axioms

Article

Conditional Expanding of Functions by q-Lidstone Series

Maryam Al-Towailb [1,*] and Zeinab S. I. Mansour [2]

[1] Department of Computer Science and Engineering, College of Applied Studies and Community Service, King Saud University, Riyadh 11451, Saudi Arabia
[2] Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt
* Correspondence: mtowaileb@ksu.edu.sa

Abstract: This paper characterizes those functions given by convergent q-Lidstone series expansion. We give the necessary and sufficient conditions so that the entire function $f(z)$ has such an expansion, in which case convergence is uniform on compact sets.

Keywords: q-Lidstone series; q-Lidstone polynomials; asymptotic expansion; convergence series

MSC: 05A30; 41A58; 41A60; 40A05

1. Introduction

In 1929, Lidstone [1] introduced a generalization of Taylor's theorem that approximates an entire function f in a neighborhood of two points instead of one, that is

$$f(x) = \sum_{n=0}^{\infty} \left[f^{(2n)}(1)\Lambda_n(x) + f^{(2n)}(0)\Lambda_n(1-x) \right], \tag{1}$$

where $\Lambda_n(x)$ is the unique polynomial of degree $2n+1$ determined by the conditions

$$\Lambda_0(x) = x; \quad \Lambda_n(0) = \Lambda_n(1) = 0 \quad (n \geq 1);$$
$$\Lambda_n''(x) = \Lambda_{n-1}(x) \quad (n \geq 1),$$

In [2], Boas and Buck show that any entire function of order less than π has an absolutely convergent Lidstone series representation. Buckholtz et al. [3,4] proved that the condition

$$f^{(n)}(0) = o(\pi^n) \quad (n \to \infty)$$

is both necessary and sufficient for (1) to hold. Several authors, including Boas [5], Poritsky [6], Schoenberg [7], Whittaker [8], and Widder [9], have presented different necessary and sufficient conditions for the representation of functions by Lidstone series (1).

Recently, this type of series was generalized by q-calculus. The quantum calculus (q-calculus or Jackson calculus [10]) is an extension of the traditional calculus. There are many new developments and applications of the q-calculus in many areas, such as ordinary fractional calculus, optimal control problems, solutions of the q-difference equations, q-integral equations, q-fractional integral inequalities, q-transform analysis, and many more (see, e.g., [11–15]).

Ismail and Mansour [16] introduced a q-analog of Lidstone series by the form

$$f(z) = \sum_{n=0}^{\infty} \left[D_{q^{-1}}^{2n} f(1)A_n(z) - D_{q^{-1}}^{2n} f(0)B_n(z) \right], \tag{2}$$

check for updates

Citation: Al-Towailb, M.; Mansour, Z.S.I. Conditional Expanding of Functions by q-Lidstone Series. *Axioms* **2023**, *12*, 22. https://doi.org/10.3390/axioms12010022

Academic Editor: Gradimir V. Milovanović

Received: 15 November 2022
Revised: 19 December 2022
Accepted: 20 December 2022
Published: 25 December 2022

where $D_{q^{-1}}$ is the Jackson q-difference operator defined by

$$D_{q^{-1}}f(z) = \begin{cases} \dfrac{f(z) - f(\frac{z}{q})}{z(1-\frac{1}{q})}, & z \neq 0; \\ f'(0), & z = 0, \end{cases}$$

provided that f is differentiable at zero (see [17]). The polynomials $(A_n)_n$ and $(B_n)_n$ are q-analogs of the Lidstone polynomials $(\Lambda_n(z))_n$ and $(\Lambda_n(1-z))_n$, and defined by the generating functions

$$\frac{E_q(zw) - E_q(-zw)}{E_q(w) - E_q(-w)} = \sum_{n=0}^{\infty} A_n(z) w^{2n}, \tag{3}$$

$$\frac{E_q(zw)E_q(-w) - E_q(-zw)E_q(w)}{E_q(w) - E_q(-w)} = \sum_{n=0}^{\infty} B_n(z) w^n, \tag{4}$$

respectively, where $E_q(z)$ is Jackson's q-exponential function defined by

$$E_q(z) = \prod_{k=0}^{\infty}(1 + (1-q)q^k z), \quad 0 < q < 1, \; z \in \mathbb{C}. \tag{5}$$

Moreover,

$$A_0(z) = z, \quad B_0(z) = z - 1, \tag{6}$$

and for $n \in \mathbb{N}$, $A_n(z)$ and $B_n(z)$ satisfy the q-difference equation

$$D^2_{q^{-1}} y_n(z) = y_{n-1}(z) \quad \text{with} \quad y_n(0) = y_n(1) = 0. \tag{7}$$

It is worth mentioning that Al-Towailb [18] has constructed another q-type Lidstone theorem by expanding a class of entire functions in terms of q-derivatives of even orders at 0 and q-derivatives of odd orders at 1. Moreover, in [19], we introduced the q-Lidstone expansion in (2) for the symmetric q-difference operator δ_q defined by

$$\delta_q f(z) = f(q^{\frac{1}{2}}z) - f(q^{-\frac{1}{2}}z)$$

(see [20]). For detailed properties of the q-analogs of Lidstone series and polynomials, readers may also refer to the literature (see [21–23]).

Our aim is to provide necessary and sufficient conditions for expanding an entire function, $f(z)$, in a q-Lidstone expansion of the form

$$f(z) = s_0 A_0(z) - s_1 B_0(z) + s_2 A_1(z) - s_3 B_1(z) + \dots, \tag{8}$$

where $\{s_n\}_n$ is a complex sequence defined by

$$s_{2n} = D^{2n}_{q^{-1}} f(1), \quad s_{2n+1} = D^{2n}_{q^{-1}} f(0) \quad (n \in \mathbb{N}). \tag{9}$$

We also give a sufficient condition for expanding entire functions in a different arrangement of series (8). More precisely, we will provide a sufficient condition on f so that

$$\sum_{n=0}^{\infty} D^{2n}_{q^{-1}} f(1) A_n(z) - \sum_{n=0}^{\infty} D^{2n}_{q^{-1}} f(0) B_n(z) \tag{10}$$

converges to $f(z)$, uniformly on each compact subset of the plane.

This paper is organized as follows: The following section gives the essential notions and some basic results on q-analysis. In Section 3, we derive the asymptotic of q-Lidstone polynomials $A_n(z)$ and $B_n(z)$ for sufficiently large n. Section 4 provides a necessary and sufficient condition on an entire function f so that the q-Lidstone series expansion converges to $f(z)$ uniformly on compact subsets of the plane. Conclusions are drawn in Section 5.

2. Preliminaries

In this section, we recall some notions, definitions, and basic results in the area of q-calculus for $0 < q < 1$ which we need in our investigations (see [17,24]).

Let $n \in \mathbb{N}_0$ (the set of non-negative integers). The q-shifted fractional $(a;q)_n$ is defined by

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j), \quad (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad (a \in \mathbb{C}).$$

The q-numbers $[n]_q$ and q-factorial $[n]_q!$ are defined as:

$$[n]_q = \frac{1-q^n}{1-q}, \quad [n]_q! = [n]_q[n-1]_q \ldots [1]_q.$$

Jackson's q-trigonometric functions $\mathrm{Sin}_q z$ and $\mathrm{Cos}_q z$ are defined by

$$\begin{aligned}\mathrm{Sin}_q z &:= \frac{E_q(iz)-E_q(-iz)}{2i} = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(2n+1)}}{(q;q)_{2n+1}}(z(1-q))^{2n+1},\\ \mathrm{Cos}_q z &:= \frac{E_q(iz)+E_q(-iz)}{2} = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(2n-1)}}{(q;q)_{2n}}(z(1-q))^{2n},\end{aligned} \tag{11}$$

and q-analogs of the hyperbolic functions $\mathrm{Sinh}_q z$ and $\mathrm{Cosh}_q z$ are given by

$$\mathrm{Sinh}_q(z) := -i\,\mathrm{Sin}_q(iz), \quad \mathrm{Cosh}_q(z) := \mathrm{Cos}_q(iz). \tag{12}$$

In [25], Ismail proved that the zeros of the second Jackson q-Bessel functions are real and simple. Moreover, since the q-sine and q-cosine functions in (11) has a representation in terms of the second Jackson q-Bessel function of order $1/2$ and $-1/2$, respectively, cf. [17], their zeros are real and simple. Therefore, from now on, we use $\{\xi_k\}_{k\in\mathbb{N}}$ to denote the positive zeros of $\mathrm{Sin}_q z$ arranged in increasing order of magnitude. Based on the results in [25–27], El-Guindy and Mansour in [28] pointed out that the real positive zeros of $\mathrm{Sin}_q z$ have the asymptotic

$$\xi_k \sim Aq^{-2k}, \quad A := \frac{q^{\frac{-3}{2}}}{1-q} \text{ as } k \to \infty. \tag{13}$$

Furthermore, they introduced the following asymptotic result.

