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Article

Certain q-Analogue of Fractional Integrals and Derivatives Involving Basic Analogue of the Several Variable Aleph-Function

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Abstract: Using Mellin-Barnes contour integrals, we aim at suggesting a q-analogue (q-extension) of the several variable Aleph-function. Then we present Riemann Liouville fractional q-integral and q-differential formulae for the q-extended several variable Aleph-function. Using the q-analogue of the Leibniz rule for the fractional q-derivative of a product of two basic functions, we also provide a formula for the q-extended several variable Aleph-function, which is expressed in terms of an infinite series of the q-extended several variable Aleph-function. Since the three main formulas presented in this article are so general, they can be reduced to yield a number of identities involving q-extended simpler special functions. In this connection, we choose only one main formula to offer some of its particular instances involving diverse q-extended special functions, for example, the q-extended I-function, the q-extended H-function, and the q-extended Meijer's G-function. The results presented here are hoped and believed to find some applications, in particular, in quantum mechanics.

Keywords: Mellin-Barnes contour integrals; fractional calculus; fractional q-calculus; q-several variable Aleph-function; q-several variable I-function; q-Leibniz rule; q-extended H-function; q-extended Meijer's G-function

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1. Introduction and Preliminaries

Since the concept of fractional calculus emerged in 1695 as a result of a notable communication between de L'Hôpital and Leibniz, fractional calculus has shown a stronger capacity than classical calculus for exact and efficient reflection of complex real-world occurrences. During the preceding four decades, fractional calculus has attracted a great deal of attention and found numerous applications in a range of scientific fields (see, e.g., [1–3]).

The fractional q-calculus is a q-extension of the conventional fractional calculus (see, e.g., [4–7]). Al-Salam [8] explored certain fractional q-integral and q-derivative operators. Al-Salam [9] presented the q-analogues of Cauchy's formulas for multiple integrals. Furthermore, Agarwal [10] investigated some fractional q-integral and q-derivative operators, similar to those in [8]. Many authors have offered image formulas of various q-special functions under fractional q-calculus operators (see, e.g., [11–15]). Purohit and Yadav [16] introduced and investigated q-extensions of the Saigo's fractional integral operators. Kumar et al. [17] derived the fractional order q-integrals and q-derivatives for a two variable basic counterpart to the Aleph-function and considered a related application and the q-extension of the corresponding Leibniz rule. Many researchers have used these fractional

q-calculus operators to evaluate the general class of q-polynomials, the basic analogue of Fox's H-function, the basic analogue of the I-function, and various other q-special functions (see, e.g., [12–15,18]).

In this paper, Mellin-Barnes contour integrals are used to introduce a q-analogue of the several variable Aleph-function, which is surely the first attempt. Then, we give the fractional Riemann–Liouville q-integral and q-differential formulas for the q-analogue of the several variable Aleph-function. Using the q-analogue of the Leibniz rule for the fractional q-analogue derivative of a product of two basic functions, we also provide a formula for the q-analogue of the several variable Aleph-function (see [19]), which is expressed as an infinite series of the q-extended several variable Aleph-function. The three principal formulae related with the q-analogue of the several variable Aleph-function, provided in this article, are sufficiently broad that they may be reduced to a number of identities using simpler special functions. Finally, we choose a single principal formula to illustrate some of its particular cases, involving q-analogues of some special functions, such as q-analogues of the I-function (see [20]), the H-function (see [21]), and Meijer's G-function (see [22]).

Here and in the sequel, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, integers and positive integers, respectively. Furthermore, let $\mathbb{Z}_{\geqslant 0} := \mathbb{N} \cup \{0\}$, $\mathbb{Z}_{\leqslant 0} := \mathbb{Z} \setminus \mathbb{N}$, and $\mathbb{Z}_{<0} := \mathbb{Z}_{\leqslant 0} \setminus \{0\}$.

We recall some definitions and notations for q-theory and q-calculus. The q-number of $a \in \mathbb{C}$ is given by

$$[a]_q = \frac{1-q^a}{1-q} \quad (q \in \mathbb{C} \setminus \{1\};\ q^a \neq 1). \tag{1}$$

It is found that

$$\lim_{q \to 1} \frac{1-q^a}{1-q} = a.$$

The q-analogue (or q-extension) of $n!$ then is defined by

$$[n]_q! := \begin{cases} 1 & \text{if } \ n = 0, \\ [n]_q\,[n-1]_q \cdots [2]_q\,[1]_q & \text{if } \ n \in \mathbb{N}, \end{cases} \tag{2}$$

from which the q-binomial coefficient (or the Gaussian polynomial analogous to $\binom{n}{k}$) is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q!\,[k]_q!} \quad (n,\, k \in \mathbb{Z}_{\geqslant 0};\ 0 \leqslant k \leqslant n). \tag{3}$$

The q-binomial coefficient in (3) can be generalized as follows:

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q := \frac{[\alpha]_{q;k}}{[k]_q!} \quad (\alpha \in \mathbb{C};\ k \in \mathbb{Z}_{\geqslant 0}), \tag{4}$$

where $[\alpha]_{q;k}$ is defined by

$$[\alpha]_{q;k} := [\alpha]_q\,[\alpha-1]_q \cdots [\alpha-k+1]_q \quad (\alpha \in \mathbb{C};\ k \in \mathbb{Z}_{\geqslant 0}). \tag{5}$$

The q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{k=0}^{n-1}\left(1 - aq^k\right) & (n \in \mathbb{N}), \end{cases} \tag{6}$$

where $a, q \in \mathbb{C}$, and it is supposed that $a \neq q^{-m}$ $(m \in \mathbb{Z}_{\geqslant 0})$. It is easily seen from (3) and (6) that

$$(q;q)_n = (1-q)^n\,[n]_q! \quad (n \in \mathbb{Z}_{\geqslant 0}). \tag{7}$$

The q-shifted factorial for negative subscript is defined by

$$(a;q)_{-n} := \frac{1}{\prod\limits_{k=1}^{n} \left(1 - aq^{-k}\right)} \quad (n \in \mathbb{Z}_{\geqslant 0}), \tag{8}$$

which gives

$$(a;q)_{-n} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n \, q^{\binom{n}{2}}}{(q/a;q)_n} \quad (n \in \mathbb{Z}_{\geqslant 0}). \tag{9}$$

We also denote

$$(a;q)_\infty := \prod_{k=0}^{\infty} \left(1 - aq^k\right) \quad (a,\, q \in \mathbb{C};\ |q| < 1). \tag{10}$$

It follows from (6), (8) and (10) that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad (n \in \mathbb{Z}), \tag{11}$$

which can be extended to $n = \alpha$ as follows:

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \quad (\alpha \in \mathbb{C};\ |q| < 1), \tag{12}$$

where the principal value of q^α is taken.

