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The Single Axiomatization on CCRL-Fuzzy Rough Approximation Operators and Related Fuzzy Topology

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Abstract: It is well known that lattice-valued rough sets are important branches of fuzzy rough sets. The axiomatic characterization and related topology are the main research directions of lattice-valued rough sets. For $L = (L, \circledast)$, a complete co-residuated lattice (CCRL), Qiao recently defined an L-fuzzy lower approximation operator ($LFLAO$) on the basis of the L-fuzzy relation. In this article, we give a further study on Qiao's $LFLAO$ around the axiomatic characterization and induced L-topology. Firstly, we investigate and discuss three new $LFLAO$ generated by $\circledast$-transitive, $\circledast$-Euclidean and $\circledast$-mediated L-fuzzy relations. Secondly, we utilize a single axiom to characterize the $LFLAO$ generated by serial, symmetric, reflexive, $\circledast$-transitive and $\circledast$-mediate L-fuzzy relations and their compositions. Thirdly, we present a method to generate Alexandrov L-topology ($ALTPO$) from $LFLAO$ and construct a bijection between $ALTPO$ and $\circledast$-preorder (i.e., reflexive and $\circledast$-transitive L-fuzzy relation) on the same underlying set.

Keywords: fuzzy approximation operator; co-residuated lattice; axiomatization; L-topology

MSC: 06B23; 03E72; 06D20

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1. Introduction

Rough-set theory [1] was put forward by Pawlak. This theory plays a vital role in handling the uncertainty, granularity and incompleteness of knowledge in information systems. Classical rough sets with strict equivalence conditions restrict the development of rough sets, so people introduced generalized rough sets to avoid this situation. For the past few years, many experts and scholars have studied various types of generalized rough sets [2,3]. It is well known that one pair of approximation operators is the basic concept of rough-set theory. Generally, we have two methods to study the approximation operators. One is the constructive method which constructs the approximation operators from relations, coverings, neighborhoods and so on [2]. The other is the axiomatic method. An abstract operator is given firstly, and then we look for single axiom or axiom sets s.t. the operator happens to be the approximate operator from the construction method [3]. In addition, the topologies induced by the approximation operators are also vital content of rough-set theory [2,4]. The most well-known result may be the existence of bijection between Alexandrov topology (i.e., quasi-discrete topology) and a preorder relation based on the same underlying set.

The theoretical development of fuzzy rough sets has made steady progress. At the beginning, taking $[0,1]$ as the degree of membership, scholars introduced various fuzzy rough sets; see [5–14]. Later, researchers discovered that the degree of membership might not be expressed in a linear order, so many lattice structures were proposed to replace $[\bot, \top]$. Among them, the complete (co) residuated lattices are closely focused on, since they can be treated as truth tables of diverse generalized multiple-valued logic [15–17]. It is well known that complete (co) residuated lattices have been widely put to use; see [10,18–36].

Just like the classic rough sets, the construction and axiomatization of L-fuzzy approximation operators and the related topology are also important directions in L-fuzzy rough sets, where $L = (L, *)$ is a complete residuated lattice (CRL) and $L = (L, \circledast)$ is a complete co-residuated lattice (CCRL).

- The work on $L = (L, *)$. For L-fuzzy relations (L-fr), Radzikowska introduced the pair of L-fuzzy upper and approximation operators. The basic properties of the L-fuzzy approximation operators generated by serial, reflexive, symmetric, transitive and Euclidean L-fr were also studied. Then, Wang [29] characterized Radzikowska's operators by using axiom set; She [18,27] improved Wang's work and characterized the related L-fuzzy approximation operators by a single axiom. Pang [23,24] defined and characterized the L-fuzzy approximation operators generated by mediated, Euclidean and adjoint L–fr. Hao [20] discussed the L-topological structure associated with L-fr and verified that there is a bijection between L-preorder (i.e., reflexive and transitive L-fuzzy relation) and Alexandrov L-topology. Ma further established the connections between L-closure and L-interior operators and L-fuzzy approximation operators. Zhao [35] introduced L-fuzzy variable-precision rough sets based on L-fr. Qiao [26] and Wang [28] further proposed the granular, variable-precision, L-fuzzy rough sets by the fuzzy granule associated with L-fr. Belohlavek built the connection between L-fuzzy rough sets and concept lattices. Han [19] discussed some categories of approximate-type systems generated by L-fr. In the above mentioned L-fuzzy rough set, the L-fr is based on the classical set. Quite recently, by considering the L-fr based on L-fuzzy sets, Wei [30] developed a general L-fuzzy rough set from both constructive and axiomatic methods. For L-fuzzy covering, Li [21] introduced and described several L-fuzzy approximation operators. Based on the L-fr generated by L-fuzzy covering, Jiang [37] proposed a covering-based variable-precision, L-fuzzy rough set and applied it in the study of multi-attribute decision-making problems when $L = [0, 1]$. For L-fuzzifying neighborhood-systems and L-fuzzy neighborhood-systems, Li [22] and Zhao [32,33] investigated two types of L-fuzzy rough sets and described them by one axiom each. Furthermore, Song [38] and Zhao [34] researched the lattice structure and L-topological structure associated with Zhao's L-fuzzy rough sets. For (L, M)-fuzzy neighborhood systems, El-Saady [39] established the (L, M)-fuzzy rough sets, which unified Li's L-fuzzy rough sets [22] and Zhao's L-fuzzy rough sets [32,33] into one framework.
- The work on $L = (L, \circledast)$. For L-fr, Qiao [25] defined and characterized a new L-fuzzy lower approximation operator on the basis of $\circledast$. He also proved that his reflexive L-fuzzy lower approximation operator induced an $ALTPO$ in his sense. In [26], Qiao further proposed a variable-precision, L-fuzzy lower approximation operator. In [40], the author introduced an L-fuzzy upper approximation operator through $\rightsquigarrow$, the co-implication w.r.t. $\circledast$. We verified that Qiao's L-fuzzy lower approximation operator is dual to our L-fuzzy upper approximation operator for some special L.

Motivations, Innovativeness and Contributions

From the above review, it is easy to find that compared with the CRL-fuzzy rough sets, the CCRL-fuzzy rough sets are still far from perfect. Therefore, from the perspective of theoretical development, the CCRL-fuzzy rough sets should continue to be studied in depth. In the paper, inspired by the following three aspects, we present further research on Qiao's L-fuzzy lower approximation operator ($LFLAO$) [25].

- The approximation operators, generated, respectively, by transitive (TR), Euclidean (EU) and mediate (ME) relations are important in the classical rough-set theory. In [25], Qiao defined TR and EU L-fr through $*$ but not $\circledast$. Obviously, such indirect definition brings inconvenience to the research and limits the scope of theory. In addition, Qiao did not define ME L-fr. The first aims were to define directly $\circledast$-TR, $\circledast$-EU and $\circledast$-ME L-frs and discuss the related $LFLAO$.
- The single-axiom description of the approximation operators is an amusive topic in various general rough sets [8,12,13,18,23,29,30]. In the literature, Qiao did dot present the single axiomatic description of his lower approximation operator. The second aim

was to use a single axiom to describe Qiao's $LFLAO$ produced through serial (SR), symmetric (SY), reflexive (RF), $\otimes$-TR and $\otimes$-ME L-frs.

- The construction of bijection between (fuzzy) Alexandrov topologies and (fuzzy) preorders is meaningful in (fuzzy) rough sets [20,30]. The corresponding result has not been established on Qiao's L-fuzzy rough sets. The third aim was to redefine Alexandrov L-topologies and construct a bijection between them and $\otimes$-preorders.

In our opinion, the innovation of this paper includes the following two aspects. On the one hand, there is very little work on CCRL-fuzzy rough sets, and our results make a meaningful supplement in this respect. On the other hand, we characterize the proposed CCRL-fuzzy rough sets from an internal aspect (axiomatic characterization) and an external aspect (the bijection to $ALTPO$).

In Section 2, we give some basic concepts and Qiao's $LFLAO$ as preliminaries. In Section 3, we investigate three new $LFLAO$ produced through $\otimes$-TR, $\otimes$-EU and $\otimes$-ME L-frs and characterize them through axiom sets. In Section 4, we use a single axiom each to describe five (including the three mentioned above) $LFLAO$s and their combinations. In Section 5, we redefine the concept of $ALTPO$ and then construct a bijection between $ALTPO$ and $\otimes$-preorder through $LFLAO$. In Section 6, we present conclusions.

In order to express conveniently, we give the following abbreviation table (Table 1).

Table 1. Abbreviation table.

Unabbreviated Form	Abbreviated Form
completeresiduated lattice.	CRL.
complete co-residuated lattice.	CCRL.
L-fuzzy approximation space.	$LFASPC$.
L-fuzzy lower approximation operator.	$LFLAO$.
L-topology.	$LTPO$.
Alexandrov L-topology.	$ALTPO$.
L-fuzzy relation.	L-fr.
serial.	SR.
symmetric.	SY.
reflexive.	RF.
similar.	SI.
transitive.	TR.
Euclidean.	EU.
mediate.	ME.
equivalent.	EQ.

As we all know, rough sets and their fuzzy generalizations are widely used in many fields [41–43]. In recent years, the multi-attribute decision-making method based on fuzzy rough sets has been a hot topic in both rough sets and decision-making [44–46]. When $L = [0, 1]$, the CRL-fuzzy rough sets have already demonstrated their applications in medical diagnosis [47], attribute reduction [48] and decision analysis [49]. Likewise, the CCRL-fuzzy rough sets have good application prospects in the above fields. However, the main purpose of this paper is to improve and expand the theoretical framework of fuzzy rough sets. As for the applications, we leave them for the future.

2. Preliminaries

In this section, we will recall some basic notions and notation used in this paper.

2.1. Basic Concepts

In this subsection, we review the basic concepts and properties of CCRL for later use.