Lemma 1. *Let α and β be real positive numbers such that $\alpha + \beta < 2$. Let A be the constant in (13) and γ is any positive number less than α and β. Then, for $q^{-\alpha}\xi_n < R_n < q^{\beta}\xi_{n+1}$,*

$$\left|\frac{1}{\mathrm{Sinh}_q(z)}\right| = o\Big(\Big(\frac{Aq^\beta}{(1+q^\gamma)}\Big)^n R_n^{-n}\Big),\ |z| = R_n,\ n \to \infty.$$

Furthermore, in [28], the authors proved that

$$\sum_{n=1}^{\infty}\frac{1}{\mathrm{Sin}_q'(\xi_n)} = -\frac{1}{2}. \tag{14}$$

The following lemma proves the absolute convergence of the series on (14).

Lemma 2. *If $(\xi_n)_n$ are the set of positive zeros of $\mathrm{Sin}_q(z)$, then for each $n \in \mathbb{N}$,*

$$|\mathrm{Sin}_q'(\xi_n)| \geq 2q^{-2(n+1)^2}q^{2(\epsilon_n+1)(n+1)}\frac{1}{(q;q)_{2n+3}}\prod_{j=1}^{\infty}(1-q^{4j})^3, \tag{15}$$

for some positive numbers ξ_n *such that* $\lim_{n\to\infty} \epsilon_n = 0$. *Moreover,* $\sum_{n=1}^{\infty} \frac{1}{\mathrm{Sin}'_q(\xi_n)}$ *is an absolutely convergent series.*

Proof. First, note that for any constant c, we have

$$\sum_{k=0}^{\infty}(-1)^k q^{k(2k+1)} \frac{c}{[2k+1]_q!}\xi_n^{2k} = c\frac{\mathrm{Sin}_q \xi_n}{\xi_n} = 0.$$

So, we obtain

$$\mathrm{Sin}'_q \xi_n = \sum_{k=0}^{\infty}(-1)^k q^{k(2k+1)} \frac{2k+1}{[2k+1]_q!}\xi_n^{2k} = \sum_{k=0}^{\infty}(-1)^k q^{k(2k+1)} \frac{2(k-n-1)}{[2k+1]_q!}\xi_n^{2k}. \tag{16}$$

From (13), we may assume that $\xi_n = Aq^{-2n+\epsilon_n} < 1$, with $A = \frac{q^{-3/2}}{(1-q)}$ and $\lim_{n\to\infty} \epsilon_n = 0$. Therefore,

$$\begin{aligned}
\mathrm{Sin}'_q \xi_n &= \sum_{k=0}^{\infty}(-1)^k q^{k(2k+1)} \frac{2(k-n-1)}{[2k+1]_q!} A^{2k} q^{-4nk+2k\epsilon_n} \\
&= (1-q)q^{-2(n+1)^2} \sum_{k=0}^{\infty}(-1)^k \frac{2(k-n-1)}{(q;q)_{2k+1}} q^{2(k-n-1)^2} q^{2k+2k\epsilon_n} \\
&= t_n \sum_{m=-(n+1)}^{\infty} (-1)^m \frac{2m}{(q^{2n+3};q)_{2m+1}} q^{2m^2} q^{2(\epsilon_n+1)m} = t_n S_n,
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
t_n &:= (-1)^{n+1} q^{-2(n+1)^2} q^{2(\epsilon_n+1)(n+1)} \frac{1}{(q^2;q)_{2n+2}}; \\
S_n &:= \sum_{m=-(n+1)}^{\infty} (-1)^m \frac{2m}{(q^{2n+3};q)_{2m+1}} q^{2m(m+1)+2\epsilon_n}.
\end{aligned} \tag{18}$$

Let p be a positive integer; then,

$$S_n = \sum_{m=-(n+1)}^{-(2p+2)} F_{m,n} + \sum_{m=-(2p+1)}^{2p} F_{m,n} + \sum_{m=2p+1}^{\infty} F_{m,n},$$

where

$$F_{m,n} := (-1)^m \frac{2m}{(q^{2n+3};q)_{2m+1}} q^{2m(m+1)+2\epsilon_n}.$$

Consequently,

$$|S_n| \geq \left| \sum_{m=-(2p+1)}^{2p} F_{m,n} \right| - \sum_{m=-(n+1)}^{-(2p+2)} |F_{m,n}| - \sum_{m=2p+1}^{\infty} |F_{m,n}|. \tag{19}$$

Since the estimate in (19) is independent of p, the last two sums on (19) tend to zero as $n \to \infty$. Therefore,

$$\liminf_{n\to\infty} |S_n| \geq \lim_{p\to\infty} \left| 2 \sum_{m=-(2p+1)}^{2p} (-1)^m m q^{2m(m+1)} \right|.$$

However,

$$\begin{aligned}\sum_{m=-(2p+1)}^{2p}(-1)^m m q^{2m(m+1)} &= \sum_{-(2p+1)}^{-1}(-1)^m m q^{2m(m+1)} + \sum_{m=0}^{2p}(-1)^m m q^{2m(m+1)} \\ &= \sum_{1}^{2p+1}(-1)^{m-1} m q^{2m(m+1)} + \sum_{m=0}^{2p}(-1)^m m q^{2m(m+1)} \\ &= \sum_{0}^{2p}(-1)^m(2m+1)q^{2m(m+1)}.\end{aligned}$$

Thus,

$$\liminf_{n\to\infty}|S_n| \geq 2\left|\sum_{0}^{\infty}(-1)^m(2m+1)q^{2m(m+1)}\right| = \prod_{j=1}^{\infty}(1-q^{4j})^3, \tag{20}$$

where we used the identity due to Jacobi, see ([29], p. 500) ,

$$\sum_{i=0}^{\infty}(-1)^i(2i+1)q^{\frac{i(i+1)}{2}} = \prod_{i=1}^{\infty}(1-q^i)^3 > 0. \tag{21}$$

So, from (18) and (17), we deduce that $|\mathrm{Sin}_q'\xi_n| \geq 2\prod_{i=1}^{\infty}(1-q^{4i})^3 t_n$, i.e., the series $\sum_{n=0}^{\infty}\frac{1}{\mathrm{Sin}_q'\xi_n}$ is absolutely convergent. □

Let $\mathcal{F}$ denote the class of all entire functions f that satisfy

$$D_{q^{-1}}^n f(0) = o(\xi_1^n) \quad (n\to\infty),$$

where ξ_1 is the smallest positive zero of $\mathrm{Sin}_q(z)$. We define a norm of $\mathcal{F}$ by

$$\|f\|_{\mathcal{F}} := \sup_{n\in\mathbb{N}_0}\frac{|(D_{q^{-1}}^n f)(0)|}{\xi_1^n}. \tag{22}$$

It is worth noting that if f is analytic at zero, then

$$\frac{f^{(n)}(0)}{n!} = q^{\frac{n(n-1)}{2}}\frac{D_{q^{-1}}^n f(0)}{[n]_q!} \quad (n\in\mathbb{N}_0). \tag{23}$$

Therefore, $(\mathcal{F}, \|f\|_{\mathcal{F}})$ is a normed space.

Lemma 3. *If $f\in\mathcal{F}$, then*

$$|f(z)| \leq \|f\|_{\mathcal{F}}\, E_q(\xi_1|z|)\ (z\in\mathbb{C}),$$

where $E_q(z)$ is Jackson's q-exponential function defined in (5).