The q-gamma function $\Gamma_q(a)$ is given by (see, e.g., [5] (p. 16); see also [23] (p. 490))

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty (1-q)^{a-1}} = \frac{(q;q)_{a-1}}{(1-q)^{a-1}} \tag{13}$$
$$(a \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0};\ 0 < q < 1).$$

It is found from (11) and (13) that

$$(a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)} \quad (a \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0};\ 0 < q < 1;\ n \in \mathbb{Z}). \tag{14}$$

The following notations are used:

$$(x-y)_n := \begin{cases} 1 & (n = 0) \\ (x-y)(x-q\,y)\cdots(x-q^{n-1}\,y) & (n \in \mathbb{N}). \end{cases} \tag{15}$$

It is found from (15) that

$$(x-y)_n = x^n (y/x;q)_n \quad (n \in \mathbb{Z}_{\geqslant 0};\ x \in \mathbb{C} \setminus \{0\}). \tag{16}$$

Generally,

$$(x-y)_\nu := x^\nu \, (y/x;q)_\nu = x^\nu \, \frac{(y/x;q)_\infty}{(y/x\, q^\nu;q)_\infty} \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0};\ x \in \mathbb{C} \setminus \{0\}), \tag{17}$$

the second equality of which follows from (12).

Jackson [24] proposed an integral represented and defined by

$$\int_a^b f(t)\, \mathrm{d}_q t := \int_0^b f(t)\, \mathrm{d}_q t - \int_0^a f(t)\, \mathrm{d}_q t, \tag{18}$$

where

$$\int_0^x f(t)\, \mathrm{d}_q t := x(1-q) \sum_{k=0}^{\infty} q^k f\left(xq^k\right), \tag{19}$$

provided that the series at the right-hand side of (19) converges at $x = a$ and $x = b$.

If $x \in \mathbb{R}^+$, the q-integral of f on $[x, \infty)$ is defined by

$$\int_x^{\infty} f(t)\, \mathrm{d}_q t = x(1-q) \sum_{k=1}^{\infty} q^{-k} f\left(xq^{-k}\right). \tag{20}$$

A q-integral of f on $[0, \infty)$ is defined by

$$\int_0^{\infty} f(t)\, \mathrm{d}_q t = (1-q) \sum_{k=-\infty}^{\infty} q^k f\left(q^k\right). \tag{21}$$

Both (19) and (20) are inverse operations of the q-derivative

$$D_q f(x) = \frac{f(xq) - f(x)}{x(q-1)}. \tag{22}$$

For the q-integrals given above and others, one may consult [5] (Section 1.11), [25] (Section 1.3), [26] (Chapter 19)).

The q-analogue of the Riemann–Liouville fractional integral operator of a function $f(x)$ is given by (see [8,25] (Equation (4.24)))

$$\begin{aligned} I_q^{\alpha}\{f(x)\} &= \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - tq)_{\alpha-1} f(t)\, \mathrm{d}_q t \\ &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (tq/x; q)_{\alpha-1} f(t)\, \mathrm{d}_q t \end{aligned} \tag{23}$$

$$(\Re(\alpha) > 0;\ |q| < 1).$$

The q-analogue of the Kober fractional integral operator of a function $f(x)$ is defined as (see [10] (Equation (1)))

$$I_q^{\eta,\alpha}\{f(x)\} = \frac{x^{-\eta-\alpha}}{\Gamma_q(\alpha)} \int_0^x (x - tq)_{\alpha-1} t^{\eta} f(t)\, \mathrm{d}_q t \quad (\Re(\alpha) > 0, \eta \in \mathbb{R}, |q| < 1). \tag{24}$$

Using (17) and (19) in (24) yields (see [10] (Equation (2)))

$$I_q^{\eta,\alpha}\{f(x)\} = \frac{1-q}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} q^k \left(q^{k+1}; q\right)_{\alpha-1} q^{k\eta} f\left(x\, q^k\right) \quad (\eta \in \mathbb{R}, |q| < 1), \tag{25}$$

which may be valid for all $\alpha \in \mathbb{C}$ and $\left(q^{k+1}; q\right)_{\alpha-1} = \left(1 - q^{k+1}\right)_{\alpha-1}$.

Setting $\eta = 0$ in (24), in view of (23), we obtain

$$I_q^{0,\alpha}\{f(x)\} = x^{-\alpha}\, I_q^{\alpha}\{f(x)\}. \tag{26}$$

The q-analogue of the Weyl fractional integral operator is given by (see [8] (Equation (2.1)))

$$K_q^{\alpha}\{f(x)\} = \frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)} \int_x^{\infty} (t - x)_{\alpha-1} f\left(tq^{1-\alpha}\right) \mathrm{d}_q t \tag{27}$$

$$(\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0},\ |q| < 1),$$

and $K_q^0\{f(x)\} = f(x)$.

The q-analogue of the generalized Weyl fractional integral operator is defined as (see [8] (Equation (3.2)); see also [18,27] (Equation (5)))

$$K_q^{\eta,\alpha}\{f(x)\} = \frac{q^{-\eta}x^{\eta}}{\Gamma_q(\alpha)} \int_x^{\infty} (t-x)_{\alpha-1} t^{-\eta-\alpha} f\left(t\,q^{1-\alpha}\right) \mathrm{d}_q t \tag{28}$$

$$(\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0},\ \eta \in \mathbb{C},\ |q| < 1).$$

Employing (20) in (28) offers (see [27] (Equation (6)))

$$K_q^{\eta,\alpha}\{f(x)\} = \frac{1-q}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} q^{k\eta} \left(1-q^{k+1}\right)_{\alpha-1} f\left(x\,q^{-k-\alpha}\right). \tag{29}$$

It follows from (25) and (26) that

$$I_q^{\alpha}\{f(x)\} = \frac{x^{\alpha}(1-q)}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} q^k (1-q^{k+1})_{\alpha-1} f\left(xq^k\right). \tag{30}$$

We find (see [13]), in view of (12) and (13), that

$$\begin{aligned} I_q^{\alpha}\{x^{\lambda-1}\} &= \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha)}\, x^{\lambda+\alpha-1} \\ &= (1-q)^{\alpha}\, \frac{(q^{\lambda+\alpha};q)_{\infty}}{(q^{\lambda};q)_{\infty}}\, x^{\lambda+\alpha-1} \\ &= \frac{(1-q)^{\alpha}}{(q^{\lambda};q)_{\alpha}}\, x^{\lambda+\alpha-1}. \end{aligned} \tag{31}$$

Indeed, setting $f(x) = x^{\lambda-1}$ in (30), with the aid of (17), we obtain

$$\begin{aligned} I_q^{\alpha}\{x^{\lambda-1}\} &= \frac{x^{\lambda+\alpha-1}}{\Gamma_q(\lambda)}\,(1-q) \sum_{k=0}^{\infty} q^{\lambda k}\, \frac{\left(q^{k+1};q\right)_{\infty}}{(q^{k+\alpha};q)_{\infty}} \\ &= \frac{x^{\lambda+\alpha-1}}{\Gamma_q(\lambda)}\, B_q(\lambda,\alpha), \end{aligned} \tag{32}$$

where $B_q(\lambda,\alpha)$ is the q-Beta function (see, e.g., [23] (p. 495)). Now, recalling (see, e.g., [23] (p. 495))

$$B_q(\lambda,\alpha) = \frac{\Gamma_q(\lambda)\,\Gamma_q(\alpha)}{\Gamma_q(\lambda+\alpha)} \tag{33}$$

to use in (32) yields the desired identity (31).

