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Article

On Certain Estimates for Parabolic Marcinkiewicz Integrals Related to Surfaces of Revolution on Product Spaces and Extrapolation

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Abstract: In this paper, appropriate L^p bounds for particular classes of parabolic Marcinkiewicz integrals along surfaces of revolution on product spaces are obtained. These bounds allow us to use Yano's extrapolation argument to obtain the L^p boundedness of the aforesaid integral operators under weak conditions on the kernels. These conditions on the kernels are the best possible among their respective classes. In this work, several previously known results on Marcinkiewicz integrals are fundamentally improved and extended.

Keywords: extrapolation; rough parabolic integrals; surfaces of revolution; product spaces

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1. Introduction

In this article, we assume that $d \geq 2$ and $\mathbb{S}^{d-1}$ are the unit sphere in the d-dimensional Euclidean space $\mathbb{R}^d$ equipped with the normalized Lebesgue surface measure $d\varrho = d\varrho_d(\cdot)$.

Let $\{\alpha_j\}_{j=1}^d$ be fixed numbers belong to the closed interval $[1, \infty)$, and let $\Omega : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ be a mapping given by $\Omega(v, \kappa) = \sum_{j=1}^{d} v_j^2 \kappa^{-2\alpha_j}$, where $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$. For any fixed $v \in \mathbb{R}^d$, one can easily check that $\Omega(v, \kappa)$ is the decreasing function in $\kappa > 0$. Accordingly, the equation $\Omega(v, \kappa) = 1$ has a unique solution represented by $\kappa(v) \equiv \kappa$. Fabes and Riviére [1] proved that $\kappa(v)$ is metric in $\mathbb{R}^d$, and called $(\mathbb{R}^d, \kappa)$ the mixed homogeneity space related to $\{\alpha_j\}_{j=1}^d$.

For $\kappa > 0$, let D_κ be referred to as the diagonal $d \times d$ matrix

$$D_\kappa = diag(\kappa^{\alpha_1}, \kappa^{\alpha_2}, \ldots, \kappa^{\alpha_d}).$$

For the space $(\mathbb{R}^d, \kappa)$, we consider the following transformation:

$$\begin{aligned} v_1 &= \kappa^{\alpha_1} \cos \omega_1 \ldots \cos \omega_{d-2} \cos \omega_{d-1}, \\ v_2 &= \kappa^{\alpha_2} \cos \omega_1 \ldots \cos \omega_{d-2} \sin \omega_{d-1}, \\ &\vdots \\ v_{d-1} &= \kappa^{\alpha_{d-1}} \cos \omega_1 \sin \omega_2, \\ v_d &= \kappa^{\alpha_d} \sin \omega_1. \end{aligned}$$

Therefore, $dv = \kappa^{\alpha-1} J_d(\omega) d\kappa d\sigma(\omega)$, where $\kappa^{\alpha-1} J_d(\omega)$ is the Jacobian of the transforms,

$$\alpha = \sum_{j=1}^{d} \alpha_j, \quad J_d(\omega) = \sum_{j=1}^{d} \alpha_j v_j^2, \quad and \quad \omega = D_{\kappa^{-1}} v \in \mathbb{S}^{d-1}.$$

In [1], the authors proved that $J_d(\omega)$ is a $C^\infty(\mathbb{S}^{d-1})$ function and that there exists a real constant C_d satisfying $1 \leq J_d(\omega) \leq C_d$.

Let $\mho$ be a measurable real valued function which is defined on $\mathbb{R}^d$ and belongs to the space $L^1(\mathbb{S}^{d-1})$ with the following properties:

$$\int_{\mathbb{S}^{d-1}} \mho(\omega) J_d(\omega) d\varrho(\omega) = 0 \quad \text{and} \quad \mho(D_\kappa \omega) \;=\; \mho(\omega), \;\; \forall \kappa > 0.$$

In [2], Ding, Xue and Yabuta studied parabolic Marcinkiewicz integrals $\mu_\mho$ given by

$$\mu_\mho(g)(x) = \left(\int_0^\infty |F_{\mho,s}(g)(x)|^2 \frac{ds}{s^3} \right)^{1/2},$$

where

$$F_{\mho,s}(g)(x) = \int_{\kappa(v) \leq s} g(x-v) \frac{\mho(v)}{\kappa^{\alpha-1}(v)} dv.$$

In addition, they established the L^p boundedness of $\mu_\mho$ for all $p \in (1, \infty)$ whenever $\mho \in L^q(\mathbb{S}^{d-1})$ with $q > 1$. Thereafter, the L^p boundedness of the operator $\mu_\mho$ under various assumptions on the kernels was investigated by many authors (see for instance [3–6]).

We indicate that the parabolic singular integral operator which is related to the integral operator $\mu_\mho$ is given by

$$T_\mho(g)(x) = p.v. \int_{\mathbb{R}^d} g(x-v) \frac{\mho(v)}{\kappa^\alpha(v)} dv.$$

The operator $T_\mho$ was studied by many researchers for a long time (we refer the readers to consult [1,7,8] among others).

The investigation of the Marcinkiewicz integral on product domains was considered by many authors (see for instance [9–12]).

For $k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, let $\alpha_k \geq 1$ and $\beta_j \geq 1$ be fixed numbers; and for $\eta = a_1 + ia_2, \lambda = b_1 + ib_2$ $(a_1, a_2, b_1, b_2 \in \mathbb{R} \;\; with \;\; a_1, b_1 > 0)$, let

$$K^{\kappa_1,\kappa_2}_{\mho,h}(v,u) = \frac{\mho(v,u) h(\kappa_1(v), \kappa_2(u))}{\kappa_1^{\alpha-\eta}(v) \kappa_2^{\beta-\lambda}(u)},$$

where $\alpha = \sum_{k=1}^{m} \alpha_j$, $\beta = \sum_{j=1}^{n} \beta_j$, h is a measurable mapping on $\mathbb{R}_+ \times \mathbb{R}_+$, and $\mho$ is a real-valued measurable mapping on $\mathbb{R}^m \times \mathbb{R}^n$, integrable over $\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}$ and satisfies the following:

$$\mho(D_{\kappa_1} v, D_{\kappa_2} u) \;=\; \mho(v,u), \;\; \forall \kappa_1, \kappa_2 > 0, \tag{1}$$

$$\int_{\mathbb{S}^{m-1}} \mho(v,.) J_m(v) d\varrho(v) \;=\; \int_{\mathbb{S}^{n-1}} \mho(.,u) J_n(u) d\varrho(u) = 0. \tag{2}$$

For convenient functions $\psi, \phi : \mathbb{R}_+ \to \mathbb{R}$, we consider the parabolic Marcinkiewicz operator

$$\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}(g)(\overline{x}, \overline{y}) = \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} |F_{s,t}(g)(\overline{x}, \overline{y})|^2 \frac{dsdt}{st} \right)^{1/2}, \tag{3}$$

where

$$F_{s,t}(g)(\overline{x}, \overline{y}) = \frac{1}{s^\eta t^\lambda} \iint_{\Lambda(v,u)} K^{\kappa_1,\kappa_2}_{\mho,h}(v,u) g(x-v, x_{m+1} - \phi(\kappa_1(v)), y-u, y_{n+1} - \psi(\kappa_2(u))) dvdu,$$

$g \in \mathcal{S}(\mathbb{R}^{m+1} \times \mathbb{R}^{n+1})$, $\Lambda(v,u) = \{(v,u) : \kappa_1(v) \leq s, \kappa_2(u) \leq t\}$, and $(\overline{x}, \overline{y}) = (x, x_{m+1}, y, y_{n+1})$.

