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On the Algebraic Independence of the Values of Functions Associated with Hypergeometric Functions †

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† The results of the work are obtained in the framework of the state contract of the Ministry of Education and Science of the Russian Federation (project no. FSWF-2020-0022).

Abstract: Functions that are integrals of products of generalized hypergeometric functions of some kind are considered. The conditions for the representability of these functions as a polynomial in hypergeometric ones are found. The necessary and sufficient conditions for the algebraic independence of such functions are established.

Keywords: generalized hypergeometric functions; algebraic independence; Siegel's method; E-functions

MSC: 11J91

1. Introduction

Let $\mathbb{A}$ be the set of all algebraic numbers, and let $\overline{|\alpha|}$ be the maximum of the absolute values of the algebraic number α and all of its conjugates in the field $\mathbb{A}$. Let $\mathbb{C}(z)$ be the field of all rational functions with complex coefficients, and let $K[z_1, \ldots, z_n]$ be the ring of all polynomials in $z_1, \ldots, z_n$ with coefficients in a ring K.

One of the main methods of transcendental number theory is the Siegel–Shidlowsky method (see [1–3]). With its help, the transcendence and algebraic independence of the values of the so-called E-functions (a subclass of entire functions of the first order, which are closed with respect to differentiation and integration) are proved.

K. Siegel [1] called the entire function

$$f(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}, \quad c_n \in \mathbb{A},$$

an E-function if:

1. for any $\varepsilon > 0$ $\overline{|c_n|} = O(n^{\varepsilon n}), \quad n \to \infty$;
2. for any $\varepsilon > 0$, the least common denominator of $c_1, \ldots, c_n$ is $O(n^{\varepsilon n}), \quad n \to \infty$;
3. $f(z)$ satisfies a linear differential equation with the coefficients in $\mathbb{C}(z)$.

An example of E-functions is a class of hypergeometric E-functions ${}_l\varphi_q(\vec{\nu}; \vec{\lambda}; \alpha z^{q-l})$, where

$${}_l\varphi_q(\vec{\nu}; \vec{\lambda}; z) = {}_{l+1}F_q\left(\begin{matrix} 1, \nu_1, \ldots, \nu_l \\ \lambda_1, \ldots, \lambda_q \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(\nu_1)_n \ldots (\nu_l)_n}{(\lambda_1)_n \ldots (\lambda_q)_n} z^n,$$

$0 \leqslant l < q$, $(\nu)_0 = 1$, $(\nu)_n = \nu(\nu+1)\ldots(\nu+n-1)$, $\vec{\nu} = (\nu_1, \ldots, \nu_l) \in \mathbb{Q}^l$, $\vec{\lambda} \in (\mathbb{Q} \setminus \mathbb{Z}_{\leqslant 0})^q$, $\alpha \in \mathbb{A}$.

There is a large number of works in which the algebraic independence of the values of various sets of hypergeometric functions at algebraic points is established (see [3]).

In [1], K. Siegel proved that every polynomial $P \in \mathbb{A}[z, f_1(z), \ldots, f_n(z)]$, where $f_1(z), \ldots, f_n(z)$ are hypergeometric E-functions or the functions obtained from them by

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Citation: Gorelov, V. On the Algebraic Independence of the Values of Functions Associated with Hypergeometric Functions. *Axioms* **2023**, *12*, 36. https://doi.org/10.3390/axioms12010036

Academic Editor: Federico G. Infusino

Received: 28 November 2022
Revised: 23 December 2022
Accepted: 24 December 2022
Published: 28 December 2022

replacing z with αz for $\alpha \in \mathbb{A}$, is an E-function satisfying a linear differential equation with the coefficients in $\mathbb{C}(z)$. In the same article ([1], §2), K. Siegel formulated a conjecture that the converse statement is also true.

In the author's articles (see [4]) that were published in 2000–2005, the Siegel conjecture was proved for E-functions satisfying linear homogeneous differential equations of an order not higher than 2, and, in some cases, inhomogeneous equations of the second order.

