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Fejér-Type Midpoint and Trapezoidal Inequalities for the Operator (ω_1, ω_2)-Preinvex Functions

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Abstract: In this work, we obtain some new integral inequalities of the Hermite–Hadamard–Fejér type for operator (ω_1, ω_2)-preinvex functions. The bounds for both left-hand and right-hand sides of the integral inequality are established for operator (ω_1, ω_2)-preinvex functions of the positive self-adjoint operator in the complex Hilbert spaces. We give the special cases to our results; thus, the established results are generalizations of earlier work. In the last section, we give applications for synchronous (asynchronous) functions.

Keywords: Hermite–Hadamard inequalities; Hermite–Hadamard–Fejér inequalities; (ω_1, ω_2)-preinvexity; self-adjoint operators; positive operators; functions of self-adjoint operators; Hölder inequality; synchronous (asynchronous) functions

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1. Introduction

In the field of inequality theory, which has many application areas in mathematical analysis and applied mathematics, researchers have proven dozens of inequality types. We will start by introducing two inequalities that stand out with their aesthetic structures, applications, and functionality among these types of inequalities.

One of the basic concepts used in many of the studies in the field of inequalities is a special function class with applications in statistics, convex programming, numerical analysis, and many other fields. The Hermite–Hadamard inequality, which is created by using convex functions, and have a very intricate structure with inequalities, is given here.

For any convex function, $\breve{g} : I \subseteq \mathbb{R} \to \mathbb{R}$, and for $\mu_1, \mu_2 \in I$ with $\mu_1 < \mu_2$, the following two-sided inequality holds true:

$$\breve{g}\left(\frac{\mu_1 + \mu_2}{2}\right) \leq \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \breve{g}(\grave{u}) d\grave{u} \leq \frac{\breve{g}(\mu_1) + \breve{g}(\mu_2)}{2}, \tag{1}$$

where $\mu_1, \mu_2 \in \mathbb{R}$ and $\mu_1 < \mu_2$. The inequality (1) is called the classical Hermite–Hadamard inequality.

The Hermite–Hadamard–Fejér inequality, which is the general form of the inequality (1) and has been proved by using a weight function, is presented as follows (see [1]).

For any convex function $\breve{g} : I \subseteq \mathbb{R} \to \mathbb{R}$ and for an integrable function $\hbar(\grave{u}) : [\ell_1, \ell_2] \longrightarrow \mathbb{R}$, which is symmetric about $\frac{\ell_1+\ell_2}{2}$, it is asserted that

$$\breve{g}\left(\frac{\ell_1 + \ell_2}{2}\right) \int_{\ell_1}^{\ell_2} \hbar(\grave{u})d\grave{u} \leqq \int_{\ell_1}^{\ell_2} \breve{g}(t)\hbar(\grave{u})d\grave{u} \leqq \frac{\breve{g}(\ell_1) + \breve{g}(\ell_2)}{2} \int_{\ell_1}^{\ell_2} \hbar(\grave{u})d\grave{u}, \tag{2}$$

where $\hbar(\grave{u})$ is a weight function.

Researchers working on these two famous inequalities have obtained generalizations, extensions, improvements, and iterations by considering different types of convex functions, different types of derivative and integral operators, new methods, and different spaces. Hermite–Hadamard inequalities for operator convex and generalized convex functions have proposed (see, for example [2–7]). In 2015, Barani [8] developed the Hermite–Hadamard inequalities for the products of two operator preinvex functions. In 2017, Wang and Sun [7] established the Hermite–Hadamard-type inequalities for operator α-preinvex functions. In 2022, Omrani et al. [9] proposed the Hermite–Hadamard-type inequalities for operator (p, h)-convex functions.

Due to wide range and applications of Hermite–Hadamard inequalities, researchers have extended their work (see, for example [10–18]).

We now recall the operator order in $\mathbf{B}(\hat{E})$, which is the set of all-bounded linear operators on a Hilbert space $(\hat{E}; \langle ., . \rangle)$. For the self-adjoint operators $K_1, K_2 \in \mathbf{B}(\hat{E})$, we may write $K_1 \leqq K_2$ if

$$\langle K_1\grave{u}, \grave{u}\rangle \leqq \langle K_2\grave{u}, \grave{u}\rangle$$

for all $\grave{u} \in \hat{E}$. We name it the operator order.

In general, we write $\mathbb{R}_0 = [0, \infty)$ and $\mathbb{R} = (-\infty, \infty)$.

Definition 1. (see [11]) Let K_1 be a bounded self-adjoint linear operator on a complex Hilbert space $(\hat{E}; \langle ., . \rangle)$ The Gelfand map establishes a $*$-isometric isomorphism between the set $C\big(\mathrm{Sp}(K_1)\big)$ of all continuous functions defined on the spectrum of K_1, denoted $\mathrm{Sp}(K_1)$, and the $C*$-algebra $C*(K_1)$ is generated by K_1 and the identity operator $1_{\hat{E}}$ on $\hat{E}$, as follows. For any $\breve{g}_1, \breve{g}_2 \in C\big(\mathrm{Sp}(K_1)\big)$ and any $\mu_1, \mu_2 \in \mathbb{C}$, we have

(i) $\Omega\big(\mu_1\breve{g}_1 + \mu_2\breve{g}_2\big) = \mu_1\Omega(\breve{g}_1) + \mu_2\Omega(\breve{g}_2)$;
(ii) $\Omega\big(\breve{g}_1\breve{g}_2\big) = \Omega\big(\breve{g}_1\big)\Omega\big(\breve{g}_2\big)$;
(iii) $\Omega\big(\breve{g}_1\big) = \Omega\big(\breve{g}_1\big)^*$;
(iv) $\|\Omega(\breve{g}_1)\| = \|\breve{g}_1\| := \sup_{\tau \in Sp(K_1)} |\breve{g}_1(\tau)|$;
(v) $\Omega(\breve{g}_1) = 1_{\hat{E}}$ *and* $\Omega(\breve{g}_1) = K_1$, *where* $\breve{g}_1(\tau) = 1$ *and* $\breve{g}_1(\tau) = 1$ *for* $\tau \in Sp(K_1)$.

Definition 2. By using the notations of Definition 1, we now define

$$\breve{g}_1(K_1) := \Omega(\breve{g}_1)$$

for all $\breve{g}_1 \in \hat{C}\big(\mathrm{Sp}(K_1)\big)$.

If K_1 is a self-adjoint linear operator and it is also bounded, and if $\breve{g}_1 \in \mathrm{Sp}(K_1)$ is a real-valued function for any $\tau \in \mathrm{Sp}(K_1)$, then

$$\breve{g}_1(\tau) \geqq 0 \Rightarrow \breve{g}_1(K_1) \geqq 0.$$

If $\breve{g}_1$ and $\breve{g}_2$ are real-valued functions on $\mathrm{Sp}(K_1)$, where

$$\breve{g}_1(\tau) \leqq \breve{g}_2(\tau)$$

for any $\tau \in \mathrm{Sp}(K_1)$, then
$$\breve{g}_1(K_1) \leqq \breve{g}_2(K_1)$$
in the operator order of $\mathbf{B}(\hat{E})$.

Definition 3. (see [3]) Let $\breve{g}_1$ be a real-valued function defined on the interval I, where $I \subseteq \mathbb{R}$. Then, $\breve{g}_1$ is called operator convex if

$$\breve{g}_1\big((1-\tau)K_1 + \tau K_2\big) \leqq (1-\tau)\breve{g}_1(K_1) + \tau\breve{g}_1(K_2).$$

If $\breve{g}_1$ is considered operator concave, then the above inequality will be reversed.

Definition 4. (see [4]) Let $I \subseteq \mathbb{R}_0$ and let $\hat{E}$ be a convex subset of $\mathbf{B}(\hat{E})^+_{\mathrm{sa}}$. Then, a continuous function $\breve{g}_1 : I \to \mathbb{R}$ is said to be operator s-convex on the interval I for operator in $\hat{E}$ if

$$\breve{g}_1\big(\tau K_1 + (1-\tau)K_2\big) \leqq \tau^s\breve{g}_1(K_1) + (1-\tau)^s\breve{g}_1(K_2)$$

in the operator order $\mathbf{B}(\hat{E})$ for all $\tau \in [0,1]$, where K_1 and K_2 are positive operators in $\hat{E}$ and their spectra are contained in the interval I and $s \in [0,1]$.