When we consider the case $\alpha_1 = \cdots = \alpha_m = 1$ and $\beta_1 = \cdots = \beta_n = 1$, we get that $\kappa_1(v) = |v|$, $\kappa_2(u) = |u|$, $\alpha = m$, $\beta = n$, and $(\mathbb{R}^{m+1} \times \mathbb{R}^{n+1}, \kappa_1, \kappa_2) = (\mathbb{R}^{m+1} \times \mathbb{R}^{n+1}, |\cdot|, |\cdot|)$. In this case, we denote $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}$ by $\mathfrak{M}_{\phi,\psi,\mho,h}$. Furthermore, when we take $h \equiv 1$, $\eta = 1 = \lambda$, $\phi(s) = s$, and $\psi(t) = t$, then the operator $\mathfrak{M}_{\phi,\psi,\mho,h}$ reduces to the classical Marcinkiewicz integral on product spaces, which is denoted by $\mathfrak{M}_\mho$. Many researchers were interested in studying the operator $\mathfrak{M}_\mho$. For instance, Ding in [13] proved the L^2 boundedness of $\mathfrak{M}_\mho$ if $\mho \in L(\log L)^2(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$. However, the authors of [14] showed that $\mathfrak{M}_\mho$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for all $p \in (1, \infty)$ under the same condition $\mho \in L(\log L)^2(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$. Later, Choi in [15] improved the last results for the special case $p = 2$. Precisely, he confirmed the $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ boundedness of $\mathfrak{M}_\mho$ provided that $\mho \in L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$. In [16], the authors proved the L^p ($1 < p < \infty$) boundedness of the operator $\mathfrak{M}_\mho$ if $\mho$ belongs to $L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$. Furthermore, they found that the condition $\mho \in L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ is optimal in the sense that if we replace the space $L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ by the space $L(\log L)^\varepsilon(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $0 < \varepsilon < 1$, then the operator $\mathfrak{M}_\mho$ will not be bounded on $L^2(\mathbb{R}^m \times \mathbb{R}^n)$.

Al-Qassem in [17] established the L^p boundedness of $\mathfrak{M}_\mho$ for all $p \in (1, \infty)$ under the assumption $\mho$ belongs to the certain block space $B_q^{(0,0)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $q > 1$. Moreover, he proved that the space $B_q^{(0,0)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ is optimal in the sense that we cannot replace it by the space $B_q^{(0,\varepsilon)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for any $\varepsilon \in (-1, 0)$ so that the operator $\mathfrak{M}_\mho$ is still bounded on $L^2(\mathbb{R}^m \times \mathbb{R}^n)$. For more information about the development and applications of the operator $\mathfrak{M}_\mho$, one can refer to [2,16–18], among other references.

The results in [16] were generalized by Al-Salman in [9] in which he proved the L^p boundedness of $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,1}$ for all $1 < p < \infty$ under the conditions $\mho \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, $\phi(\kappa_1) = \kappa_1$, and $\psi(\kappa_2) = \kappa_2$. Very recently, this result was improved in [12], in which the authors satisfied the L^p boundedness of $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}$ for all $|1/2 - 1/p| < \min\{1/\mu', 1/2\}$, provided that $\phi(\kappa_1) = \kappa_1$, $\psi(\kappa_2) = \kappa_2$, $h \in \Upsilon_\mu(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $\mu > 1$, and $\mho \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cup B_q^{(0,0)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for some $q > 1$, where $\Upsilon_\mu(\mathbb{R}_+ \times \mathbb{R}_+)$ (for $\mu > 1$) refers to the class of all functions h that are defined on $\mathbb{R}_+ \times \mathbb{R}_+$, are measurable and satisfy

$$\|h\|_{\Upsilon_\mu(\mathbb{R}_+ \times \mathbb{R}_+)} = \sup_{k,j \in \mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} |h(\kappa_1, \kappa_2)|^\mu \frac{d\kappa_1 d\kappa_2}{\kappa_1 \kappa_2} \right)^{1/\mu} < \infty.$$

The consideration of the L^p mapping properties of rough integral operators related to surfaces has been given a great deal of attention by many mathematicians (see for example [19–21] and the references therein.)

In this article, we let $\mathbb{I}$ denote the collection of all non-negative $C^1(\mathbb{R}_+)$ mappings ϑ that satisfy the following properties:

(a) ϑ is strictly increasing and ϑ' is monotone on $\mathbb{R}_+$,
(b) $\vartheta(\kappa) \leq M_1 \vartheta(2\kappa)$ for a fixed constant $M_1 \in (0, 1)$ and $\vartheta(\kappa) \geq M_2 \vartheta(2\kappa)$ for a constant $M_2 \in (0, M_1]$,
(c) $\vartheta(\kappa) \leq M_3 \kappa \vartheta'(\kappa)$ on $\mathbb{R}_+$ for a fixed constant $M_3 \in (\frac{-1}{\ln(M_2)}, \infty)$.

Additionally, we let $\mathbb{D}$ denote the collection of all non-negative $C^1(\mathbb{R}_+)$ mappings ϑ that satisfy the following properties:

(a) ϑ is strictly decreasing and ϑ' is monotone on $\mathbb{R}_+$,
(b) $\vartheta(2\kappa) \leq M_1 \vartheta(\kappa)$ for a fixed constant $M_1 \in (0, 1)$ and $\vartheta(2\kappa) \geq M_2 \vartheta(\kappa)$ for a constant $M_2 \in (0, M_1]$,
(c) $\vartheta(\kappa) \leq M_3 |\kappa \vartheta'(\kappa)|$ on $\mathbb{R}_+$ for a fixed constant $M_3 \in (\frac{-1}{\ln(M_2)}, \infty)$.

We indicate here that the collections $\mathbb{D}$ and $\mathbb{I}$ were established and introduced in [18]. Some model examples for mappings belong to $\mathbb{D}$ are $\vartheta(\kappa) = \kappa^{-\nu} e^{-\iota\kappa}$ with $\nu \geq 0$ and $\iota \geq 0$, and functions belonging to $\mathbb{I}$ are $\vartheta(\kappa) = \kappa^\nu e^{\iota\kappa}$ for $\nu \geq 0$ and $\iota \geq 0$.

In view of the results in [9,12] on the boundedness of the operator $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}$ along the curve $(\overline{x},\overline{y}) = (\overline{v},\overline{u})$ and of the results on the boundedness of the rough operators along surfaces of revolution, we are prompted to ask the following natural question: is the rough parabolic operator $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}$ along surfaces of revolutions bounded?