A CCRL [25] is an algebra structure $(L, \sqcap, \sqcup, \circledast, \bot, \top)$ s.t. $(L, \sqcap, \sqcup, \bot, \top)$ is a complete lattice, $(L, \circledast)$ is a commutative monoid and $\forall \mu, \mu_k (k \in K) \in L$:

$$\mu \circledast \bot = \bot, \mu \circledast \bigsqcap_{k \in K} \mu_k = \bigsqcap_{k \in K} (\mu \circledast \mu_k).$$

A binary operator $\looparrowright$ on L with

$$\forall \mu, \nu \in L, \mu \looparrowright \nu = \bigsqcap \{\omega \in L | \mu \circledast \omega \geq \nu\}$$

is called the co-implication about $\circledast$.

CCRL contains right-continuous triangular models as special cases. Here are the three most important ones.

(1) The standard max operator $\mu \circledast_{SMO} \nu = \max\{\mu, \nu\}$;
(2) The probabilistic sum $\mu \circledast_{PS} \nu = \mu + \nu - \mu \cdot \nu$;
(3) The bounded sum $\mu \circledast_{BS} \nu = \min\{1, \mu + \nu\}$.

Proposition 1 ([26], Proposition 2.1)**.** *Let L be a CCRL.*

(1) $\nu \leq \mu \circledast \omega \Leftrightarrow \mu \looparrowright \nu \leq \omega$;
(2) $\bot \looparrowright \mu = \mu$, $\top \circledast \mu = \top$;
(3) $\mu \geq \nu \looparrowright \mu$;
(4) $\mu \geq \nu \Leftrightarrow \mu \looparrowright \nu = \bot$, *especially* $\mu \looparrowright \mu = \bot$;
(5) $\mu \looparrowright (\nu \looparrowright \omega) = \nu \looparrowright (\mu \looparrowright \omega) = (\mu \circledast \nu) \looparrowright \omega$;
(6) $\mu \circledast (\mu \looparrowright \nu) \geq \nu$;
(7) $(\mu \looparrowright \nu) \circledast (\nu \looparrowright \omega) \geq \mu \looparrowright \omega$;
(8) $\mu \circledast (\nu \looparrowright \omega) \geq \nu \looparrowright (\mu \circledast \omega)$;
(9) $\mu \looparrowright \bigsqcup_{k \in K} \nu_k = \bigsqcup_{k \in K} (\mu \looparrowright \nu_k)$;
(10) $(\bigsqcap_{k \in K} \nu_k) \looparrowright \mu = \bigsqcup_{k \in K} (\mu \looparrowright \nu_k)$.

Let us assume that W is a nonempty set. Every mapping $f : W \longrightarrow L$ is called an LF-set on W. All LF-sets on W are signed as L^W. Take $\mu \in L$. The symbol μ is also used to represent the constant LF-set valued as μ. Put $B \subseteq W$. The symbol $\top_B$ represents an LF-set with $\top_B(w) = 1$ whenever $w \in B$ and $\top_B(w) = 0$ whenever $w \notin B$.

A mapping $\neg : L \longrightarrow L$ is termed an involutive negation whenever it is non-increasing and $\neg\neg\mu = \mu$ for any $\mu \in L$. It is well known that $\forall \mu_k (k \in K) \in L$,

$$\neg \bigsqcup_{k \in K} \mu_k = \bigsqcap_{k \in K} \neg u_k, \neg \bigsqcap_{k \in K} \mu_k = \bigsqcup_{k \in K} \neg \mu_k.$$

For $B_k (k \in K) \in L^W$, $B \in L^W$ and $\mu \in L$, we define LF-sets $\bigsqcup_{k \in K} B_k$, $\bigsqcap_{k \in K} B_k$, $\mu \circledast B$, $\mu \looparrowright B$, $B^\neg$ point-wisely.

Definition 1 ([25], Definition 2.1)**.** *For two sets W and Z, each LF-set* $\mathcal{S} : W \times Z \longrightarrow L$ *refers to an L-fuzzy relation (L-fr) from W to Z.*

(1) $\mathcal{S}$ *is called serial (SR) whenever* $\forall \alpha \in W$, $\bigsqcup_{\beta \in Z} \mathcal{S}(\alpha, \beta) = \top$.
Moreover, let $W = Z$.
(2) $\mathcal{S}$ *is called symmetric (SY) whenever* $\forall \alpha, \beta \in W$, $\mathcal{S}(\alpha, \beta) = \mathcal{S}(\beta, \alpha)$.
(3) $\mathcal{S}$ *is called reflexive (RF) whenever* $\forall \alpha \in W$, $\mathcal{S}(\alpha, \alpha) = \top$.
(4) $\mathcal{S}$ *is called similar (SI) provided it is reflexive and symmetric.*

Definition 2. *For an L-fr* $\mathcal{S} : W \times Z \longrightarrow L$, *the triple* $(W, Z, \mathcal{S})$ *refers to an L-fuzzy approximation space (LFASPC). Whenever* $W = Z$, $(W, Z, \mathcal{S})$ *is simplified as* $(W, \mathcal{S})$.

2.2. Qiao's L-Fuzzy Lower Approximation Operator via $\otimes$ and Their Axiomatic Set Characterizations

In this subsection, we recall Qiao's L-fuzzy lower approximation operators generated by (resp., SR, SY and RF) L-fr and their axiomatic set characterizations from [25].

Definition 3 ([25], Definition 3.1 and [40], Definition 4)**.** *Let $(W, Z, \mathcal{S})$ be an LFASPC. Then, the functions $\underline{\mathcal{S}}, \overline{\mathcal{S}} : L^Z \longrightarrow L^W$ defined by: $\forall B \in L^Z, \forall \alpha \in W$,*

$$\underline{\mathcal{S}}(B)(\alpha) = \bigsqcap_{\beta \in Z} \Big(\mathcal{S}^{\neg}(\alpha, \beta) \circledast B(\beta) \Big),$$

$$\overline{\mathcal{S}}(B)(\alpha) = \bigsqcup_{\beta \in Z} \Big(\mathcal{S}^{\neg}(\alpha, \beta) \looparrowright B(\beta) \Big)$$

are termed an L-fuzzy lower approximation operator ($LFLAO$) and an L-fuzzy upper approximation operator of $(W, Z, \mathcal{S})$, separately.

Example 1. *Let $W = \{\alpha_1, \alpha_2, \alpha_3\}$, $L = [0,1]$ and $\neg\mu = 1 - \mu$ for $\mu \in [0,1]$. Put $\mathcal{S}$ as a L-fr on W with*

$$\mathcal{S} = \begin{bmatrix} 1 & 0.8 & 0.2 \\ 0.8 & 1 & 0.4 \\ 0.2 & 0.4 & 1 \end{bmatrix}.$$

Let $B = \frac{0.9}{\alpha_1} + \frac{0.7}{\alpha_2} + \frac{0.5}{\alpha_3}$.
(1) For $\circledast = \circledast_{SMO}$, its co-implication $\looparrowright$ is

$$\mu \looparrowright \nu = \begin{cases} \mu, & \mu < \nu, \\ 0, & \mu \geq \nu. \end{cases}$$

Then,

$$\underline{\mathcal{S}}(B) = \frac{0.6}{\alpha_1} + \frac{0.6}{\alpha_2} + \frac{0.5}{\alpha_3},$$
$$\overline{\mathcal{S}}(B) = \frac{0.9}{\alpha_1} + \frac{0.9}{\alpha_2} + \frac{0.9}{\alpha_3}.$$

(2) For $\circledast = \circledast_{PS}$, its co-implication $\looparrowright$ is given by

$$\mu \looparrowright \nu = \begin{cases} 0, & \mu = 1, \\ \max\{0, \frac{\nu - \mu}{1 - \mu}\}, & \text{otherwise}. \end{cases}$$

Then,

$$\underline{\mathcal{S}}(B) = \frac{0.9}{\alpha_1} + \frac{0.7}{\alpha_2} + \frac{0.5}{\alpha_3},$$
$$\overline{\mathcal{S}}(B) = \frac{0.9}{\alpha_1} + \frac{0.7}{\alpha_2} + \frac{0.5}{\alpha_3}.$$

(3) For $\circledast = \circledast_{BS}$, its co-implication $\looparrowright$ is given by $\mu \looparrowright \nu = \max\{0, \nu - \mu\}$. Then,

$$\underline{\mathcal{S}}(B) = \frac{0.76}{\alpha_1} + \frac{0.7}{\alpha_2} + \frac{0.5}{\alpha_3},$$

$$\overline{\mathcal{S}}(B) = \frac{0.9}{\alpha_1} + \frac{0.7}{\alpha_2} + \frac{0.5}{\alpha_3}.$$

The following proposition gives the fundamental properties of $LFLAO$.

Proposition 2 ([25], Proposition3.3)**.** *Let $(W, Z, \mathcal{S})$ be an LFASPC and $\mu \in L$, $B, C, B_k (k \in K) \in L^Z$.*

(1) *When $B \leq C$, then $\underline{\mathcal{S}}(B) \leq \underline{\mathcal{S}}(C)$.*

(2) $\underline{\mathcal{S}}(\top) = \top$.
(3) *For any* $\alpha \in W, \beta \in Z$, $\underline{\mathcal{S}}(\top_{Z-\{\beta\}})(\alpha) = \mathcal{S}^{\neg}(\alpha,\beta)$.
(4) $\mu \leq \underline{\mathcal{S}}(\mu)$.
(5) $\underline{\mathcal{S}}(\bigsqcap_{k\in K} B_k) = \bigsqcap_{k\in K} \underline{\mathcal{S}}(B_k)$.
(6) $\underline{\mathcal{S}}(\mu \circledast B) = \mu \circledast \underline{\mathcal{S}}(B)$.

Qiao also considered the *LFLAO* along with SR, RF and SY L-fr, respectively.

Proposition 3 ([25], Propositions 3.4, 3.5, 3.6)**.** *Assume that* $(W, Z, \mathcal{S})$ *is an LFASPC.*

(1) $\mathcal{S}$ *is SR iff* $\underline{\mathcal{S}}(\mu) = \mu$, $\forall \mu \in L$.
(2) $\mathcal{S}$ *is RF iff* $\forall B \in L^W$, $\underline{\mathcal{S}}(B) \leq B$.
(3) $\mathcal{S}$ *is SY iff* $\underline{\mathcal{S}}(\top_{W-\{\beta\}})(\alpha) = \underline{\mathcal{S}}(\top_{W-\{\alpha\}})(\beta)$, $\forall \alpha, \beta \in W$.