2. q-Analogue of the Several Variable Aleph-Function

Dutta and Arora [28] introduced and investigated a q-analogue of the one variable Aleph-function defined by means of Mellin-Barnes type contour integral. Ahmad et al. [29] applied (13) or (36) to the q-analogue of the one variable Aleph-function in [28] to give an alternative definition for the q-analogue of the one variable Aleph-function. Sahni et al. [30] introduced and investigated the q-analogue of the several variable I-function. Kumar et al. [17] presented and explored a q-analogue of the two variable Aleph-function (see [31]). By modifying the techniques employed in the cited works here, we introduce a q-analogue of the several variable Aleph-function as given in Definition 1.

For simplicity, we put

$$G(q^a) := \left[\prod_{j=0}^{\infty}\left(1-q^{a+j}\right)\right]^{-1} = \frac{1}{(q^a;q)_{\infty}}. \tag{34}$$

It is remarked that

(i) $G(q^a)$ has simple poles at $a=-n$ $(n\in\mathbb{Z}_{\geqslant 0})$ with their residues

$$\operatorname*{Res}_{a=-n} G(q^a)=\frac{1}{(q^{-n};q)_n\,(q;q)_\infty\,\log q^{-1}}. \tag{35}$$

(ii) In view of (13),

$$G(q^a)=\frac{(1-q)^{a-1}}{(q;q)_\infty}\,\Gamma_q(a) \tag{36}$$
$$(a\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0};\ 0<q<1).$$

Definition 1. *Let ℓ, r, $r^{(k)}\in\mathbb{N}$ $(k=1,\ldots,\ell)$. A q-analogue of the ℓ-variable Aleph-function $\aleph(z_1,\cdots,z_\ell;q)$ is defined as follows:*

$$\begin{aligned}
&\aleph(z_1,\ldots,z_\ell;q):=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_\ell,n_\ell}_{\mu_i,\nu_i,\tau_i,r:\mu_{i^{(1)}},\nu_{i^{(1)}},\tau_{i^{(1)}},r^{(1)};\cdots;\mu_{i^{(\ell)}},\nu_{i^{(\ell)}},\tau_{i^{(\ell)}},r^{(\ell)}}\left(\begin{array}{c|}z_1\\ \vdots\ ;q\\ z_\ell\end{array}\right.\\
&\quad\left[\left(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(\ell)}\right)\right]_{1,\mathfrak{n}},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(\ell)}\right)\right]_{\mathfrak{n}+1,\mu_i}:\left[\left(c_j^{(1)}\right),\left(\gamma_j^{(1)}\right)\right]_{1,n_1},\\
&\qquad\qquad\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(\ell)}\right)\right]_{1,\nu_i}:\left[\left(d_j^{(1)}\right),\left(\delta_j^{(1)}\right)\right]_{1,m_1},\\
&\quad\left.\begin{array}{l}\left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,\mu_i^{(1)}};\cdots;\left[\left(c_j^{(\ell)}\right),\left(\gamma_j^{(\ell)}\right)_{1,n_\ell}\right],\left[\tau_{i^{(\ell)}}\left(c_{ji^{(\ell)}}^{(\ell)},\gamma_{ji^{(\ell)}}^{(\ell)}\right)\right]_{n_\ell+1,\mu_i^{(\ell)}}\\ \left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,\nu_i^{(1)}};\cdots;\left[\left(d_j^{(\ell)}\right),\left(\delta_j^{(\ell)}\right)_{1,m_\ell}\right],\left[\tau_{i^{(\ell)}}\left(d_{ji^{(\ell)}}^{(\ell)},\delta_{ji^{(\ell)}}^{(\ell)}\right)\right]_{m_\ell+1,\nu_i^{(\ell)}}\end{array}\right)\\
&=\frac{1}{(2\pi\omega)^\ell}\int_{L_1}\cdots\int_{L_\ell}\pi^\ell\psi(s_1,\cdots,s_\ell;q)\prod_{k=1}^{\ell}\theta_k(s_k;q)\,z_k^{s_k}\mathrm{d}s_1\cdots\mathrm{d}s_\ell,
\end{aligned}\tag{37}$$

where $\omega=\sqrt{-1}$,

$$\psi(s_1,\cdots,s_\ell;q):=\frac{\prod\limits_{j=1}^{\mathfrak{n}}G\left(q^{1-a_j+\sum\limits_{k=1}^{\ell}\alpha_j^{(k)}s_k}\right)}{\sum\limits_{i=1}^{r}\tau_i\left\{\prod\limits_{j=\mathfrak{n}+1}^{\mu_i}G\left(q^{a_{ji}-\sum\limits_{k=1}^{\ell}\alpha_{ji}^{(k)}s_k}\right)\prod\limits_{j=1}^{\nu_i}G\left(q^{1-b_{ji}+\sum\limits_{k=1}^{\ell}\beta_{ji}^{(k)}s_k}\right)\right\}}, \tag{38}$$

and

$$\begin{aligned}\theta_k(s_k;q)&:=\frac{\prod\limits_{j=1}^{m_k}G\left(q^{d_j^{(k)}-\delta_j^{(k)}s_k}\right)\prod\limits_{j=1}^{n_k}G\left(q^{1-c_j^{(k)}+\gamma_j^{(k)}s_k}\right)}{\sum\limits_{i^{(k)}=1}^{r^{(k)}}\tau_{i^{(k)}}\left\{\prod\limits_{j=m_k+1}^{\nu_{i^{(k)}}}G\left(q^{1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k}\right)\prod\limits_{j=n_k+1}^{\mu_{i^{(k)}}}G\left(q^{c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k}\right)\right\}}\\&\quad\times\frac{1}{G(q^{1-s_k})\sin(\pi s_k)},\end{aligned}\tag{39}$$

provided that

(i) *here and elsewhere, an empty product is interpreted as unity;*

(ii) $z_1,\ldots,z_\ell\in\mathbb{C}\setminus\{0\}$;

(iii) $\mathfrak{n}$, μ_i, ν_i, m_k, n_k, $\mu_{i^{(k)}}$, $\nu_{i^{(k)}}\in\mathbb{Z}_{\geqslant 0}$ *which satisfy* $0\leqslant\mathfrak{n}\leqslant\mu_i$, $0\leqslant m_k\leqslant\nu_{i^{(k)}}$, *and* $0\leqslant n_k\leqslant\mu_{i^{(k)}}$;

(iv) τ_i, $\tau_{i^{(1)}},\ldots,\tau_{i^{(\ell)}}\in\mathbb{R}^+$;