Proof. From the Maclaurin series expansion and Equation (23), we have

$$f(z) = \sum_{m=0}^{\infty}\frac{f^{(m)}(0)}{m!}z^m = \sum_{m=0}^{\infty} q^{\frac{m(m-1)}{2}}\frac{D_{q^{-1}}^m f(0)}{[m]_q!}z^m. \tag{24}$$

Then, from (22), we obtain

$$|f(z)| = \Big| \sum_{m=0}^{\infty} \frac{D_{q^{-1}}^m f(0)}{\xi_1^m} q^{\frac{m(m-1)}{2}} \frac{(\xi_1 z)^m}{[m]_q!} \Big|$$
$$\leq \|f\|_{\mathcal{F}} \sum_{m=0}^{\infty} q^{\frac{m(m-1)}{2}} \frac{(\xi_1 |z|)^m}{[m]_q!} = \|f\|_{\mathcal{F}} E_q(\xi_1 |z|).$$

This completes the proof. □

In the following, we introduce some essential properties for the q-Lidstone polynomials $A_n(z)$ and $B_n(z)$ defined in (3) and (4), respectively.

Let $\{p_n(z;q)\}_n$ be the sequence defined by

$$p_{2n}(z;q) = A_n(z), \quad p_{2n+1}(z;q) = -B_n(z), \tag{25}$$

and $\{L_n\}_n$ be the sequence of linear functionals defined by

$$L_{2n}(f) := D_{q^{-1}}^{2n} f(1), \quad L_{2n+1}(f) := D_{q^{-1}}^{2n} f(0). \tag{26}$$

Then, the expansion (2) can be written in the form

$$f(z) = \sum_{n=0}^{\infty} L_n(f) p_n(z;q). \tag{27}$$

By using (24), we can write L_n as

$$L_0(f) = f(1), \quad L_n(f) = \sum_{m=n-1}^{\infty} D_{q^{-1}}^m f(0)\, L_{nm} \ (n \in \mathbb{N}), \tag{28}$$

where $L_{nm} = \delta_{m,n-1}$ for odd n, where $\delta_{i,j}$ is the Kronecker's delta, and

$$L_{nm} = \begin{cases} \dfrac{q^{\frac{(m-n)(m-n-1)}{2}}}{[m-n]_q!}, & m \geq n; \\ 0, & m < n, \end{cases}$$

for even n. Consequently,

$$0 \leq L_{nm} < \frac{1}{[m-n]_q!}, \quad \text{for } 0 \leq n \leq m. \tag{29}$$

From (28), we have

$$L_m(z^j) = \begin{cases} q^{-\frac{j(j-1)}{2}} L_{jm}, & m \leq j+1, \\ 0, & m > j+1. \end{cases}$$

Therefore, by using (27), we obtain

$$\frac{q^{\frac{j(j-1)}{2}} z^j}{[j]_q!} = \sum_{m=0}^{j+1} L_{jm}\, p_m(z;q) \quad (j \in \mathbb{N}_0). \tag{30}$$

Remark 1. *From (6), (7) and (25), we have*

$$p_0(0) = 0,\ p_1(0) = 1 \ \text{and} \ p_n(0) = 0 \ \text{for all} \ n > 1.$$

Moreover, from (7)*, we conclude that* $D^2_{q^{-1}} p_n(z;q) = p_{n-2}(z;q)$*. Therefore, by mathematical induction on* k*, we can verify that*

$$D^{2k}_{q^{-1}} p_n(z;q) = p_{n-2k}(z;q) \quad (n \geq 2k). \tag{31}$$

This implies $L_n(p_k) = \delta_{n,k}$*, and*

$$D^{2k}_{q^{-1}} p_n(0) = p_{n-2k}(0), \quad D^{2k+1}_{q^{-1}} p_n(0) = D_{q^{-1}} p_{n-2k}(0). \tag{32}$$

3. Asymptotic Expansions for the q-Lidstone Polynomials

In this section, we use the contour integration and Darboux method (see [30]) for determining the asymptotic behavior of the q-Lidstone polynomials $A_n(z)$ and $B_n(z)$ defined in (3) and (4) for sufficiently large n.

Consider the generating function

$$G_q(z,w) := \frac{\operatorname{Sinh}_q(zw)}{w^{m+1}\operatorname{Sinh}_q(w)}. \tag{33}$$

We choose a contour Γ_n, as shown in Figure 1, such that $G_q(z,w)$ is analytic in an open set containing Γ_n and its interior, except for the poles inside Γ_n.

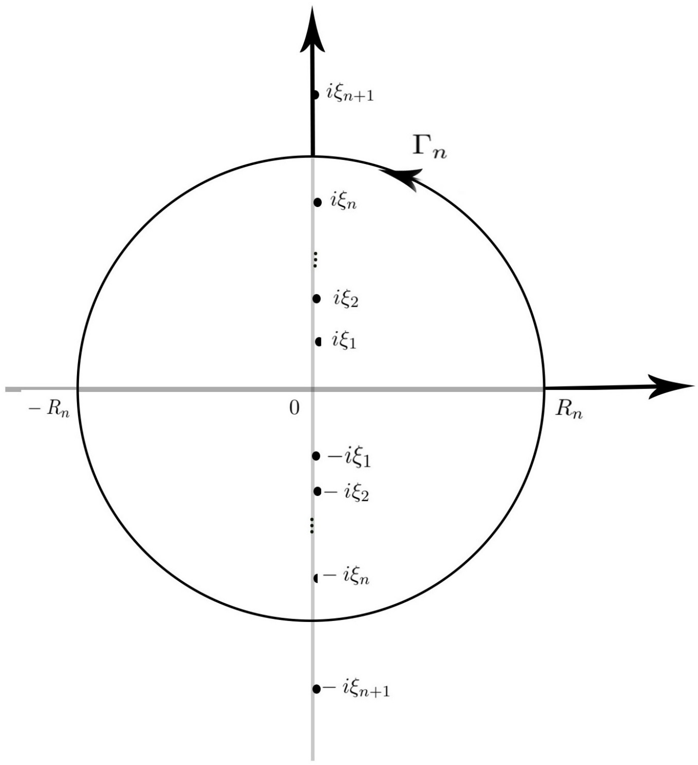

Figure 1. The contour Γ_n in Lemma 1.

Proposition 1. *Let* $\{\xi_k\}_{k\in\mathbb{N}}$ *be the sequence of the positive zeros of* $\operatorname{Sin}_q(z)$*. Then*

$$\oint_{\Gamma_n} G_q(z,w)dw = (2\pi i)\Big[\tilde{A}_m(z) + 2\cos(\frac{\pi}{2}m)\sum_{k=1}^{n} \frac{\operatorname{Sin}_q(\xi_k z)}{(\xi_k)^{m+1}\operatorname{Sin}'_q(\xi_k)}\Big], \tag{34}$$

where $G_q(z,w)$ *is defined in* (33)*, and* $\{\tilde{A}_m\}_m$ *is the sequence of function defined by*

$$\tilde{A}_{2m}(z) = A_m(z), \quad \tilde{A}_{2m+1}(z) = 0.$$

Proof. The function $G_q(z,w)$ has a pole of order $m+1$ at 0 and simple poles at $\pm i\xi_k$ for $k = 1,2,\ldots,n$. So, by using the Cauchy residue theorem, we obtain

$$\frac{1}{2\pi i}\oint_{\Gamma_n} G_q(z,w)dw = \frac{1}{m!}\frac{d^m}{dz^m}\Big[\frac{\operatorname{Sinh}_q(zw)}{\operatorname{Sinh}_q(w)}\Big](0) + \sum_{k=1}^{n}\operatorname{Res}(G_q,\pm i\xi_k). \tag{35}$$

From Equation (3), we obtain

$$\frac{1}{m!}\frac{d^m}{dz^m}\Big[\frac{\operatorname{Sinh}_q(zw)}{\operatorname{Sinh}_q(w)}\Big](0) = \begin{cases} A_r(z), & \text{m = 2r;} \\ 0, & \text{m = 2r + 1.}\end{cases} \tag{36}$$