Qiao further described the *LFLAO* by axiom sets.

Definition 4 ([25], Proposition 4.2)**.** *A mapping* $\Theta : L^Z \longrightarrow L^W$ *is also termed an LFLAO whenever* $\Theta = \underline{\mathcal{S}}$ *for some* L-fr $\mathcal{S}$.

Theorem 1 ([25], Proposition 4.1)**.** $\Theta : L^Z \longrightarrow L^W$ *is an LFLAO iff*

(L1) $\Theta(\bigsqcap_{k\in K} B_k) = \bigsqcap_{k\in K} \Theta(B_k)$ *for any* $B_k (k \in K) \in L^Z$,
(L2) $\Theta(\mu \circledast B) = \mu \circledast \Theta(B)$ *for any* $B \in L^Z$ *and* $\mu \in L$.

Definition 5. *A mapping* $\Theta : L^Z \longrightarrow L^W$ *is called a SR (resp., SY, RF and SI) LFLAO whenever* $\Theta = \underline{\mathcal{S}}$ *for some SR (resp., SY, RF and SI)* L-fr $\mathcal{S}$.

Theorem 2 ([25], Propositions 4.6, 4.7, 4.9, 4.24)**.** *Let* $\Theta : L^Z \longrightarrow L^W$ *be a mapping.*

(1) Θ *is an SR LFLAO iff* Θ *fulfills* (L1) , (L2) *and*
(L3) $\Theta(\mu) = \mu$ *for any* $\mu \in L$.
Furthermore, let $W = Z$.
(2) Θ *is a SY LFLAO iff* Θ *fulfills* (L1), (L2) *and*
(L4) $\Theta(\top_{W-\{\beta\}} \circledast \mu)(\alpha) = \Theta(\top_{W-\{\alpha\}} \circledast \mu)(\nu), \forall \alpha, \beta \in W, \mu \in L$.
(3) Θ *is a RF LFLAO iff* Θ *fulfills* (L1), (L2) *and*
(L5) $B \geq \Theta(B)$ *for any* $B \in L^W$.
(4) Θ *is a SI LFLAO iff* Θ *fulfills* (L1), (L2), (L4) *and* (L5).

3. The Approximation Operators Produced through Three Special L-fr and Their Axiomatic Set Descriptions

In this section, we will define $\circledast$-TR, $\circledast$-EU and $\circledast$-ME conditions for L-fr. Then, we describe the associated *LFLAO* through axiom sets, respectively.

Definition 6. *Suppose that* $(W, \mathcal{S})$ *is an LFASPC and* $\forall \alpha, \beta, \rho \in W$,

(1) $\mathcal{S}$ *is called* $\circledast$*-TR whenever* $\mathcal{S}^{\neg}(\alpha,\beta) \circledast \mathcal{S}^{\neg}(\beta,\rho) \geq \mathcal{S}^{\neg}(\alpha,\rho)$.
(2) $\mathcal{S}$ *is called* $\circledast$*-EU whenever* $\mathcal{S}^{\neg}(\alpha,\beta) \circledast \mathcal{S}^{\neg}(\alpha,\rho) \geq \mathcal{S}^{\neg}(\beta,\rho)$.
(3) $\mathcal{S}$ *is called* $\circledast$*-ME whenever* $\bigsqcap_{\beta\in W} \left[\mathcal{S}^{\neg}(\alpha,\beta) \circledast \mathcal{S}^{\neg}(\beta,\rho)\right] \leq \mathcal{S}^{\neg}(\alpha,\rho)$.
(4) $\mathcal{S}$ *is called* $\circledast$*-PR provided it is RF and* $\circledast$*-TR.*
(5) $\mathcal{S}$ *is called* $\circledast$*-EQ provided it is RF, SY and* $\circledast$*-TR.*

It is not difficult to see that RF implies $\circledast$-ME.

Remark 1. *(1) For* $L = \{\bot, \top\}$*, the* L-fr $\mathcal{S}$ *degenerates into a classical binary relation. Then,* $\mathcal{S}$ *is* $\circledast$*-ME iff* $\forall \alpha, \rho \in W$, $\bigsqcup_{\beta\in W} \left[\mathcal{S}(\alpha,\beta) \sqcap \mathcal{S}(\beta,\rho)\right] \geq \mathcal{S}(\alpha,\rho)$*; i.e.,* $\forall (\alpha,\rho) \in \mathcal{S}, \exists \beta \in W$ *s.t.* $(\alpha,\beta) \in \mathcal{S}$

and $(\beta,\rho)\in\mathcal{S}$*, which is the definition of ME condition in [50] (Definition 2) for a classical binary relation.*

(2) Let $\circledast=\sqcup$*. Then, it is easily noted:*

$\mathcal{S}$ *is* $\circledast$*-TR* $\Longleftrightarrow\forall\alpha,\beta,\rho\in W,\ \mathcal{S}(\alpha,\beta)\sqcap\mathcal{S}(\beta,\rho)\leq\mathcal{S}(\alpha,\rho)$.

$\mathcal{S}$ *is* $\circledast$*-EU* $\Longleftrightarrow\forall\alpha,\beta,\rho\in W,\ \mathcal{S}(\alpha,\beta)\sqcap\mathcal{S}(\alpha,\rho)\leq\mathcal{S}(\beta,\rho)$.

Both of them are well-known.

In the following, when verifying $p\Longleftrightarrow q$, we use the symbol "$\Longrightarrow$." (resp., "$\Longleftarrow$.") to remind the reader that we will verify $p\Longrightarrow q$ (resp., $p\Longleftarrow q$).

Proposition 4. *Let* $(W,Z,\mathcal{S})$ *be an LFASPC. Then,*

(1) $\mathcal{S}$ *is* $\circledast$*-EU iff* $\underline{\mathcal{S}}(B)\leq\underline{\mathcal{S}^{op}}\,\underline{\mathcal{S}}(B)$ *for all* $B\in L^W$*, where* $\mathcal{S}^{op}(\alpha,\beta)=\mathcal{S}(\beta,\alpha)$.

(2) $\mathcal{S}$ *is* $\circledast$*-TR iff* $\underline{\mathcal{S}}(B)\leq\underline{\mathcal{S}}\,\underline{\mathcal{S}}(B)$ *for all* $B\in L^W$.

(3) $\mathcal{S}$ *is* $\circledast$*-ME iff* $\underline{\mathcal{S}}(B)\geq\underline{\mathcal{S}}\,\underline{\mathcal{S}}(B)$ *for all* $B\in L^W$.

Proof. The proof of (1) and (2) is analogous to [25]. Thus, we omit them.

(3) $\Longrightarrow$. For any $\alpha\in W$,

$$\begin{aligned}
&\underline{\mathcal{S}}\,\underline{\mathcal{S}}(B)(\alpha)\\
=\ &\bigsqcap_{\beta\in W}\Big(\mathcal{S}^{\neg}(\alpha,\beta)\circledast\underline{\mathcal{S}}(B)(\beta)\Big)\\
=\ &\bigsqcap_{\beta\in W}\Big(\mathcal{S}^{\neg}(\alpha,\beta)\circledast\bigsqcap_{\rho\in W}\Big[\mathcal{S}^{\neg}(\beta,\rho)\circledast B(\rho)\Big]\Big)\\
=\ &\bigsqcap_{\beta,\rho\in W}\Big(\mathcal{S}^{\neg}(\alpha,\beta)\circledast\Big[\mathcal{S}^{\neg}(\beta,\rho)\circledast B(\rho)\Big]\Big)\\
=\ &\bigsqcap_{\beta,\rho\in W}\Big(\Big[\mathcal{S}^{\neg}(\alpha,\beta)\circledast\mathcal{S}^{\neg}(\beta,\rho)\Big]\circledast B(\rho)\Big)\\
=\ &\bigsqcap_{\rho\in W}\Big(\Big[\bigsqcap_{\beta\in W}\big(\mathcal{S}^{\neg}(\alpha,\beta)\circledast\mathcal{S}^{\neg}(\beta,\rho)\big)\Big]\circledast B(\rho)\Big)\\
\overset{\circledast-\mathrm{ME}}{\leq}\ &\bigsqcap_{\rho\in W}\Big(\mathcal{S}^{\neg}(\alpha,\rho)\circledast B(\rho)\Big)\\
=\ &\underline{\mathcal{S}}(B)(\alpha).\quad\square
\end{aligned}$$

$\Longleftarrow$. $\forall\alpha,\rho\in W$,

$$\begin{aligned}
&\underline{\mathcal{S}}\,\underline{\mathcal{S}}(1_{W-\{\rho\}})(\alpha)\\
=\ &\bigsqcap_{\beta\in W}\Big(\mathcal{S}^{\neg}(\alpha,\beta)\circledast\underline{\mathcal{S}}(1_{W-\{\rho\}})(\beta)\Big)\\
=\ &\bigsqcap_{\beta\in W}\Big(\mathcal{S}^{\neg}(\alpha,\beta)\circledast\bigsqcap_{\xi\in W}[\mathcal{S}^{\neg}(\beta,\xi)\circledast\top_{W-\{\rho\}}(\xi)]\Big)\\
=\ &\bigsqcap_{\beta\in W}\Big(\mathcal{S}^{\neg}(\alpha,\beta)\circledast\mathcal{S}^{\neg}(\beta,\rho)\Big),
\end{aligned}$$

$$\begin{aligned}
&\underline{\mathcal{S}}(\top_{W-\{\rho\}})(\alpha)\\
=\ &\bigsqcap_{\beta\in W}\Big(\mathcal{S}^{\neg}(\alpha,\beta)\circledast(\top_{W-\{\rho\}}(\beta))\Big)\\
=\ &\mathcal{S}^{\neg}(\alpha,\rho).
\end{aligned}$$

It follows by $\underline{\mathcal{S}}\underline{\mathcal{S}}(\top_{W-\{\rho\}})(\alpha) \leq \underline{\mathcal{S}}(\top_{W-\{\rho\}})(\alpha)$, we obtain

$$\prod_{\beta \in W} [\mathcal{S}^{\neg}(\alpha,\beta) \circledast \mathcal{S}^{\neg}(\beta,\rho)] \leq \mathcal{S}^{\neg}(\alpha,\rho),$$

i.e., $\mathcal{S}$ is $\circledast$-ME.