(v) *the values of* $\alpha_j^{(k)}$, $\alpha_{ji}^{(k)}$, $\beta_{ji}^{(k)}$, $\delta_j^{(k)}$, $\delta_{ji^{(k)}}^{(k)}$, $\gamma_j^{(k)}$ and $\gamma_{ji^{(k)}}^{(k)}$ *are assumed to be positive for standardization purposes, the definition of the basic analogue of the several variable Aleph-function, provided above, will still make sense, even if some of these values are zero;*

(vi) $a_j^{(k)}$, $a_{ji}^{(k)}$, $b_{ji}^{(k)}$, $d_j^{(k)}$, $d_{ji}^{(k)}$, $c_j^{(k)}$ *and* $c_{ji^{(k)}}^{(k)}$ *are assumed to be complex numbers.*

(vii) *the contours* L_k *in the complex* s_k*-planes* $(k = 1, \ldots, \ell)$ *are of the Mellin-Barnes type, running from* $-\omega\infty$ to $\omega\infty$ *(if necessary) with indentations, such that all the poles of* $G\left(q^{d_j^{(k)} - \delta_j^{(k)} s_k}\right)$ $(j = 1, \ldots, m_k)$ *are separated from those of* $G\left(q^{1-c_j^{(k)} + \gamma_j^{(k)} s_k}\right)$ $(j = 1, \ldots, n_k)$ *and* $G\left(q^{1-a_j + \sum_{i=1}^{\ell} \alpha_j^{(k)} s_i}\right)$ $(j = 1, \ldots, \mathfrak{n})$.

(viii) *for large values of* $|s_k|$, *the integrals converge if* $\Re(s_k \log(z_k) - \log\sin(\pi s_k)) < 0$ $(k = 1, \ldots, \ell)$.

For simplicity and convenience, the following notations are used:

$$V := m_1, n_1; \cdots ; m_\ell, n_\ell; \tag{40}$$

$$W := \mu_{i^{(1)}}, \nu_{i^{(1)}}, \tau_{i^{(1)}}, r^{(1)}; \cdots ; \mu_{i^{(\ell)}}, \nu_{i^{(\ell)}}, \tau_{i^{(\ell)}}, r^{(\ell)}; \tag{41}$$

$$A := \left[\left(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(\ell)}\right)\right]_{1,\mathfrak{n}}, \left[\tau_i\left(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(\ell)}\right)\right]_{\mathfrak{n}+1,\mu_i}; \tag{42}$$

$$B := \left[\tau_i\left(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(\ell)}\right)\right]_{1,\nu_i}; \tag{43}$$

$$\begin{aligned} C := &\left[\left(c_j^{(1)}; \gamma_j^{(1)}\right)\right]_{1,n_1}, \left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,\mu_{i^{(1)}}}; \\ &\cdots ; \left[\left(c_j^{(\ell)}; \gamma_j^{(\ell)}\right)\right]_{1,n_\ell}, \left[\tau_{i^{(\ell)}}\left(c_{ji^{(\ell)}}^{(\ell)}; \gamma_{ji^{(\ell)}}^{(\ell)}\right)\right]_{n_\ell+1,\mu_{i^{(\ell)}}}; \end{aligned} \tag{44}$$

$$\begin{aligned} D := &\left[\left(d_j^{(1)}; \delta_j^{(1)}\right)\right]_{1,m_1}, \tau_{i^{(1)}}\left[\left(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,\nu_{i^{(1)}}}; \\ &\cdots ; \left[\left(d_j^{(\ell)}; \delta_j^{(\ell)}\right)\right]_{1,m_\ell}, \left[\tau_{i^{(\ell)}}\left(d_{ji^{(\ell)}}^{(\ell)}; \delta_{ji^{(\ell)}}^{(\ell)}\right)\right]_{m_\ell+1,\nu_{i^{(\ell)}}}. \end{aligned} \tag{45}$$

It is noted in passing that the q-analogue of the ℓ-variable Aleph-function $\aleph(z_1, \cdots, z_\ell; q)$ in Definition 1, when $\ell = 2$, is easily seen to reduce to the q-analogue of the 2-variable Aleph-function $\aleph(z_1, z_2; q)$ in [17].

3. Main Results

This section will establish Riemann–Liouville fractional q-integral and q-differential formulae for the q-extended several variable Aleph-function.

Theorem 1. *Let* $\Re(\eta) > 0$, $\Re(\lambda + \eta) > 0$, $|q| < 1$, $\rho_j \in \mathbb{N}$ $(j = 1, \ldots, r)$, $\Re\Big(s_k \log(z_k) - \log\sin(\pi s_k)\Big) < 0$ $(k = 1, \ldots, r)$ *and* $I_q^\eta\{\cdot\}$ *be the Riemann Liouville fractional q-integral operator* (23). *Furthermore, restrictions and notations in Section 2 are assumed to be satisfied. Then the following formula holds true:*

$$\begin{aligned} I_q^\eta &\left\{ x^{\lambda-1} \aleph^{0,\mathfrak{n}:V}_{\mu_i,\nu_i,\tau_i;R:W} \left(\begin{array}{c} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{array} ; q \,\middle|\, \begin{array}{c} A : C \\ \\ B : D \end{array} \right) \right\} = (1-q)^\eta \, x^{\lambda+\eta-1} \\ &\times \aleph^{0,\mathfrak{n}+1:V}_{\mu_i+1,\nu_i+1,\tau_i;R:W} \left(\begin{array}{c} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{array} ; q \,\middle|\, \begin{array}{c} (1-\lambda; \rho_1, \cdots, \rho_r), A : C \\ \\ B, (1-\lambda-\eta; \rho_1, \cdots, \rho_r) : D \end{array} \right). \end{aligned} \tag{46}$$

Proof. Let $\mathcal{I}$ be the left-hand side of Equation (46). By making use of (23) and (37), we obtain

$$\begin{aligned}\mathcal{I} =& I_q^{\eta}\left\{x^{\lambda-1}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\pi^r\psi(s_1,\cdots,s_r;q)\right.\\ &\left.\times\prod_{i=1}^{r}\theta_i(s_i;q)z_i^{s_i}x^{\sum_{i=1}^{r}\rho_i s_i}\,\mathrm{d}_q s_1\cdots\mathrm{d}_q s_r\right\}.\end{aligned}\tag{47}$$

Interchanging the order of integrals in (47), which may be verified under the restrictions in Section 2, we get

$$\begin{aligned}\mathcal{I} =& \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\pi^r\psi(s_1,\cdots,s_r;q)\\ &\times\prod_{i=1}^{r}\theta_i(s_i;q)z_i^{s_i}\,I_q^{\eta}\left\{x^{\sum_{i=1}^{r}\rho_i s_i+\lambda-1}\right\}\mathrm{d}_q s_1\cdots\mathrm{d}_q s_r,\end{aligned}$$

which, upon using (31), yields

$$\begin{aligned}\mathcal{I} =& \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\pi^r\psi(s_1,\cdots,s_r;q)\\ &\times\prod_{i=1}^{r}\theta_i(s_i;q)z_i^{s_i}\,\frac{\left(q^{\sum_{i=1}^{r}\rho_i s_i+\lambda+\eta};q\right)_\infty}{\left(q^{\rho_i s_i+\lambda};q\right)_\infty}\,\mathrm{d}_q s_1\cdots\mathrm{d}_q s_r.\end{aligned}\tag{48}$$