For a fixed $k \in \{1,2,\ldots,n\}$,

$$\begin{aligned}\operatorname{Res}(G_q, i\xi_k) &= \lim_{w\to i\xi_k}\frac{(w-i\xi_k)}{w^{m+1}\operatorname{Sinh}_q(w)}\operatorname{Sinh}_q(zw)\\ &= \frac{\operatorname{Sinh}_q(i\xi_k z)}{(i\xi_k)^{m+1}}\lim_{w\to i\xi_k}\frac{(w-i\xi_k)}{\operatorname{Sin}_q(w)}\\ &= \frac{i\operatorname{Sin}_q(\xi_k z)}{(i\xi_k)^{m+1}\operatorname{Sin}_q'(\xi_k)}.\end{aligned}$$

Similarly, $\operatorname{Res}(G_q,-i\xi_k) = \dfrac{(-i)\operatorname{Sin}_q(\xi_k z)}{(-i\xi_k)^{m+1}\operatorname{Sin}_q'(\xi_k)}$. Therefore,

$$\sum_{k=1}^{n}\operatorname{Res}(G_q,\pm i\xi_k) = 2\cos(\frac{\pi}{2}m)\sum_{k=1}^{n}\frac{\operatorname{Sin}_q(\xi_k z)}{(\xi_k)^{m+1}\operatorname{Sin}_q'(\xi_k)}, \tag{37}$$

and the result follows by substituting (36) and (37) into (35). □

The following proposition proves that the contour integration on Γ_n in (34) vanishes as $n\to\infty$.

Proposition 2. *Let $\{\xi_k\}_{k\in\mathbb{N}}$ be the sequence of the positive zeros of $\operatorname{Sin}_q(z)$, and let α and β be real positive numbers such that $\alpha+\beta<2$. Then, there exists $m_0\in\mathbb{N}$ such that for $q^{-\alpha}\xi_n < R_n < q^{\beta}\xi_{n+1}$*

$$\lim_{n\to\infty}\oint_{\Gamma_n:|w|=R_n}\frac{\operatorname{Sinh}_q(zw)}{w^{m+1}\operatorname{Sinh}_q(w)}dw = 0 \quad (|z|\le 1,\ m\ge m_0).$$

Proof. It suffices to prove that

$$\lim_{n\to\infty}\oint_{\Gamma_n}\frac{E_q(zw)}{w^{m+1}\operatorname{Sinh}_q(w)}dw = 0 \quad (|z|\le 1).$$

From (5), if $|z|\le 1$ and $|w| = R_n$, then

$$|E_q(zw)| = \prod_{k=0}^{\infty}\Big|1+q^k(1-q)zw\Big| \le \prod_{k=0}^{2n+1}(1+q^kR_n)\prod_{k=2n+2}^{\infty}(1+q^kR_n). \tag{38}$$

From the asymptotic (13), $\lim_{k\to\infty}\dfrac{\xi_k}{Aq^{-2k}} = 1$, where $A = \frac{q^{-3/2}}{1-q}$. Consequently, for any $\epsilon>0$ there exists $k_0\in\mathbb{N}$ such that

$$(1-\epsilon)Aq^{-2k} < \xi_k < (1+\epsilon)Aq^{-2k} \ \text{ for all } \ k\ge k_0. \tag{39}$$

Now, set $\delta = \frac{q^{-\beta}}{A(1+\epsilon)}$. Since $R_n < q^{\beta}\xi_{n+1}$, then from (39), we obtain

$$\begin{aligned}\prod_{k=0}^{2n+1}(1+q^k R_n) &\leq \prod_{k=0}^{2n+1}(1+q^{\beta+k}\xi_{n+1}) \leq \prod_{k=0}^{2n+1}(1+\frac{q^{k-2n-2}}{\delta}) \\ &\leq \frac{q^{-(n+1)(2n+1)}}{\delta^{2n+2}}(-q\delta;q)_{2n+1}.\end{aligned} \tag{40}$$

Furthermore, we can verify that

$$\prod_{k=2n+2}^{\infty}(1+q^k R_n) \leq (-\frac{1}{\delta};q)_\infty. \tag{41}$$

Combining the inequalities (38), (40), (41) and using (39), we conclude that

$$\begin{aligned}|E_q(zw)| &\leq \frac{q^{-(n+1)(2n+1)}}{\delta^{2n+2}}(-q\delta;q)_\infty(-\frac{1}{\delta};q)_\infty \\ &\leq \frac{q^{-1}}{\delta^{2n+1}}\left(\frac{q^\alpha R_n}{A(1-\epsilon)}\right)^{n+3/2}(-q\delta;q)_\infty(-\frac{1}{\delta};q)_\infty.\end{aligned}$$

Therefore, from Lemma 1, if $c = \frac{Aq^\beta}{(1+q^\gamma)}$, $\gamma < \min\{\alpha,\beta\}$, and

$$\alpha_n := \frac{c^{-n}R_n^n}{|\mathrm{Sinh}_q(w)|},\ |w| = R_n,$$

then $\lim_{n\to\infty}\alpha_n = 0$. Therefore,

$$\left|\frac{E_q(zw)}{w^{m+1}\,\mathrm{Sinh}_q(w)}\right| \leq \alpha_n R_n^{3/2}\left(\frac{q^{\alpha+3\beta}A^2(1+\epsilon)^2}{(1-\epsilon)(1+q^\gamma)}\right)^n\frac{(-q\delta;q)_\infty(-\frac{1}{\delta};q)_\infty}{q\delta},$$

and then

$$\left|\oint_{\Gamma_n}\frac{E_q(zw)}{w^{m+1}\mathrm{Sinh}_q(w)}dw\right| \leq 2\pi\alpha_n R_n^{-m+3/2}\left(\frac{q^{\alpha+3\beta}A^2(1+\epsilon)^2}{(1-\epsilon)(1+q^\gamma)}\right)^n\frac{(-q\delta;q)_\infty(-\frac{1}{\delta};q)_\infty}{q\delta}. \tag{42}$$

Hence, there exists $m_0 \in \mathbb{N}$ such that the limit of the right-hand side of (42) as $n \to \infty$ is zero for all $m \geq m_0$. This yields the required result. □

Propositions 1 and 2 lead to the following expansion of the polynomials $(A_m(z))_m$.

Theorem 1. *Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a sequence of the positive zeros of $\mathrm{Sin}_q(z)$. Then, there exists $m_0 \in \mathbb{N}$ such that*

$$A_m(z) = 2(-1)^{m+1}\sum_{k=1}^{\infty}\frac{\mathrm{Sin}_q(\xi_k z)}{\xi_k^{2m+1}\mathrm{Sin}'_q(\xi_k)},\quad \text{for } |z| \leq 1, \tag{43}$$

for all $m \geq m_0$, where $(A_m)_m$ is the sequence of the q-Lidstone polynomials which is defined in (3). In particular,

$$A_m(z) = 2(-1)^{m+1}\frac{\mathrm{Sin}_q(\xi_1 z)}{\xi_1^{2m+1}\,\mathrm{Sin}'_q(\xi_1)} + o\left(\frac{1}{\xi_2^{2m+1}}\right)\quad (m\to\infty). \tag{44}$$

Proof. From Proposition 1, we obtain

$$\frac{1}{2\pi i}\oint_{\Gamma_n}G_q(z,w)dw = A_m(z) + 2(-1)^{m+1}\sum_{k=1}^{n}\frac{\mathrm{Sin}_q(\xi_k z)}{\xi_k^{m+1}\,\mathrm{Sin}'_q(\xi_k)}.$$

Letting $n \to \infty$ and using Proposition 2, we obtain (43). The proof of (44) follows directly from (43). □

The Darboux method (see [30], pp. 309–310) states that if $f(z) = \sum a_n z^n$ is a meromorphic function in $|z| > 0$, r is the distance from the origin to the nearest pole, and if there exists a function g such that