Definition 5. A bivariate function $\Im : \hat{E} \times \hat{E} \to \mathbb{R}$ is said to satisfy the condition $(\hat{C})$ if, for every $\mu_1, \mu_2 \in \hat{E}$ and $\tau \in [0,1]$, we have

$$\Im\big(\mu_1, \mu_2 + \tau\Im(\mu_1, \mu_2)\big) = -\tau\Im(\mu_1, \mu_2)$$

and
$$\Im\big(\mu_1, \mu_2 + \tau\Im(\mu_1, \mu_2)\big) = (1-\tau)\Im(\mu_1, \mu_2).$$
We note that, for every $\mu_1, \mu_2 \in \hat{E}$ and every $\mu_1, \mu_2 \in [0,1]$, we find from the condition $(\hat{C})$ that
$$\Im\big(\mu_1 + \mu_2\Im(\mu_1, \mu_2), \mu_1 + \mu_1\Im(\mu_1, \mu_2)\big) = (\mu_2 - \mu_1)\Im(\mu_1, \mu_2).$$

Definition 6. A general form of the classical Beta function $\mathrm{B}(\alpha, \beta)$, which is known as the incomplete Beta function, is defined by

$$\mathrm{B}_z(\alpha, \beta) = \mathrm{B}(z; \alpha, \beta) := \int_0^z \tau^{\alpha-1}(1-\tau)^{\beta-1}\, d\tau$$

$$\big(\Re(\alpha) > 0;\ \min\{\Re(\alpha), \Re(\beta)\} > 0 \ \text{ and } \ z = 1\big).$$

Definition 7. Let $\hat{E} \subseteq \mathbf{B}(\hat{E})^+_{\mathrm{sa}}$ be an invex set with respect to $\Im : \hat{E} \times \hat{E} \to \mathbf{B}(\hat{E})^+_{\mathrm{sa}}$ and $\breve{g}_1 : \mathbb{R} \to \mathbb{R}$. Then, the continuous function $\breve{g}_1$ is called operator preinvex with respect to $\Im$ if
$$\breve{g}_1\big(K_1 + \tau\Im(K_2, K_1)\big) \leqq (1-\tau)\breve{g}_1(K_1) + \tau\breve{g}_1(K_2)$$
in the operator order in $\mathbf{B}(\hat{E})$ for all $K_1, K_2 \in \hat{E}$ and $\tau \in [0,1]$.

We now recall each of the following known results.

Theorem 1. (see [3], Theorem 3) *Let $\hat{E}$ be an invex subset of $\mathrm{B}(\hat{E})^+_{sa}$ and let $\Im$ be a function, where $\Im : \hat{E} \times \hat{E} \to \mathbf{B}(\hat{E})^+_{sa}$ and $\breve{g}_1 : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is a continuous function on the interval I. Suppose also that the set $\hat{E}$ satisfies the condition $(\hat{C})$ on the set $\hat{E}$. If the function $\breve{g}_1$ is said to be operator preinvex on $\Im$-path P_{K_1C} with spectra of K_1 and C contained in V, then the following result holds true*:

$$\breve{g}_1\left(\frac{K_1 + K_2}{2}\right) \leqq \int_0^1 \breve{g}_1\big(K_1 + \tau\Im(K_2, K_1)\big)d\tau \leqq \frac{\breve{g}_1(K_1) + \breve{g}_1(K_2)}{2}$$

for every $K_1, K_2 \in \hat{E}$, $C = K_1 + \Im(K_2, K_1)$ *and* $\tau \in (0,1]$.

Theorem 2. (see [6], Theorem 2.5) *Let* $\hat{E}$ *be an invex subset of* $\mathbf{B}(\hat{E})^+_{sa}$ *and let* $\Im$ *be a function, where* $\Im : \hat{E} \times \hat{E} \to \mathbf{B}(\hat{E})^+_{sa}$ *and* $\breve{g}_1 : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ *is a continuous function on the interval* I. *Suppose also that the set* $\hat{E}$ *satisfies the condition* $(\hat{C})$ *on the set* $\hat{E}$. *If, for* $s \in (0,1]$, *the function* $\breve{g}_1$ *is said to be s-preinvex preinvex on the* $\Im$*-path* P_{K_1C} *with spectra of* K_1 *and* C *are contained in* V, *then the following result holds true*:

$$2^{s-1}\,\breve{g}_1\left(\frac{K_1+K_2}{2}\right) \leqq \int_0^1 \breve{g}_1(K_1+\tau\Im(K_2,K_1))d\tau \leqq \frac{\breve{g}_1(K_1)+\breve{g}_1(K_2)}{s+1}$$

for every $K_1, K_2 \in \hat{E}$ *and* $C = K_1 + \Im(K_2, K_1)$ *and* $\tau \in (0,1]$.

Theorem 3. (see [5], Theorem 3.1) *Let* $\hat{E}$ *be an invex subset of* $\mathbf{B}(\hat{E})^+_{sa}$ *and let* $\Im$ *be a function, where* $\Im : \hat{E} \times \hat{E} \to \mathbf{B}(\hat{E})^+_{sa}$ *and* $\breve{g}_1 : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ *is a continuous function on the interval* I. *Suppose also that the condition* $(\hat{C})$ *is satisfied on* $\hat{E}$. *If, for* $\alpha \in (0,1]$, *the function* $\breve{g}_1$ *is the operator* α*-preinvex on the* $\Im$*-path* P_{K_1C} *with spectra of* K_1 *and* C *are contained in* V, *then the following inequality holds true*:

$$\breve{g}_1\left(\frac{K_1+K_2}{2}\right) \leqq \int_0^1 \breve{g}_1(K_1+\tau\Im(K_2,K_1))d\tau \leqq \frac{\alpha\breve{g}_1(K_1)+\breve{g}_1(K_2)}{\alpha+1},$$

for every $K_1, K_2 \in \hat{E}$, $C = K_1 + \Im(K_2, K_1)$ *and* $\tau \in (0,1]$.

Each of the following lemmas will be needed in our investigation.

Lemma 1. (see [19], Lemma 1) *Let* $\hat{E} \subseteq \mathbb{R}_0$ *be an invex subset of* $\mathbb{R}$ *and let* $\Im$ *be a function such that* $\Im : \hat{E} \times \hat{E} \to \mathbb{R}$ *and, for* $\mu_1, \mu_2 \in \hat{E}\mathbb{R}$, *where* $\mu_1 < \mu_1 + \Im(\mu_2, \mu_1)$. *If* $\breve{g}_1 \in L([\mu_1, \mu_1 + \Im(\mu_2, \mu_1)])$ *is a differentiable function on* $\hat{E}$. *If* $\hbar_1 : [\mu_1, \mu_1 + \Im(\mu_2, \mu_1)] \to \mathbb{R}_0$ *is an integrable mapping, then the following results holds true*:

$$\begin{aligned} &\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \breve{g}_1(\grave{u})\hbar_1(\grave{u})d\grave{u} \\ &\qquad - \frac{1}{\Im(\mu_2,\mu_1)}\breve{g}_1\left(\mu_1+\frac{\Im(\mu_2,\mu_1)}{2}\right)\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \hbar_1(\grave{u})d\grave{u} \\ &= \Im(\mu_2,\mu_1)\int_0^1 K(\tau)\breve{g}_1(\tau)d\tau, \end{aligned}$$

where

$$K(\tau) = \begin{cases} \int_0^\tau \hbar_1(\mu_1+u\Im(\mu_2,\mu_1))du & (\tau \in [0,\frac{1}{2})) \\ -\int_\tau^1 \hbar_1(\mu_1+u\Im(\mu_2,\mu_1))du & (\tau \in [\frac{1}{2},1]). \end{cases}$$

Lemma 2. (see [19], Lemma 2) *Let* $\hat{E} \subseteq \mathbb{R}_0$ *be an invex subset of* $\mathbb{R}$ *and let* $\Im$ *be a function such that* $\Im : \hat{E} \times \hat{E} \to \mathbb{R}$ *and, for* $\mu_1, \mu_2 \in \hat{E}\mathbb{R}$, *where* $\mu_1 < \mu_1 + \Im(\mu_2, \mu_1)$. *If* $\breve{g}_1 \in L([\mu_1, \mu_1 + \Im(\mu_2, \mu_1)])$ *is a differentiable function on* $\hat{E}$ *and if* $\hbar_1 : [\mu_1, \mu_1 + \Im(\mu_2, \mu_1)] \to \mathbb{R}_0$ *is an integrable mapping, which is symmetric with respect to* $\mu_1 + \frac{1}{2}\Im(\mu_2, \mu_1)$, *then the following results holds true*:

$$\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\breve{g}_1(\grave{u})\hbar_1(\grave{u})d\grave{u} - \left(\frac{\breve{g}_1(\mu_1)+\breve{g}_1(\mu_1+\Im(\mu_2,\mu_1))}{2\Im(\mu_2,\mu_1)}\right)\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\hbar_1(\grave{u})d\grave{u}$$
$$= \frac{\Im(\mu_2,\mu_1)}{2}\int_0^1 K(\tau)\breve{g}_1(\mu_1+\tau\Im(\mu_2,\mu_1))d\tau,$$

where

$$K(\tau) = \int_\tau^1 \hbar_1(\mu_1+u\Im(\mu_2,\mu_1))du - \int_0^\tau \hbar_1(\mu_1+u\Im(\mu_2,\mu_1))du \qquad (\tau\in[0,1]).$$

In this article, we give some new Hermite–Hadamard–Fejér-type inequalities for the operator (ω_1,ω_2)-preinvex functions. We shall also demonstrate how our main findings in this article provide generalizations of some results in earlier studies.