Now, by interpreting the q-Mellin-Barnes multiple contour integrals in terms of the basic analogue of the several variable Aleph-function in Section 2, we get the desired result (46). □

In view of (49), Theorem 1 easily gives Theorem 2, which provides the Riemann–Liouville fractional q-derivative of the q-analogue of several variable Aleph-function. Since the solution conditions do not change for a fractional integral, the most plausible idea of a fractional derivative is to apply derivatives of real non-negative integer order $n = \lfloor\Re(\eta)\rfloor + 1$ to a fractional integral, which is always possible, but the swapping of derivative and integral is forbidden in the general case. Here is n times q-analogue of derivative (22):

$$D_q^{\eta}\{f(x)\} := \left(\frac{f(qx)-f(x)}{x\,(q-1)}\right)^{(n)}\left\{I_q^{n-\eta}f(x)\right\}.\tag{49}$$

By this definition (49), a fractional integral (or derivative) of any complex order η is valid for $-\infty < \Re(\eta) < \infty$. Thus, not only power laws, but even Aleph-functions can become objects of a fractional derivative (or integral) of any complex order η except for $|\eta| = \infty$.

As another trial, Agarwal [10] defined a q-fractional derivative as follows (see also [25] (p. 114)):

$$\begin{aligned}D_q^{\alpha}\{f(x)\} := I_q^{-\alpha}\{f(x)\} &= \frac{1}{\Gamma_q(-\alpha)}\int_0^x (x-tq)_{-\alpha-1}f(t)\,\mathrm{d}_q t\\ &= \frac{x^{-\alpha-1}}{\Gamma_q(-\alpha)}\int_0^x (tq/x;q)_{-\alpha-1}f(t)\,\mathrm{d}_q t\end{aligned}$$

$$(\Re(\alpha) < 0;\ |q| < 1).$$

This may be an encrypted writing of a fractional integral only.

Theorem 2. *Let* $-\infty < \Re(\eta) < \infty$, $n = \lfloor\Re(\eta)\rfloor + 1$, $\Re(\lambda+\eta-n) > 0$, $0 < |q| < \infty$, $\rho_j \in \mathbb{N}$ $(j = 1,\ldots,r)$, $\Re\Big(s_k\log(z_k) - \log\sin(\pi s_k)\Big) < 0$ $(k = 1,\ldots,r)$ *and* $D_q^{\eta}\{\cdot\}$ *be the Riemann*

Liouville fractional q-derivative operator (49). *Furthermore, restrictions and notations in Section* 2 *are assumed to be satisfied. Then the following formula holds true:*

$$D_q^\eta\left\{x^{\lambda-1}\aleph^{0,\mathfrak{n}:V}_{\mu_i,\nu_i,\tau_i;R:W}\left(\begin{array}{c}z_1x^{\rho_1}\\ \vdots\\ z_rx^{\rho_r}\end{array};q\left|\begin{array}{c}A:C\\ \\ B:D\end{array}\right.\right)\right\}=(1-q)^{-\eta}\,x^{\lambda-\eta-1}$$
$$\times\,\aleph^{0,\mathfrak{n}+1:V}_{\mu_i+1,\nu_i+1,\tau_i;R:W}\left(\begin{array}{c}z_1x^{\rho_1}\\ \vdots\\ z_rx^{\rho_r}\end{array};q\left|\begin{array}{c}(1-\lambda;\rho_1,\cdots,\rho_r),A:C\\ \\ B,(1-\lambda+\eta;\rho_1,\cdots,\rho_r):D\end{array}\right.\right). \tag{50}$$

The results (47) and (50) demonstrate with (49) that for $\lambda=1$ the following identity is valid: $I_q^\eta\left(D_q^\eta(f(x))\right)=f(x)$, where the arbitrary function $f(x)$ is shown to be a q-analogue of the several variable Aleph-function. This idea goes back to Riemann in 1847, when he defined an iterated integral for positive integer η only to avoid integration constants.

4. Leibniz Type Rule for Derivatives and their Extensions and Applications

The classical Leibniz rule or formula of elementary calculus is

$$D^n\{u(x)v(x)\}=\sum_{k=0}^{n}\binom{n}{k}\{D^kv(x)\}\{D^{n-k}u(x)\}, \tag{51}$$

where $n\in\mathbb{Z}_{\geqslant 0}$, u and v are assumed to be n-fold differentiable on some interval. A number of extensions of (51) and their applications have been explored (see, e.g., [25,32,33] (Chapter 6), [34–37] (pp. 73–79), [38]). Liouville [36] presented the Leibniz rule for fractional q-derivatives (see also [25] (Equation (6.1)))

$$D_q^\eta\{u(x)v(x)\}=\sum_{k=0}^{\infty}\frac{\Gamma(\eta+1)}{\Gamma(\eta-k+1)\,k!}\,D_q^{\eta-k}\{u(x)\}D_q^k\{v(x)\}, \tag{52}$$

where $\eta\in\mathbb{C}\setminus\mathbb{Z}_{<0}$. Watanabe [32] extended (52) as follows:

$$D_q^\eta\{u(x)v(x)\}=\sum_{k=-\infty}^{\infty}\frac{\Gamma(\eta+1)}{\Gamma(\eta-\xi-k+1)\,\Gamma(\xi+k+1)}\,D_q^{\eta-\xi-k}\{u(x)\}D_q^{\xi+k}\{v(x)\}, \tag{53}$$

where $\eta\in\mathbb{C}\setminus\mathbb{Z}_{<0}$ and $\xi\in\mathbb{C}$ holds fixed.

Remark 1. *The formula* (52) *is a very slightly corrected version of* [25] (Equation (6.1)), *where* $k!$ *at the denominator of the summation on its right side may be missed. The case* $\xi=0$ *of* (53) *produces* (52). *Osler* [33] *presented the precise convergence conditions of the series in* (52) *for the functions* $u(x)$ *and* $v(x)$ *(see [33] (p. 664)) by strengthening the contention in Watanabe* [32]: *the series in* (52) *converges wherever* $u(x)$ *and* $v(x)$ *are analytic centered at* 0.