Definition 7. *A mapping $\Theta : L^W \longrightarrow L^W$ is termed a $\circledast$-TR (resp., $\circledast$-ME, $\circledast$-PR, $\circledast$-EQ) LFLAO whenever $\Theta = \underline{\mathcal{S}}$ for some $\circledast$-TR (resp., $\circledast$-ME, $\circledast$-PR, $\circledast$-EQ) L-fr $\mathcal{S}$.*

From Proposition 4 and Theorem 1, the following theorem can be deduced naturally:

Theorem 3 (One L-fr)**.** *Let $\Theta : L^W \longrightarrow L^W$ be a mapping.*

(1) *Θ is a $\circledast$-TR LFLAO iff Θ fulfills* (L1) *and* (L2) *and*
(L6) *$\Theta(B) \leq \Theta\Theta(B)$ for all $B \in L^W$.*
(2) *Θ is a $\circledast$-ME L-FLAO iff Θ fulfills* (L1) *and* (L2) *and*
(L7) *$\Theta(B) \geq \Theta\Theta(B)$ for all $B \in L^W$.*

From Theorems 2 and 3, the next theorems can be deduced naturally:

Theorem 4 (The combination of two L-frs)**.** *Let $\Theta : L^W \longrightarrow L^W$ be a mapping.*

(1) *Θ is an SR and SY LFLAO iff Θ fulfills* (L1), (L2), (L3) *and* (L4).
(2) *Θ is an SR and $\circledast$-TR LFLAO iff Θ fulfills* (L1), (L2), (L3) *and* (L6).
(3) *Θ is a (SR) and $\circledast$-ME LFLAO iff Θ fulfills* (L1), (L2), (L3) *and* (L7).
(4) *Θ is a $\circledast$-PR LFLAO iff Θ fulfills* (L1), (L2), (L5) *and* (L6). *Furthermore, the "≥" in* (L6) *can be changed as "=".*
(5) *Θ is a SY and $\circledast$-TR LFLAO iff Θ fulfills* (L1), (L2), (L4) *and* (L6).
(6) *Θ is a SY and $\circledast$-ME LFLAO iff Θ fulfills* (L1), (L2), (L4) *and* (L7).
(7) *Θ is a $\circledast$-TR and $\circledast$-ME LFLAO iff Θ fulfills* (L1), (L2), (L6) *and* (L7).

Theorem 5 (The combination of three L-frs)**.** *Let $\Theta : L^W \longrightarrow L^W$ be a mapping.*

(1) *Θ is an SR, SY and $\circledast$-TR LFLAO iff Θ fulfills* (L1), (L2), (L3), (L4) *and* (L6).
(2) *Θ is an SR, SY and $\circledast$-ME LFLAO iff Θ fulfills* (L1), (L2), (L3), (L4) *and* (L7).
(3) *Θ is an SR, $\circledast$-TR and $\circledast$-ME LFLAO iff Θ fulfills* (L1), (L2), (L3), (L6) *and* (L7).
(4) *Θ is a $\circledast$-EQ LFLAO iff Θ fulfills* (L1), (L2), (L4), (L5) *and* (L6).
(5) *Θ is a SY, $\circledast$-TR and $\circledast$-ME LFLAO iff Θ fulfills* (L1), (L2), (L4), (L6) *and* (L7).

4. The Single-Axiom Description of *LFLAO*

In this section, we describe the mentioned *LFLAO* by single axiom.

4.1. One L-fr

In this subsection, we use a single axiom to describe the *LFLAO* generated by one L-fr.

Theorem 6. *$\Theta : L^Z \longrightarrow L^W$ is an LFLAO iff it fulfills*
(LG) *for any $\mu_k \in L, B_k \in L^Z (k \in K)$,*

$$\Theta\Big(\prod_{k \in K} (\mu_k \circledast B_k)\Big) = \prod_{k \in K} \Big(\mu_k \circledast \Theta(B_k)\Big).$$

Proof. We certify (L1) + (L2)$\Longleftrightarrow$ (LG).

$\Longrightarrow$.

$$\Theta\Big(\bigsqcap_{k\in K}(\mu_k \circledast B_k)\Big) \overset{(L1)}{=} \bigsqcap_{k\in K}\Theta\Big(\mu_k \circledast B_k\Big) \overset{(L2)}{=} \bigsqcap_{k\in K}\Big(\mu_k \circledast \Theta(B_k)\Big).$$

$\Longleftarrow$. Take $\mu_k \equiv \bot$ in (LG). We have $\Theta\Big(\bigsqcap_{k\in K} B_k\Big) \overset{(LG)}{=} \bigsqcap_{k\in K}\Theta(B_k)$,; i.e., (L1) holds.

Take $\mu_k \equiv \mu$ and $B_k \equiv B$ in (LG). We have $\Theta(\mu \circledast B) \overset{(LG)}{=} \mu \circledast \Theta(B)$,; i.e., (L2) holds. □

Theorem 7. *$\Theta : L^Z \longrightarrow L^W$ is an SR LFLAO iff it fulfills*
(LSR) *for any $\mu \in L$, $\mu_k \in L$, $B_k \in L^Z (k \in K)$,*

$$\Theta\Big(\bigsqcap_{k\in K}[\mu \wedge (\mu_k \circledast B_k)]\Big) = \bigsqcap_{k\in K}\Big(\mu \wedge [\mu_k \circledast \Theta(B_k)]\Big).$$

Proof. We certify (L1) + (L2) + (L3)$\Longleftrightarrow$ (LSR).
$\Longrightarrow$.

$$\Theta\Big(\bigsqcap_{k\in K}[\mu \wedge (\mu_k \circledast B_k)]\Big) \overset{(L1)}{=} \bigsqcap_{k\in K}\Big(\Theta(\mu) \wedge \Theta(\mu_k \circledast B_k)\Big) \overset{(L2,L3)}{=\!=} \bigsqcap_{k\in K}\Big(\mu \wedge [\mu_k \circledast \Theta(B_k)]\Big).$$

$\Longleftarrow$. Let $\mu = \bot$, $\mu_k \equiv \bot$ and $B_k \equiv \bot$ in (LSR), and we have $\Theta(\bot) = \bot \wedge [\bot \circledast \Theta(\bot)] = \bot$.
Take $\mu_k \equiv \top$ and $B_k \equiv \bot$ in (LSR). We have

$$\begin{aligned}\Theta(\mu) &= \Theta(\mu \wedge \top) \\ &= \Theta\Big(\bigsqcap_{k\in K}[\mu \wedge (\top \circledast \bot)]\Big) \\ &\overset{(LSR)}{=} \bigsqcap_{k\in K}\Big(\mu \wedge [\top \circledast \Theta(\bot)]\Big) \\ &= \mu,\end{aligned}$$

i.e., (L3) holds.
Take $\mu = \top$ in (LSR). We obtain (LG), which ensures (L1) and (L2). □

Lemma 1 ([25], Proposition 4.2)**.** $\forall B \in L^W$, $B = \bigsqcap_{\beta\in W}(\top_{W-\{\beta\}} \circledast B(\beta))$.

Theorem 8. *$\Theta : L^W \longrightarrow L^W$ is a SY LFLAO iff it fulfills*
(LSY) *for any $\alpha, \beta \in W$, $B \in L^W$ and $\mu_k \in L$, $B_k \in L^W (k \in K)$,*

$$\Theta(B)(\alpha) \sqcap \Theta\Big(\bigsqcap_{k\in K}(\mu_k \circledast B_k)\Big) = \bigsqcap_{\beta\in W}\Theta\Big(\top_{W-\{\alpha\}} \circledast B(\beta)\Big)(\beta) \sqcap \bigsqcap_{k\in K}\Big(\mu_k \circledast \Theta(B_k)\Big),$$

where $\beta \in B$ represents $B(\beta) > \bot$.

Proof. We certify (L1)+(L2) + (L4)$\Longleftrightarrow$ (LSY).

$\Longrightarrow$. For $B \in L^W$, note that

$$\begin{aligned} \prod_{\beta \in W} \Theta\Big(\top_{W-\{\alpha\}} \circledast B(\beta)\Big)(\beta) &\overset{(L4)}{=} \prod_{\beta \in B} \Theta\Big(\top_{W-\{\beta\}} \circledast B(\beta)\Big)(\alpha) \\ &\overset{(L1)}{=} \Theta\Big(\prod_{\beta \in B} \big(\top_{W-\{\beta\}} \circledast B(\beta)\big)\Big)(\alpha) \\ &\overset{\text{Lemma 1}}{=\!=} \Theta(B)(\alpha), \end{aligned}$$

through (L1)+(L2), (LSY) holds.

$\Longleftarrow$. Put $B = \bot, K = \varnothing$ in (LSY). From $\prod \varnothing = \top$, it yields that

$$\forall \alpha \in W, \Theta(\bot)(\alpha) \sqcap \Theta(\top) = \top.$$

This implies $\Theta(\top) = \top$.

$\forall \alpha, \beta \in W$, $\mu \in L$, take $B = \top_{W-\{\beta\}} \circledast \mu$ and $K = \varnothing$ in (LSY). Note that $B(\beta) = \mu$ and $B(\rho) = \top$ for any $\rho \in W, \rho \neq \beta$, so

$$\begin{aligned} \Theta(\top_{W-\{\beta\}} \circledast \mu)(\alpha) &= \prod_{\rho \in B} \Theta\Big(\top_{W-\{\alpha\}} \circledast B(\rho)\Big)(\rho) \\ &= \Theta(\top_{W-\{\alpha\}} \circledast \mu)(\beta). \end{aligned}$$

Thus, (L4) holds.

Take $B = \top$ in (LSY). It follows by $\Theta(\top) = \top$ that we obtain (LG), which ensures (L1) and (L2). □

Theorem 9. *$\Theta : L^W \longrightarrow L^W$ is a RF LFLAO iff it fulfills*

(LRF) *for any $\mu_k \in L, B_k \in L^W (k \in K)$,*

$$\prod_{k \in K} \Big((\mu_k \circledast B_k) \sqcap (\mu_k \circledast \Theta(B_k))\Big) = \Theta\Big(\prod_{k \in K} (\mu_k \circledast B_k)\Big).$$

Proof. We certify (L1)+(L2) + (L5)$\Longleftrightarrow$ (LRF).