1. g is analytic in $0 < |z| < r$;
2. $f - g$ is continuous on $0 < |z| \leq r$;
3. the coefficients b_n in the Laurent expansion of the function g have known asymptotic behavior as $n \to \infty$;

then,

$$a_n = b_n + o(r^{-n}) \quad \text{as } n \to \infty. \tag{45}$$

Moreover, if $f - g$ is m times continuously differentiable on the circle $|z| = r$, then

$$a_n = b_n + o(r^{-n} n^{-m}) \quad \text{as } n \to \infty. \tag{46}$$

Ismail and Mansour [16] proved that the q-Lidstone polynomial $B_n(z)$ can be written as

$$B_n(z) = \frac{2^{2n+1}}{[2n+1]_q!} B_{2n+1}(z/2; q), \tag{47}$$

where $(B_n(z;q))_n$ are the q-Bernoulli polynomials defined by

$$\frac{w E_q(zw) E_q(-w/2)}{E_q(w/2) - E_q(-w/2)} = \sum_{n=0}^{\infty} B_n(z;q) \frac{w^n}{[n]_q!}. \tag{48}$$

In [31], the authors used the Darboux method with (45) to provide the following asymptotic behavior of the q-Bernoulli polynomials $B_n(z;q)$ defined in (48) for sufficiently large n by

$$\begin{aligned} B_{2n}(z;q) &= \frac{(-1)^{n+1} [2n]_q! \operatorname{Cos}_q(2\xi_1 z) \operatorname{Cos}_q(\xi_1)}{(1-q)(2\xi_1)^{2n} \operatorname{Sin}_q'(\xi_1)} + o(\xi_1^{-2n}), \\ B_{2n+1}(z;q) &= \frac{(-1)^{n+1} [2n+1]_q! \operatorname{Sin}_q(2\xi_1 z) \operatorname{Cos}_q(\xi_1)}{(1-q)(2\xi_1)^{2n+1} \operatorname{Sin}_q'(\xi_1)} + o(\xi_1^{-(2n+1)}). \end{aligned} \tag{49}$$

We can improve the o term in (49) by using (46) to obtain

$$\begin{aligned} B_{2n}(z;q) &= \frac{(-1)^{n+1} [2n]_q! \operatorname{Cos}_q(2\xi_1 z) \operatorname{Cos}_q(\xi_1)}{(1-q)(2\xi_1)^{2n} \operatorname{Sin}_q'(\xi_1)} + o(\xi_1^{-2n} (2n)^{-m}), \\ B_{2n+1}(z;q) &= \frac{(-1)^{n+1} [2n+1]_q! \operatorname{Sin}_q(2\xi_1 z) \operatorname{Cos}_q(\xi_1)}{(1-q)(2\xi_1)^{2n+1} \operatorname{Sin}_q'(\xi_1)} + o(\xi_1^{-(2n+1)} (2n+1)^{-m}) \end{aligned} \tag{50}$$

as $n \to \infty$, where $m \in \mathbb{N}_0$ is fixed. Consequently, from Equation (47), we obtain the n large asymptotic of the q-Lidstone polynomial $B_n(z)$ as in the following result.

Proposition 3. *The large n asymptotic of the q-Lidstone polynomials $B_n(z)$ defined in (4) is*

$$B_n(z) = \frac{(-1)^{n+1} \operatorname{Sin}_q(\xi_1 z) \operatorname{Cos}_q(\xi_1)}{(1-q)(\xi_1)^{2n+1} \operatorname{Sin}_q'(\xi_1)} + o(\xi_1^{-2n} (2n)^{-m}), \tag{51}$$

where ξ_1 is the smallest positive zero of $\operatorname{Sin}_q(z)$, and m is any non-negative integer.

Lemma 4. *If $(p_n)_n$ is the sequence of polynomials defined in (25), then there exists a positive constant M such that*

$$\left|D_{q^{-1}}^{k} p_n(0)\right| \leq \frac{M}{(\xi_1)^{n-k}}, \tag{52}$$

where ξ_1 is the smallest positive zero of $\operatorname{Sin}_q(\cdot)$.

Proof. According to (25) and Theorem 1, we obtain

$$D_{q^{-1}}^{2r} p_{2n}(z;q) = D_{q^{-1}}^{2r} A_n(z) = 2(-1)^{n+1} \sum_{k=1}^{\infty} (-1)^r \frac{\xi_k^{2r} \operatorname{Sin}_q(\xi_k z)}{(\xi_k)^{2n} \operatorname{Sin}_q'(\xi_k)},$$

$$D_{q^{-1}}^{2r+1} p_{2n}(z;q) = D_{q^{-1}}^{2r+1} A_n(z) = 2(-1)^{n+1} \sum_{k=1}^{\infty} (-1)^r \frac{\xi_k^{2r+1} \operatorname{Cos}_q(\xi_k z)}{(\xi_k)^{2n} \operatorname{Sin}_q'(\xi_k)}.$$

Therefore, $D_{q^{-1}}^{2r} p_{2n}(z;q)|_{z=0} = 0$, and

$$\begin{aligned} \left|D_{q^{-1}}^{2r+1} p_{2n}(z;q)|_{z=0}\right| &= \left|2(-1)^{n+1} \sum_{k=1}^{\infty} (-1)^r \frac{\xi_k^{2r+1}}{(\xi_k)^{2n} \operatorname{Sin}_q'(\xi_k)}\right| \\ &\leq \frac{2}{\xi_1^{2n-2r-1}} \sum_{k=1}^{\infty} \left|\frac{1}{\operatorname{Sin}_q'(\xi_k)}\right|. \end{aligned}$$

From Lemma 2, the series $\sum_{k=1}^{\infty} \left|\frac{1}{\operatorname{Sin}_q'(\xi_k)}\right|$ is convergent. Hence, there exists a constant $M_1 > 0$ such that

$$\left|D_{q^{-1}}^{r} p_{2n}(0;q)\right| = \left|D_{q^{-1}}^{r} A_n(0)\right| \leq \frac{M_1}{\xi_1^{2n-r}}. \tag{53}$$

In [16], Ismail and Mansour introduced an expansion of $E_q(zw)$ as

$$E_q(zw) = \sum_{m=0}^{\infty} B_m(z) w^{2m} + E_q(w) \sum_{m=0}^{\infty} A_m(z) w^{2m}.$$

Using the series representation of $E_q(zw)$, we can verify that

$$B_m(z) = -q^{\binom{2m}{2}} \frac{z^{2m}}{[2m]_q!} + \sum_{k=0}^{m} \frac{q^{\binom{2m-2k}{2}}}{[2m-2k]_q!} A_k(z),$$

$$D_{q^{-1}}^{r} B_m(0) = -\delta_{r,2m} + \sum_{k=0}^{m} \frac{q^{\binom{2m-2k}{2}}}{[2m-2k]_q!} D_{q^{-1}}^{r} A_k(0).$$

Therefore, $D_{q^{-1}}^{2m} B_m(0) = 1$, and from (53), we obtain

$$\begin{aligned} |D_{q^{-1}}^{r} B_m(0)| &\leq M \sum_{k=0}^{m} \frac{q^{\binom{2m-2k}{2}}}{[2m-2k]_q!} \frac{1}{\xi_1^{2k-r}} \\ &\leq \frac{M}{\xi_1^{2m-r}} \sum_{j=0}^{m} \frac{q^{\binom{2j}{2}}}{[2j]_q!} \frac{1}{\xi_1^{2j}} \\ &\leq \frac{M}{\xi_1^{2m-r}} E_q\left(\frac{1}{\xi_1}\right) \quad (r < m). \end{aligned}$$

Thus, there exists a constant $M_1 > 0$ such that

$$\left|D_{q^{-1}}^{r} p_{2m+1}(0;q)\right| = |D_{q^{-1}}^{r} B_m(0)| \leq \frac{M_1}{\xi_1^{2m-r}}. \tag{54}$$

Hence, the lemma follows from (53) and (54). $\square$

4. The Main Results

In this section, we provide the necessary and sufficient condition in order that the entire function $f(z)$ has the expansion (8).