In 2005, the author [5] proved a weakened version of the Siegel conjecture for all linear differential equations of an order not higher than 2, including inhomogeneous ones. In the same article, it was suggested that, in general, Siegel's hypothesis is incorrect. As examples of rebuttals, it was suggested to consider the E-functions

$$V(z) = V_{\lambda,\alpha}(z) = e^{\alpha z} \int_0^z e^{-\alpha t} \varphi_\lambda(t) dt = z + \left(\frac{1}{\lambda+1} + \alpha \right) \frac{z^2}{2} + \dots, \tag{1}$$

satisfying the equations

$$y'' + (-\alpha - 1 + \lambda/z)y' + (\alpha - \lambda\alpha/z)y = \lambda/z, \tag{2}$$

where $\alpha \in \mathbb{A}$, $\lambda \in \mathbb{Q}$, $-\lambda \notin \mathbb{N}$, and $\varphi_\lambda(z)$ is the function introduced by A.B. Shidlowsky (see [3], ch. 5, §2),

$$\varphi_\lambda(z) = {}_1F_1\left(\begin{matrix} 1 \\ \lambda+1 \end{matrix} \middle| z \right) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{(\lambda+1)\dots(\lambda+n)} = \lambda z^{-\lambda} e^z \int^z t^{\lambda-1} e^{-t} dt.$$

If $\lambda > 0$, then the lower integration bound is 0. Otherwise, the integral denotes the antiderivative for the function $t^{\lambda-1}e^{-t}$, which is obtained after multiplying all of the terms of the Taylor decomposition of e^{-t} by $t^{\lambda-1}$ and the termwise integration with a constant of integration equal to zero (see [3], ch. 5, §2, formula (21)).

Recently, J. Frezan and P. Jossen [6], relying on the works of I. Andre ([7,8]), N. Katz [9], T. Rivoal and S. Fischler [10], and other mathematicians, strictly proved that some E-functions are not polynomials in hypergeometric E-functions. The resulting refutation of Siegel's conjecture makes it relevant to develop methods for studying the algebraic and number-theoretic properties of functions that are not algebraically expressed through hypergeometric E-functions.

2. On Algebraic Identities between the Functions $V(z)$, $\varphi_\lambda(z)$, and e^z

The function $\varphi_\lambda(z)$, introduced by A.B. Shidlowsky, satisfies the equation

$$y' = (1 - \lambda/z)y + \lambda/z \tag{3}$$

and can be understood as an "inhomogeneous analogue" of the function e^z.

Lemma 1 ([3], ch. 5, §3; Lemma 4 in [11]). *Let $\lambda_0 \in \mathbb{Z}_{\geqslant 0}$, $\lambda_1, \dots, \lambda_m \in \mathbb{C} \setminus \mathbb{Z}$, $m \geqslant 0$, and $\lambda_i - \lambda_j \notin \mathbb{Z}$, $i \neq j$; the numbers $\beta_1, \dots, \beta_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, $\alpha_1, \dots, \alpha_n \in \mathbb{C} \setminus \{0\}$, $\alpha_i \neq \alpha_j$, $i \neq j$. Then, the $(m+1)n$ functions*

$$\varphi_{\lambda_0}(\beta_i z),\ \varphi_{\lambda_j}(\alpha_i z), \quad j = 1, \dots, m,\ i = 1, \dots, n \tag{4}$$

are algebraically independent over $\mathbb{C}(z)$.

Taking into account the identities $\varphi_0(z) = e^z$ and

$$\varphi_\lambda(z) = \frac{z^l}{(\lambda+1)\dots(\lambda+l)} \varphi_{\lambda+l}(z) + 1 + \sum_{n=1}^{l-1} \frac{z^n}{(\lambda+1)\dots(\lambda+n)}, \tag{5}$$

where $l \in \mathbb{N}$, the conditions of Lemma 1 are necessary and sufficient.

From Lemma 7 in [12], a stronger version of Lemma 1 follows.