The main goal of this paper is to give an affirmative answer to the above question. In fact, we have the following:

Theorem 1. *Assume that h belongs to $\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)$ with $\mu>1$ and assume that $\mho$ belongs to the space $L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ with $1<q\le 2$. Suppose that ϕ,ψ are in $\mathbb{I}$ or $\mathbb{D}$. Then, there is a positive constant C_p such that*

$$\left\|\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}(g)\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \le C_p\frac{\mu}{(\mu-1)(q-1)}\|\mho\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\|g\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}$$

for $|1/2-1/p|<\min\{1/\mu',1/2\}$.

By the estimates in Theorem 1 and Yano's extrapolation argument(see [7,22]), we establish the following result:

Theorem 2. *Assume that $\mho$ satisfies the conditions (1)–(2), and assume that h, ϕ and ψ are given as in Theorem 1.*
(i) If $\mho\in B^{(0,0)}_q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $q>1$, then the inequality

$$\begin{aligned}&\left\|\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}(g)\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\ \le\ & C_p\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\|g\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\left(1+\|\mho\|_{B^{(0,0)}_q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\right)\end{aligned}$$

holds for all $|1/2-1/p|<\min\{1/\mu',1/2\}$;
(ii) If $\mho\in L(\log L)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$, then the inequality

$$\left\|\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}(g)\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \le C_p\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\|g\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\left(1+\|\mho\|_{L(logL)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\right)$$

holds for all $|1/2-1/p|<\min\{1/\mu',1/2\}$.

Noteworthy is the fact that, in Theorem 2, the boundedness of the operator $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}$ is obtained for the full range of p, i.e., $1<p<\infty$, whenever $\mu\ge 2$. However, when $1<\mu<2$, we satisfy that $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}$ is bounded only for $\frac{2\mu}{3\mu-1}<p<\frac{2\mu}{2-\mu}$. A natural question arising here is whether we obtain the L^p boundedness of $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}$ for $1<p<\infty$ whenever $\mu\in(1,2)$. We shall answer this question in the next theorem.

Theorem 3. *Assume that ϕ and ψ belong to $\mathbb{I}$ or $\mathbb{D}$ and that $\mho$ satisfies the conditions (1) and (2). Let $h\in\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\mu\in(1,2]$.*

(i) If $\mho\in L(\log L)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})\cup B^{(0,0)}_q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $q>1$, then $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}$ is bounded on $L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$ for all $p\in[2,\infty)$.

(ii) If $\mho\in L(\log L)^2(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})\cup B^{(0,1)}_q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $q>1$, then $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}$ is bounded on $L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$ for all $p\in(1,2)$.

From Theorem 3, we see that the boundedness of $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}$ is satisfied whenever the condition on $\mho$ is optimal only for $p\in[2,\infty)$. However, for the case $p\in(1,2)$, the boundedness of $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}$ is obtained, but the condition on $\mho$ is not optimal.

Marcinkiewicz integrals operators are parts of the class of Littlewood-Paley g-functions. The theory of Marcinkiewicz integrals has a long history. This theory is of vast scope and utility due its role in dealing with many important problems arising in such parts of analysis as partial differential equations and several complex variables. Recent efforts in dealing with these operators have been mostly focused on finding the weakest possible kernel conditions under which L^p boundedness holds.

Henceforward, the constant C signifies a positive real number that could be different at each occurrence but is independent of all essential variables.

2. Some Lemmas

This section is devoted to establishing some lemmas that will be needed to prove the main results of this paper. Let us first recall the following lemma from [7].

Lemma 1. *Suppose that ϕ belongs to $\mathbb{I}$ or $\mathbb{D}$. For a suitable mapping g, we let the maximal function $\mathcal{M}_{\phi}^{\omega}$ be defined on $\mathbb{R}^{d+1}$ by*

$$\mathcal{M}_{\phi}^{\omega} g(\overline{x}) = \sup_{j\in\mathbb{Z}} \left| \int_{2^j}^{2^{j+1}} g(x - D_{\kappa}\omega, x_{d+1} - \phi(\kappa)) \frac{d\kappa}{\kappa} \right|.$$

Then for $p > 1$, there exists a positive constant C_p such that

$$\left\| \mathcal{M}_{\phi}^{\omega}(g) \right\|_{L^p(\mathbb{R}^{d+1})} \leq C_p \|g\|_{L^p(\mathbb{R}^{d+1})}.$$

Lemma 2. *Assume that ϕ, ψ are in $\mathbb{I}$ or $\mathbb{D}$. Define the maximal function $\mathcal{M}_{\phi,\psi}^{\omega,\nu}$ on $\mathbb{R}^{m+1} \times \mathbb{R}^{n+1}$ by*

$$\mathcal{M}_{\phi,\psi}^{\omega,\nu} g(\overline{x}, \overline{y}) = \sup_{k,j\in\mathbb{Z}} \left| \int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} g(x - D_{\kappa_1}\omega, x_{m+1} - \phi(\kappa_1), y - D_{\kappa_2}\nu, y_{n+1} - \psi(\kappa_2)) \frac{d\kappa_1}{\kappa_1} \frac{d\kappa_2}{\kappa_2} \right|.$$

Then there is a constant $C_p > 0$ such that the inequality

$$\left\| \mathcal{M}_{\phi,\psi}^{\omega,\nu}(g) \right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \leq C_p \|g\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}$$

holds for all $g \in L^p(\mathbb{R}^{m+1} \times \mathbb{R}^{n+1})$ with $1 < p \leq \infty$.

Proof. It is well known that $\mathcal{M}_{\phi,\psi}^{\omega,\nu} g(\overline{x}, \overline{y}) \leq \mathcal{M}_{\psi}^{\nu} \circ \mathcal{M}_{\phi}^{\omega} g(\overline{x}, \overline{y})$, where $\mathcal{M}_{\phi}^{\omega} g(\overline{x}, \overline{y}) = \mathcal{M}_{\phi}^{\omega} g(\cdot, \overline{y})(\overline{x})$, $\mathcal{M}_{\psi}^{\nu} g(\overline{x}, \overline{y}) = \mathcal{M}_{\psi}^{\nu} g(\overline{x}, \cdot)(\overline{y})$ and $\circ$ refers to the composition of the operators. Hence, by Lemma 1 we have

$$\left\| \mathcal{M}_{\phi,\psi}^{\omega,\nu}(g) \right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \leq C_p \left\| \mathcal{M}_{\psi}^{\nu}\left(\mathcal{M}_{\phi}^{\omega}(g) \right) \right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \leq C_p \|g\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}.$$

□

We shall need the following from [4]:

Lemma 3. *Let γ denote the distinct numbers of $\{\alpha_j\}$ with $j \in \{1, 2, \ldots, d\}$, and let $\delta \in [0, 1]$. Then for $x, \xi \in \mathbb{R}^d$, there exists $C >$ such that*

$$\left| \int_1^2 e^{-iD_{\kappa}x\cdot\xi} \frac{d\kappa}{\kappa} \right| \leq C|x \cdot \xi|^{-\frac{\delta}{\gamma}}.$$