$\Longrightarrow$.

$$\begin{aligned} \prod_{k \in K} \Big((\mu_k \circledast B_k) \sqcap (\mu_k \circledast \Theta(B_k)\Big) &\overset{(L5)}{=} \prod_{k \in K} \Big(\mu_k \circledast \Theta(B_k)\Big) \\ &\overset{(L1)+(L2)}{=} \Theta\Big(\prod_{k \in K} (\mu_k \circledast B_k)\Big). \end{aligned}$$

$\Longleftarrow$. Take $\mu_k \equiv \bot$ and $B_k \equiv B$ in (LRF). We have

$$\begin{aligned} B \sqcap \Theta(B) &= (\bot \circledast B) \sqcap (\bot \circledast \Theta(B)) \\ &\overset{(LRF)}{=} \Theta(\bot \circledast B) \\ &= \Theta(B). \end{aligned}$$

Thus, $B \geq \Theta(B)$; i.e., (L5) holds up. Then, utilizing (L5) in (LRF), (LG) holds, which ensures (L1) and (L2). □

Theorem 10. *$\Theta : L^Z \longrightarrow L^W$ is a $\circledast$-TR LFLAO iff it fulfills*

(LTR) *for any $\mu_k \in L, B_k \in L^W (k \in K)$,*

$$\prod_{k \in K} \Big([\mu_k \circledast \Theta\Theta(B_k)] \sqcap [\mu_k \circledast \Theta(B_k)]\Big) = \Theta\Big(\prod_{k \in K} (\mu_k \circledast B_k)\Big).$$

Proof. We certify (L1)+(L2) + (L6)$\Longleftrightarrow$ (LTR).

$\Longrightarrow$.

$$\bigsqcap_{k\in K}\Big([\mu_k \circledast \Theta\Theta(B_k)] \sqcap [\mu_k \circledast \Theta(B_k)]\Big) \overset{\text{(L6)}}{=} \bigsqcap_{k\in K}\Big(\mu_k \circledast \Theta(B_k)\Big) \overset{\text{(L1)+(L2)}}{=} \Theta\Big(\bigsqcap_{k\in K}(\mu_k \circledast B_k)\Big).$$

$\Longleftarrow$. Put $\mu_k \equiv \bot, B_k \equiv B$ in (LTR); then,

$$\begin{aligned}\Theta\Theta(B) \sqcap \Theta(B) &= [\bot \circledast \Theta\Theta(B)] \sqcap [\bot \circledast \Theta(B)] \\ &\overset{\text{(LTR)}}{=} \Theta(\bot \circledast B) \\ &= \Theta(B).\end{aligned}$$

Thus, $\Theta\Theta(B) \geq \Theta(B)$, i.e., (L6) holds up. Later, utilizing (L6) in (LTR), we get (LG), which ensures (L1) and (L2). □

Theorem 11. $\Theta : L^W \longrightarrow L^W$ *is a* $\circledast$*-ME LFLAO iff it fulfills*
(LME) *for any* $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\bigsqcap_{k\in K}\Big([\mu_k \circledast \Theta\Theta(B_k)] \sqcup [\mu_k \circledast \Theta(B_k)]\Big) = \Theta\Big(\bigsqcap_{k\in K}(\mu_k \circledast B_k)\Big).$$

Proof. We certified (L1)+(L2) + (L7)$\Longleftrightarrow$ (LME).
$\Longrightarrow$.

$$\bigsqcap_{k\in K}\Big([\mu_k \circledast \Theta\Theta(B_k)] \sqcup [\mu_k \circledast \Theta(B_k)]\Big) \overset{\text{(L7)}}{=} \bigsqcap_{k\in K}\Big(\mu_k \circledast \Theta(B_k)\Big) \overset{\text{(L1)+(L2)}}{=} \Theta\Big(\bigsqcap_{k\in K}(\mu_k \circledast B_k)\Big).$$

$\Longleftarrow$. Take $\mu_k \equiv \bot$ and $B_k \equiv B$ in (LME). We have

$$\begin{aligned}&\Theta\Theta(B) \sqcup \Theta(B) \\ =\ &[\bot \circledast \Theta\Theta(B)] \sqcup [\bot \circledast \Theta(B)] \\ \overset{\text{(LME)}}{=}\ &\Theta(\bot \circledast B) = \Theta(B).\end{aligned}$$

Thus, $\Theta\Theta(B) \leq \Theta(B)$; i.e., (L7) holds up. Later utilizing (L7) in (LME), we have (LG), which ensures (L1) and (L2). □

4.2. Combination of Two L-frs

In this subsection, we use a single axiom to describe the *LFLAO* generated by the combination of two *L*-*fr*s.

Theorem 12. $\Theta : L^W \longrightarrow L^W$ *is a SI LFLAO iff it fulfills*
(LSM) *for any* $\alpha, \beta \in W$, $B \in L^W$, $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\begin{aligned}&\Theta(B)(\alpha) \sqcap \Theta\Big(\bigsqcap_{k\in K}(\mu_k \circledast B_k)\Big) \\ =\ &\bigsqcap_{\beta\in B}\Theta\Big(\top_{W-\{\alpha\}} \circledast B(\beta)\Big)(\beta) \sqcap \bigsqcap_{k\in K}\Big([\mu_k \circledast B_k] \sqcap [\mu_k \circledast \Theta(B_k)]\Big).\end{aligned}$$

Proof. We certify (LSY) + (L5)$\Longleftrightarrow$ (LSM).
$\Longrightarrow$. Straightforward.
$\Longleftarrow$. Put $B = \bot, K = \varnothing$ in (LSM); one gets $\Theta(\top) = \top$.
Put $B = \top$ in (LSM). From $\Theta(\top) = \top$, one gets (LRF), and thus (L5) holds.

Utilizing (L5) in (LSM), one obtains (LSY). □

Theorem 13. $\Theta : L^W \longrightarrow L^W$ *is a* $\circledast$*-PR LFLAO iff it fulfills*
(LFO) *for any* $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\prod_{k\in K}\Big([\mu_k \circledast \Theta\Theta(B_k)] \sqcap [\mu_k \circledast \Theta(B_k)] \sqcap [\mu_k \circledast B_k]\Big) = \Theta\Big(\prod_{k\in K}(\mu_k \circledast B_k)\Big).$$

Proof. We certify (LTR) + (L5)$\Longleftrightarrow$ (LFO).
$\Longrightarrow$. Straightforward.
$\Longleftarrow$. Put $\mu_k \equiv \bot, B_k \equiv B$ in (LFO); one has $\Theta\Theta(B) \sqcap \Theta(B) \sqcap B = \Theta(B)$, so (L5) holds.
We apply (L5) in (LFO); then, (LTR) holds up. □

Theorem 14. $\Theta : L^W \longrightarrow L^W$ *is an SR and SY LFLAO iff it fulfills*
(LSR-SY) *for any* $\alpha, \beta \in W$, $B \in L^W$, $\mu \in L$, $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\begin{aligned} &\Theta(B)(\alpha) \sqcap \Theta\Big(\prod_{k\in K}[\mu \sqcap (\mu_k \circledast B_k)]\Big) \\ = &\prod_{\beta\in B}\Theta\Big(\top_{W-\{\alpha\}} \circledast B(\beta)\Big)(\beta) \sqcap \prod_{k\in K}\Big(\mu \sqcap [\mu_k \circledast \Theta(B_k)]\Big). \end{aligned}$$

Proof. We certify (LSY) + (L3)$\Longleftrightarrow$ (LSR-SY).
$\Longrightarrow$. Straightforward.
$\Longleftarrow$. Put $B = \bot, K = \varnothing$ in (LSR-SY), and one gets $\Theta(\top) = \top$.
Put $B = \top$, $\mu_k \equiv \top$ in (LSR-SY), from $\Theta(\top) = \top$, and one gets (L3).
Put $\mu = \top$ in (LSR-SY), and then (LSY) holds up. □

Theorem 15. $\Theta : L^W \longrightarrow L^W$ *is an SR and* $\circledast$*-TR LFLAO iff it fulfills*
(LSR-TR) *for any* $\mu \in L$, $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\begin{aligned} &\prod_{k\in K}\Big(\mu \sqcap ([\mu_k \circledast \Theta\Theta(B_k)] \sqcap [\mu_k \circledast \Theta(B_k)])\Big) \\ = &\Theta\Big(\prod_{k\in K}[\mu \sqcap (\mu_k \circledast B_k)]\Big). \end{aligned}$$

Proof. We certify (LSR) + (L6)$\Longleftrightarrow$ (LSR-TR).
$\Longrightarrow$. Straightforward.
$\Longleftarrow$. Put $\mu = \top$, $\mu_k \equiv \bot, B_k \equiv B$ in (LSR-TR); one gets (L6).
Using (L6) in (LSR-TR), (LSR) holds up. □

Theorem 16. $\Theta : L^W \longrightarrow L^W$ *is an SR and* $\circledast$*-ME LFLAO iff it fulfills*
(LSR-ME) *for any* $\mu \in L$, $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\begin{aligned} &\prod_{k\in K}\Big(\mu \sqcap \Big[\big(\mu_k \circledast \Theta\Theta(B_k)\big) \vee \big(\mu_k \circledast \Theta(B_k)\big)\Big]\Big) \\ = &\Theta\Big(\prod_{k\in K}\Big[\mu \sqcap (\mu_k \circledast B_k)\Big]\Big). \end{aligned}$$

Proof. We certify (LSR) + (L7)$\Longleftrightarrow$ (LSR-ME).
$\Longrightarrow$. Straightforward.
$\Longleftarrow$. Put $\mu = \top$, $\mu_k \equiv \bot, B_k \equiv B$ in (LSR-ME); (L7) can be deduced naturally.
Using (L7) in (LSR-ME), (LSR) holds. □

Theorem 17. $\Theta : L^W \longrightarrow L^W$ *is a SY and* $\circledast$*-TR LFLAO iff it fulfills*

(LSY-TR) *for any* $\alpha, \beta \in W$, $B \in L^W$, $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\begin{aligned} &\Theta(B)(\alpha) \sqcap \Theta\Big(\bigsqcap_{k \in K}(\mu_k \circledast B_k)\Big) \\ = &\bigsqcap_{\beta \in B} \Theta\Big(\top_{W-\{\alpha\}} \circledast B(\beta)\Big)(\beta) \sqcap \bigsqcap_{k \in K}\Big([\mu_k \circledast \Theta\Theta(B_k)] \sqcap [\mu_k \circledast \Theta(B_k)]\Big). \end{aligned}$$

Proof. We certify (LSY) + (L6)$\Longleftrightarrow$ (LSY-TR).