Lemma 2. *Under the conditions of Lemma 1, the functions* (4) *are algebraically independent over* $\mathbb{C}$ *together with the functions* $z^{\xi_1}, \ldots, z^{\xi_p}$*, where the numbers* $\xi_1, \ldots, \xi_p \in \mathbb{C}$ *are linearly independent over* $\mathbb{Q}$*,* $\xi_1 \in \mathbb{Q}$*,* $p \in \mathbb{N}$*.*

Theorem 1. *The following identities hold:*

$$V_{0,\,\alpha}(z) = e^{\alpha z}\int_0^z e^{-\alpha t}e^t dt = \frac{1}{1-\alpha}(e^z - e^{\alpha z}), \quad \alpha \neq 1; \quad V_{0,\,1}(z) = ze^z; \tag{6}$$

$$V_{\lambda,\,1}(z) = e^z\int_0^z e^{-t}\varphi_\lambda(t)dt = \frac{z}{1-\lambda}\varphi_\lambda(z) + \frac{\lambda}{1-\lambda}(1-e^z), \quad \lambda \neq 1; \tag{7}$$

$$V_{1/2,\,2}(z) = e^{2z}\int_0^z e^{-2t}\varphi_{1/2}(t)\,dt = z\varphi_{1/2}^2(z); \tag{8}$$

$$V_{3/2,\,2}(z) = -\frac{4}{3}z^3\varphi_{3/2}^2(z) - (4z^2+2z)\varphi_{3/2}(z) - 3z - 3 + 3e^{2z}; \tag{9}$$

$$V_{-1/2,\,2}(z) = \frac{1}{4z}\varphi_{-1/2}^2(z) - (1+\frac{1}{2z})\varphi_{-1/2}(z) + \frac{1}{4z} + e^{2z}. \tag{10}$$

All other functions $V_{k+1/2,\,2}(z)$*,* $k \in \mathbb{Z}$*, are also uniquely represented as* $c\,e^{2z} + P_0$*, where* $c \in \mathbb{Q}\setminus\{0\}$*,* P_0 *is a second-power polynomial in* $\varphi_{k+1/2}(z)$ *with the coefficients in* $\mathbb{Q}[z, z^{-1}]$*.*

Proof. Identities (6) are proved through direct computation. The validity of identity (8) can be established as follows. If we multiply both parts of it by e^{-2z}, differentiate with respect to z, and divide by e^{-2z}, then, taking (3) into account, we get the correct identity of the form $\varphi_{1/2}(z) = \varphi_{1/2}(z)$. By performing these actions in reverse order, due to $V_{1/2,\,2}(0) = 0$, we get (8).

Identity (7) is proved similarly to (8).

From identity (5), for $l = 1$, by using integration by parts and (3) successively, we get

$$V_{\lambda,2}(z) = e^{2z}\int_0^z e^{-2t}\varphi_\lambda(t)dt = \frac{1}{\lambda+1}e^{2z}\int_0^z e^{-2t}t\varphi_{\lambda+1}(t)dt - \frac{1}{2}(1-e^{2z}),$$

$$I = \int_0^z e^{-2t}t\varphi_{\lambda+1}(t)dt = -\frac{1}{2}\int_0^z t\varphi_{\lambda+1}(t)de^{-2t} =$$

$$= -\frac{1}{2}\left(e^{-2t}t\varphi_{\lambda+1}(t)\Big|_0^z - \int_0^z e^{-2t}(\varphi_{\lambda+1}(t) + t\varphi'_{\lambda+1}(t))dt\right)$$

$$= \frac{1}{2}I - \frac{\lambda}{2}\int_0^z e^{-2t}\varphi_{\lambda+1}(t)dt - \frac{1}{2}e^{-2z}z\varphi_{\lambda+1}(z) + \frac{\lambda+1}{4}(1-e^{-2z}),$$

$$I = -\lambda\int_0^z e^{-2t}\varphi_{\lambda+1}(t)dt - e^{-2z}z\varphi_{\lambda+1}(z) + \frac{\lambda+1}{2}(1-e^{-2z}),$$

$$V_{\lambda,\,2}(z) = -\frac{\lambda}{\lambda+1}V_{\lambda+1,\,2}(z) - \frac{1}{\lambda+1}z\varphi_{\lambda+1}(z) - 1 + e^{2z}, \tag{11}$$

$$V_{\lambda+1,\,2}(z) = -\frac{\lambda+1}{\lambda}V_{\lambda,\,2}(z) - \frac{1}{\lambda}z\varphi_{\lambda+1}(z) + \frac{\lambda+1}{\lambda}(-1+e^{2z}). \tag{12}$$