Let $\tau \geq 2$. We define the family of measures $\{\varrho_{K_{\mho,h}^{\kappa_1,\kappa_2}s,t} := \varrho_{s,t} : s, t \in \mathbb{R}_+\}$ and its related maximal operators ϱ_h^* and $M_{h,\tau}$ on $\mathbb{R}^{m+1} \times \mathbb{R}^{n+1}$ by

$$\iint_{\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}} g d\varrho_{s,t} = \frac{1}{s^{\eta}t^{\lambda}} \int_{1/2s\leq\kappa_1(v)\leq s} \int_{1/2t\leq\kappa_2(u)\leq t} g(\overline{v}, \overline{u}) K_{\mho,h}^{\kappa_1,\kappa_2}(v, u) du dv,$$

$$\varrho_h^* g(\overline{v},\overline{u}) = \sup_{s,t\in\mathbb{R}_+} ||\varrho_{s,t}| * g(\overline{v},\overline{u})|,$$

and

$$M_{h,\tau} g(\overline{v},\overline{u}) = \sup_{j,k\in\mathbb{Z}} \int_{\tau^j}^{\tau^{j+1}} \int_{\tau^k}^{\tau^{k+1}} ||\varrho_{s,t}| * g(\overline{v},\overline{u})| \frac{dsdt}{st},$$

where $|\varrho_{s,t}|$ is defined similarly to $\varrho_{s,t}$ with replacing $\mho h$ by $|\mho h|$.

To prove Theorem 1, we need to establish the following lemmas.

Lemma 4. *Let $\mho \in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ with $q>1$ and satisfy the conditions (1) and (2). For $s,t>0$ and suitable mappings ϕ,ψ, let*

$$\begin{aligned}\mathcal{H}(\kappa_1,\kappa_2) &= \iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}} e^{-i\{D_{s\kappa_1}v\cdot\xi+\phi(s\kappa_1(v))\xi_{m+1}+D_{t\kappa_2}u\cdot\zeta+\psi(t\kappa_2(u))\zeta_{n+1}\}} \\ &\times \mho(v,u)J_m(v)J_n(u)d\varrho(v)d\varrho(u).\end{aligned}$$

Then, there exist positive constants C and δ with $0<\delta<\min\{\frac{\gamma_1}{2q'},\frac{\gamma_2}{2q'},\frac{\gamma_1}{\alpha},\frac{\gamma_2}{\beta},\frac{1}{2}\}$ such that

$$\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{H}(\kappa_1,\kappa_2)|^2 \frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2} \leq C\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} |D_s\xi|^{\pm\frac{\delta}{\gamma_1 q'}} |D_t\zeta|^{\pm\frac{\delta}{\gamma_2 q'}}, \tag{4}$$

where $a^{\pm b}=\min\{a^b,a^{-b}\}$ and γ_1,γ_2 denote the distinct numbers of $\{\alpha_i\}$, $\{\beta_j\}$, respectively.

Proof. We shall prove the lemma only for the case $1<q\leq 2$ since $L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}) \subseteq L^2(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for all $q\geq 2$. Thanks to the Schwartz inequality, we know that

$$\begin{aligned}\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{H}(\kappa_1,\kappa_2)|^2 \frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2} &\leq C\int_{\mathbb{S}^{n-1}}\Big(\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{m-1}} \mathcal{G}(\xi,v,x) \\ &\times \mho(v,u)\overline{\mho(x,u)}J_n(v)J_n(x)d\varrho(v)d\varrho(x)\Big)J_m(u)d\varrho(u),\end{aligned}$$

where $\mathcal{G}(\xi,v,x)=\int_1^2 e^{-iD_{\frac{s}{2}\kappa_1}\xi\cdot(v-x)}\frac{d\kappa_1}{\kappa_1}$. Let $\rho=\frac{D_{\frac{s}{2}}\xi}{|D_{\frac{s}{2}}\xi|}$. Then using Lemma 3, we obtain

$$\begin{aligned}\mathcal{G}(\xi,v,x) &\leq C\mid D_{\frac{s}{2}}\xi\cdot(v-x)\mid^{-\delta/\gamma_1} \leq C2^{\alpha\delta/\gamma_1}(\mid\rho\cdot(v-x)\mid\mid D_s\xi\mid)^{-\delta/\gamma_1} \\ &\leq C\mid D_s\xi\mid^{-\delta/\gamma_1}(\mid\rho\cdot(v-x)\mid)^{-\delta/\gamma_1},\end{aligned}$$

where $0<\delta<\min\{\frac{1}{2},\frac{\gamma_1}{\alpha}\}$. This in turn by Hölder's inequality implies

$$\begin{aligned}\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{H}(\kappa_1,\kappa_2)|^2 \frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2} &\leq C|D_s\xi|^{-\frac{\delta}{q'\gamma_1}}\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \\ &\times \left(\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{m-1}} |\rho\cdot(v-x)|^{-\frac{\delta q'}{\gamma_1}} d\varrho(v)d\varrho(x)\right)^{1/q'}.\end{aligned}$$

Now, if we choose $0<\delta<\frac{\gamma_1}{2q'}$, we deduce that the last integral is finite, and hence

$$\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{H}(\kappa_1,\kappa_2)|^2 \frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2} \leq C\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|D_s\xi|^{-\frac{\delta}{\gamma_1 q'}}. \tag{5}$$

Similarly, we obtain

$$\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{H}(\kappa_1,\kappa_2)|^2\frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2} \leq C\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|D_t\zeta|^{-\frac{\delta}{\gamma_2 q'}}. \tag{6}$$

Now, to prove the other estimates in (4), we need to use conditions (1) and (2) and a simple change of variable to obtain

$$\begin{aligned}
&\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{H}(\kappa_1,\kappa_2)|^2\frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2}\\
\leq\ & C\int_{1/2}^{1}\int_{1/2}^{1}\left(\iint_{\mathbf{S}^{m-1}\times\mathbb{S}^{n-1}}|e^{-iD_{s\kappa_1}\xi\cdot v}-1||\mho(v,u)J_m(v)J_n(u)|d\varrho(v)d\varrho(u)\right)^2\frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2}\\
\leq\ & C\|\mho\|^2_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|D_s\xi|^2.
\end{aligned}$$

Thus, when the last estimate is combined with the estimate $\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{H}(\kappa_1,\kappa_2)|^2\frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2} \leq C\|\mho\|^2_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}$, we obtain that

$$\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{H}(\kappa_1,\kappa_2)|^2\frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2} \leq C\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|D_s\xi|^{\frac{\delta}{\gamma_1 q'}}. \tag{7}$$

Similarly, we know that

$$\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{H}(\kappa_1,\kappa_2)|^2\frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2} \leq C\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|D_t\zeta|^{\frac{\delta}{\gamma_2 q'}}. \tag{8}$$

Consequently by (5)–(8), we obtain all the estimates in the lemma and hence the proof is complete. □

Lemma 5. *Assume that $h \in \Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\mu > 1$, $\mho \in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $q > 1$, $\tau \geq 2$, and ψ, ϕ are given as in Theorem 1. Then for some positive constant C we have the following estimates*

$$\|\varrho_{s,t}\| \leq C\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}, \tag{9}$$

$$\begin{aligned}
\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}|\hat{\varrho}_{s,t}(\xi,\zeta)|^2\frac{dsdt}{st} \leq\ & C\ln^2(\tau)\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\\
\times\ & |D_{\tau^k}\xi|^{\pm\frac{2\delta}{\gamma_1 q' w}}|D_{\tau^j}\zeta|^{\pm\frac{2\delta}{\gamma_2 q' w}}
\end{aligned} \tag{10}$$

for all $j, k \in \mathbb{Z}$, where δ is the same as in Lemma 4, $w = \max\{2, \mu'\}$ and $\|\varrho_{s,t}\|$ is the total variation of $\varrho_{s,t}$.