2. Main Results

We begin this section by recalling the following definition.

Definition 8. (see [20]) Let $\hat{E}$ be an invex subset of $\mathbf{B}(\hat{E})^+_{\text{sa}}$ with respect to $\Im : \hat{E}\times\hat{E}\to\mathbf{B}(\hat{E})^+_{\text{sa}}$. Furthermore, let $\breve{g}_1$ be a continuous function such that $\breve{g}_1 : \hat{E}\to\mathbb{R}$ is said to be the operator (ω_1,ω_2)-preinvex on $\hat{E}$ if

$$\breve{g}_1\big(K_1+\tau\Im(K_2,K_1)\big) \leqq \tau^{\omega_1}(1-\tau)^{\omega_2}\breve{g}_1(K_1)+(1-\tau)^{\omega_1}\tau^{\omega_2}\breve{g}_1(K_2), \tag{3}$$

in the operator order $\mathbf{B}(\hat{E})^+_{\text{sa}}$, for all $\tau\in[0,1]$ and, for every positive operators K_1 and K_2 in $\hat{E}$, their spectra are contained in C.

Lemma 3. *Let $\hat{E}\subseteq\mathbb{R}_0$ be the invex subset of $\mathbf{B}(\hat{E})^+_{sa}$ and assume there is a function $\Im$, where $\Im : \hat{E}\times\hat{E}\to\mathbf{B}(\hat{E})^+_{sa}$ and $\breve{g}_1 : \hat{E}\to\mathbb{R}$ is a continuous function. If the condition $(\hat{C})$ is fulfilled on $\hat{E}$, then for $C = K_1+\Im(K_2,K_1)$ for all $K_1,K_2\in\hat{E}$, the function $\breve{g}_1$ is an operator (ω_1,ω_2)-preinvex with respect to $\Im$ on $\Im$-path P_{K_1C} and $\hat{E}$ contains the spectra of C if and only if the function $\Omega_{\grave{u},K_1,K_2} : [0,1]\to\mathbb{R}$, defined by*

$$\Omega_{\grave{u},K_1,K_2}(\tau) := \langle\breve{g}_1\big(K_1+\tau\Im(K_2,K_1)\big)\grave{u},\grave{u}\rangle,$$

is (ω_1,ω_2)-convex in the interval $[0,1]$ for all $\grave{u}\in\hat{E}$ with $\|\grave{u}\|=1$.

Proof. Let $\grave{u}\in\hat{E}$ with $\|\grave{u}\|=1$ and let $\Omega_{\grave{u},K_1,K_2} : [0,1]\to\mathbb{R}$ be (ω_1,ω_2)-convex on $[0,1]$. For all

$$\hat{C}_1 := K_1+\omega_1\Im(K_2,K_1)\in P_{K_1C} \quad\text{and}\quad \hat{C}_2 := K_1+\omega_2\Im(K_2,K_1)\in P_{K_1C},$$

we fix $\tau\in[0,1]$ and utilize the condition $\hat{C}$ as follows:

$$\begin{aligned}\left\langle \breve{g}_1\left(\hat{C}_1+\tau\Im(\hat{C}_2,\hat{C}_1)\right)\grave{u},\grave{u}\right\rangle &= \left\langle \breve{g}_1\left(K_1+\omega_1\Im(K_2,K_1)+\tau(\omega_2-\omega_1)\Im(K_2,K_1)\right)\grave{u},\grave{u}\right\rangle \\ &= \left\langle \breve{g}_1\left(K_1+(1-\tau)\omega_1\Im(K_2,K_1)+\tau\omega_2\Im(K_2,K_1)\right)\grave{u},\grave{u}\right\rangle \\ &= \left\langle \breve{g}_1\left(K_1+((1-\tau)\omega_1+\tau\omega_2)\Im(K_2,K_1)\right)\grave{u},\grave{u}\right\rangle \\ &= \Omega_{\grave{u},K_1,K_2}\left((1-\tau)\omega_1+\tau\omega_2\right) \\ &\leqq \tau^{\omega_1}(1-\tau)^{\omega_2}\Omega_{\grave{u},K_1,K_2}(\omega_1)+(1-\tau)^{\omega_1}\tau^{\omega_2}\Omega_{\grave{u},K_1,K_2}(\omega_2) \\ &= \tau^{\omega_1}(1-\tau)^{\omega_2}\left\langle \breve{g}_1\left(K_1+\omega_1\Im(K_2,K_1)\right)\grave{u},\grave{u}\right\rangle \\ &\quad +(1-\tau)^{\omega_1}\tau^{\omega_2}\left\langle \breve{g}_1\left(K_1+\omega_2\Im(K_2,K_1)\right)\grave{u},\grave{u}\right\rangle \\ &= \tau^{\omega_1}(1-\tau)^{\omega_2}\left\langle \breve{g}_1(\hat{C}_1)\grave{u},\grave{u}\right\rangle+(1-\tau)^{\omega_1}\tau^{\omega_2}\left\langle \breve{g}_1(\hat{C}_2)\grave{u},\grave{u}\right\rangle.\end{aligned}$$

Hence $\breve{g}_1$ is operator (ω_1,ω_2)-preinvex.

Conversely, let $K_1,K_2\in\hat{E}$ and let $\breve{g}_1$ be operator (ω_1,ω_2)-preinvex with respect to $\Im$ on $\Im$-path P_{K_1C}. Suppose that $\omega_1,\omega_2\in[0,1]$ and $\grave{u}\in\hat{E}$ with $\|\grave{u}\|=1$. Then, we have

$$\begin{aligned}\Omega_{\grave{u},K_1,K_2}\left((1-\tau)\omega_1+\tau\omega_2\right) &:= \left\langle \breve{g}_1\left(K_1+\left((1-\tau)\omega_1+\tau\omega_2\right)\Im(K_2,K_1)\right)\grave{u},\grave{u}\right\rangle \\ &= \left\langle \breve{g}_1\left(K_1+\omega_1\Im(K_2,K_1)+\tau(\omega_2-\omega_1)\Im(K_2,K_1)\right)\grave{u},\grave{u}\right\rangle \\ &= \langle \breve{g}_1(K_1+\omega_1\Im(K_2,K_1) \\ &\quad +\tau\Im(K_1+\omega_2\Im(K_2,K_1),K_1+\omega_1\Im(K_2,K_1)))\grave{u},\grave{u}\rangle \\ &\leqq \tau^{\omega_1}(1-\tau)^{\omega_2}\langle \breve{g}_1(K_1+\omega_1\Im(K_2,K_1)\grave{u},\grave{u}\rangle \\ &\quad +(1-\tau)^{\omega_1}\tau^{\omega_2}\langle \breve{g}_1(K_1+\omega_2\Im(K_2,K_1)\grave{u},\grave{u}\rangle \\ &= \tau^{\omega_1}(1-\tau)^{\omega_2}\Omega(\omega_1)+(1-\tau)^{\omega_1}\tau^{\omega_2}\Omega(\omega_2),\end{aligned}$$

which shows that $\Omega_{\grave{u},K_1,K_2}$ is (ω_1,ω_2)-convex on $[0,1]$. □

Theorem 4. *Let $\hat{E}\subseteq\mathbb{R}_0$ be the invex subset of $\mathbf{B}(\hat{E})^+_{sa}$ and assume there is a function $\Im$, where $\Im:\hat{E}\times\hat{E}\to\mathbf{B}\,(\hat{E})^+_{sa}$ and $\breve{g}_1:\hat{E}\to\mathbb{R}$ is a continuous function. If the condition $(\hat{C})$ is fulfilled on $\hat{E}$, then for $C=K_1+\Im(K_2,K_1)$ for all $K_1,K_2\in\hat{E}$, the function $|\Omega'|$ is an operator (ω_1,ω_2)-preinvex with respect to $\Im$ on $\Im$-path P_{K_1C} and that $\hat{E}$ contains the spectra of C. Then, for all $\mu_1,\mu_2\in(0,1)$ where $\mu_1<\mu_2$ and for all $\grave{u}\in\hat{E}$, where $\|\grave{u}\|=1$, the following inequality holds true*:

$$\begin{aligned}&\left|\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\left\langle\left(\int_0^{\tau}\breve{g}_1(K_1+u\Im(K_2,K_1))du\right)\grave{u},\grave{u}\right\rangle\hbar_1(\grave{u})d\grave{u}\right. \\ &\qquad\left.-\frac{1}{\Im(\mu_2,\mu_1)}\int_0^{\mu_1+\frac{\Im(\mu_2,\mu_1)}{2}}\langle\breve{g}_1(K_1+u\Im(K_2,K_1)\grave{u},\grave{u}\rangle du\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\hbar_1(\grave{u})d\grave{u}\right| \\ &\leqq \|\hbar_1\|_\infty\int_{\mu_1}^{\mu_1+\frac{1}{2}\Im(\mu_2,\mu_1)}\left[\beta\left(1-\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)}:1+\omega_1,1+\omega_2\right)\right. \\ &\qquad\left.-\beta\left(\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)}:1+\omega_1,1+\omega_2\right)\right]\left|\left\langle\breve{g}_1\left(K_1+\mu_1\Im(K_2,K_1)\right)\grave{u},\grave{u}\right\rangle\right| \\ &\qquad\cdot\left[\beta\left(1-\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)}:1+\omega_2,1+\omega_1\right)-\beta\left(\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)}:1+\omega_2,1+\omega_1\right)\right] \\ &\qquad\cdot\left|\left\langle\breve{g}_1\left(K_1+\mu_2\Im(K_2,K_1)\right)\grave{u},\grave{u}\right\rangle\right|d\grave{u}.\end{aligned}\tag{4}$$

It is also asserted that

$$
\begin{aligned}
&\Bigg\| \frac{1}{\Im(\mu_2,\mu_1)} \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \int_0^{\tau} \breve{g}_1\big(K_1+u\Im(K_2,K_1)\big)du\, \hbar_1(\grave{u})d\grave{u} \\
&\quad - \frac{1}{\Im(\mu_2,\mu_1)} \int_0^{\mu_1+\frac{\Im(\mu_2,\mu_1)}{2}} \breve{g}_1\big(K_1+u\Im(K_2,K_1)\big)du \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \hbar_1(\grave{u})d\grave{u} \Bigg\| \\
&\leqq \|\hbar_1\|_\infty \int_{\mu_1}^{\mu_1+\frac{1}{2}\Im(\mu_2,\mu_1)} \bigg[\beta\bigg(1-\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)} : 1+\omega_1, 1+\omega_2\bigg) \\
&\quad -\beta\bigg(\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)} : 1+\omega_1, 1+\omega_2\bigg)\bigg] \big\|\breve{g}_1\big(K_1+\mu_1\Im(K_2,K_1)\big)\big\| \\
&\quad \cdot \bigg[\beta\bigg(1-\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)} : 1+\omega_2, 1+\omega_1\bigg) - \beta\bigg(\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)} : 1+\omega_2, 1+\omega_1\bigg)\bigg] \\
&\quad \cdot \big\|\breve{g}_1\big(K_1+\mu_2\Im(K_2,K_1)\big)\big\|.
\end{aligned} \tag{5}
$$

Proof. Let $K_1, K_2 \in \hat{E}$ and suppose that, for all $\mu_1, \mu_2 \in (0,1)$, where $\mu_1 < \mu_2$. For $\grave{u} \in \hat{E}$ with $\|\grave{u}\| = 1$, there is a function $\Omega : [\mu_1, \mu_2] \subseteq [0,1] \to \mathbb{R}_0$ given by

$$
\Omega(\tau) := \bigg\langle \bigg(\int_0^{\tau} \breve{g}_1\big(K_1+u\Im(K_2,K_1)\big)du \bigg)\grave{u}, \grave{u} \bigg\rangle. \tag{6}
$$

Now, by applying the properties of integrals of operator-valued functions and continuity in the inner product, we obtain

$$
\bigg\langle \bigg(\int_0^{\tau} \breve{g}_1(K_1+u\Im(K_2,K_1)du \bigg)\grave{u}, \grave{u} \bigg\rangle = \int_0^{\tau} \big\langle \breve{g}_1\big(K_1+u\Im(K_2,K_1)\big)\grave{u}, \grave{u} \big\rangle du. \tag{7}
$$

Since $\breve{g}_1\big(K_1+u\Im(K_2,K_1)\big) \geqq 0$, $\Omega(\tau) \geqq 0$ for all $\tau \in [\mu_1, \mu_2]$, we have

$$
\Omega'(\tau) = \big\langle \breve{g}_1\big(K_1+\tau\Im(K_2,K_1)\big)\grave{u}, \grave{u} \big\rangle \geqq 0. \tag{8}
$$

Hence

$$
\big|\Omega'(\tau)\big| = \Omega'(\tau).
$$

Furthermore, since $\breve{g}_1$ is operator (ω_1, ω_2)-preinvex with respect to $\Im$ on $\Im$-path P_{K_1C} and Ω' is (ω_1, ω_2)-convex. Now, by using Lemma 1, we obtain

$$
\begin{aligned}
&\bigg| \frac{1}{\Im(\mu_2,\mu_1)} \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \Omega(\grave{u})\hbar_1(\grave{u})d\grave{u} \\
&\quad - \frac{1}{\Im(\mu_2,\mu_1)} \Omega\bigg(\mu_1+\frac{\Im(\mu_2,\mu_1)}{2}\bigg) \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \hbar_1(\grave{u})d\grave{u} \bigg|
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
&\leqq \Im(\mu_2,\mu_1) \int_0^{1/2} \bigg(\int_0^{\tau} |\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|du \bigg) \\
&\quad \cdot \big[\tau^{\omega_1}(1-\tau)^{\omega_2}\big|\Omega'(\mu_1)\big| + (1-\tau)^{\omega_1}\tau^{\omega_2}\big|\Omega'(\mu_2)\big|\big]d\tau \\
&\quad + \Im(\mu_2,\mu_1) \int_{1/2}^{1} \bigg(\int_{\tau}^{1} |\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|du \bigg) \\
&\quad \cdot \big[\tau^{\omega_1}(1-\tau)^{\omega_2}\big|\Omega'(\mu_1)\big| + (1-\tau)^{\omega_1}\tau^{\omega_2}\big|\Omega'(\mu_2)\big|\big]d\tau \\
&= I_1 + I_2.
\end{aligned} \tag{10}
$$

Thus, after changing order of integration, we obtain

$$
\begin{aligned}
I_1 &= \Im(\mu_2,\mu_1)\int_0^{1/2}\left(\int_0^{\tau}|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|du\right)\\
&\qquad\cdot\left[\tau^{\omega_1}(1-\tau)^{\omega_2}\left|\Omega'(\mu_1)\right|+(1-\tau)^{\omega_1}\tau^{\omega_2}\left|\Omega'(\mu_2)\right|\right]d\tau\\
&= \Im(\mu_2,\mu_1)\int_0^{1/2}|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|\\
&\qquad\cdot\left(\int_u^{1/2}\left[\tau^{\omega_1}(1-\tau)^{\omega_2}\left|\Omega'(\mu_1)\right|+(1-\tau)^{\omega_1}\tau^{\omega_2}\left|\Omega'(\mu_2)\right|\right]d\tau\right)du\\
&= \Im(\mu_2,\mu_1)\int_0^{1/2}|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|\\
&\qquad\cdot\left[\left(\beta\left(\frac{1}{2}:1+\omega_1,1+\omega_2\right)-\beta(u:1+\omega_1,1+\omega_2)\right)\left|\Omega'(\mu_1)\right|\right.\\
&\qquad\left.+\left(\beta\left(\frac{1}{2}:1+\omega_2,1+\omega_1\right)-\beta(u:1+\omega_2,1+\omega_1)\right)\left|\Omega'(\mu_2)\right|\right]du.
\end{aligned}
$$

Using the change of variable given by $\grave{u}=\mu_1+u\Im(\check{a}_2,\check{a}_1)$ for $u\in[0,1]$, we find that

$$
\begin{aligned}
I_1 &= \int_{\mu_1}^{\mu_1+\frac{1}{2}\Im(\mu_2,\mu_1)}|\hbar_1(\grave{u})|\\
&\cdot\left[\left(\beta\left(\frac{1}{2}:1+\omega_1,1+\omega_2\right)-\beta\left(\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)}:1+\omega_1,1+\omega_2\right)\right)\left|\Omega'(\mu_1)\right|\right.\\
&\left.+\left(\beta\left(\frac{1}{2}:1+\omega_2,1+\omega_1\right)-\beta\left(\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)}:1+\omega_2,1+\omega_1\right)\right)\left|\Omega'(\mu_2)\right|\right]d\grave{u}. \quad (11)
\end{aligned}
$$