From Formula (12), by induction on n, we easily obtain the formula

$$V_{\frac{2n+1}{2},\,2}(z) = (-1)^n(2n+1)V_{\frac{1}{2},\,2}(z) +$$

$$+(-1)^{n+1}(2n+1)\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{2n-1}\right)e^{2z} + L, \qquad n = 1, 2, \ldots,$$

where L is a linear combination of the functions 1, $z\varphi_{3/2}(z), \ldots, z\varphi_{(2n+1)/2}(z)$ with the coefficients in $\mathbb{Q}$. From here, by using Formulas (5) and (8), for $V_{(2n+1)/2,\,2}(z)$, we obtain the expression described in the statement of the theorem. This expression is defined uniquely by virtue of Lemma 1.

Similarly to (11), we get

$$V_{-\frac{2n+1}{2},\,2}(z) = (-1)^n(2n+1)V_{\frac{1}{2},\,2}(z) +$$
$$+(-1)^n(2n+1)\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^n}{2n+1}\right)e^{2z} + L, \qquad n = 0, 1, 2, \ldots,$$

where L is a linear combination of the functions 1, $z\varphi_{-1/2}(z), \ldots,$ and $z\varphi_{-(2n+1)/2}(z)$, with the coefficients in $\mathbb{Q}$. From this, we come again to the statement of the theorem. Formulas (9) and (10) are special cases of the reasoning used. Theorem 1 is proved. □

Theorem 2. *Under the conditions of Lemma 1, the function $V_{\lambda,\alpha}(z)$, $\lambda \notin \mathbb{Z}$, can be expressed as a polynomial P in functions (4) with the coefficients in $\mathbb{C}[z, z^{-1}]$ only in the case of $m = n = 1$, $\lambda_1 - \lambda \in \mathbb{Z}$, $\alpha_1 = 1$, $\alpha \in \{1, 2\}$, $2\lambda \in \mathbb{Z}$ for $\alpha = 2$, $P = ce^{\alpha z} + P_0$, where $c \in \mathbb{C}$, and P_0 is a polynomial not higher than the second power in the function $\varphi_\lambda(z)$. If $\lambda \in \mathbb{Z}_{\geqslant 0}$, then for $\alpha \neq 1$, P is a linear form with the coefficients in $\mathbb{C}[z, z^{-1}]$ of the functions e^z and $e^{\alpha z}$, and for $\alpha = 1$, it is of the functions e^z and 1. In addition, if $\lambda \in \mathbb{Z}_{\geqslant 0}$, $\alpha \neq 1$, then $\lambda = 0$.*

Proof. We express, if necessary, the function $\varphi_{\lambda_0}(z)$ via e^z (according to the identities (5) and $\varphi_0(z) = e^z$), and we assume that

$$V_{\lambda,\alpha}(z) = e^{\alpha z}\int_0^z e^{-\alpha t}\varphi_\lambda(t)dt = P = P_1 e^{\gamma_1 z} + \cdots + P_s e^{\gamma_s z} + P_0,$$

where $s \geqslant 0$, $P_0, \ldots, P_s$ are non-zero polynomials in the functions $\varphi_{\lambda_j}(\alpha_i z)$, $j \neq 0$; the numbers $\gamma_1, \ldots, \gamma_s \in \mathbb{C} \setminus \{0\}$ are different. Multiplying this equality by $e^{-\alpha z}$, differentiating with respect to z, and dividing by $e^{-\alpha z}$, we get

$$\varphi_\lambda(z) = (P_1' + (\gamma_1 - \alpha)P_1)e^{\gamma_1 z} + \cdots + (P_s' + (\gamma_s - \alpha)P_s)e^{\gamma_s z} + P_0' - \alpha P_0. \tag{13}$$