$\Longrightarrow$. Straightforward.

$\Longleftarrow$. Put $B = \perp, K = \varnothing$ in (LSY-TR); then, $\Theta(\top) = \top$ holds up.

Put $B = \top$, $\mu_k \equiv \perp, B_k \equiv B$ in (LSY-TR), and then by using $\Theta(\top) = \top$, (L6) can be deduced naturally.

We apply (L6) in (LSY-TR), and then (LSY) holds. □

Theorem 18. $\Theta : L^W \longrightarrow L^W$ *is a SY and* $\circledast$*-ME LFLAO iff it fulfills*

(LSY-ME) *for any* $\alpha, \beta \in W$, $B \in L^W$, $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\begin{aligned} &\Theta(B)(\alpha) \sqcap \Theta\Big(\bigsqcap_{k \in K}(\mu_k \circledast B_k)\Big) \\ = &\bigsqcap_{\beta \in B} \Theta\Big(\top_{W-\{\alpha\}} \circledast B(\beta)\Big)(\beta) \sqcap \bigsqcap_{k \in K}\Big([\mu_k \circledast \Theta\Theta(B_k)] \vee [\mu_k \circledast \Theta(B_k)]\Big). \end{aligned}$$

Proof. We certify (LSY) + (L7)$\Longleftrightarrow$ (LSY-ME).

$\Longrightarrow$. Straightforward.

$\Longleftarrow$. Put $B = \perp, K = \varnothing$ in (LSY-ME), and then $\Theta(\top) = \top$ holds.

Put $B = \top$, $\mu_k \equiv \perp, B_k \equiv B$ in (LSY-ME), and then by using $\Theta(\top) = \top$, (L7) holds.

We apply (L7) in (LSY-ME); (LSY) can be deduced naturally. □

Theorem 19. $\Theta : L^W \longrightarrow L^W$ *is a* $\circledast$*-TR and* $\circledast$*-ME LFLAO iff it fulfills*

(LTR-ME) *for any* $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\bigsqcap_{k \in K}\Big(\mu_k \circledast \Theta\Theta(B_k)\Big) = \Theta\Big(\bigsqcap_{k \in K}(\mu_k \circledast B_k)\Big).$$

Proof. We certify (L1)+(L2) + (L6)+(L7)$\Longleftrightarrow$ (LTR-ME).

$\Longrightarrow$. Straightforward.

$\Longleftarrow$. Put $\mu_k \equiv \perp, B_k \equiv B$ in (LTR-ME); then, $\Theta\Theta(B) = \Theta(B)$ can be deduced naturally. Hence, (L6) and (L7) hold.

From (L6)+(L7), one gets $\Theta\Theta(B_k) = \Theta(B_k)(\forall k \in K)$. Then, by using it in (LTR-ME), (L1)+(L2) can be deduced naturally. □

4.3. Combination of Three L-frs

In this subsection, we use a single axiom to describe the *LFLAO* generated by the combination of three L-frs.

Theorem 20. $\Theta : L^W \longrightarrow L^W$ *is a* $\circledast$*-EQ LFLAO iff it fulfills*

(UEQ) *For any* $\alpha, \beta \in W$, $B \in L^W$, $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\begin{aligned} &\Theta(B)(\alpha) \sqcap \Theta\Big(\bigsqcap_{k \in K}(\mu_k \circledast B_k)\Big) \\ = &\bigsqcap_{\beta \in B} \Theta\Big(\top_{W-\{\alpha\}} \circledast B(\beta)\Big)(\beta) \sqcap \bigsqcap_{k \in K}\Big([\mu_k \circledast \Theta\Theta(B_k)] \sqcap [\mu_k \circledast \Theta(B_k)] \sqcap [\mu_k \circledast B_k]\Big). \end{aligned}$$

Proof. We certify (LSM) + (L6)$\Longleftrightarrow$ (UEQ).

$\Longrightarrow$. Straightforward.

$\Longleftarrow$. Put $B = \perp, K = \varnothing$ in (UEQ); one gets $\Theta(\top) = \top$.

Put $B = \top$, $\mu_k \equiv \bot, B_k \equiv B$ in (UEQ). Then, from $\Theta(\top) = \top$, (L6) can be deduced naturally. Using (L6) in (UEQ), (LSM) holds. □

Theorem 21. $\Theta : L^W \longrightarrow L^W$ *is an SR, SY and* $\circledast$*-TR LFLAO iff it fulfills*
(LSR-SY-TR) *For any* $\alpha, \beta \in W$, $B \in L^W$, $\mu \in L$, *any* $\mu_k \in L, B_k \in L^W (k \in K)$:

$$\Theta(B)(\alpha) \sqcap \Theta\Big(\bigsqcap_{k\in K}[\mu \sqcap (\mu_k \circledast B_k)]\Big)$$
$$= \bigsqcap_{\beta\in B} \Theta\Big(\top_{W-\{\alpha\}} \circledast B(\beta)\Big)(\beta) \sqcap \bigsqcap_{k\in K} \Big(\mu \sqcap [\mu_k \circledast \Theta\Theta(B_k)] \sqcap [\mu_k \circledast \Theta(B_k)]\Big).$$

Proof. We certify (LSR-SY) + (L6)$\Longleftrightarrow$ (LSR-SY-TR).
$\Longrightarrow$. Straightforward.
$\Longleftarrow$. Put $B = \bot, K = \varnothing$ in (LSR-SY-TR); then $\Theta(\top) = \top$ holds up.
Put $B = \top$, $\mu = \top$, $\mu_k \equiv \bot, B_k \equiv B$ in (LSR-SY-TR); then from $\Theta(\top) = \top$, (L6) can be deduced naturally.
By applying (L6) in (LSR-SY-TR), (LSR-SY) holds. □

Theorem 22. $\Theta : L^W \longrightarrow L^W$ *is an SR, SY and* $\circledast$*-ME LFLAO iff it fulfills*
(LSR-SY-ME) *for any* $\alpha, \beta \in W$, $B \in L^W$, $\mu \in L$, *any* $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\Theta(B)(\alpha) \sqcap \Theta\Big(\bigsqcap_{k\in K}\Big[\mu \sqcap (\mu_k \circledast B_k)\Big]\Big)$$
$$= \bigsqcap_{\beta\in B} \Theta\Big(\top_{W-\{\alpha\}} \circledast B(\beta)\Big)(\beta) \sqcap \bigsqcap_{k\in K} \Big(\mu \sqcap \Big[\big(\mu_k \circledast \Theta\Theta(B_k)\big) \vee \big(\mu_k \circledast \Theta(B_k)\big)\Big]\Big).$$

Proof. We certify (LSR-SY) + (L7)$\Longleftrightarrow$ (LSR-SY-ME).
$\Longrightarrow$. Straightforward.
$\Longleftarrow$. Put $B = \bot, K = \varnothing$ in (LSY-SY-ME); then $\Theta(\top) = \top$ holds up.
Put $B = \top$, $\mu = \top$, $\mu_k \equiv \bot$, $B_k \equiv B$ in (LSR-SY-ME); then through $\Theta(\top) = \top$, (L7) can be deduced naturally.
By applying (L7) in (LSR-SY-ME), (LSR-SY) holds. □

Theorem 23. $\Theta : L^W \longrightarrow L^W$ *is an SR,* $\circledast$*-TR and* $\circledast$*-ME LFLAO iff it fulfills*
(LSR-TR-ME) *for any* $\mu \in L$, $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\Theta\Big(\bigsqcap_{k\in K}\Big[\mu \sqcap (\mu_k \circledast B_k)\Big]\Big) = \bigsqcap_{k\in K}\Big(\mu \sqcap \Big[\mu_k \circledast \Theta\Theta(B_k)\Big]\Big).$$

Proof. We certify (LTR-ME) + (L3)$\Longleftrightarrow$ (LSR-TR-ME).
$\Longrightarrow$. Straightforward.
$\Longleftarrow$. Put $\mu = \top$ in (LSR-TR-ME); then, (LTR-ME) can be deduced naturally.
Put $\mu_k \equiv \top$ in (LSR-TR-ME); then, (L3) holds up. □

Theorem 24. $\Theta : L^W \longrightarrow L^W$ *is a SY,* $\circledast$*-TR and* $\circledast$*-ME LFLAO iff it fulfills*
(LSY-TR-ME) *for any* $\alpha, \beta \in W$, $B \in L^W$, $\mu_k \in L, B_k \in L^W (k \in K)$,

$$\Theta(B)(\alpha) \sqcap \Theta\Big(\bigsqcap_{k\in K}(\mu_k \circledast B_k)\Big)$$
$$= \bigsqcap_{\beta\in B} \Theta\Big(\top_{W-\{\alpha\}} \circledast B(\beta)\Big)(\beta) \sqcap \bigsqcap_{k\in K}\Big(\mu_k \circledast \Theta\Theta(B_k)\Big).$$

Proof. We certify (LTR-ME) + (L4)$\Longleftrightarrow$ (LSY-TR-ME).
$\Longrightarrow$. Straightforward.
$\Longleftarrow$. Put $B = \bot$ and $K = \varnothing$ in (LSY-TR-ME); then $\Theta(\top) = \top$ holds up.
Put $B = \top$ in (LSY-TR-ME); then from $\Theta(\top) = \top$, (LTR-ME) holds up.