Similarly, on changing the integration order and considering the fact that $\hbar_1$ is symmetric with respect to $\mu_1+\frac{1}{2}\Im(\check{a}_2,\check{a}_1)$, we obtain

$$
\begin{aligned}
I_2 &= \Im(\mu_2,\mu_1)\int_{1/2}^{1}\left(\int_0^{\tau}|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|du\right)\\
&\qquad\cdot\left[\tau^{\omega_1}(1-\tau)^{\omega_2}\left|\Omega'(\mu_1)\right|+(1-\tau)^{\omega_1}\tau^{\omega_2}\left|\Omega'(\mu_2)\right|\right]d\tau\\
&= \Im(\mu_2,\mu_1)\int_{1/2}^{1}|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|\\
&\qquad\cdot\left(\int_{1/2}^{u}\left[\tau^{\omega_1}(1-\tau)^{\omega_2}\left|\Omega'(\mu_1)\right|+(1-\tau)^{\omega_1}\tau^{\omega_2}\left|\Omega'(\mu_2)\right|\right]d\tau\right)du\\
&= \Im(\mu_2,\mu_1)\int_{1/2}^{1}|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|\\
&\qquad\cdot\left[\left(\beta(u:1+\omega_1,1+\omega_2)-\beta\left(\frac{1}{2}:1+\omega_1,1+\omega_2\right)\right)\left|\Omega'(\mu_1)\right|\right.\\
&\qquad\left.+\left(\beta(u:1+\omega_2,1+\omega_1)-\beta\left(\frac{1}{2}:1+\omega_2,1+\omega_1\right)\right)\left|\Omega'(\mu_2)\right|\right]du.
\end{aligned}
$$

By changing the variable $\grave{u}=\mu_1+(1-u)\Im(\check{a}_2,\check{a}_1)$ for $u\in[0,1]$, we have

$$
\begin{aligned}
I_2 &= \int_{\mu_1}^{\mu_1+\frac{1}{2}\Im(\mu_2,\mu_1)}|\hbar_1(\grave{u})|\\
&\cdot\left[\left(\beta\left(1-\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)}:1+\omega_1,1+\omega_2\right)-\beta\left(\frac{1}{2}:1+\omega_1,1+\omega_2\right)\right)\left|\Omega'(\mu_1)\right|\right.\\
&\left.+\left(\beta\left(1-\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)}:1+\omega_2,1+\omega_1\right)-\beta\left(\frac{1}{2}:1+\omega_2,1+\omega_1\right)\right)\left|\Omega'(\mu_2)\right|\right]d\grave{u}. \quad (12)
\end{aligned}
$$

We now add (11) and (12) and utilize the fact that

$$\|\hbar_1\|_\infty = \sup_{\grave{u}\in[\check{a}_1,\, \check{a}_1+\Im(\check{a}_2,\check{a}_1)]} |\hbar_1(\grave{u})|.$$

We then obtain

$$\begin{aligned} I_1 + I_2 = \|\hbar_1\|_\infty \int_{\mu_1}^{\mu_1+\frac{1}{2}\Im(\mu_2,\mu_1)} &|\Omega'(\mu_1)| \\ \cdot &\left[\beta\left(1 - \frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)} : 1+\omega_1, 1+\omega_2\right) - \beta\left(\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)} : 1+\omega_1, 1+\omega_2\right)\right] \\ + &|\Omega'(\mu_2)|\left[\beta\left(1 - \frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)} : 1+\omega_2, 1+\omega_1\right) - \beta\left(\frac{\grave{u}-\mu_1}{\Im(\mu_2,\mu_1)} : 1+\omega_2, 1+\omega_1\right)\right] d\grave{u}. \end{aligned}$$

Finally, by using the results of (6)–(8), we obtain (4) and, upon taking the supremum on both sides of (4) with $\|\grave{u}\| = 1$, we obtain (5). □

Remark 1. For $\omega_1 = s$ and $\omega_2 = 0$ in (5), we have

$$\begin{aligned} &\left\| \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \int_0^\tau \breve{g}_1(K_1 + u\Im(K_2,K_1))du\, \hbar_1(\grave{u})d\grave{u} \right. \\ &\qquad \left. - \int_0^{\mu_1+\frac{1}{2}\Im(\mu_2,\mu_1)} \breve{g}_1(K_1 + u\Im(K_2,K_1))du \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \hbar_1(\grave{u})d\grave{u} \right\| \\ &\leqq (\Im(\mu_2,\mu_1))^2 \|\hbar_1\|_\infty \left[\frac{2^{s+1}-1}{(s+1)(s+2)2^{s+1}}\right] \\ &\qquad \cdot \left[\|\breve{g}_1(K_1 + \mu_1\Im(K_2,K_1))\| + \|\breve{g}_1(K_1 + \mu_2\Im(K_2,K_1))\|\right]. \end{aligned}$$

Remark 2. For $s = 1$ in (5), we have

$$\begin{aligned} &\left\| \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \int_0^\tau \breve{g}_1(K_1 + u\Im(K_2,K_1))du\, \hbar_1(\grave{u})d\grave{u} \right. \\ &\qquad \left. \cdot - \int_0^{\mu_1+\frac{1}{2}\Im(\mu_2,\mu_1)} \breve{g}_1(K_1 + u\Im(K_2,K_1))du \cdot \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \hbar_1(\grave{u})d\grave{u} \right\| \\ &\leqq \frac{(\Im(\mu_2,\mu_1))^2 \|\hbar_1\|_\infty}{8} \left[\|\breve{g}_1(K_1 + \mu_1\Im(K_2,K_1))\| + \|\breve{g}_1(K_1 + \mu_2\Im(K_2,K_1))\|\right]. \end{aligned}$$

Theorem 5. *Let $\hat{E} \subseteq \mathbb{R}_0$ be the invex subset of $\mathbf{B}(\hat{E})^+_{sa}$ and assume there is a function $\Im$, where $\Im : \hat{E} \times \hat{E} \to \mathbf{B}(\hat{E})^+_{sa}$ and $\breve{g}_1 : \hat{E} \to \mathbb{R}$ is a continuous function. If the condition $(\hat{C})$ is fulfilled on $\hat{E}$, then for $C = K_1 + \Im(K_2, K_1)$ for all $K_1, K_2 \in \hat{E}$, the function $|\Omega'|^b$ is an operator (ω_1, ω_2)-preinvex with respect to $\Im$ on $\Im$-path P_{K_1C} and that $\hat{E}$ contains the spectra of C. Then, for all $\mu_1, \mu_2 \in (0,1)$ where $\mu_1 < \mu_2$ and for all $\grave{u} \in \hat{E}$, where $\|\grave{u}\| = 1$, the following inequality holds true*:

$$
\begin{aligned}
&\left|\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\left\langle\left(\left(\int_0^{\tau}\breve{g}_1\left(K_1+u\Im(K_2,K_1)\right)du\right)\grave{u},\grave{u}\right\rangle\hbar_1(\grave{u})d\grave{u}\right.\\
&\left.-\frac{1}{\Im(\mu_2,\mu_1)}\int_0^{\mu_1+\frac{1}{2}\Im(\mu_2,\mu_1)}\langle\breve{g}_1\left(K_1+u\Im(K_2,K_1)\right)\grave{u},\grave{u}\rangle du\cdot\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\hbar_1(\grave{u})d\grave{u}\right|\\
&\leqq\Im(\mu_2,\mu_1)\left(\int_0^1|K(\tau)|^a d\tau\right)^{\frac{1}{a}}[2\beta(1+\omega_1,1+\omega_2)\\
&\cdot\left|\left\langle\breve{g}_1\left(K_1+\mu_1\Im(K_2,K_1)\right)\grave{u},\grave{u}\right\rangle\right|^b+\left|\left\langle\breve{g}_1\left(K_1+\mu_2\Im(K_2,K_1)\right)\grave{u},\grave{u}\right\rangle\right|^b\Big]^{\frac{1}{b}}.
\end{aligned}
\tag{13}
$$