Proof. By the definition of $\varrho_{s,t}$, we immediately obtain (9). Furthermore, by Hölder's inequality and a simple change in variables, we have

$$\begin{aligned}|\hat{\varrho}_{s,t}(\xi,\zeta)| &\leq C\int_{\frac{1}{2}s}^{s}\int_{\frac{1}{2}t}^{t}|h(\kappa_1,\kappa_2)|\left|\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}e^{-i\{D_{\kappa_1}v\cdot\xi+\phi(\kappa_1(v))\xi_{m+1}+D_{\kappa_2}u\cdot\zeta+\psi(\kappa_2(u))\zeta_{n+1}\}}\right.\\ &\times \left. J_m(v)J_n(v)\mho(v,u)d\varrho(v)d\varrho(u)\right|\frac{d\kappa_1 d\kappa_1}{\kappa_1\kappa_2}\\ &\leq C\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\left(\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(\kappa_1,\kappa_2)|^{\mu'}\frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2}\right)^{1/\mu}.\end{aligned}$$

It is easy to see that if $\mu\in(1,2]$, we obtain

$$|\hat{\varrho}_{s,t}(\xi,\zeta)| \leq \|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\|\mho\|^{(1-2/\mu')}_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\left(\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(\kappa_1,\kappa_2)|^2\frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2}\right)^{1/\mu'}.$$

However, if $\mu>2$, using Hölder's inequality we obtain

$$|\hat{\varrho}_{s,t}(\xi,\zeta)| \leq \|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\left(\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(\kappa_1,\kappa_2)|^2\frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2}\right)^{1/2}.$$

Hence, in either case of μ, we have

$$|\hat{\varrho}_{s,t}(\xi,\zeta)| \leq C\|\mho\|^{(w-2)/\mu'}_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\left(\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(\kappa_1,\kappa_2)|^2\frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2}\right)^{1/w},$$

where $w=\max\{2,\mu'\}$. Thus, by Lemma 4, we obtain

$$|\hat{\varrho}_{s,t}(\xi,\zeta)|^2 \leq C\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}|D_s\xi|^{\pm\frac{2\delta}{w\gamma_1 q'}}|D_t\zeta|^{\pm\frac{2\delta}{w\gamma_2 q'}}.$$

In addition, since $\tau^k\leq s\leq\tau^{k+1}$ and $\tau^j\leq t\leq\tau^{j+1}$, we directly obtain that

$$|\hat{\varrho}_{s,t}(\xi,\zeta)|^2 \leq C\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}|D_{\tau^k}\xi|^{-\frac{2\delta}{w\gamma_1 q'}}|D_{\tau^j}\zeta|^{-\frac{2\delta}{w\gamma_2 q'}} \tag{11}$$

and

$$\begin{aligned}|\hat{\varrho}_{s,t}(\xi,\zeta)|^2 &\leq C\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}|D_{\tau^{k+1}}\xi|^{+\frac{2\delta}{w\gamma_1 q'}}|D_{\tau^{j+1}}\zeta|^{+\frac{2\delta}{w\gamma_2 q'}}\\ &\leq C\tau^{\max\{\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_n\}\frac{2\delta}{wq'}(\frac{1}{\gamma_1}+\frac{1}{\gamma_2})}\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\\ &\times |D_{\tau^k}\xi|^{+\frac{2\delta}{w\gamma_1 q'}}|D_{\tau^j}\zeta|^{+\frac{2\delta}{w\gamma_2 q'}}\\ &\leq C2^{2\delta\max\{\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_n\}(\frac{1}{\gamma_1}+\frac{1}{\gamma_2})}\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\\ &\times |D_{\tau^k}\xi|^{+\frac{2\delta}{w\gamma_1 q'}}|D_{\tau^j}\zeta|^{+\frac{2\delta}{w\gamma_2 q'}}\\ &\leq C\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}|D_{\tau^k}\xi|^{+\frac{2\delta}{w\gamma_1 q'}}|D_{\tau^j}\zeta|^{+\frac{2\delta}{w\gamma_2 q'}}.\end{aligned} \tag{12}$$

Consequently, by combining (11) with (12), we obtain (10). □

Lemma 6. *Let $\mho \in L^1(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$, $h \in \Upsilon_\mu(\mathbb{R}_+ \times \mathbb{R}_+)$ with $\mu > 1$ and ψ, ϕ be given as in Theorem 1. Then, for every $\mu' < p \leq \infty$, there exists a positive constant C_p such that for all $g \in L^p(\mathbb{R}^{m+1} \times \mathbb{R}^{n+1})$,*

$$\|\varrho_h^*(g)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \leq C_p \|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)} \|\mho\|_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|g\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}.$$

Proof. By Hölder's inequality, we obtain

$$\begin{aligned} ||\varrho_{s,t}| * g(\bar{x},\bar{y})| &\leq C\|h\|_{\Upsilon_\mu(\mathbf{R}_+\times\mathbb{R}_+)} \|\mho\|^{1/\mu}_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \left(\frac{1}{st} \int_{\frac{s}{2}}^{s} \int_{\frac{t}{2}}^{t} \int_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}} |\mho(v,u)| \right. \\ \times \quad & \left. |g(x - D_{\kappa_1} v, x_{m+1} - \phi(\kappa_1(v)), y - D_{\kappa_2} u, y_{n+1} - \psi(\kappa_2(u)))|^{\mu'} d\varrho(v) d\varrho(u) d\kappa_1 d\kappa_2 \right)^{1/\mu'}. \end{aligned}$$

Hence, using Minkowski's inequality for integrals together along with Lemma 2, we obtain

$$\begin{aligned} \|\varrho_h^*(g)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} &\leq C\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)} \|\mho\|^{1/\mu}_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \\ \times & \left(\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}} |\mho(v,u)| \|\mathcal{M}^{v,u}_{\phi,\psi}(|g|^{\mu'})\|_{L^{(p/\mu')}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} d\varrho(v) d\varrho(u) \right)^{1/\mu'} \\ \leq & \; C\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)} \|\mho\|_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|\mathcal{M}^{v,u}_{\phi,\psi}(|g|)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \\ \leq & \; C_p \|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)} \|\mho\|_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|g\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}. \end{aligned}$$

□

Lemma 7. *We assume that $h \in \Upsilon_\mu(\mathbb{R}_+ \times \mathbb{R}_+)$ with $\mu > 1$, $\mho \in L^q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $1 < q \leq 2$ and ψ, ϕ belong to $\mathbb{D}$ or $\mathbb{I}$. Then, for any functions $\{\mathcal{B}_{k,j}(\cdot,\cdot),\ j,k \in \mathbb{Z}\}$ on $\mathbb{R}^{m+1} \times \mathbb{R}^{n+1}$, a positive constant C_p exists such that the inequality*

$$\begin{aligned} &\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\tau^j}^{\tau^{j+1}} \int_{\tau^k}^{\tau^{k+1}} \left| \varrho_{s,t} * \mathcal{B}_{j,k} \right|^2 \frac{dsdt}{st} \right)^{1/2} \right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \\ &\leq C_p \ln(\tau) \|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{B}_{j,k} \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n_1})} \end{aligned}$$

holds for all $|1/p - 1/2| < \min\{1/\mu', 1/2\}$.