Put $B = \top_{W-\{\beta\}} \circledast \mu$, $K = \varnothing$ in (LSY-TR-ME); then (L4) holds up. □

5. The Bijection between ⊛-PR and Alekandrov L-Topologies

In this section, we redefine Alexandrov L-topologies and construct a bijection between them and ⊛-PR on the same underlying set.

Definition 8. *A family $\mathcal{T} \subseteq L^W$ refers to an L-topology (LTPO) on W whenever*
(LO1) $\bot, \top \in \mathcal{T}$,
(LO2) $\bigsqcup_{i\in\Delta} B_i \in \mathcal{T}$ for any $\{B_i\}_{i\in\Delta} \subseteq \mathcal{T}$,
(LO3) $B \sqcap C \in \mathcal{T}$ for any $B, C \in \mathcal{T}$.
$\mathcal{T}$ is called stratified whenever
(SO) $\mu \looparrowright B \in \mathcal{T}$ for any $\mu \in L$, $B \in \mathcal{T}$.
$\mathcal{T}$ is called co-stratified whenever
(SA) $\mu \circledast B \in \mathcal{T}$ for any $\mu \in L$, $B \in \mathcal{T}$.
A stratified and co-stratified $\mathcal{T}$ is called Alexandrov L-topology (ALTPO) whenever
(AO) $\bigsqcap_{i\in\Delta} B_i \in \mathcal{T}$ for any $\{B_i\}_{i\in\Delta} \subseteq \mathcal{T}$.

Remark 2. *In [25], Qiao defined an ALTPO as an LTPO satisfying (AO). It is well-known that there is a bijection between crisp Alexandrov topologies and preorders on the same underlying set. However, Qiao did not establish a bijection between his Alexandrov L-topologies and L-preorders. As we will see in Lemma 2 and Theorem 26, in the proof of the bijection between Alexandrov L-topologies and ⊛-PR, the stratified and co-stratified conditions are needed. In this sense, Qiao's ALTPO seems weaker. Hence, we redefine the concept of ALTPO.*

The following theorem proves for any L-fr on W, we can define an $ALTPO$ through $LFLAO$.

Theorem 25. *Let $(W, \mathcal{S})$ be an LFASPC. Then,*

$$\mathcal{T}_{\mathcal{S}} = \{B \in L^W | B \leq \underline{\mathcal{S}}(B)\}$$

is an ALTPO.

Proof. (LO1) It can be obtained through Proposition 2.
(LO2) Let $\{B_i\}_{i\in\Delta} \subseteq \mathcal{T}_{\mathcal{S}}$, we have

$$\bigsqcup_{i\in\Delta} B_i \leq \bigsqcup_{i\in\Delta} \underline{\mathcal{S}}(B_i) \leq \underline{\mathcal{S}}\Big(\bigsqcup_{i\in\Delta} B_i\Big),$$

which means $\bigsqcup_{i\in\Delta} B_i \in \mathcal{T}_{\mathcal{S}}$.

(AO) Let $\{B_i\}_{i\in\Delta} \subseteq \mathcal{T}_{\mathcal{S}}$, by Proposition 2 (5), $\bigsqcap_{i\in\Delta} B_i \leq \bigsqcap_{i\in\Delta} \underline{\mathcal{S}}(B_i) = \underline{\mathcal{S}}\Big(\bigsqcap_{i\in\Delta} B_i\Big)$, which means $\bigsqcap_{i\in\Delta} B_i \in \mathcal{T}_{\mathcal{S}}$.

(SA) Let $B \in \mathcal{T}_{\mathcal{S}}$ and $\alpha \in L$. Then, $B \leq \underline{\mathcal{S}}(B)$. It follows that

$$\underline{\mathcal{S}}(\alpha \circledast B) \overset{\text{Proposition 2(6)}}{=\!=\!=\!=} \alpha \circledast \underline{\mathcal{S}}(B) \geq \alpha \circledast B,$$

which means $\alpha \circledast B \in \mathcal{T}_{\mathcal{S}}$

(SO) Let $B \in \mathcal{T}_{\mathcal{S}}$ and $\alpha \in L$. Then, $B \leq \underline{\mathcal{S}}(B)$. For any $\alpha \in W$,

$$\begin{aligned}
& \underline{\mathcal{S}}(\alpha \looparrowright B)(\alpha) \\
= & \bigsqcap_{\beta \in W} \left(\mathcal{S}^{\neg}(\alpha,\beta) \otimes [\alpha \looparrowright B(\beta)]\right) \\
\geq & \bigsqcap_{\beta \in W} \left(\alpha \looparrowright [\mathcal{S}^{\neg}(\alpha,\beta) \otimes B(\beta)]\right) \\
\geq & \ \alpha \looparrowright \bigsqcap_{\beta \in W} \left(\mathcal{S}^{\neg}(\alpha,\beta) \otimes B(\beta)\right) \\
= & \ \alpha \looparrowright \underline{\mathcal{S}}(B)(\alpha) \\
\geq & \ \alpha \looparrowright B(\alpha),
\end{aligned}$$

which means $\alpha \looparrowright B \in \mathcal{T}_{\mathcal{S}}$.

Therefore, $\mathcal{T}_{\mathcal{S}}$ is an $ALTPO$. □

Remark 3. *In [25], for a reflexive L-fr $\mathcal{S}$ on W, Qiao proved that the family $\mathcal{T}_{\mathcal{S}}^{Q} = \{B \in L^W | B = \underline{\mathcal{S}}(B)\}$ forms an ALTPO in his sense. In this case, it is easily seen that $\mathcal{T}_{\mathcal{S}}^{Q} = \mathcal{T}_{\mathcal{S}}$ since $\forall B \in L^W$ and $\underline{\mathcal{S}}(B) \leq B$ by Proposition 3 (2).*

Example 2. *We assume W, L, $\neg$ and $\mathcal{S}$ are consistent with in Example 1. Obviously, $\mathcal{S}$ is reflexive.*

Put $B = \frac{0.9}{\alpha_1} + \frac{0.7}{\alpha_2} + \frac{0.5}{\alpha_3}$.

(1) *For $\otimes = \otimes_{SMO}$, we have $\underline{\mathcal{S}}(B) \neq B$. Thus, $B \notin \mathcal{T}_{\mathcal{S}}$.*
(2) *For $\otimes = \otimes_{PS}$, we have $\underline{\mathcal{S}}(B) = B$. Thus, $B \in \mathcal{T}_{\mathcal{S}}$.*
(3) *For $\otimes = \otimes_{BS}$, we have $\underline{\mathcal{S}}(B) \neq B$. Thus, $B \notin \mathcal{T}_{\mathcal{S}}$.*

Next, we will show that for any $LTPO$ on W, we can define a $\otimes$-(PR) on W.

Definition 9. *Suppose that $\mathcal{T}$ is an LTPO on W. Then, $\mathcal{S}_{\mathcal{T}} \in L^{W \times W}$:*

$$\forall \alpha, \beta \in W, \mathcal{S}_{\mathcal{T}}^{\neg}(\alpha,\beta) = \bigsqcup_{B \in \mathcal{T}} (B(\beta) \looparrowright B(\alpha)).$$

is called the L-fr generated by $\mathcal{T}$.

Example 3. *We assume W, L, $\neg$ and $\mathcal{S}$ are consistent with Example 1.*

Put $\mathcal{T} = \{0, 1, B\}$, where $B = \frac{0.9}{\alpha_1} + \frac{0.7}{\alpha_2} + \frac{0.5}{\alpha_3}$.

(1) *For $\otimes = \otimes_{SMO}$, we have*

$$\mathcal{S}_{\mathcal{T}} = \begin{bmatrix} 1 & 0.1 & 0.1 \\ 1 & 1 & 0.3 \\ 1 & 0.3 & 1 \end{bmatrix}.$$

(2) *For $\otimes = \otimes_{PS}$, we have*

$$\mathcal{S}_{\mathcal{T}} = \begin{bmatrix} 1 & \frac{1}{3} & 0.2 \\ 1 & 1 & 0.6 \\ 1 & 1 & 1 \end{bmatrix}.$$

(3) *For $\otimes = \otimes_{BS}$, we have*

$$\mathcal{S}_{\mathcal{T}} = \begin{bmatrix} 1 & 0.8 & 0.6 \\ 1 & 1 & 0.8 \\ 1 & 1 & 1 \end{bmatrix}.$$

Proposition 5. *Suppose that $\mathcal{T}$ is an LTPO on W. Then, $\mathcal{S}_{\mathcal{T}}$ is a $\otimes$-PR.*

Proof. First, for $\alpha \in W$,

$$\mathcal{S}_{\mathcal{T}}^{\neg}(\alpha,\alpha) = \bigsqcup_{B\in\mathcal{T}} (B(\alpha) \nrightarrow B(\alpha)) = \perp,$$

i.e., $\mathcal{S}_{\mathcal{T}}(\alpha,\alpha) = \top$, which means $\mathcal{S}_{\mathcal{T}}$ is reflexive.

Second, let $\alpha, \beta, \rho \in W$, and then

$$\begin{aligned}
& \mathcal{S}_{\mathcal{T}}^{\neg}(\alpha,\beta) \circledast \mathcal{S}_{\mathcal{T}}^{\neg}(\beta,\rho) \\
= \; & \left[\bigsqcup_{B\in\mathcal{T}} \Big(B(\beta) \nrightarrow B(\alpha)\Big)\right] \circledast \left[\bigsqcup_{B\in\mathcal{T}} \Big(B(z) \nrightarrow B(\beta)\Big)\right] \\
\geq \; & \bigsqcup_{B\in\mathcal{T}} \left[\Big(B(\beta) \nrightarrow B(\alpha)\Big) \circledast \bigsqcup_{B\in\mathcal{T}} \Big(B(z) \nrightarrow B(\beta)\Big)\right] \\
\geq \; & \bigsqcup_{B\in\mathcal{T}} \left[\Big(B(\beta) \nrightarrow B(\alpha)\Big) \circledast \Big(B(\rho) \nrightarrow B(\beta)\Big)\right] \\
\geq \; & \bigsqcup_{B\in\mathcal{T}} \Big(B(\rho) \nrightarrow B(\alpha)\Big) \\
= \; & \mathcal{S}_{\mathcal{T}}^{\neg}(\alpha,\rho),
\end{aligned}$$

which means $\mathcal{S}_{\mathcal{T}}$ is $\circledast$-TR. □

Theorems 26 and 27 illustrate that there is a bijection between $ALTPO$ and $\circledast$-PR on W. We fix a lemma first.