It is also asserted that

$$
\begin{aligned}
&\left\|\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\int_0^{\tau}\breve{g}_1\left(K_1+u\Im(K_2,K_1)\right)du\ \hbar_1(\grave{u})d\grave{u}\right.\\
&\qquad\left.-\frac{1}{\Im(\mu_2,\mu_1)}\int_0^{\mu_1+\frac{\Im(\mu_2,\mu_1)}{2}}\breve{g}_1\left(K_1+u\Im(K_2,K_1)\right)du\cdot\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\hbar_1(\grave{u})d\grave{u}\right\|\\
&\quad\leqq\Im(\mu_2,\mu_1)\left(\int_0^1|K(\tau)|^a d\tau\right)^{\frac{1}{a}}[2\beta(1+\omega_1,1+\omega_2)\\
&\qquad\cdot\left|\breve{g}_1\left(K_1+\mu_1\Im(K_2,K_1)\right)\right|^b+\left|\breve{g}_1\left(K_1+\mu_2\Im(K_2,K_1)\right)\right|^b\Big]^{\frac{1}{b}},
\end{aligned}
\tag{14}
$$

where

$$\frac{1}{a}+\frac{1}{b}=1.$$

Proof. After applying Hölder's inequality on Lemma 1, we obtain

$$
\begin{aligned}
&\left|\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\Omega(\grave{u})\hbar_1(\grave{u})d\grave{u}\right.\\
&\qquad\left.-\frac{1}{\Im(\mu_2,\mu_1)}\Omega\left(\mu_1+\frac{\Im(\mu_2,\mu_1)}{2}\right)\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\hbar_1(\grave{u})d\grave{u}\right|\\
&\quad\leqq\Im(\mu_2,\mu_1)\left(\int_0^1|K(\tau)|^a d\tau\right)^{\frac{1}{a}}\\
&\qquad\cdot\left(\int_0^1\left|\Omega'\left((1-\tau)\omega_1+\tau\omega_2\right)\right|^b d\tau\right)^{\frac{1}{b}}.
\end{aligned}
$$

From the (ω_1,ω_2)-convexity of $|\Omega'|^b$, we have

$$\left|\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\Omega(\grave{u})\hbar_1(\grave{u})d\grave{u}\right.$$
$$\left.-\frac{1}{\Im(\mu_2,\mu_1)}\Omega\left(\mu_1+\frac{\Im(\mu_2,\mu_1)}{2}\right)\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\hbar_1(\grave{u})d\grave{u}\right|$$
$$\leqq \Im(\mu_2,\mu_1)\left(\int_0^1|K(\tau)|^a d\tau\right)^{\frac{1}{a}}$$
$$\cdot\left(\int_0^1\left[\tau^{\omega_1}(1-\tau)^{\omega_2}\left|\Omega'(\mu_1)\right|^b+(1-\tau)^{\omega_1}\tau^{\omega_2}\left|\Omega'(\mu_2)\right|^b\right]d\tau\right)^{\frac{1}{b}}$$
$$=\Im(\mu_2,\mu_1)\left(\int_0^1|K(\tau)|^a d\tau\right)^{\frac{1}{a}}$$
$$\cdot\left[\beta(1+\omega_1,1+\omega_2)\left(\left|\Omega'(\mu_1)\right|^b+\left|\Omega'(\mu_2)\right|^b\right)\right]^{\frac{1}{b}}.$$

Finally, by applying the results in (6)–(8), we obtain (13) and, upon taking the supremum on both sides of (13) with $\|\grave{u}\|=1$, we obtain (14). □

Theorem 6. *Let $\hat{E}\subseteq\mathbb{R}_0$ be the invex subset of $\mathbf{B}(\hat{E})^+_{sa}$ and assume there is a function $\Im$, where $\Im:\hat{E}\times\hat{E}\to\mathbf{B}\,(\hat{E})^+_{sa}$ and $\breve{g}_1:\hat{E}\to\mathbb{R}$ is a continuous function. If the condition $(\hat{C})$ is fulfilled on $\hat{E}$, then for $C=K_1+\Im(K_2,K_1)$ for all $K_1,K_2\in\hat{E}$, the function $|\Omega'|$ is an operator (ω_1,ω_2)-preinvex with respect to $\Im$ on $\Im$-path P_{K_1C} and that $\hat{E}$ contains the spectra of C. Then, for all $\mu_1,\mu_2\in(0,1)$ where $\mu_1<\mu_2$ and for all $\grave{u}\in\hat{E}$, where $\|\grave{u}\|=1$, the following inequality holds true*:

$$\left|\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\left\langle\left(\int_0^\tau\breve{g}_1(K_1+u\Im(K_2,K_1))du\right)\grave{u},\grave{u}\right\rangle\hbar_1(\grave{u})d\grave{u}\right.$$
$$-\frac{1}{2\Im(\mu_2,\mu_1)}\left[\left\langle\left(\int_0^{\mu_1}\breve{g}_1(K_1+u\Im(K_2,K_1))du\right)\grave{u},\grave{u}\right\rangle\right.$$
$$\left.\left.+\left\langle\left(\int_0^{\mu_1+\Im(\mu_2,\mu_1)}\breve{g}_1(K_1+u\Im(K_2,K_1))du\right)\grave{u},\grave{u}\right\rangle\right]\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\hbar_1(\grave{u})d\grave{u}\right|$$
$$\leqq\beta(\omega_1+1,\omega_2+1)\left[\left|\langle\breve{g}_1(K_1+\mu_1\Im(K_2,K_1))\grave{u},\grave{u}\rangle\right|\right.$$
$$\left.+\left|\langle\breve{g}_1(K_1+\mu_2\Im(K_2,K_1))\grave{u},\grave{u}\rangle\right|\right]\|\hbar_1\|_\infty. \tag{15}$$

It is also asserted that

$$\left\|\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\int_0^\tau\breve{g}_1(K_1+u\Im(K_2,K_1))du\,\hbar_1(\grave{u})d\grave{u}\right.$$
$$-\frac{1}{2\Im(\mu_2,\mu_1)}\left[\int_0^{\mu_1}\breve{g}_1(K_1+u\Im(K_2,K_1))du\right.$$
$$\left.\left.+\int_0^{\mu_1+\Im(\mu_2,\mu_1)}\breve{g}_1(K_1+u\Im(K_2,K_1))du\right]\cdot\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\hbar_1(\grave{u})d\grave{u}\right\|$$
$$\leqq\beta(\omega_1+1,\omega_2+1)\left[\left\|\breve{g}_1(K_1+\mu_1\Im(K_2,K_1))\right\|\right.$$
$$\left.+\left\|\breve{g}_1(K_1+\mu_2\Im(K_2,K_1))\right\|\right]\|\hbar_1\|_\infty. \tag{16}$$

Proof. From Lemma 2 and the fact of the (ω_1,ω_2)-convexity of Ω', we have

$$\begin{aligned}
&\left|\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\breve{g}_1(\grave{u})\hbar_1(\grave{u})d\grave{u}\right.\\
&\qquad\left.-\left(\frac{\breve{g}_1(\mu_1)+\breve{g}_1(\mu_1+\Im(\mu_2,\mu_1))}{2\Im(\mu_2,\mu_1)}\right)\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\hbar_1(\grave{u})d\grave{u}\right|\\
&\leqq \Im(\mu_2,\mu_1)\int_0^1\left(\int_0^\tau|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|du\right)\\
&\qquad\cdot\left[\tau^{\omega_1}(1-\tau)^{\omega_2}\left|\Omega'(\mu_1)\right|+(1-\tau)^{\omega_1}\tau^{\omega_2}\left|\Omega'(\mu_2)\right|\right]d\tau\\
&\qquad+\Im(\mu_2,\mu_1)\int_0^1\left(\int_\tau^1|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|du\right)\\
&\qquad\cdot\left[\tau^{\omega_1}(1-\tau)^{\omega_2}\left|\Omega'(\mu_1)\right|+(1-\tau)^{\omega_1}\tau^{\omega_2}\left|\Omega'(\mu_2)\right|\right]d\tau\\
&= I_1+I_2,
\end{aligned}$$

which, upon changing the order of integration, yields

$$\begin{aligned}
I_1 &= \Im(\mu_2,\mu_1)\int_0^1\left(\int_0^\tau|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|du\right) \qquad (17)\\
&\qquad\cdot\left[\tau^{\omega_1}(1-\tau)^{\omega_2}\left|\Omega'(\mu_1)\right|+(1-\tau)^{\omega_1}\tau^{\omega_2}\left|\Omega'(\mu_2)\right|\right]d\tau\\
&= \Im(\mu_2,\mu_1)\int_0^1|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|\\
&\qquad\cdot\int_u^1\left[\tau^{\omega_1}(1-\tau)^{\omega_2}\left|\Omega'(\mu_1)\right|+(1-\tau)^{\omega_1}\tau^{\omega_2}\left|\Omega'(\mu_2)\right|\right]d\tau du. \qquad (18)
\end{aligned}$$

Similarly, on changing the integration order and considering $\hbar_1$ is symmetric with respect to $\mu_1+\frac{1}{2}\Im(\breve{a}_2,\breve{a}_1)$, we obtain

$$\begin{aligned}
I_2 &= \Im(\mu_2,\mu_1)\int_0^1\left(\int_\tau^1|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|du\right)\\
&\qquad\cdot\left[\tau^{\omega_1}(1-\tau)^{\omega_2}\left|\Omega'(\mu_1)\right|+(1-\tau)^{\omega_1}\tau^{\omega_2}\left|\Omega'(\mu_2)\right|\right]d\tau\\
&= \int_0^1|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|\\
&\qquad\cdot\int_0^u\left[\tau^{\omega_1}(1-\tau)^{\omega_2}\left|\Omega'(\mu_1)\right|+(1-\tau)^{\omega_1}\tau^{\omega_2}\left|\Omega'(\mu_2)\right|\right]d\tau\, du. \qquad (19)
\end{aligned}$$