If $\lambda \notin \mathbb{Z}$, then according to Lemma 1 $P_k' + (\gamma_k - \alpha)P_k = 0$, i.e., $P_k = c_k e^{(\gamma_k - \alpha)z}$, $c_k \neq 0$. Hence, in view of Lemma 1 $\gamma_k = \alpha$, $P_k = c_k$, $s = 1$. If $\lambda \in \mathbb{Z}_{\geqslant 0}$, $\alpha \neq 1$, then another case $\gamma_k = 1$, $s = 2$ is possible.

Thus, it is proved that for $\lambda \notin \mathbb{Z}$,

$$V_{\lambda,\alpha}(z) = P = ce^{\alpha z} + P_0, \tag{14}$$

and for $\lambda \in \mathbb{Z}_{\geqslant 0}$,

$$V_{\lambda,\alpha}(z) = P = ce^{\alpha z} + P_1 e^z + P_0, \tag{15}$$

where $c \in \mathbb{C}$, P_0, P_1 are polynomials in the functions $\varphi_{\lambda_j}(\alpha_i z)$, $j \neq 0$.

Let the equality (14) be

$$P_0 = P_k\varphi_{\lambda_1}^k(\alpha_1 z) + P_{k-1}\varphi_{\lambda_1}^{k-1}(\alpha_1 z) + \cdots + P_1\varphi_{\lambda_1}(\alpha_1 z) + P_*,$$

where $k \geqslant 1$, $P_*, P_1, \ldots, P_k$ are polynomials in the functions $\varphi_{\lambda_j}(\alpha_i z)$, except for $\varphi_{\lambda_1}(\alpha_1 z)$, with the coefficients in $\mathbb{C}[z, z^{-1}]$.

Then, by multiplying the equality (14) by $e^{-\alpha z}$, differentiating with respect to z, and dividing by $e^{-\alpha z}$, we get

$$\varphi_\lambda(z) = \left(P_k' + \left(k\alpha_1 - \alpha - \frac{k\lambda_1}{z}\right)P_k\right)\varphi_{\lambda_1}^k(\alpha_1 z) +$$

$$+\left(P_{k-1}' + \left((k-1)\alpha_1 - \alpha - \frac{(k-1)\lambda_1}{z}\right)P_{k-1} + \frac{k\lambda_1}{z}P_k\right)\varphi_{\lambda_1}^{k-1}(\alpha_1 z) + \dots .$$

If $\lambda_1 - \lambda \notin \mathbb{Z}$, then according to Lemma 1, the coefficients at $\varphi_{\lambda_1}^t(\alpha_1 z)$ should be zero. Hence,

$$P_k' = \left(\alpha - k\alpha_1 + \frac{k\lambda_1}{z}\right)P_k, \quad P_k = c_k e^{(\alpha-k\alpha_1)z} z^{k\lambda_1}, \quad c_k \neq 0.$$

According to Lemma 1, this is only possible if $\alpha = k\alpha_1,\ k\lambda_1 \in \mathbb{Z}$. Hence,

$$P_{k-1}' = \left(\alpha_1 + \frac{(k-1)\lambda_1}{z}\right)P_{k-1} - c_k k\lambda_1 z^{k\lambda_1 - 1}, \quad P_{k-1} \neq 0.$$

Solving this equation with the method of variation of parameters, we get

$$P_{k-1} = c_0 z^{k\lambda_1}\varphi_{\lambda_1}(\alpha_1 z) + c_1 e^{\alpha_1 z} z^{k\lambda_1} z^{-\lambda_1}, \quad c_0, c_1 \in \mathbb{C},$$

which contradicts Lemma 2. Therefore, the polynomial P_0 does not depend on the functions $\varphi_{\lambda_j}(\alpha_i z)$, except, perhaps, $\varphi_\lambda(z)$, and in this case, $\lambda_1 - \lambda \in \mathbb{Z}$, $k \leqslant 2$, $\alpha_1 = 1,\ \alpha \in \{1, 2\}$, $2\lambda \in \mathbb{Z}$ for $\alpha = 2$.