Proof. We will follow a similar argument employed in [23]. Since $\Upsilon_\mu(\mathbb{R}_+ \times \mathbb{R}_+) \subseteq \Upsilon_2(\mathbb{R}_+ \times \mathbb{R}_+)$ for all $\mu \geq 2$, it suffices to prove the lemma for the case $1 < \mu \leq 2$. Thus, we have $|1/2 - 1/p| < 1/\mu'$. Now, if $2 \leq p < \frac{2\mu}{2-\mu}$, then by duality, there is a function $G \in L^{(p/2)'}(\mathbb{R}^{m+1} \times \mathbb{R}^{n+1})$ which is non-negative and satisfies $\|G\|_{L^{(p/2)'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \leq 1$ and

$$\begin{aligned} &\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\tau^j}^{\tau^{j+1}} \int_{\tau^k}^{\tau^{k+1}} \left| \varrho_{s,t} * \mathcal{B}_{j,k} \right|^2 \frac{dsdt}{st} \right)^{1/2} \right\|^2_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \\ &= \iint_{\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}} \sum_{j,k\in\mathbb{Z}} \int_{\tau^j}^{\tau^{j+1}} \int_{\tau^k}^{\tau^{k+1}} \left| \varrho_{s,t} * \mathcal{B}_{j,k}(\bar{x},\bar{y}) \right|^2 \frac{dsdt}{st} G(\bar{x},\bar{y}) d\bar{x} d\bar{y}. \end{aligned}$$

Thanks to Schwartz's inequality, we deduce that

$$\begin{aligned}\left|\varrho_{s,t} * \mathcal{B}_{j,k}(\overline{x},\overline{y})\right|^2 \quad &\leq \quad C\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^{\mu}_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\left(\int_{\frac{1}{2}t}^{t}\int_{\frac{1}{2}s}^{s}\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{m-1}}\right.\\ &\times \quad \left|\mathcal{B}_{j,k}(x - D_{\kappa_1}v, x_{m+1} - \phi(\kappa_1(v)), y - D_{\kappa_2}u, y_{n+1} - \psi(\kappa_2(u)))\right|^2\\ &\times \quad \left.|\mho(v,u)||h(\kappa_1,\kappa_2)|^{2-\mu}d\varrho(v)d\varrho(u)\frac{d\kappa_1 d\kappa_2}{\kappa_1\kappa_2}\right).\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|\varrho_{s,t} * \mathcal{B}_{j,k}\right|^2\frac{dsdt}{st}\right)^{1/2}\right\|^2_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \leq C\|h\|^{\mu}_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\\ \times \quad &\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbf{S}^{n-1})}\iint_{\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}}\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{B}_{j,k}(\overline{x},\overline{y})\right|^2\right)M_{|h|^{2-\mu},\tau}\widetilde{G}(-\overline{x},-\overline{y})d\overline{x}d\overline{y}\\ \leq \quad &C\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^{\mu}_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\sum_{j,k\in\mathbb{Z}}\left|\mathcal{B}_{j,k}\right|^2\right\|_{L^{(p/2)}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\ \times \quad &\left\|M_{|h|^{2-\mu},\tau}(\widetilde{G})\right\|_{L^{(p/2)'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})},\end{aligned}$$

where $\widetilde{G}(-\overline{x},-\overline{y}) = G(\overline{x},\overline{y})$. Since $h \in \Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)$, we obtain $|h|^{2-\mu} \in \Upsilon_{\frac{\mu}{2-\mu}}(\mathbb{R}_+\times\mathbb{R}_+)$, and since $\left(\frac{p}{2}\right)' > \left(\frac{\mu}{2-\mu}\right)'$, we obtain, by Lemma 6 and Hölder's inequality

$$\begin{aligned}&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|\varrho_{s,t} * \mathcal{B}_{j,k}\right|^2\frac{dsdt}{st}\right)^{1/2}\right\|^2_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\ \leq \quad &C\ln^2(\tau)\|h\|^{\mu}_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{B}_{j,k}\right|^2\right)^{1/2}\right\|^2_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\ \times \quad &\left\|\varrho^*{}_{|h|^{2-\mu}}(\widetilde{G})\right\|_{L^{(p/2)'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\ \leq \quad &C_p\ln^2(\tau)\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{B}_{j,k}\right|^2\right)^{1/2}\right\|^2_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}.\end{aligned}$$

On the other hand, if $\frac{2\mu}{3\mu-2} < p < 2$, then by the duality, there is a set of functions $X = X_{j,k}(\overline{x},\overline{y},s,t)$ defined on $\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}\times\mathbb{R}_+\times\mathbb{R}_+$ with

$$\left\|\left\|\|X_{j,k}\|_{L^2([\tau^k,\tau^{k+1}]\times[\tau^j,\tau^{j+1}],\frac{dsdt}{st})}\right\|_{l^2}\right\|_{L^{p'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \leq 1$$

such that

$$\begin{aligned}
&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|\varrho_{s,t}*\mathcal{B}_{j,k}\right|^2\frac{dsdt}{st}\right)^{1/2}\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
&= \iint_{\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}}\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left(\varrho_{s,t}*\mathcal{B}_{j,k}(\overline{x},\overline{y})\right)X_{j,k}(\overline{x},\overline{y},s,t)\frac{dsdt}{st}d\overline{x}d\overline{y}\\
&\leq C_p\ln(\tau)\left\|(\Gamma(X))^{1/2}\right\|_{L^{p'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{B}_{j,k}\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}, \qquad (13)
\end{aligned}$$

where

$$\Gamma(X)(\overline{x},\overline{y})=\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|\varrho_{s,t}*X_{j,k}(\overline{x},\overline{y},s,t)\right|^2\frac{dsdt}{st}.$$

Again, since $\frac{p'}{2}>1$, then by the duality, a function $Z\in L^{(p'/2)'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$ exists that satisfies $\|Z\|_{L^{(p'/2)'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\leq 1$ and

$$\begin{aligned}
&\left\|(\Gamma(X))^{1/2}\right\|^2_{L^{p'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
&= \sum_{j,k\in\mathbb{Z}}\iint_{\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|\varrho_{s,t}*X_{j,k}(\overline{x},\overline{y},s,t)\right|^2\frac{dsdt}{st}Z(\overline{x},\overline{y})d\overline{x}d\overline{y}\\
&\leq C\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^{\mu}_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\varrho^*{}_{|h|^{2-\mu}}(Z)\right\|_{L^{(p'/2)'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
&\times\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|X_{j,k}(\cdot,\cdot,s,t)\right|^2\frac{dsdt}{st}\right)\right\|_{L^{(p'/2)}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
&\leq C\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}.
\end{aligned}$$

Therefore, by the last inequality, together with (13), the desired inequality holds for the case $\frac{2\mu}{3\mu-2}<p<2$ which in turn finishes the proof of this lemma. □

By adapting the same technique employed in proving Lemma 4 in [7] to the product space setting, it is easy to show the following result.