First, we define the sequence $\{\sigma_n\}_n$ by

$$\sigma_{2n} = 2\frac{(-1)^n}{\xi_1^{2n+1}\operatorname{Sin}_q'(\xi_1)}, \quad \sigma_{2n+1} = \frac{(-1)^{n+1}\operatorname{Cos}_q(\xi_1)}{(1-q)\xi_1^{2n+1}\operatorname{Sin}_q'(\xi_1)}. \tag{55}$$

From Theorem 1, Proposition 3, and the definition of $p_n(z;q)$ in (25), we obtain

$$\frac{p_n(z;q)}{\sigma_n} = \begin{cases} \operatorname{Sin}_q(\xi_1 z) + o((\frac{\xi_1}{\xi_2})^n), & n \text{ is even;} \\ \\ \operatorname{Sin}_q(\xi_1 z) + o(\frac{1}{n^m}), & n \text{ is odd,} \end{cases} \tag{56}$$

where m is a fixed positive integer. In the following, we choose $m \geq 2$.

Theorem 2. *Let $(s_n)_n$ be a sequence of complex numbers. Suppose that the series*

$$\sum_{n=0}^{\infty} s_n p_n(z;q) \tag{57}$$

converges to $f(z)$ for a single value $z_0 \in (0,1)$. Then, it converges to $f(z)$ uniformly on compact subsets of $\mathbb{C}$. Furthermore,

$$s_n = L_n(f) \quad (n \in \mathbb{N}), \tag{58}$$

where $(L_n(f))_n$ are defined as in (28), and the function f satisfies the condition

$$D_{q^{-1}}^n f(0) = o(\xi_1^n) \quad \text{as } n \to \infty. \tag{59}$$

Proof. Suppose that the series (57) converges at z_0. Then, from (56),

$$0 = \lim_{n\to\infty} s_n p_n(z_0;q) = \lim_{n\to\infty} \frac{p_n(z_0;q)}{\sigma_n}\sigma_n s_n = \operatorname{Sin}_q(\xi_1 z_0) \lim_{n\to\infty} \sigma_n s_n.$$

Since $0 < z_0 < 1$, $\operatorname{Sin}_q(\xi_1 z_0) \neq 0$, and then $\lim_{n\to\infty} \sigma_n s_n = 0$. By using (56) and Weierstrass M-test, we conclude that the series

$$s(z) = \sum_{n=0}^{\infty} \sigma_n s_n \Big[\frac{p_n(z;q)}{\sigma_n} - \operatorname{Sin}_q(\xi_1 z)\Big] \tag{60}$$

converges uniformly on compact sets. Since the series $s(z)$ converges at z_0, we have

$$f(z_0) - s(z_0) = \operatorname{Sin}_q(\xi_1 z_0) \sum_{n=0}^{\infty} \sigma_n s_n.$$

Therefore, $\sum_{n=0}^{\infty} \sigma_n s_n$ is convergent. This implies the uniform convergence of $\sum_{n=0}^{\infty} \sigma_n s_n \operatorname{Sin}_q(\xi_1 z)$ on compact subsets of the complex plane. In view of (60), we obtain the series (57) converges uniformly on each compact subsets of $\mathbb{C}$. Combining (9) with (26) we have already verified Equation (58).

Now, from (31) we have

$$(D_{q^{-1}}^{2k} f)(z) = \sum_{n=0}^{\infty} s_n (D_{q^{-1}}^{2k} p_n)(z;q) = \sum_{n=2k}^{\infty} s_n \, p_{n-2k}(z;q)$$
$$= \sum_{n=2k}^{\infty} (\sigma_{n-2k} s_n) \frac{p_{n-2k}(z;q)}{\sigma_{n-2k}}, \tag{61}$$

and from (55), we obtain

$$\frac{\sigma_{n-2k}}{\sigma_n} = (-1)^k \frac{\xi_1^{2k}}{\mathrm{Sin}_q'(\xi_1)}. \tag{62}$$

By substituting (62) into (61), we obtain

$$(D_{q^{-1}}^{2k} f)(z) = (-1)^k \frac{\xi_1^{2k}}{\mathrm{Sin}_q'(\xi_1)} \sum_{n=2k}^{\infty} (\sigma_n s_n) \frac{p_{n-2k}(z;q)}{\sigma_{n-2k}}.$$

So, if $h_n(z;q) := \dfrac{p_n(z;q)}{\sigma_n} - \mathrm{Sin}_q(\xi_1 z)$, then

$$(-1)^k \frac{\mathrm{Sin}_q'(\xi_1)}{\xi_1^{2k}} D_{q^{-1}}^{2k} f(z) = \sum_{n=2k}^{\infty} (\sigma_n s_n) h_{n-2k}(z;q) + \mathrm{Sin}_q(\xi_1 z) \sum_{n=2k}^{\infty} (\sigma_n s_n). \tag{63}$$

From the asymptotic (56), we conclude that there exists a constant $K > 0$ such that

$$|h_n(z;q)| < K \left(\frac{\xi_1}{\xi_2}\right)^n, \quad \text{for } |z| < 1 \text{ and } n \in \mathbb{N}_0. \tag{64}$$

Therefore, from (63) and (64) we have

$$\left| \frac{\mathrm{Sin}_q'(\xi_1)}{\xi_1^{2k}} D_{q^{-1}}^{2k} f(z) \right| \leq \frac{K\xi_2}{\xi_2 - \xi_1} \big(\sup_{n \geq 2k} |\sigma_n s_n| \big) + \left| \mathrm{Sin}_q(\xi_1 z) \right| \left| \sum_{n=2k}^{\infty} (\sigma_n s_n) \right|.$$

Consequently, the sequence $(F_k(z;q))_k = (\xi_1^{-2k} D_{q^{-1}}^{2k} f(z))_k$ is uniformly convergent on $|z| < 1$. Therefore, the sequence of functions $(D_{q^{-1}} F_k(z;q))_k$ also convergent uniformly on $|z| < 1$. Furthermore,

$$\lim_{n \to \infty} F_n(0;q) = 0 = \lim_{n \to \infty} (D_{q^{-1}} F_n)(0;q),$$

which implies the function f satisfies the condition (59) and the proof is complete. □

Theorem 3. *If $f \in \mathcal{F}$, then the series*

$$\sum_{n=0}^{\infty} L_n(f) p_n(z;q)$$

converges to f uniformly on compact subsets of $\mathbb{C}$.

Proof. Since the convergence with respect to the uniform norm is uniform convergence, it is enough to prove that

$$\lim_{k \to \infty} \left\| f - \sum_{n=0}^{k} L_n(f) p_n(z;q) \right\| = 0,$$

where $\|\cdot\|$ is defined in (22). From (28) and (30), we obtain

$$\begin{aligned}\sum_{n=0}^{k} L_n(f)p_n(z;q) &= \sum_{n=0}^{k}\left(\sum_{j=n-1}^{\infty} D^j_{q^{-1}}f(0)L_{nj}\right)p_n(z;q)\\ &= \sum_{j=0}^{\infty} D^j_{q^{-1}}f(0)\sum_{n=0}^{\min(k,j+1)} L_{nj}\,p_n(z;q)\\ &= \sum_{j=0}^{k-1} D^j_{q^{-1}}f(0)\sum_{n=0}^{j+1} L_{nj}\,p_n(z;q) + \sum_{j=k}^{\infty} D^j_{q^{-1}}f(0)\sum_{n=0}^{k} L_{nj}\,p_n(z;q)\\ &= \sum_{j=0}^{k-1} q^{\frac{j(j-1)}{2}} D^j_{q^{-1}}f(0)\,\frac{z^j}{[j]_q!} + \sum_{j=k}^{\infty} D^j_{q^{-1}}f(0)\sum_{n=0}^{k} L_{nj}p_n(z;q).\end{aligned} \tag{65}$$

Using the Maclaurin's expansion (24) and Equation (65), we obtain

$$\left\|f - \sum_{n=0}^{k} L_n(f)p_n(z;q)\right\| = \left\|\sum_{j=k}^{\infty} q^{\frac{j(j-1)}{2}} D^j_{q^{-1}}f(0)\,\frac{z^j}{[j]_q!} - \sum_{j=k}^{\infty} D^j_{q^{-1}}f(0)\sum_{n=0}^{k} L_{nj}p_n(z;q)\right\|. \tag{66}$$

