}(Z_n)-r)Z_n}{r^2Z_n^2}\right)\right| \\
&\leq\left|\mathbb{E}\left(\frac{(\bar{m}(Z_n)-r)Y_n}{r^2q^{n+1}}\right)\right| \\
&=\frac{1}{r}\mathbb{E}(S_n|(\bar{m}(Z_n)-r)|).
\end{aligned}$$

From Assumption (A2), i.e., the concavity of $p(x)$, we have

$$|\mathbb{E}S_{n+1}-\mathbb{E}S_n|=\frac{1}{r}\mathbb{E}(p(Y_n)S_n)\leq\frac{1}{r}p(\mathbb{E}Y_n)\mathbb{E}(S_n)=\frac{1}{r}p(q^n\mathbb{E}S_n)\mathbb{E}(S_n).$$

Note that $q>1$, hence by applying Lemma 2, it follows that $\lim_{n\to\infty}\mathbb{E}S_n$ exists and

$$b:=\lim_{n\to\infty}\mathbb{E}S_n<+\infty.$$

Note that b lies on the starting state Z_0. Since $\mathbb{P}(Z_0=z_0)=1$, then by using Lemma 2 it is easy to observe that $b>0$ if $\mathbb{E}S_0=1/z_0$ large enough. Therefore, by a similar argument as stated in [12], we can observe $b>0$ only if $z_0>0$.

On the other hand, noting that

$$S_n-\mathbb{E}(S_{n+1}|\mathcal{F}_n)=\int_0^{+\infty}\frac{1}{t^{n+1}}(\mathrm{e}^{-rsZ_n}-\mathrm{e}^{-Z_nF(Z_n,s)})\mathrm{d}s>0,$$

we declare that $\{S_n,\mathcal{F}_n\}$ is a non-negative super martingale. Using Theorem 1 we speculate that there exists a random variable S such that

$$\lim_{n\to\infty}S_n=S,\ \text{ a.s..}$$

By Fatou's Lemma we claim that

$$\mathbb{E}S\leq\lim_{n\to\infty}\mathbb{E}S_n<+\infty.$$

Accordingly, we complete the proof. □

Theorem 4. *Let $\{Z_n\}$ be a GJP, Assumptions (A1)–(A5) hold and $\mathbb{P}(Z_0=z_0)=1$. Then,*

$$S=\lim_{n\to\infty}S_n,\ \text{ in }\ L^2,$$

and

$$\mathbb{P}(S>0)>0.$$

Proof. Recall a simple calculation

$$c^2\int_0^{+\infty} s\mathrm{e}^{-cs}\mathrm{d}s = 1,\quad c>0.$$

First, we observe that

$$\begin{aligned}\mathbb{E}(Y_{n+1}^2|\mathcal{F}_n) &= \mathbb{E}\left(\frac{1}{Z_{n+1}^2}|\mathcal{F}_n\right)\\ &= \mathbb{E}\left(\int_0^{+\infty} s\mathrm{e}^{-sZ_{n+1}}\mathrm{d}s|Z_n\right)\\ &= \int_0^{+\infty} s\mathbb{E}(\mathrm{e}^{-sZ_{n+1}}|Z_n)\mathrm{d}s\\ &= \int_0^{+\infty} s\mathrm{e}^{-Z_nF(Z_n,s)}\mathrm{d}s\\ &= \int_0^{+\infty} s\mathrm{e}^{-\frac{1}{Y_n}F\left(\frac{1}{Y_n},s\right)}\mathrm{d}s.\end{aligned}$$

Hence, one sees that

$$\begin{aligned}&|\mathbb{E}S_n^2-\mathbb{E}S_{n+1}^2|\\ =\;& \frac{1}{q^{2n+2}}\left|\mathbb{E}\int_0^{+\infty} s\mathrm{e}^{-\frac{s}{qY_n}}\mathrm{d}s-\mathbb{E}\int_0^{+\infty} s\mathrm{e}^{-\frac{1}{Y_n}F\left(\frac{1}{Y_n},s\right)}\mathrm{d}s\right|\\ \leq\;& \frac{1}{q^{2n+2}}\left|\mathbb{E}\left(\int_0^{+\infty} s(\mathrm{e}^{-rsZ_n}-\mathrm{e}^{-Z_n\bar{m}(Z_n)s})\mathrm{d}s\right)\right|.\end{aligned}\tag{5}$$