Assume that u is continuous on $[0,X]$ $(X\in\mathbb{R}^+)$ and v is analytic on $[0,X]$. Then the Leibniz formula for fractional integrals is given as follows (see, e.g., [37] (p. 75)): For $\eta>0$ and $0<x\leq X$,

$$I_q^\eta\{u(x)v(x)\}=\sum_{k=0}^{\infty}\binom{-\eta}{k}\,I_q^{\eta+k}\{u(x)\}\,D_q^k\{v(x)\}. \tag{54}$$

Agarwal [34] provided the Leibniz rule for the fractional q-derivatives for a product of two analytic functions, which is recalled in the following lemma (see also [38] (Equation (1))).

Lemma 1. *Let the same restrictions for (54), except for one, be assumed. Here, both u and v are analytic on* $[0, X]$. *Then*

$$D_q^{\eta}\{u(x)v(x)\} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}} [q^{-\eta}; q]_n}{(q;q)_n} D_q^{\eta-n}\{u(xq^n)\} D_q^n\{v(x)\}. \tag{55}$$

The following theorem establishes a Riemann–Liouville fractional q-derivative of a product of two functions $u(x)$ and $v(x)$ in (57).

Theorem 3. *Let* $\Re(\eta) < 0$, $|q| < 1$, $\rho_j \in \mathbb{N}$ $(i = 1, \ldots, r)$ *and* $\Re(s_k \log(z_k) - \log\sin(\pi s_k)) < 0$ $(k = 1, \ldots, r)$. *Furthermore, restrictions and notations in* Section 2 *are assumed to be satisfied. Then the following Riemann–Liouville fractional q-derivative formula holds true:*

$$\begin{aligned}
&\aleph^{0,\mathfrak{n}+1:V}_{\mu_i+1,\nu_i+1,\tau_i;R:W}\left(\begin{array}{c} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{array} ; q \left| \begin{array}{c} (1-\lambda;\rho_1,\cdots,\rho_r), A : C \\ \\ B, (1-\lambda+\eta;\rho_1,\cdots,\rho_r) : D \end{array} \right.\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n\lambda + \frac{n(n-1)}{2}} [q^{-\eta}; q]_n}{(q;q)_n (q^{\lambda};q)_{n-\eta}} \\
&\times \aleph^{0,\mathfrak{n}+1:V}_{\mu_i+1,\nu_i+1,\tau_i;R:W}\left(\begin{array}{c} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{array} ; q \left| \begin{array}{c} (0;\rho_1,\cdots,\rho_r), A : C \\ B, (n;\rho_1,\cdots,\rho_r) : D \end{array} \right.\right).
\end{aligned} \tag{56}$$

Proof. By choosing

$$u(x) := x^{\lambda-1} \text{ and } v(x) := \aleph^{0,\mathfrak{n}:V}_{\mu_i,\nu_i,\tau_i;R:W}\left(\begin{array}{c} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{array} ; q \left| \begin{array}{c} A : C \\ \\ B : D \end{array} \right.\right) \tag{57}$$

to use Lemma 1, we get

$$\begin{aligned}
&D_{x,q}^{\eta}\left\{ x^{\lambda-1} \aleph\left(\begin{array}{c} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{array} ; q \left| \begin{array}{c} A : C \\ \\ B : D \end{array} \right.\right)\right\} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}} [q^{-\eta}; q]_n}{(q;q)_n} D_{x,q}^{\eta-n}\left\{(xq^n)^{\lambda-1}\right\} D_{x,q}^n\{\aleph(z_1 x^{\rho_1}, \cdots, z_r x^{\rho_r}; q)\}.
\end{aligned} \tag{58}$$

From (31) and (49), we obtain

$$D_{x,q}^{\eta-n}\left\{(xq^n)^{\lambda-1}\right\} = \frac{q^{n(\lambda-1)} (1-q)^{n-\eta}}{(q^{\lambda};q)_{n-\eta}} x^{\lambda+n-\eta-1}. \tag{59}$$

Setting $\lambda = 1$ in (50) and replacing η with n, we derive

$$\begin{aligned}
&D_{x,q}^{n}\{\aleph(z_1 x^{\rho_1}, \cdots, z_r x^{\rho_r}; q)\} \\
&= (1-q)^{-n} x^{-n} \aleph^{0,\mathfrak{n}+1:V}_{\mu_i+1,\nu_i+1,\tau_i;R:W}\left(\begin{array}{c} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{array} ; q \left| \begin{array}{c} (0;\rho_1,\cdots,\rho_r), A : C \\ \\ B, (n;\rho_1,\cdots,\rho_r) : D \end{array} \right.\right).
\end{aligned} \tag{60}$$

Finally, substituting (59) and (60) for the right-hand member of (58) and replacing the left-hand member of (58) by the right-hand side in (50), and simplifying the resulting identity, we are led to the desired formula (56). □

5. Particular Cases

This section discusses some specific instances of Theorem 3.

Corollary 1. *Let the restrictions in Section 2 and Theorem 3 be accordingly and suitably modified. Then*

$$
\begin{aligned}
&I^{0,\mathfrak{n}+1:V}_{\mu_i+1,\nu_i+1;R:W}\left(\begin{array}{c} z_1 x^{\rho_1} \\ \vdots \\ z_\ell x^{\rho_\ell}\end{array} ; q \left| \begin{array}{c} (1-\lambda;\rho_1,\cdots,\rho_\ell), A_1 : C_1 \\ \\ B_1, (1-\lambda+\eta;\rho_1,\cdots,\rho_\ell) : D_1 \end{array}\right.\right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k\lambda+\frac{k(k-1)}{2}} [q^{-\eta};q]_k}{(q;q)_k (q^\lambda;q)_{k-\eta}} \\
&\times I^{0,\mathfrak{n}+1:V}_{\mu_i+1,\nu_i+1;R:W}\left(\begin{array}{c} z_1 x^{\rho_1} \\ \vdots \\ z_\ell x^{\rho_\ell}\end{array} ; q \left| \begin{array}{c} (0;\rho_1,\cdots,\rho_\ell), A_1 : C_1 \\ \\ B_1, (k;\rho_1,\cdots,\rho_\ell) : D_1 \end{array}\right.\right).
\end{aligned} \tag{61}
$$

Proof. If $\tau_i, \tau_{i^{(k)}}$ $(k = 1, \ldots, \ell) \to 1$, then the q-analogue of the several variable Aleph-function reduces to the q-analogue of the several variable I-function (see [20]; see also [30]). Furthermore, when $\tau_i, \tau_{i^{(k)}}$ $(k = 1, \ldots, \ell) \to 1$, the A, B, C and D in Section 2 are replaced, respectively, by

$$
A_1 := \left[\left(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)}\right)\right]_{1,\mathfrak{n}}, \left[\left(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)}\right)\right]_{\mathfrak{n}+1,\mu_i}; \tag{62}
$$

$$
B_1 := \left[\left(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)}\right)\right]_{1,\nu_i}; \tag{63}
$$

$$
\begin{aligned}
C_1 :=& \left[\left(c_j^{(1)};\gamma_j^{(1)}\right)\right]_{1,n_1}, \left[\left(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,\mu_{i^{(1)}}}; \\
&\cdots; \left[\left(c_j^{(\ell)};\gamma_j^{(\ell)}\right)\right]_{1,n_\ell}, \left[\left(c_{ji^{(\ell)}}^{(\ell)};\gamma_{ji^{(\ell)}}^{(r)}\right)\right]_{n_\ell+1,\mu_{i^{(\ell)}}};
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
D_1 :=& \left[\left(d_j^{(1)};\delta_j^{(1)}\right)\right]_{1,m_1}, \left[\left(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,\nu_{i^{(1)}}}; \\
&\cdots; \left[\left(d_j^{(\ell)};\delta_j^{(\ell)}\right)\right]_{1,m_\ell}, \left[\left(d_{ji^{(\ell)}}^{(\ell)};\delta_{ji^{(\ell)}}^{(\ell)}\right)\right]_{m_\ell+1,\nu_{i^{(\ell)}}}.
\end{aligned} \tag{65}
$$