After adding (17) and (19) and using the assumption that

$$\|\hbar_1\|_\infty = \sup_{\grave{u}\in[\breve{a}_1,\,\breve{a}_1+\Im(\breve{a}_2,\breve{a}_1)]}|\hbar_1(\grave{u})|,$$

we obtain

$$\begin{aligned}
I &= I_1+I_2\\
&= \int_0^1|\hbar_1(\mu_1+u\Im(\mu_2,\mu_1))|\\
&\qquad\cdot\int_0^1\left[\tau^{\omega_1}(1-\tau)^{\omega_2}\left|\Omega'(\mu_1)\right|+(1-\tau)^{\omega_1}\tau^{\omega_2}\left|\Omega'(\mu_2)\right|\right]d\tau du\\
&= \frac{\|\hbar_1\|_\infty[|\Omega'(\mu_1)|+|\Omega'(\mu_2)|]}{\Im(\mu_2,\mu_1)}\beta(\omega_1+1,\omega_2+1)\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}du\\
&= \left[\left|\Omega'(\mu_1)\right|+\left|\Omega'(\mu_2)\right|\right]\beta(\omega_1+1,\omega_2+1)\|\hbar_1\|_\infty
\end{aligned}$$

Finally, by utilizing the results in (6)–(8), we obtain (15) and, on applying the supremum on both sides of (15) with $\|\grave{u}\|=1$, we obtain (16). □

Remark 3. For $\omega_1 = s$ and $\omega_2 = 0$ in (16), we have

$$\begin{aligned}&\Bigg\|\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\int_0^{\tau}\breve{g}_1\big(K_1+u\Im(K_2,K_1)\big)du\,\hbar_1(\grave{u})d\grave{u}\\&\quad-\frac{1}{2\Im(\mu_2,\mu_1)}\bigg[\int_0^{\mu_1}\breve{g}_1\big(K_1+u\Im(K_2,K_1)\big)du\\&\quad+\bigg\langle\bigg(\int_0^{\mu_1+\Im(\mu_2,\mu_1)}\breve{g}_1\big(K_1+u\Im(K_2,K_1)\big)du\bigg)\grave{u},\grave{u}\bigg\rangle\bigg]\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\hbar_1(\grave{u})d\grave{u}\Bigg\|\\&\leqq\frac{\|\hbar_1\|_\infty}{s+1}\|\,[\breve{g}_1\big(K_1+\mu_1\Im(K_2,K_1)\big)\,\|\\&\quad+\|\breve{g}_1\big(K_1+\mu_2\Im(K_2,K_1)\big)\|].\end{aligned}$$

Remark 4. For $s = 1$, we have

$$\begin{aligned}&\Bigg\|\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\int_0^{\tau}\breve{g}_1\big(K_1+u\Im(K_2,K_1)\big)du\,\hbar_1(\grave{u})d\grave{u}\\&\quad-\frac{1}{2\Im(\mu_2,\mu_1)}\bigg[\int_0^{\mu_1}\breve{g}_1\big(K_1+u\Im(K_2,K_1)\big)du\\&\quad+\bigg\langle\int_0^{\mu_1+\Im(\mu_2,\mu_1)}\big(\breve{g}_1\big(K_1+u\Im(K_2,K_1)\big)du\,\big)\,\grave{u},\grave{u}\bigg\rangle\bigg]\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\hbar_1(\grave{u})d\grave{u}\Bigg\|\\&\leqq\frac{\|\hbar_1\|_\infty}{2}\|\,[\breve{g}_1\big(K_1+\mu_1\Im(K_2,K_1)\big)\,\|\\&\quad+\|\breve{g}_1\big(K_1+\mu_2\Im(K_2,K_1)\big)\|].\end{aligned}$$

Finally, we state and prove the following result.

Theorem 7. *Let $\hat{E} \subseteq \mathbb{R}_0$ be the invex subset of $\mathbf{B}(\hat{E})^+_{sa}$ and assume there is a function $\Im$, where $\Im : \hat{E} \times \hat{E} \to \mathbf{B}\,(\hat{E})^+_{sa}$ and $\breve{g}_1 : \hat{E} \to \mathbb{R}$ is a continuous function. If the condition $(\hat{C})$ is fulfilled on $\hat{E}$, then for $C = K_1 + \Im(K_2, K_1)$ for all $K_1, K_2 \in \hat{E}$, the function $|\Omega'|^b$ is an operator (ω_1, ω_2)-preinvex with respect to $\Im$ on $\Im$-path P_{K_1C} and that $\hat{E}$ contains the spectra of C. Then, for all $\mu_1, \mu_2 \in (0, 1)$, where $\mu_1 < \mu_2$ and for all $\grave{u} \in \hat{E}$, where $\|\grave{u}\| = 1$, the following inequality holds true*:

$$\begin{aligned}&\Bigg|\frac{1}{\Im(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\bigg\langle\bigg(\int_0^{\tau}\breve{g}_1(K_1+u\Im(K_2,K_1))du\bigg)\grave{u},\grave{u}\bigg\rangle\hbar_1(\grave{u})d\grave{u}\\&\quad-\frac{1}{2\Im(\mu_2,\mu_1)}\bigg[\bigg\langle\bigg(\int_0^{\mu_1}\breve{g}_1(K_1+u\Im(K_2,K_1))du\bigg)\grave{u},\grave{u}\bigg\rangle\\&\quad+\bigg\langle\bigg(\int_0^{\mu_1+\Im(\mu_2,\mu_1)}\breve{g}_1(K_1+u\Im(K_2,K_1))du\bigg)\grave{u},\grave{u}\bigg\rangle\bigg]\int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)}\hbar_1(\grave{u})d\grave{u}\Bigg|\\&\leqq\Im(\mu_2,\mu_1)\bigg(\int_0^1|K(\tau)|^a d\tau\bigg)^{\frac{1}{a}}\\&\quad\cdot\Big[\beta(1+\omega_1,1+\omega_2)\Big(|\langle\breve{g}_1(K_1+\mu_1\Im(K_2,K_1))\grave{u},\grave{u}\rangle|^b\\&\quad+|\langle\breve{g}_1(K_1+\mu_2\Im(K_2,K_1))\grave{u},\grave{u}\rangle|\Big)\Big]^{\frac{1}{b}}.\end{aligned}\tag{20}$$

It is also asserted that

$$\begin{aligned}
&\left\| \frac{1}{\Im(\mu_2,\mu_1)} \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \int_0^{\tau} \breve{g}_1(K_1 + u\Im(K_2,K_1))du\, \hbar_1(\grave{u})d\grave{u} \right. \\
&\qquad - \frac{1}{2\Im(\mu_2,\mu_1)} \left[\int_0^{\mu_1} \breve{g}_1(K_1 + u\Im(K_2,K_1))du \right. \\
&\qquad \left. \left. + \int_0^{\mu_1+\Im(\mu_2,\mu_1)} \breve{g}_1(K_1 + u\Im(K_2,K_1))du \right] \cdot \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \hbar_1(\grave{u})d\grave{u} \right\| \\
&\leqq \Im(\mu_2,\mu_1) \left(\int_0^1 |K(\tau)|^a d\tau \right)^{\frac{1}{a}} \\
&\qquad \cdot \Big[\beta(1+\omega_1, 1+\omega_2) \Big(\|\breve{g}_1(K_1 + \mu_1\Im(K_2,K_1))\|^b \\
&\qquad + \|\breve{g}_1(K_1 + \mu_2\Im(K_2,K_1))\|^b \Big) \Big]^{\frac{1}{b}},
\end{aligned} \tag{21}$$

where $\frac{1}{a} + \frac{1}{b} = 1$.