By the mean-value theorem, there exists a constant $\vartheta\in[0,1]$ such that

$$\begin{aligned}&\left|\mathbb{E}\left(\int_0^{+\infty} s(\mathrm{e}^{-rsZ_n}-\mathrm{e}^{-Z_n\bar{m}(Z_n)s})\mathrm{d}s\right)\right|\\ =\;& \left|\mathbb{E}\int_0^{+\infty} s(\mathrm{e}^{-(rsZ_n+\vartheta(Z_n\bar{m}(Z_n)s-rsZ_n))}(Z_n\bar{m}(Z_n)s-rsZ_n))\mathrm{d}s\right|\\ \leq\;& \left|\mathbb{E}\int_0^{+\infty} s^2Z_n\mathrm{e}^{-rsZ_n}(\bar{m}(Z_n)-r)\mathrm{d}s\right|\\ \leq\;& \mathbb{E}\frac{2|\bar{m}(Z_n)-r|Z_n}{r^3Z_n^3}.\end{aligned}\tag{6}$$

Based on (5) and (6) we obtain

$$|\mathbb{E}S_n^2-\mathbb{E}S_{n+1}^2|\leq\frac{1}{q^{2n}}\mathbb{E}\frac{2|\bar{m}(Z_n)-r|}{rZ_n^2}=\frac{2}{rq^{2n}}\mathbb{E}(Y_n^2p(Y_n)).$$

By the concavity of $l(x):=xp(\sqrt{x})$, we have

$$\begin{aligned}|\mathbb{E}S_n^2-\mathbb{E}S_{n+1}^2| &= \frac{2}{rq^{2n}}\mathbb{E}(l(Y_n^2))\\ &\leq \frac{2}{rq^{2n}}l(\mathbb{E}Y_n^2)\\ &= \frac{2}{r}\mathbb{E}S_n^2p\left(\sqrt{\mathbb{E}(S_n^2)}q^n\right).\end{aligned}$$

Since $p(x)$ is non-increasing, we obtain

$$|\mathbb{E}S_n^2-\mathbb{E}S_{n+1}^2|\leq\frac{2}{r}\mathbb{E}S_n^2p(q^n\mathbb{E}S_n).$$

According to the conclusion in Theorem 1 we obtain $b^* := \inf_n \mathbb{E}S_n > 0$. Hence, ones have

$$|\mathbb{E}S_n^2 - \mathbb{E}S_{n+1}^2| \leq \frac{2}{r}\mathbb{E}S_n^2 p(q^n b^*).$$

That is to say, we arrive at

$$\mathbb{E}S_{n+1}^2 \leq \frac{2}{r}\mathbb{E}S_n^2(1 + p(q^n b^*)).$$

From Lemma 1 we have $\sum_{n=1}^{\infty} p(b^* q^n) < +\infty$, which means that

$$\sup_n \mathbb{E}S_n^2 < +\infty$$

and thus the limit $\beta := \lim_{n\to\infty} \mathbb{E}S_n^2$ exists. Now, we construct a martingale as

$$U_n := S_n + V_n,$$

where

$$V_n := \sum_{k=0}^{n-1} \int_0^{+\infty} \frac{1}{q^{k+1}} (\mathrm{e}^{-rsZ_k} - \mathrm{e}^{-Z_k F(Z_k,s)})\mathrm{d}s.$$

Denote $\|X\|$ as the L^2-norm of the random variable X, hence, it is clear that

$$\|U_n\| \leq \|S_n\| + \|V_n\|.$$

Define

$$Q_k = \int_0^{+\infty} \frac{1}{q^{k+1}} (\mathrm{e}^{-rsZ_k} - \mathrm{e}^{-Z_k F(Z_k,s)})\mathrm{d}s.$$