Now, set

$$g_k(z) := \sum_{j=k}^{\infty} q^{\frac{j(j-1)}{2}} D^j_{q^{-1}}f(0),\ r_k(z) := \sum_{j=k}^{\infty} D^j_{q^{-1}}f(0)\sum_{n=0}^{k} L_{nj}p_n(z;q).$$

Then, from (22), we obtain

$$\begin{aligned}\|g_k(z)\| &= \sup_{j\in\mathbb{N}_0}\left|\frac{D^j_{q^{-1}}g_k(0)}{\xi_1^j}\right| = \sup_{j\geq k}\left|\frac{1}{\xi_1{}^j} D^j_{q^{-1}}f(0)\right|,\\ \|r_k(z)\| &= \sup_{j\in\mathbb{N}_0}\left|\frac{D^j_{q^{-1}}r_k(0)}{\xi_1^j}\right| = \sup_{m\in\mathbb{N}_0}\left|\frac{1}{\xi_1{}^m}\sum_{j=k}^{\infty} D^j_{q^{-1}}f(0)\sum_{n=0}^{k} L_{nj}D^m_{q^{-1}}p_n(0)\right|.\end{aligned}$$

Now, using the inequalities (29) and (52), we conclude that there exists a constant $M > 0$ such that

$$\begin{aligned}\left|\frac{1}{\xi_1{}^m}\sum_{j=k}^{\infty} D^j_{q^{-1}}f(0)\sum_{n=0}^{k} L_{nj}D^m_{q^{-1}}p_n(0)\right| &\leq M\left|\sum_{j=k}^{\infty}\frac{1}{\xi_1^j} D^j_{q^{-1}}f(0)\right|\sum_{n=0}^{k} q^{\frac{(j-n)(j-n-1)}{2}}\frac{\xi_1^{j-n}}{[j-n]_q!}\\ &< M\sup_{j\geq k}\left|\frac{1}{\xi_1^j} D^j_{q^{-1}}f(0)\right|\sum_{j=k}^{\infty}\sum_{n=-\infty}^{k} q^{\frac{(j-n)(j-n-1)}{2}}\frac{\xi_1^{j-n}}{[j-n]_q!}.\end{aligned} \tag{67}$$

Since

$$\sum_{j=k}^{\infty}\sum_{n=-\infty}^{k} q^{\frac{(j-n)(j-n-1)}{2}}\frac{\xi_1^{j-n}}{[j-n]_q!} = \sum_{m=0}^{\infty}\sum_{r=m}^{\infty} q^{\frac{r(r-1)}{2}}\frac{\xi_1^r}{[r]_q!}, \tag{68}$$

changing the order of summations yields

$$\begin{aligned}\sum_{j=k}^{\infty}\sum_{n=-\infty}^{k} q^{\frac{(j-n)(j-n-1)}{2}}\frac{\xi_1^{j-n}}{[j-n]_q!} &= \sum_{r=0}^{\infty} q^{\frac{r(r-1)}{2}}\frac{\xi_1^r}{[r]_q!}\sum_{m=0}^{r} 1\\ &= \sum_{r=0}^{\infty}(r+1)q^{\frac{r(r-1)}{2}}\frac{\xi_1^r}{[r]_q!}\\ &= E_q(\xi_1) + \phi_1(\xi_1),\end{aligned} \tag{69}$$

where $\phi_1(t) = \sum_{r=0}^{\infty} q^{\frac{r(r-1)}{2}} \frac{r\,t^r}{[r]_q!}$. By combining Equation (66) with the above estimates (67) and (69), we obtain

$$\left\|f - \sum_{n=0}^{k} L_n(f) p_n(z;q)\right\| \leq \left[1 + M(E_q(\xi_1) + \phi_1(\xi_1))\right] \sup_{j\geq k} \left|\frac{1}{\xi_1^j} D^j_{q^{-1}} f(0)\right|,$$

and this complete the proof. □

Theorem 4. *Let f be an entire function. Then, the q-Lidstone series expansion*

$$f(1)A_0(z) - f(0)B_0(z) + (D^2_{q^{-1}} f)(1)A_1(z) - (D^2_{q^{-1}} f)(0)B_1(z) + \dots \tag{70}$$

converges to f if and only if $f \in \mathcal{F}$.

Proof. The proof follows immediately from Theorems 2 and 3. □

Remark 2. *Theorem 4 is not applicable with the following arrangement of q-Lidstone series:*

$$\sum_{n=0}^{\infty} D^{2n}_{q^{-1}} f(1) A_n(z) - \sum_{n=0}^{\infty} D^{2n}_{q^{-1}} f(0) B_n(z). \tag{71}$$

Notice, in the proof of Theorem 3, we used Inequality (52) to estimate the norm of the function

$$\left\|f - \sum_{n=0}^{k} L_n(f) p_n(z;q)\right\|$$

to the quantity $\left[1 + M(E_q(\xi_1) + \phi(\xi_1))\right] \sup_{j\geq k} \left|\frac{1}{\xi_1^j}(D^j_{q^{-1}} f)(0)\right|$. This estimation is inaccurate if the q-Lidstone series has the form (2). Here, each of the two series is required to be convergent. Therefore, we can not apply the result in this case.

We end the paper by providing a sufficient condition on f such that (71) converges to $f(z)$ uniformly on each compact subset of the plane.

Theorem 5. *Let ξ_1 be the smallest positive zero of $\mathrm{Sin}_q(z)$. Suppose the series*

$$\sum_{n=0}^{\infty} \frac{D^{2n}_{q^{-1}} f(0)}{\xi_1{}^{2n}}, \quad \sum_{n=0}^{\infty} \frac{D^{2n}_{q^{-1}} f(1)}{\xi_1{}^{2n}} \tag{72}$$

are absolutely convergent. Then

$$\sum_{n=0}^{\infty} D^{2n}_{q^{-1}} f(1) A_n(z) - \sum_{n=0}^{\infty} D^{2n}_{q^{-1}} f(0) B_n(z)$$

converges uniformly on each compact subset of $\mathbb{C}$ to an entire function $f(z)$.

Proof. Let $z \in \mathbb{C}$. Then, from (44), there exists a constant $M > 0$ such that

$$\left|A_n(z) - \frac{2(-1)^n \mathrm{Sin}_q(\xi_1 z)}{\xi_1^{2n+1} \mathrm{Sin}'_q(\xi_1)}\right| \leq \frac{M}{\xi_2^{2n}} \quad (n \to \infty).$$

Therefore,

$$\sum_{n=0}^{\infty}\left|D_{q^{-1}}^{2n}f(1)A_n(z)\right| \leq \frac{2|\mathrm{Sin}_q(\xi_1 z)|}{\xi_1\mathrm{Sin}_q'(\xi_1)}\sum_{n=0}^{\infty}\left|\frac{D_{q^{-1}}^{2n}f(1)}{\xi_1{}^{2n}}\right| + M\sum_{n=0}^{\infty}\left|\frac{D_{q^{-1}}^{2n}f(1)}{\xi_1^{2n}}\right|. \tag{73}$$

Since the series in (72) are absolutely convergent, then the series $\sum\limits_{n=0}^{\infty} D_{q^{-1}}^{2n}f(1)A_n(z)$ converges uniformly on compact subsets of $\mathbb{C}$. Similarly, by using (51) we can verify that the series $\sum\limits_{n=0}^{\infty} D_{q^{-1}}^{2n}f(0)B_n(z)$ converges uniformly on each compact subsets of $\mathbb{C}$. Now, by using Equations (6), (7) and (24), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} D_{q^{-1}}^{2n}f(1)A_n(z) - \sum_{n=0}^{\infty} D_{q^{-1}}^{2n}f(0)B_n(z)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_{q^{-1}}^{2n+k}f(0)}{[k]_q!}A_n(z) - \sum_{n=0}^{\infty} D_{q^{-1}}^{2n}f(0)B_n(z)\end{aligned} \tag{74}$$