Lemma 2. *When $\mathcal{T}$ is an $ALTPO$ on W,*

$$\mathcal{T} = \{B | B \text{ satisfies (BI)} : \forall \alpha, \beta \in W, B(\alpha) \leq \mathcal{S}_{\mathcal{T}}^{\neg}(\alpha,\beta) \circledast B(\beta)\}.$$

Proof. Let $B \in \mathcal{T}$. Then, B satisfies condition (BI), since $\forall \alpha, \beta \in W$,

$$\begin{aligned}
\mathcal{S}_{\mathcal{T}}^{\neg}(\alpha,\beta) \circledast B(\beta) &= \left[\bigsqcup_{B\in\mathcal{T}} \Big(B(\beta) \nrightarrow B(\alpha)\Big)\right] \circledast B(\beta) \\
&\geq \Big(B(\beta) \nrightarrow B(\alpha)\Big) \circledast B(\beta) \\
&\geq B(\alpha).
\end{aligned}$$

Conversely, assume that B satisfies (BI). We prove below $B \in \mathcal{T}$.

For $\alpha \in W$, we define $n_\alpha : W \longrightarrow L$ by $n_\alpha = B(\alpha) \circledast \mathcal{S}_{\mathcal{T}}^{\neg}(-,\alpha)$. Then, from (BI) we know $n_\alpha \geq B$, and then $\bigsqcap_{\alpha\in W} n_\alpha \geq B$. In addition, $\forall \beta \in W$, notice that $\big(\bigsqcap_{\alpha\in W} n_\alpha\big)(\beta) \leq n_\beta(\beta) = B(\beta)$, which means $\bigsqcap_{\alpha\in W} n_\alpha \leq B$. Hence, $\bigsqcap_{\alpha\in W} n_\alpha = B$.

For $\alpha \in W$ and $B \in \mathcal{T}$, define $h_C^\alpha : W \longrightarrow L$ by $h_C^\alpha = C(\alpha) \nrightarrow C$, and then $h_C^\alpha \in \mathcal{T}$ by (SO). It follows that

$$\begin{aligned}
n_\alpha &= B(\alpha) \circledast \mathcal{S}_{\mathcal{T}}^{\neg}(-,\alpha) \\
&= B(\alpha) \circledast \left[\bigsqcup_{C\in\mathcal{T}} \Big(C(\alpha) \nrightarrow C\Big)\right] \\
&= B(\alpha) \circledast \left[\bigsqcup_{C\in\mathcal{T}} h_C^\alpha\right],
\end{aligned}$$

which means $n_\alpha \in \mathcal{T}$ from (LO2) and (SA).

Thus, we obtain from (AO) that $B = \bigsqcap_{\alpha\in W} n_\alpha \in \mathcal{T}$. □

Theorem 26. *When $\mathcal{T}$ is a $ALTPO$ on W, $\mathcal{T}_{\mathcal{S}_{\mathcal{T}}} = \mathcal{T}$.*

Proof. Let $B \in L^W$; then,

$$\begin{aligned} & B \in \mathcal{T} \\ \overset{\text{Lemma 2}}{\Longleftrightarrow} \quad & \forall \alpha, \beta \in W, B(\alpha) \leq \mathcal{S}^{\neg}{}_{\mathcal{T}}(\alpha, \beta) \circledast B(\beta) \\ \Longleftrightarrow \quad & \forall \alpha \in W, B(\alpha) \leq \bigsqcap_{\beta \in W} \Big(\mathcal{S}^{\neg}_{\mathcal{T}}(\alpha, \beta) \circledast B(\beta) \Big) \\ \Longleftrightarrow \quad & \forall \alpha \in W, B(\alpha) \leq \underline{\mathcal{S}_{\mathcal{T}}}\big(B\big)(\alpha) \\ \Longleftrightarrow \quad & B \leq \underline{\mathcal{S}_{\mathcal{T}}}(B) \\ \Longleftrightarrow \quad & B \in \mathcal{T}_{\mathcal{S}_{\mathcal{T}}}, \end{aligned}$$

which means $\mathcal{T}_{\mathcal{S}_{\mathcal{T}}} = \mathcal{T}$. □

Theorem 27. *If $\mathcal{S}$ is a $\circledast$-PR on W, then $\mathcal{S}_{\mathcal{T}_{\mathcal{S}}} = \mathcal{S}$.*

Proof. We only need to prove $\mathcal{S}^{\neg}_{\mathcal{T}_{\mathcal{S}}} = \mathcal{S}^{\neg}$.

Let $B \in \mathcal{T}_{\mathcal{S}}$, and note that

$$\begin{aligned} B \leq \underline{\mathcal{S}}(B) \quad \Longleftrightarrow \quad & \forall \alpha \in W, B(\alpha) \leq \bigsqcap_{\beta \in W} \Big(\mathcal{S}^{\neg}(\alpha, \beta) \circledast B(\beta) \Big) \\ \Longleftrightarrow \quad & \forall \alpha, \beta \in W, B(\alpha) \leq \neg \mathcal{S}(\alpha, \beta) \circledast B(\beta) \\ \Longleftrightarrow \quad & \forall \alpha, \beta \in W, B(\beta) \nrightarrow B(\alpha) \leq \mathcal{S}^{\neg}(\alpha, \beta). \end{aligned}$$

Hence, $\mathcal{S}^{\neg}_{\mathcal{T}_{\mathcal{S}}}(\alpha, \beta) = \bigsqcup_{B \in \mathcal{T}_{\mathcal{S}}} \Big(B(\beta) \nrightarrow B(\alpha) \Big) \leq \mathcal{S}^{\neg}(\alpha, \beta)$. Conversely, due to $\circledast$-TR, for $\beta \in W$, $\forall \alpha, \rho \in W$,

$$\begin{aligned} & \mathcal{S}^{\neg}(\alpha, \beta) \leq \mathcal{S}^{\neg}(\alpha, \rho) \circledast \mathcal{S}^{\neg}(\rho, \beta) \\ \Longleftrightarrow \quad & \forall \alpha \in W, \mathcal{S}^{\neg}(\alpha, \beta) \leq \bigsqcap_{\rho \in W} \Big(\mathcal{S}^{\neg}(\alpha, \rho) \circledast \mathcal{S}^{\neg}(\rho, \beta) \Big) \\ \Longleftrightarrow \quad & \mathcal{S}^{\neg}(-, \beta) \leq \underline{\mathcal{S}}\Big(\mathcal{S}^{\neg}(-, \beta) \Big), \end{aligned}$$

which means $\mathcal{S}^{\neg}(-, \beta) \in \mathcal{T}_{\mathcal{S}}$. It follows that

$$\begin{aligned} \mathcal{S}^{\neg}_{\mathcal{T}_{\mathcal{S}}}(\alpha, \beta) \quad & = \quad \bigsqcup_{B \in \mathcal{T}_{\mathcal{S}}} \Big(B(\beta) \nrightarrow B(\alpha) \Big), \text{ take } B = \mathcal{S}^{\neg}(-, \beta), \\ & \geq \quad \Big(\mathcal{S}^{\neg}(\beta, \beta) \nrightarrow \mathcal{S}^{\neg}(\alpha, \beta) \Big) \\ & \overset{\text{RF}}{=} \quad \bot \nrightarrow \mathcal{S}^{\neg}(\alpha, \beta) \\ & = \quad \mathcal{S}^{\neg}(\alpha, \beta). \end{aligned}$$

Therefore, $\mathcal{S}^{\neg}_{\mathcal{T}_{\mathcal{S}}} = \mathcal{S}^{\neg}$. In turn there will be $\mathcal{S}_{\mathcal{T}_{\mathcal{S}}} = \mathcal{S}$. □

6. Concluding Remarks

In classical rough sets, the approximation operators generated by serial, mediate, reflexive, symmetric, transitive and Euclidean binary relations are extensively studied, since they correspond to different modal logic systems. Furthermore, the axiomatic characterization provides an internal description for approximate operators, and the one-to-one correspondence with Alexander topology provides an external description for reflexive and transitive approximation operators. Note that Qiao's L-fuzzy lower approximation [25] has not been fully discussed about the above three aspects. Therefore, we give a further study on Qiao's L-fuzzy lower approximation and obtain the following results. Firstly, the $\circledast$-TR, $\circledast$-EU and $\circledast$-ME conditions for L-fuzzy relation were introduced, and the associated L-fuzzy lower approximation operators were characterized by axiomatic set. Secondly, the single axiom characterizations on $LFLAO$ associated with SR, SY, RF, ⊗-ME and ⊗-TR L-fuzzy relations and their combinations were presented, respectively. Thirdly, by the $LFLAO$, a one-to-one correspondence between ⊗-preorder and Alexandrov L-topology was obtained.

In the future, we shall further research CCRL-fuzzy rough sets from the following four angles. Firstly, notice that the CCRL-fuzzy rough sets based on L-fuzzy covering and L-fuzzy neighborhood systems are important branches of fuzzy-rough-set theory. Nowadays, both of them have not been studied. Hence, we will consider these two kinds of L-fuzzy rough sets in order to enrich and improve the theoretical framework of fuzzy rough sets. Secondly, it is known that the variable-precision fuzzy rough sets [51] and multi-granularity fuzzy rough sets [17] have attracted much attention because of their fault-tolerant ability. However, there is no research based on CCRL at present. Thus, we will discuss the variable-precision and multi-granularity fuzzy rough sets based on CCRL. Thirdly, as is known to all, category theory is an important tool for studying mathematical structure. With the help of category theory, people can understand the given structure at a higher level. Therefore, we will study the category properties and category relations of the proposed CCRL-fuzzy rough sets. Last but not least, when $L = [0,1]$, CRL-fuzzy rough sets have been widely used in medical diagnosis, rule extraction, decision analysis and many other fields [17,45–48]. Hence, we will explore the application of CCRL-fuzzy rough sets in related fields. Especially, we will combine CCRL-fuzzy rough sets with three-way decisions.

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