Then Theorem 3 reduces to Corollary 1. □

Corollary 2. *Let the restrictions in Section 2 and Theorem 3 be accordingly and suitably modified. Then*

$$
\begin{aligned}
&\aleph^{0,n_1+1:m_2,n_2;m_3,n_3}_{\mu_i+1,\nu_i+1,\tau_i,r:\mu_{i(1)},\nu_{i(1)},\tau_{i(1)},r^{(1)};\mu_{i(2)},\nu_{i(2)},\tau_{i(2)},r^{(2)}}\left(\begin{array}{c} z_1 x^{\rho} \\ \\ z_2 x^{\sigma}\end{array} ; q \left| \begin{array}{c} (1-\lambda;\rho,\sigma), A_2 : C_2 \\ \\ B_2, (1-\lambda+\eta;\rho,\sigma) : D_2 \end{array}\right.\right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k\lambda+\frac{k(k-1)}{2}} [q^{-\eta};q]_k}{(q;q)_k (q^\lambda;q)_{k-\eta}} \\
&\times \aleph^{0,n_1+1:m_2,n_2;m_3,n_3}_{\mu_i+1,\nu_i+1,\tau_i,r:\mu_{i(1)},\nu_{i(1)},\tau_{i(1)},r^{(1)};\mu_{i(2)},\nu_{i(2)},\tau_{i(2)},r^{(2)}}\left(\begin{array}{c} z_1 x^{\rho} \\ \\ z_2 x^{\sigma}\end{array} ; q \left| \begin{array}{c} (0;\rho,\sigma), A_2 : C_2 \\ \\ B_2, (k;\rho,\sigma) : D_2 \end{array}\right.\right).
\end{aligned} \tag{66}
$$

Proof. If $\ell = 2$, the q-analogue of the ℓ-variable Aleph-function in Section 2 reduces to the q-analogue of the two variable Aleph-function (see [31]). In this case, the A, B, C and D in Section 2 are replaced, respectively, by

$$A_2 := \left(a_j, \alpha_j, A_j\right)_{1,n_1}, \left[\tau_i\left(a_{ji}, \alpha_{ji}, A_{ji}\right)\right]_{n_1+1,\mu_i}; \tag{67}$$

$$B_2 := \left[\tau_i\left(b_{ji}, \beta_{ji}, B_{ji}\right)\right]_{1,\nu_i}; \tag{68}$$

$$C_2 := \left(c_j, \gamma_j\right)_{1,n_2}, \left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}, \gamma_{ji^{(1)}}\right)\right]_{n_2+1,\mu_{i^{(1)}}}; \left(e_j, E_j\right)_{1,n_3}, \left[\tau_{i^{(2)}}\left(e_{ji^{(2)}}, \gamma_{ji^{(2)}}\right)\right]_{n_3+1,\mu_{i^{(2)}}}; \tag{69}$$

$$D_2 := \left(d_j, \delta_j\right)_{1,m_2}, \left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}, \delta_{ji^{(1)}}\right)\right]_{m_2+1,\nu_{i^{(1)}}}; \left(f_j, F_j\right)_{1,m_3}, \left[\tau_{i^{(2)}}\left(f_{ji^{(2)}}, F_{ji^{(2)}}\right)\right]_{m_3+1,\nu_{i^{(2)}}}. \tag{70}$$

Then Theorem 3 reduces to Corollary 2. □

Corollary 3. *Let the restrictions in Section 2 and Theorem 3 be accordingly and suitably modified. Then*

$$\begin{aligned} & I^{0,n_1+1:m_2,n_2;m_3,n_3}_{\mu_i+1,\nu_i+1,r:\mu_{i^{(1)}},\nu_{i^{(1)}},r^{(1)};\mu_{i^{(2)}},\nu_{i^{(2)}},r^{(2)}}\left(\begin{array}{c} z_1x^\rho \\ \\ z_2x^\sigma \end{array} ; q \left| \begin{array}{c} (1-\lambda;\rho,\sigma), A_2' : C_2' \\ \\ B_2', (1-\lambda+\eta;\rho,\sigma) : D_2' \end{array}\right.\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k\lambda+\frac{k(k-1)}{2}} \left[q^{-\eta};q\right]_k}{(q;q)_k \left(q^\lambda;q\right)_{k-\eta}} \\ &\quad \times I^{0,n_1+1:m_2,n_2;m_3,n_3}_{\mu_i+1,\nu_i+1,r:\mu_{i^{(1)}},\nu_{i^{(1)}},r^{(1)};\mu_{i^{(2)}},\nu_{i^{(2)}},r^{(2)}}\left(\begin{array}{c} z_1x^\rho \\ \\ z_2x^\sigma \end{array} ; q \left| \begin{array}{c} (0;\rho,\sigma), A_2' : C_2' \\ \\ B_2', (k;\rho,\sigma) : D_2' \end{array}\right.\right). \end{aligned} \tag{71}$$

Proof. If $\ell = 2$ in Corollary 1, then the q-analogue of the ℓ-variable I-function reduces to the q-analogue of the 2-variable I-function (see [20]). Let

$$A_2' := \left(a_j, \alpha_j, A_j\right)_{1,n_1}, \left[\left(a_{ji}, \alpha_{ji}, A_{ji}\right)\right]_{n_1+1,\mu_i}; \tag{72}$$

$$B_2' := \left[\left(b_{ji}, \beta_{ji}, B_{ji}\right)\right]_{1,\nu_i}; \tag{73}$$

$$C_2' := \left(c_j, \gamma_j\right)_{1,n_2}, \left[\left(c_{ji^{(1)}}, \gamma_{ji^{(1)}}\right)\right]_{n_2+1,\mu_{i^{(1)}}}; \left(e_j, E_j\right)_{1,n_3}, \left[\left(e_{ji^{(2)}}, \gamma_{ji^{(2)}}\right)\right]_{n_3+1,\mu_{i^{(2)}}}; \tag{74}$$

$$D_2' := \left(d_j, \delta_j\right)_{1,m_2}, \left[\left(d_{ji^{(1)}}, \delta_{ji^{(1)}}\right)\right]_{m_2+1,\nu_{i^{(1)}}}; \left(f_j, F_j\right)_{1,m_3}, \left[\left(f_{ji^{(2)}}, F_{ji^{(2)}}\right)\right]_{m_3+1,\nu_{i^{(2)}}}. \tag{75}$$