Now the series $S_1 := \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_{q^{-1}}^{2n+k}f(0)}{[k]_q!}A_n(z)$ can be rearranged as

$$S_1 = \sum_{m=0}^{\infty} D_{q^{-1}}^{2m}f(0)\sum_{n=0}^{m}\frac{q^{\binom{2m-2n}{2}}}{[2m-2n]_q!}A_n(z) + \sum_{m=0}^{\infty} D_{q^{-1}}^{2m+1}f(0)\sum_{n=0}^{m}\frac{q^{\binom{2m+1-2n}{2}}}{[2m+1-2n]_q!}A_n(z). \tag{75}$$

Combining (74) and (75) and using (30), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} D_{q^{-1}}^{2n}f(1)A_n(z) - \sum_{n=0}^{\infty} D_{q^{-1}}^{2n}f(0)B_n(z) =\\ &\sum_{m=0}^{\infty} D_{q^{-1}}^{2m}f(0)\left(\sum_{n=0}^{m}\frac{q^{\binom{2m-2n}{2}}}{[2m-2n]_q!}A_n(z) - B_m(z)\right) + \sum_{m=0}^{\infty} D_{q^{-1}}^{2m+1}f(0)\sum_{n=0}^{m}\frac{q^{\binom{2m+1-2n}{2}}}{[2m+1-2n]_q!}A_n(z)\\ &= \sum_{m=0}^{\infty} D_{q^{-1}}^{m}f(0)\frac{q^{\binom{m}{2}}}{[m]_q!}z^m = f(z).\end{aligned}$$

This proves the series (71) has the limit $f(z)$ on the plane, and we obtain the required result. □

5. Conclusions and Future Work

We proved that the q-Lidstone series expansion

$$f(1)A_0(z) - f(0)B_0(z) + D_{q^{-1}}^{2}f(1)A_1(z) - D_{q^{-1}}^{2}f(0)B_1(z) + \ldots,$$

converges to the function $f(z)$ for each complex z if and only if $f \in \mathcal{F}$, i.e.,

$$D_{q^{-1}}^{n}f(0) = o(\xi_1^n) \quad \text{as } n \to \infty,$$

where ξ_1 is the smallest positive zero of the function $\mathrm{Sin}_q(z)$. We also provided a sufficient condition on f so that

$$\sum_{n=0}^{\infty} D_{q^{-1}}^{2n}f(1)A_n(z) - \sum_{n=0}^{\infty} D_{q^{-1}}^{2n}f(0)B_n(z)$$

converges to $f(z)$, uniformly on each compact subset of the plane.

Another study to give a characterization of those functions on the plane given by absolutely convergent of q-Lidstone series expansion is in progress.

Author Contributions: The authors contributed equally and significantly in writing this article. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express appreciation to the editor and the referees for their helpful comments and suggestions that improved this article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Lidstone, G. Notes on the extension of Aitken's theorem (for polynomial interpolation) to the Everett types. *Proc. Edinb. Math. Soc.* **1929**, *2*, 16–19. [CrossRef]
2. Boas, R.P.; Buck, R.C. *Polynomial Expansions of Analytic Functions*, 2nd ed.; Springer: Berlin, Germany, 1964.
3. Buckholtz, J.D.; Shaw, J.K. On functions expandable in Lidstone series. *J. Math. Anal. Appl.* **1974**, *47*, 626–632. [CrossRef]
4. Golightly, G.O. Coefficients in sine series expansions of special entire functions. *Huston J. Math.* **1988**, *14*, 365–410.
5. Boas, R.P. Representation of functions by Lidstone series. *Duke Math. J.* **1943**, *10*, 239–245.
6. Portisky, H. On certain polynomial and other approximations to analytic functions. *Proc. Natl. Acad. Sci. USA* **1930**, *16*, 83–85. [CrossRef]
7. Schoenberg, I. On certain two-point expansions of integral functions of exponential type. *Bull. Am. Math. Soc.* **1936**, *42*, 284–288. [CrossRef]
8. Whittaker, J.M. On Lidstone' series and two-point expansions of analytic functions. *Proc. Lond. Math. Soc.* **1934**, *2*, 451–469. [CrossRef]
9. Widder, D. Completely convex functions and Lidstone series. *Trans. Am. Math. Soc.* **1942**, *51*, 387–398. [CrossRef]
10. Jackson, F.H. On q-functions and a certain difference operator. *Trans. Roy. Soc. Edinb.* **1908**, *46*, 64–72. [CrossRef]
11. Ayman Mursaleen, M.; Serra-Capizzano, S. Statistical Convergence via q-Calculus and a Korovkin's Type Approximation Theorem. *Axioms* **2022**, *11*, 70. [CrossRef]
12. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: New York, NY, USA, 2002.
13. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.* **2013**, *282*, 1–19. [CrossRef]
14. Vivas-Cortez, M.; Aamir Ali, M.; Kashuri, A.; Bashir Sial, I.; Zhang, Z. Some New Newton's Type Integral Inequalities for Co-Ordinated Convex Functions in Quantum Calculus. *Symmetry* **2020**, *12*, 1476. [CrossRef]
15. Ali, I.; Malghani, Y.A.K.; Hussain, S.M.; Khan, N.; Ro, J.-S. Generalization of k-Uniformly Starlike and Convex Functions Using q-Difference Operator. *Fractal Fract.* **2022**, *6*, 216. [CrossRef]
16. Ismail, M.; Mansour, Z.S. q-analogs of Lidstone expansion theorem, two point Taylor expansion theorem, and Bernoulli polynomials. *Anal. Appl.* **2018**, *17*, 1–47. [CrossRef]
17. Gasper, G.; Rahman, M. *Basic Hypergeometric Series*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2004.
18. AL-Towailb, M. A generalization of the q-Lidstone series. *AIMS Math. J.* **2022**, *7*, 9339–9352. [CrossRef]
19. AL-Towailb, M.; Mansour, Z.S. The q-Lidstone series involving q-Bernoulli and q-Euler polynomials generated by the third Jackson q-Bessel function. *Kjm* 2022, *accepted*.
20. Cardoso, J.L. Basic Fourier series: Convergence on and outside the q-linear grid. *J. Fourier Anal. Appl.* **2011**, *17*, 96–114. [CrossRef]
21. Mansour, Z.S.; AL-Towailb, M. The Complementary q-Lidstone Interpolating Polynomials and Applications. *Math. Comput. Appl.* **2020**, *25*, 34. [CrossRef]
22. Al-Towailb, M. A q-Difference Equation and Fourier Series Expansions of q-Lidstone Polynomials. *Symmetry* **2022**, *14*, 782. [CrossRef]
23. Mansour, Z.S.; AL-Towailb, M. q-Lidstone polynomials and existence results for q-boundary value problems. *Bound Value Probl.* **2017**, *2017*, 178. [CrossRef]
24. Jackson, F.H. A basic-sine and cosine with symbolical solutions of certain differential equations. *Proc. Edin. Math. Soc.* **1904**, *22*, 28–38. [CrossRef]
25. Ismail, M. The zeros of basic Bessel functions, the functions $J_{\nu+\alpha}(x)$ and associated orthogonal polynomials. *J. Math. Anal. Appl.* **1982**, *86*, 11–19. [CrossRef]
26. Annaby, M.H.; Mansour, M.H. On the zeros of the second and third Jackson q-Bessel functions and their associated q-Hankel transforms. *Math. Proc. Camb. Philos. Soc.* **2009**, *147*, 47–67. [CrossRef]

27. Bergweiler, W.; Hayman, W.K. Zeros of solutions of a functional equation. *Comput. Methods Funct. Theory* **2003**, *3*, 55–78. [CrossRef]
28. El Guindy, A.; Mansour, Z.M. On q-analogs of zeta functions associated with a pair of q-analogs of Bernoulli numbers and polynomials. *Quaest. Math.* **2021**, *45*, 853–895. [CrossRef]
29. Andrews, G.E.; Askey, R.; Roy, R. *Special Functions. Encyclopedia of Mathematics and Its Applications*; Cambridge University Press: Cambridge, UK, 1999.
30. Olver, F.W. *Asymptotic and Special Functions*; Academic Press: New York, NY, USA, 1974.
31. Eweis, S.Z.H.; Mansour, Z.S. A determinat approach for generalized q-Bernoulli polynomials and asymptotic results. 2022, *submitted*.