It is obvious that for any n, one has

$$\|V_n\| \leq \sum_{k=0}^{\infty} \|Q_k\|.$$

Moreover, from the estimate in the proof of Theorem 3, we have

$$|Q_n| \leq \frac{1}{q^{n+1}} \int_0^{+\infty} \mathrm{e}^{-Z_n s}|rsZ_n - F(Z_n,s)Z_n|\mathrm{d}s \leq \frac{1}{qr^2}\frac{|r - m(Z_n)|}{S_n}.$$

Since $xp^2(\sqrt{x})$ is a concave function (see Assumption (A4)), we can obtain that

$$\begin{aligned}
\sum_{k=0}^{\infty} \|Q_k\| &\leq \sum_{k=0}^{\infty} \sqrt{\mathbb{E}\left[\left(\frac{1}{qr^2} S_k p(S_k q^k)\right)^2\right]} \\
&\leq \sum_{k=0}^{\infty} \frac{1}{r} \sqrt{\mathbb{E}(S_k^2) p^2(q^k \mathbb{E}S_k)}.
\end{aligned}$$

Since $\alpha^2 := \sup_n ES_n^2 < +\infty$, then it follows that

$$\sup_n \|V_n\| \leq \sum_{k=0}^{\infty} \frac{\alpha}{r} p(\mathbb{E}S_k q^k).$$

Thus, by utilizing Lemma 1, it is not hard to verify that

$$\sum_{k=0}^{\infty} p(\mathbb{E}S_k q^k) \leq \sum_{k=0}^{\infty} p(b^* q^k) < +\infty,$$

then

$$\sup_n \|V_n\| \le \sum_{k=0}^{\infty} \|Q_k\| < +\infty,$$

which establishes that

$$\sup_n \|U_n\| \le \sup_n \|S_n\| + \sup_n \|V_n\| \le \alpha^2 + \sup_n \|V_n\| < +\infty.$$

Combining the above inequality with the fact that $\{U_n, \mathcal{F}_n\}$ is a martingale, we claim that $\{U_n\}$ has a limit in the sense of L^2 from the martingale L^p convergence theorem. On the other hand, we observe that $\{V_n\}$ also has the L^2 limit since $\left\{\sum_{k=0}^{n} \|Q_k\|\right\}$ is a Cauchy sequence. Recall that $S_n = U_n - V_n$ and we have shown that $\{S_n\}$ has the limit S in the sense of almost surely, then we have

$$S_n \to S, \text{ a.s., } L^2,$$

and

$$\lim_{n\to\infty} \mathbb{E}S_n^2 = \mathbb{E}S^2.$$

Moreover, $\lim_{n\to\infty} \mathbb{E}S_n = \mathbb{E}S > 0$, thus, $\mathbb{P}(S > 0) > 0$. That is to say, S is non-degenerate. □

3. Conclusions

Compared with the results in [12], the assumptions in the present paper do not need that $\inf_{x\ge 0} r(x) > 0$. We also even do not require that $\inf_{x\ge 0} m(x) > 0$. Intuitively, $\inf_{x\ge 0} m(x) = 0$ will make the process more likely to be extinct. Hence, $\inf_{x\ge 0} r(x) > 0$ is not a natural enough condition under the case $\mathbb{P}(Z_n \to 0) = 1$, which we consider. Moreover, the extinction rate may be different between in [12] and in this paper, since under the assumption in [12] the rate will be $\lim_{x\to 0^+} r(x)$ (if it exists). One can see that there are many cases (for example, the case that $v(x, \mathrm{d}u)$ is not depending on x) in which $\lim_{x\to 0^+} r(x) < \lim_{x\to 0^+} m(x)$. We remind that the rate in our paper $\lim_{x\to 0^+} m(x)$ appears reasonable because of $m(x) = x^{-1}\mathbb{E}(Z_{n+1}|Z_n = x)$, and further, we consider the case that the process is extinct.

Throughout our paper, under the Assumptions (A1)–(A5), we obtain an extinction rate for a GJP in the sense of almost surely and L^2, which enriches the limit theory of GJP process. Therefore, our results have potential values in applications.

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