Similarly, let the equality (15) be

$$P_1 e^z + P_0 = P_k\varphi_{\lambda_1}^k(\alpha_1 z) + P_{k-1}\varphi_{\lambda_1}^{k-1}(\alpha_1 z) + \dots + P_1\varphi_{\lambda_1}(\alpha_1 z) + P_*,$$

where $k \geqslant 1$, $P_*, P_1, \dots, P_k$ are polynomials in the functions e^z and $\varphi_{\lambda_j}(\alpha_i z)$, except for $\varphi_{\lambda_1}(\alpha_1 z)$, with the coefficients in $\mathbb{C}[z, z^{-1}]$. Let us repeat the reasoning carried out for equality (14). Here, due to the fact that P_k may depend on e^z, another case, $\alpha = k\alpha_1 + 1,\ k\lambda_1 \in \mathbb{Z},\ P_k = c_k e^z z^{k\lambda_1},\ c_k \neq 0$, arises. In this case,

$$P_{k-1} = c_0 z^{k\lambda_1} e^z \varphi_{\lambda_1}(\alpha_1 z) + c_1 e^{\alpha_1 z} e^z z^{k\lambda_1} z^{-\lambda_1}, \quad c_0, c_1 \in \mathbb{C},$$

which contradicts Lemma 2.

Thus, we can assume that in equality (15), $P_1, P_0 \in \mathbb{C}[z, z^{-1}]$. If $\lambda = m \in \mathbb{Z}_{\geqslant 0}$, then, from equality (13), with $s = 2,\ \gamma_1 = 1,\ \gamma_2 = \alpha$, taking (5) into account, we get $m!/z^m = P_1' + (1-\alpha)P_1$. However, since $P_1 = z^{n_1} + \dots + z^{n_k},\ n_1 > \dots > n_k$, then with $\alpha \neq 1$, this is possible only in the case of $m = 0$. Then, from (13), $P_0' = \alpha P_0$, from which $P_0 = 0$. Theorem 2 is proven. □

3. On the Algebraic Independence of the Functions $V(z)$ and $V'(z)$ and Their Values

Lemma 3 (Theorem 2 in [4])**.** *A function $f(z)$ is an E-function satisfying a linear differential equation of the second order (generally speaking, inhomogeneous) with coefficients in $\mathbb{C}(z)$, and the functions $f(z)$ and $f'(z)$ are algebraically dependent over $\mathbb{C}(z)$ if and only if either*

$$f(z) = P_2\varphi_k(\alpha z) + P_1\varphi_k(\sigma\alpha z) + P_0,$$

or

$$f(z) = P_2\varphi_\lambda^2(\alpha z) + P_1\varphi_\lambda(\alpha z) + P_0,$$

where $P_0, P_1, P_2 \in \mathbb{A}[z],\ k \in \mathbb{Z}_{\geqslant 0},\ \lambda, \sigma \in \mathbb{Q},\ \alpha \in \mathbb{A}$.

Lemma 4. *The E-functions $V_{\lambda,\alpha}(z)$ and $V_{\lambda,\alpha}'(z)$ are algebraically dependent over $\mathbb{C}(z)$ if and only if $\lambda = 0,\ \alpha \in \mathbb{Q},\ \lambda = 1/2,\ \alpha = 2$, or $\lambda \in \mathbb{Z}_{\geqslant 2},\ \alpha = 1$.*

Proof. The sufficiency of the conditions of Lemma 4 follows from identities (6)–(8) and Lemma 3.