Proof. On applying Hölder's inequality to Lemma 2, we can write

$$\begin{aligned}
&\left| \frac{1}{\Im(\mu_2,\mu_1)} \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \breve{g}_1(\grave{u})\hbar_1(\grave{u})d\grave{u} \right. \\
&\qquad \left. - \left(\frac{\breve{g}_1(\mu_1) + \breve{g}_1(\mu_1 + \Im(\mu_2,\mu_1))}{2\Im(\mu_2,\mu_1)} \right) \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \hbar_1(\grave{u})d\grave{u} \right| \\
&\leqq \frac{\Im(\mu_2,\mu_1)}{2} \left(\int_0^1 |K(\tau)|^a d\tau \right)^{\frac{1}{a}} \\
&\qquad \cdot \left(\int_0^1 |\Omega'((1-\tau)\omega_1 + \tau\omega_2)|^b d\tau \right)^{\frac{1}{b}}.
\end{aligned}$$

From the (ω_1, ω_2)-convexity of $|\Omega'|^b$, we obtain

$$\begin{aligned}
&\left| \frac{1}{\Im(\mu_2,\mu_1)} \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \Omega(\grave{u})\hbar_1(\grave{u})d\grave{u} \right. \\
&\qquad \left. - \frac{1}{\Im(\mu_2,\mu_1)} \Omega\left(\mu_1 + \frac{1}{2}\Im(\mu_2,\mu_1) \right) \int_{\mu_1}^{\mu_1+\Im(\mu_2,\mu_1)} \hbar_1(\grave{u})d\grave{u} \right| \\
&\leqq \frac{\Im(\mu_2,\mu_1)}{2} \left(\int_0^1 |K(\tau)|^a d\tau \right)^{\frac{1}{a}} \\
&\qquad \cdot \left(\int_0^1 \left[\tau^{\omega_1}(1-\tau)^{\omega_2} |\Omega'(\mu_1)|^b + (1-\tau)^{\omega_1}\tau^{\omega_2} |\Omega'(\mu_2)|^b \right] d\tau \right)^{\frac{1}{b}} \\
&= \Im(\mu_2,\mu_1) \left(\int_0^1 |K(\tau)|^a d\tau \right)^{\frac{1}{a}} \\
&\qquad \cdot \left[\beta(1+\omega_1, 1+\omega_2) \left(|\Omega'(\mu_1)|^b + |\Omega'(\mu_2)|^b \right) \right]^{\frac{1}{b}}.
\end{aligned}$$

Finally, by using the results in (6)–(8), we obtain (20) and then, by applying the supremum on both sides of (20) with $\|\grave{u}\| = 1$, we obtain (21). □

3. Applications for Synchronous (Asynchronous) Functions

The functions $\breve{g}_1, \hbar_1 : [K_1, K_2] \to \mathbb{R}$ are synchronous (asynchronous) on $[K_1, K_2]$ if the following inequality holds:

$$(\breve{g}_1(t) - \breve{g}_1(s))(\hbar_1(t) - \hbar_1(s)) \geq (\leq) 0,$$

for all $t, s \in [K_1, K_2]$. It is clear that, if the functions $\breve{g}_1, \hbar_1$ are monotonic and have the same monotonicity on $[K_1, K_2]$, then they are synchronous on $[K_1, K_2]$; meanwhile, if they have opposite monotonicity, they are synchronous. The following result provides a Cebyhsev-type inequality for functions of self-adjoint operators.

Theorem 8. *(see [21]) Let A be a self-adjoint operator with $Sp(K_1) \subset [x, M]$ for some real numbers $m \leq M$ if $\breve{g}_1, \hbar_1 : [m, M] \to \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\langle \breve{g}_1(K_1)\hbar_1(K_1)x, x\rangle \geq (\leq)\langle \breve{g}_1(K_1)x, x\rangle\langle \hbar_1(K_1)x, x\rangle,$$

for any $x \in H$ with $\|x\| = 1$

If $\breve{g}_1, \hbar_1$ are synchronous, then

$$N(K_1, K_2)(x) \leq M(K_1, K_2)(x) \leq Q(K_1, K_2)(x), \tag{22}$$

for any $x \in H$ with $\|x\| = 1$.

If $\breve{g}_1, \hbar_1$ are asynchronous, then reverse inequalities holds in (22)

$$N(K_1, K_2)(x) \geq M(K_1, K_2)(x) \geq Q(K_1, K_2)(x).$$

For all positive operators K_1 and K_2 on a Hilbert space H with spectra in I , we define real functions, where

$$\begin{aligned} M &= M(K_1, K_2)(x) = \langle \breve{g}_1(K_1)x, x\rangle\langle \hbar_1(K_1)x, x\rangle + \langle \breve{g}_1(K_2)x, x\rangle\langle \hbar_1(K_2)x, x\rangle. \\ N &= N(K_1, K_2)(x) = \langle \breve{g}_1(K_1)x, x\rangle\langle \hbar_1(K_2)x, x\rangle + \langle \breve{g}_1(K_2)x, x\rangle\langle \hbar_1(K_1)x, x\rangle. \end{aligned}$$

Theorem 9. *Let $\breve{g}_1, \hbar_1 : [m, M] \to \mathbb{R}^+$ operator (ω_1, ω_2)-preinvex and $K_1, K_2 \subset Sp(K_1) \cup Sp(K_2) \subset [m, M]$, $\omega_1, \omega_2 \in [0, 1]$*

(i) If $\breve{g}_1, \hbar_1$ are synchronous and $\breve{g}_1, \hbar_1 \geq 0$, then we have the following inequality

$$\begin{aligned} &\int_0^1 \langle \breve{g}_1(K_1 + u\Im(K_2, K_1))x, x\rangle\langle \hbar_1(K_1 + u\Im(K_2, K_1))x, x\rangle du \\ \leq\ &[\beta(2\omega_1 + 1, 2\omega_2 + 1) + \beta(\omega_1 + \omega_2 + 1, \omega_1 + \omega_2 + 1)]Q, \end{aligned}$$

where

$$Q := Q(K_1, K_2)(x) = \langle(\breve{g}_1(K_1)\hbar_1(K_1) + \breve{g}_1(K_2)\hbar_1(K_2))x, x\rangle.$$

Remark 5. *If $\omega_1 = 1$ and $\omega_2 = 0$, then the following inequality holds*

$$\begin{aligned} &\int_0^1 \langle \breve{g}_1(K_1 + u\Im(K_2, K_1))x, x\rangle\langle \hbar_1(K_1 + u\Im(K_2, K_1))x, x\rangle du \\ \leq\ &\frac{1}{2}Q. \end{aligned}$$

(ii) If $\breve{g}_1, \hbar_1$ are synchronous and $\breve{g}_1, \hbar_1 \geq 0$, then we have the following inequality

$$\begin{aligned}&\left\langle \breve{g}_1\left(\frac{K_1+K_2}{2}\right)x,x\right\rangle\left\langle \hbar_1(\left(\frac{K_1+K_2}{2}\right))x,x\right\rangle\\ \leq\ &\frac{1}{2}\int_0^1 \langle \breve{g}_1(K_1+u\Im(K_2,K_1))x,x\rangle\langle \hbar_1(K_1+u\Im(K_2,K_1))x,x\rangle du\\ &+\frac{1}{2}[\beta(2\omega_1+1,2\omega_2+1)+\beta(\omega_1+\omega_2+1,\omega_1+\omega_2+1)]Q,\end{aligned}$$

where

$$Q := Q(K_1,K_2)(x) = \langle(\breve{g}_1(K_1)\hbar_1(K_1)+\breve{g}_1(K_2)\hbar_1(K_2))x,x\rangle.$$

(iii) If $\breve{g}_1, \hbar_1$ are synchronous and $\breve{g}_1, \hbar_1 \geq 0$, then we have the following inequality

$$\begin{aligned}&\left\langle \breve{g}_1\left(\frac{K_1+K_2}{2}\right)x,x\right\rangle\int_0^1 \langle \hbar_1(K_1+u\Im(K_2,K_1))x,x\rangle du\\ &+\left\langle \hbar_1(\left(\frac{K_1+K_2}{2}\right))x,x\right\rangle\int_0^1 \langle \breve{g}_1(K_1+u\Im(K_2,K_1))x,x\rangle du\\ \leq\ &\frac{1}{2}\int_0^1 \langle \hbar_1(K_1+u\Im(K_2,K_1))x,x\rangle\int_0^1 \langle \breve{g}_1(K_1+u\Im(K_2,K_1))x,x\rangle du\\ &\frac{1}{2}[\beta(2\omega_1+1,2\omega_2+1)+\beta(\omega_1+\omega_2+1,\omega_1+\omega_2+1)]Q\\ &+\left\langle \breve{g}_1\left(\frac{K_1+K_2}{2}\right)x,x\right\rangle\left\langle \hbar_1(\left(\frac{K_1+K_2}{2}\right))x,x\right\rangle.\end{aligned}$$

4. Conclusions

We have developed new estimates for both the lower and the upper bounds of the Hermite–Hadamard–Fejér type inequalities for the operator (ω_1,ω_2)-preinvex functions. The main result of our work is Lemma 3. The remaining established results are based on Lemma 3. Additionally, we have provided some remarks that illustrate how the main theorems extend other results shown in the cited literature. All of the information presented here should encourage more study in this field. Interested readers can establish the fascinating results on different class of convex and generalized functions. The results can be generalized to different fields such as fractional calculus, q-calculus, interval-valued, and time-scale domains for the square operator modulus in semi Hilbert spaces (see, for example [15,22]).

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