Lemma 8. *Assume that* $h\in\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)$ *with* $1<\mu\leq 2$, $\mho\in L^q\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ *with* $1<q\leq 2$ *and* ϕ,ψ *belong to* $\mathbb{I}$ *or* $\mathbb{D}$. *Then for any functions* $\{\mathcal{B}_{k,j}(\cdot,\cdot),\ j,k\in\mathbb{Z}\}$ *on* $\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}$, *there is a positive constant* C_p *such that*

$$\begin{aligned}
&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|\varrho_{s,t}*\mathcal{B}_{j,k}\right|^2\frac{dsdt}{st}\right)^{1/2}\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
&\leq C_p\ln^2(\tau)\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{B}_{j,k}\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n}1)} \quad \textit{for } 1<p<2,
\end{aligned}$$

and

$$\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|\varrho_{s,t}*\mathcal{B}_{j,k}\right|^2\frac{dsdt}{st}\right)^{1/2}\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}$$
$$\leq C_p\ln(\tau)\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{B}_{j,k}\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n_1})}\quad for\ 2<p<\infty.$$

Proof. First, we consider the case $1<p<2$. By following the same above arguments, we obtain, by the duality, there are functions $A=A_{j,k}(\bar{x},\bar{y},s,t)$ defined on $\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}\times\mathbb{R}_+\times\mathbb{R}_+$ with

$$\left\|\left\|\|A_{j,k}\|_{L^2([\tau^k,\tau^{k+1}]\times[\tau^j,\tau^{j+1}],\frac{dsdt}{st})}\right\|_{l^2}\right\|_{L^{p'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\leq 1$$

and satisfies

$$\begin{aligned}
&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|\varrho_{s,t}*\mathcal{B}_{j,k}\right|^2\frac{dsdt}{st}\right)^{1/2}\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
=&\iint_{\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}}\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left(\varrho_{s,t}*\mathcal{B}_{j,k}(\bar{x},\bar{y})\right)A_{j,k}(\bar{x},\bar{y},s,t)\frac{dsdt}{st}d\bar{x}d\bar{y}\\
\leq&\ C_p\ln(\tau)\left\|(\mathcal{H}(A))^{1/2}\right\|_{L^{p'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{B}_{j,k}\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})},
\end{aligned}\tag{14}$$

where

$$\mathcal{H}(A)(\bar{x},\bar{y})=\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|\varrho_{s,t}*A_{j,k}(\bar{x},\bar{y},s,t)\right|^2\frac{dsdt}{st}.$$

As $\frac{p'}{2}>1$, then again by the duality, there exists a function $\mathcal{P}\in L^{(p'/2)'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$ such that $\|\mathcal{P}\|_{L^{(p'/2)'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\leq 1$ and

$$\begin{aligned}
&\left\|(\mathcal{H}(A))^{1/2}\right\|^2_{L^{p'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
=&\sum_{j,k\in\mathbb{Z}}\iint_{\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|\varrho_{s,t}*A_{j,k}(\bar{x},\bar{y},s,t)\right|^2\frac{dsdt}{st}\mathcal{P}(\bar{x},\bar{y})d\bar{x}d\bar{y}\\
\leq&\ C\|\mho\|_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Upsilon_1(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\varrho^*_{|h|}(\widetilde{\mathcal{P}})\right\|_{L^{(p'/2)'}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
\times&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|A_{j,k}(\cdot,\cdot,s,t)\right|^2\frac{dsdt}{st}\right)\right\|_{L^{(p'/2)}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
\leq&\ C\ln^2(\tau)\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}.
\end{aligned}\tag{15}$$

Therefore, using inequalities (14) and (15), we end the proof of this lemma. □

3. Proof of the Main Results

Proof of Theorem 1. The proof of this theorem mainly depends on the approaches used in [4,11,16,23], which have their roots in [24]. For an $\mu > 1$, let $h \in Y_\mu(\mathbb{R}_+ \times \mathbb{R}_+)$. Then, by Minkowski's inequality, we obtain

$$\begin{aligned}
\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}(g)(\overline{x},\overline{y}) &= \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\sum_{j,k=0}^{\infty}\frac{1}{s^\eta t^\lambda}\int_{2^{-j-1}t<\kappa_2(u)\le 2^{-j}t}\int_{2^{-k-1}s<\kappa_1(v)\le 2^{-k}s}\right.\right.\\
&\times \left.\left. K^{\kappa_1,\kappa_2}_{\mho,h}(v,u)g(x-v,x_{m+1}-\phi(\kappa_1(v)),y-u,y_{n+1}-\psi(\kappa_2(u)))dvdu\right|^2\frac{dsdt}{st}\right)^{1/2}\\
&\le \sum_{j,k=0}^{\infty}\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\frac{1}{s^\eta t^\lambda}\int_{2^{-j-1}t<\kappa_2(u)\le 2^{-j}t}\int_{2^{-k-1}s<\kappa_1(v)\le 2^{-k}s}\right.\right.\\
&\times \left.\left. K^{\kappa_1,\kappa_2}_{\mho,h}(v,u)g(x-v,x_{m+1}-\phi(\kappa_1(v)),y-u,y_{n+1}-\psi(\kappa_2(u)))dvdu\right|^2\frac{dsdt}{st}\right)^{1/2}\\
&\le \frac{2^{a_1+b_1}}{(2^{a_1}-1)(2^{b_1}-1)}\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|\varrho_{s,t}*g(\overline{x},\overline{y})|^2\frac{dsdt}{st}\right)^{1/2}. \qquad (16)
\end{aligned}$$