Let us prove the necessity of these conditions. If the E-functions $V(z)$ and $V'(z)$ are algebraically dependent, then, according to Lemma 3,

$$V(z) = e^{\alpha z}\int_0^z e^{-\alpha t}\varphi_\lambda(t)dt = P_2\varphi_k(\alpha_1 z) + P_1\varphi_k(\sigma\alpha_1 z) + P_0 \quad (16)$$

or

$$V(z) = e^{\alpha z}\int_0^z e^{-\alpha t}\varphi_\lambda(t)dt = P_2\varphi_{\lambda_1}^2(\alpha_1 z) + P_1\varphi_{\lambda_1}(\alpha_1 z) + P_0, \quad (17)$$

where $P_0, P_1, P_2 \in \mathbb{A}[z],\ k \in \mathbb{Z}_{\geqslant 0},\ \lambda_1, \sigma \in \mathbb{Q},\ \alpha_1 \in \mathbb{A}$.

The following reasoning can be somewhat simplified by using the statement of Theorem 2, but here, the proof is carried out directly.

If equality (16) holds, then, in view of (5), we can assume that $k = 0$, $\varphi_k(z) = e^z$, $P_0, P_1, P_2 \in \mathbb{A}[z, z^{-1}]$, $\sigma \notin \{0; 1\}$. Then, by multiplying the equality (16) by $e^{-\alpha z}$, differentiating with respect to z, and dividing by $e^{-\alpha z}$, we get

$$\varphi_\lambda(z) = \big(P_2' + (\alpha_1 - \alpha)P_2\big)e^{\alpha_1 z} + \big(P_1' + (\sigma\alpha_1 - \alpha)P_1\big)e^{\sigma\alpha_1 z} + P_0' - \alpha P_0. \quad (18)$$

According to Lemma 1, $\lambda = m \in \mathbb{Z}_{\geqslant 0}$. In addition, one of the brackets is zero on the right-hand side of equality (18). Without loss of generality, consider that $P_1' + (\sigma\alpha_1 - \alpha)P_1 = 0$. Then, $P_1 = 0$ or $\alpha = \sigma\alpha_1$, $P_1 \in \mathbb{A}$. In both cases, taking (5) into account, equality (18) has the form

$$\frac{m!}{z^m}e^z - \frac{m!}{z^m}\left(1 + \sum_{n=1}^{m-1}\frac{z^n}{n!}\right) = \big(P_2' + (\alpha_1 - \alpha)P_2\big)e^{\alpha_1 z} + P_0' - \alpha P_0.$$

Applying Lemma 1 again, we get $\alpha_1 = 1$. In the case of $\alpha = \sigma\alpha_1$, we have $P_2' + (1-\sigma)P_2 = m!/z^m$. Since $P_2 \in \mathbb{A}[z, z^{-1}]$, $\sigma \neq 1$, this is possible only if $m = 0$, $P_2 = 1/(1-\sigma)$, $\varphi_\lambda(z) = \varphi_0(z) = e^z$. In the case of $P_1 = 0$, similarly, we get $\alpha \neq 1$, $\lambda = m = 0$, or $\alpha = 1$, $\lambda = m \neq 1$.

If equality (17) holds, then by multiplying it by $e^{-\alpha z}$, differentiating with respect to z, dividing by $e^{-\alpha z}$, and taking (3) into account, we get

$$\varphi_\lambda(z) = \left(P_2' + \left(2\alpha_1 - \alpha - \frac{2\lambda_1}{z}\right)P_2\right)\varphi_{\lambda_1}^2(\alpha_1 z) +$$

$$+\left(P_1' + \left(\alpha_1 - \alpha - \frac{\lambda_1}{z}\right)P_1 + \frac{2\lambda_1}{z}P_2\right)\varphi_{\lambda_1}(\alpha_1 z) + P_0' - \alpha P_0 + \frac{\lambda_1}{z}P_1.$$

From here, $\lambda - \lambda_1 \in \mathbb{Z}$. Then, in view of (5), we can assume that in (17) $\lambda_1 = \lambda$, $P_0, P_1, P_2 \in \mathbb{A}[z, z^{-1}]$. If $\lambda_1 = \lambda \in \mathbb{Z}_{\geqslant 0}$, then we return to the case of (16). Let $\lambda \notin \mathbb{Z}_{\geqslant 0}$. Then, $\alpha_1 = 1$,

$$P_2' + \left(2 - \alpha - \frac{2\lambda}{z}\right)P_2 = 0, \quad P_1' + \left(1 - \alpha - \frac{\lambda}{z}\right)P_1 + \frac{2\lambda}{z}P_2 = 1,$$

implying that $P_2 = 0$ or $\alpha = 2$, $P_2' = \frac{2\lambda}{z}P_2$, $P_2 = cz^{2\lambda}$, $c \in \mathbb{C}\setminus\{0\},\ 2\lambda \in \mathbb{Z}$.

If $P_2 = 0$, then

$$P_1' = \left(\alpha - 1 + \frac{\lambda}{z}\right)P_1 + 1, \quad P_0' - \alpha P_0 + \frac{\lambda}{z}P_1.$$

Hence, in the case of $\alpha \neq 1$,

$$P_1 = \frac{1}{1-\alpha} + \frac{c_1}{z} + \cdots + \frac{c_m}{z^m}, \quad c_i \in \mathbb{A},\ m \in \mathbb{N}.$$

However, then, $-mc_m = \lambda c_m$, $\lambda = -m$, which is not possible. In the case of $\alpha = 1$ $P_1 = z/(1-\lambda)$, $P_0' = P_0 + \lambda/(1-\lambda)$, $P_0 = -\lambda/(1-\lambda)$, which contradicts (7).

If $P_2 \neq 0$, $P_1 = 0$, then $2c\lambda z^{2\lambda-1} = 1$, $\lambda = 1/2$, $c = 1$, $P_0 = 0$, and we come to (8). Hence, only the case $P_2 \neq 0$, $P_1 \neq 0$, $\alpha = 2$, $\lambda - 1/2 \in \mathbb{Z}$, $\lambda \neq 1/2$ remains. In view of Lemma 3 and Theorem 1, Lemma 4 is proven. □

Theorem 3. *Let $\lambda \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$, $\alpha \in \mathbb{A}$, $\xi \in \mathbb{A} \setminus \{0\}$. Then, for the algebraic independence of the numbers $V_{\lambda,\alpha}(\xi)$ and $V'_{\lambda,\alpha}(\xi)$, it is necessary and sufficient that $(\lambda, \alpha) \neq (1/2, 2)$, $(\lambda, \alpha) \neq (0, r)$, and $(\lambda, \alpha) \neq (k, 1)$, where $r \in \mathbb{Q}$, $k \in \mathbb{Z}_{\geqslant 2}$.*

The proof follows from Lemma 4 and Shidlovsky's theorem ([3], ch. 3).

4. Conclusions

1. One of the conditions for the applicability of the method of the article in [6] is the algebraic independence of the investigated function, $f(z)$, from $f'(z)$ and $f''(z)$ over $\mathbb{C}(z)$. However, in view of (2), this condition for the function $V_{\lambda,\alpha}(z)$ is not met. Nevertheless, Theorem 1, as follows from Theorem 2 and Lemma 4, describes all cases in which the function $V_{\lambda,\alpha}(z)$ can be represented as a polynomial in the functions $\varphi_{\lambda_1}(z)$ and $e^{\alpha_1 z}$ with the coefficients in $\mathbb{Q}[z, z^{-1}]$.
2. In addition to the identities of Theorem 1, we can also note the identity

$$V_{\lambda,\,0}(z) = \int_0^z \varphi_\lambda(t)dt = \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)(\lambda+1)\dots(\lambda+n)} = z\ {}_1\varphi_2(1; 2, \lambda+1; z).$$

3. Theorem 3 provides necessary and sufficient conditions for the algebraic independence of the values of functions, not all of which, apparently, are algebraically expressed in terms of hypergeometric functions. By using the methods in the articles of [4,12], Theorem 3 can be generalized to larger sets of functions.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

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