Let $\tau = 2^{q'\gamma'}$. Then, we know that $\ln(\tau) \le C\frac{q\mu}{(1-q)(1-\mu)}$. In addition, for $j \in \mathbb{Z}$, let $\{T_j\}_{-\infty}^{\infty}$ be a smooth partition of unity which is defined on $(0,\infty)$ and adapted to the interval $\mathcal{I}_j = [\tau^{-1-i}, \tau^{1-i}]$. Precisely, we have the following:

$$\begin{aligned}
T_j &\in C^\infty,\ 0 \le T_j \le 1,\ \sum_{j\in\mathbb{Z}} T_j^2(\kappa) = 1,\\
\text{supp }(T_i) &\subseteq \mathcal{I}_j,\ and\ \left|\frac{d^r T_j(\kappa)}{d\kappa^r}\right| \le \frac{C_r}{\kappa^r},
\end{aligned}$$

where C_r does not depend on the lacunary sequence $\{\tau^j; j \in \mathbb{Z}\}$. We define the multiplier operators $M_{j,k}$ on $\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}$ by $(\widehat{M_{j,k}(g)})(\overline{\xi},\overline{\zeta}) = T_k(\kappa_1(\xi))T_j(\kappa_2(\zeta))\hat{g}(\overline{\xi},\overline{\zeta})$. Thus, for any $g \in \mathcal{S}(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})$, we obtain $g(\overline{x},\overline{y}) = \sum\limits_{j,k\in\mathbb{Z}}(M_{j+l_2,k+l_1}(g))(\overline{x},\overline{y})$. This leads, by Minkowski's inequality, to

$$\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|\varrho_{s,t}*g(\overline{x},\overline{y})|^2\frac{dsdt}{st}\right)^{1/2} \le C\sum_{l_1,l_2\in\mathbb{Z}}\mathcal{N}_{l_2,l_1}(g)(\overline{x},\overline{y}), \qquad (17)$$

where

$$\begin{aligned}
\mathcal{N}_{l_2,l_1}(g)(\overline{x},\overline{y}) &= \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|\mathcal{V}_{l_2,l_1}(g)(\overline{x},\overline{y},s,t)|^2\frac{dsdt}{st}\right)^{1/2},\\
\mathcal{V}_{l_2,l_1}(g)(\overline{x},\overline{y},s,t) &= \sum_{j,k\in\mathbb{Z}}\varrho_{s,t}*M_{j+l_2,k+l_1}*g(\overline{x},\overline{y})\chi_{[\tau^k,\tau^{k+1})\times[\tau^j,\tau^{j+1})}(s,t).
\end{aligned}$$

Therefore, to prove Theorem 1, it is enough to show that

$$\begin{aligned}
&\|\mathcal{N}_{l_2,l_1}(g)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \qquad (18)\\
&\le C_p\ln(\tau)2^{-\frac{\varepsilon}{2}(|l_1|+|l_2|)}\|\mho\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\|h\|_{Y_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\|g\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}
\end{aligned}$$

for any p satisfying $|1/2-1/p| < \min\{1/\mu', 1/2\}$ and for some $\varepsilon > 0$.

Let us first estimate the L^2-norm for $\mathcal{N}_{l_2,l_1}(g)$. By using Plancherel's Theorem, Fubini's Theorem, Lemma 5, and similar procedures as those employed in [11], we obtain

$$
\begin{aligned}
&\|\mathcal{N}_{l_2,l_1}(g)\|^2_{L^2(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
\leq\;& \sum_{j,k\in\mathbb{Z}}\iint_{\Theta_{j+l_2,k+l_1}}\left(\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left|\hat{\varrho}_{s,t}(\bar{\xi},\bar{\zeta})\right|^2\frac{dsdt}{st}\right)\left|\hat{g}(\bar{\xi},\bar{\zeta})\right|^2 d\bar{\xi}d\bar{\zeta}\\
\leq\;& C_p\ln^2(\tau)\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\\
\times\;& \sum_{j,k\in\mathbb{Z}}\iint_{\Theta_{j+l_2,k+l_1}}\left|D_{\tau^k}\xi\right|^{\pm\frac{2\delta}{\gamma_1 q' w}}\left|D_{\tau^j}\zeta\right|^{\pm\frac{2\delta}{\gamma_2 q' w}}\left|\hat{g}(\bar{\xi},\bar{\zeta})\right|^2 d\bar{\xi}d\bar{\zeta}\\
\leq\;& C_p\ln^2(\tau)\,2^{-\varepsilon(|l_1|+|l_2|)}\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\sum_{j,k\in\mathbb{Z}}\iint_{\Theta_{j+l_2,k+l_1}}\left|\hat{g}(\bar{\xi},\bar{\zeta})\right|^2 d\bar{\xi}d\bar{\zeta}\\
\leq\;& C_p\ln^2(\tau)\,2^{-\varepsilon(|l_1|+|l_2|)}\|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbf{S}^{n-1})}\|h\|^2_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\|g\|^2_{L^2(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})},
\end{aligned}\tag{19}
$$

where $\Theta_{j,k}=\{(\bar{\xi},\bar{\zeta})\in\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}:(\kappa_1(\xi),\kappa_2(\zeta))\in\mathcal{I}_k\times\mathcal{I}_j\}$ and $\varepsilon\in(0,1)$.

On the other hand, the L^p-norm for $\mathcal{N}_{l_2,l_1}(g)$ is estimated as follows: by invoking Lemma 7 together with the Littlewood–Paley theory and using (3.20) in [11] we obtain

$$
\begin{aligned}
&\|\mathcal{N}_{l_2,l_1}(g)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
\leq\;& C\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left(\left|\varrho_{s,t}*M_{j+l_2,k+l_1}*g\right|\right)^2\frac{dsdt}{st}\right)^{1/2}\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
\leq\;& C_p\ln(\tau)\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|M_{j+l_2,k+l_1}*g\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
\leq\;& C_p\frac{\mu}{(q-1)(\mu-1)}\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\|g\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}.
\end{aligned}\tag{20}
$$

Now, we interpolate between (19) and (20), and immediately obtain (18). This finishes the proof of Theorem 1.

Finally, the proof of Theorem 3 can be obtained by following the above arguments, invoking Lemma 8 instead of Lemma 7 and then adapting Yano's extrapolation method. Precisely, using Lemma 8, we get that

$$
\begin{aligned}
&\|\mathcal{N}_{l_2,l_1}(g)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
\leq\;& C\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\tau^j}^{\tau^{j+1}}\int_{\tau^k}^{\tau^{k+1}}\left(\left|\varrho_{s,t}*M_{j+l_2,k+l_1}*g\right|\right)^2\frac{dsdt}{st}\right)^{1/2}\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
\leq\;& C_p\ln(\tau)\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|M_{j+l_2,k+l_1}*g\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
\leq\;& C_p\frac{\mu}{(q-1)(\mu-1)}\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\|g\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}
\end{aligned}\tag{21}
$$

for all $2<p<\infty$, and

$$
\begin{aligned}
&\|\mathcal{N}_{l_2,l_1}(g)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}\\
\leq\;& C_p\frac{\mu^2}{(q-1)^2(\mu-1)^2}\|\mho\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Upsilon_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\|g\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})}
\end{aligned}\tag{22}
$$

for all $1 < p < 2$. Consequently, when we interpolate (19) with (21) and (22), we obtain (18). The proof of Theorem 3 is complete.

4. Conclusions

In this work, we obtained suitable L^p estimates for a certain class of parabolic Marcinkiewicz integral operators $\mathfrak{M}^{\kappa_1,\kappa_2}_{\phi,\psi,\mho,h}$ when $\mho \in L^q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $q > 1$. Using these estimates together with Yano's extrapolation argument, we proved the L^p boundedness of the aforesaid operator under very weak assumptions on $\mho$. Actually, we proved our results when $\mho \in L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \cup B_q^{(0,0)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for some $q > 1$ which are considered to be the best possible in their respective classes. Furthermore, we established the L^p boundedness of our operator for the full range $1 < p < \infty$ under stronger conditions on $\mho$. Our results improve as well as extend numerous known results in the Marcinkiewicz operators.

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