Then Corollary 1 reduces to Corollary 3. □

Corollary 4. *Let the restrictions in Section 2 and Theorem 3 be accordingly and suitably modified. Then*

$$\begin{aligned} & H^{0,n_1+1:m_2,n_2;m_3,n_3}_{\mu_1+1,\nu_1+1:\mu_2,\nu_2;\mu_3,\nu_3}\left(\begin{array}{c} z_1x^\rho \\ \\ z_2x^\sigma \end{array} ; q \left| \begin{array}{c} (1-\lambda;\rho,\sigma), A_2'' : C_2'' \\ \\ B_2'', (1-\lambda+\eta;\rho,\sigma) : D_2'' \end{array}\right.\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k\lambda+\frac{k(k-1)}{2}} \left[q^{-\eta};q\right]_k}{(q;q)_k \left(q^\lambda;q\right)_{k-\eta}} \\ &\quad \times H^{0,n_1+1:m_2,n_2;m_3,n_3}_{\mu_1+1,\nu_1+1:\mu_2,\nu_2;\mu_3,\nu_3}\left(\begin{array}{c} z_1x^\rho \\ \\ z_2x^\sigma \end{array} ; q \left| \begin{array}{c} (0;\rho,\sigma), A_2'' : C_2'' \\ \\ B_2'', (k;\rho,\sigma) : D_2'' \end{array}\right.\right). \end{aligned} \tag{76}$$

Proof. Let $r = r^{(1)} = r^{(2)} = 1$. Then the q-analogue of the two variable I-function in Corollary 3 reduces to the q-analogue of the two variable H-function (see [39]). In this case, let

$$A_2'' := \left(a_i, \alpha_i, A_i\right)_{1,\mu_1}; \tag{77}$$

$$B_2'' := (b_i, \beta_i, B_i)_{1,\nu_1}; \tag{78}$$

$$C_2'' := (e_i, E_i)_{1,\mu_2}, (g_i, G_i)_{1,\mu_3}; \tag{79}$$

$$D_2'' := (f_i, F_i)_{1,\nu_2}, (h_i, H_i)_{1,\nu_3}. \tag{80}$$

Then Corollary 3 reduces to Corollary 4. □

Corollary 5. *Let the restrictions in Section 2 and Theorem 3 be accordingly and suitably modified. Then*

$$\begin{aligned} &G^{0,n_1+1:m_2,n_2;m_3,n_3}_{\mu_1+1,\nu_1+1:\mu_2,\nu_2;\mu_3,\nu_3}\left(\begin{matrix} z_1 x^\rho \\ \\ z_2 x^\sigma \end{matrix} ;\, q \;\middle|\; \begin{matrix} (1-\lambda;\rho,\sigma), (a_j)_{1,\mu_1} : (e_j)_{1,\mu_2}, (g_j)_{1,\mu_3} \\ \\ (b_j)_{1,\nu_1}, (1-\lambda+\eta;\rho,\sigma) : (f_j)_{1,\nu_2}, (h_j)_{1,\nu_3} \end{matrix}\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\, q^{k\lambda + \frac{k(k-1)}{2}}\, [q^{-\eta};q]_k}{(q;q)_k\, (q^\lambda;q)_{k-\eta}} \\ &\quad \times G^{0,n_1+1:m_2,n_2;m_3,n_3}_{\mu_1+1,\nu_1+1:\mu_2,\nu_2;\mu_3,\nu_3}\left(\begin{matrix} z_1 x^\rho \\ \\ z_2 x^\sigma \end{matrix} ;\, q \;\middle|\; \begin{matrix} (0;\rho,\sigma), (a_j)_{1,\mu_1} : (e_j)_{1,\mu_2}, (g_j)_{1,\mu_3} \\ \\ (b_j)_{1,\nu_1}, (k;\rho,\sigma) : (f_j)_{1,\nu_2}, (h_j)_{1,\nu_3} \end{matrix}\right). \end{aligned} \tag{81}$$

Proof. Take

$$(\alpha_i)_{1,\mu_1} = (A_i)_{1,\mu_1} = (\beta_i)_{1,\nu_1} = (B_i)_{1,\nu_1} = (E_i)_{1,\mu_2} = (G_i)_{1,\mu_3} = (F_i)_{1,\nu_2} = (H_i)_{1,\nu_3} = 1$$

in Corollary 4. Then the q-analogue of the two variable H-function in Corollary 4 reduces to the q-analogue of the two variable Meijer's G-function (see [22]). Then Corollary 4 reduces to Corollary 5. □

Corollary 6. *Let the restrictions in Section 2 and Theorem 3 be accordingly and suitably modified. Then*

$$\begin{aligned} &\aleph^{m,n+1}_{\mu_i+1,\nu_i+1,\tau_i,r}\left(zx^\rho;\, q \;\middle|\; \begin{matrix} (1-\lambda;\rho), (a_j,A_j)_{1,n} \cdots [\tau_i(a_{ji},A_{ji})]_{n+1,\mu_i} \\ (b_j,B_j)_{1,m} \cdots [\tau_i(b_{ji},B_{ji})]_{m+1,\nu_i}, (1-\lambda+\eta;\rho) \end{matrix}\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\, q^{k\lambda + \frac{k(k-1)}{2}}\, [q^{-\eta};q]_k}{(q;q)_k\, (q^\lambda;q)_{k-\eta}} \\ &\quad \times \aleph^{m,n+1}_{\mu_i+1,\nu_i+1,\tau_i,r}\left(zx^\rho;\, q \;\middle|\; \begin{matrix} (0;\rho), (a_j,A_j)_{1,n} \cdots [\tau_i(a_{ji},A_{ji})]_{n+1,\mu_i} \\ (b_j,B_j)_{1,m} \cdots [\tau_i(b_{ji},B_{ji})]_{m+1,\nu_i}, (k;\rho) \end{matrix}\right). \end{aligned} \tag{82}$$

Proof. The q-analogue of the two variable Aleph-function reduces to the q-analogue of the one variable Aleph-function, which is defined by Ahmad et al. [29] (see also the case $r = 1$ in Theorem 3). Then Theorem 3 reduces to Corollary 6. □

Corollary 7. *Let the restrictions in Section 2 and Corollary 6 with $\tau_i \to 1$ be accordingly and suitably modified. Then*

$$\begin{aligned} &I^{m,n+1}_{\mu_i+1,\nu_i+1,r}\left(zx^\rho;\, q \;\middle|\; \begin{matrix} (1-\lambda;\rho), (a_j,A_j)_{1,n}, \cdots, (a_{ji},A_{ji})_{n+1,\mu_i} \\ (b_j,B_j)_{1,m}, \cdots, (b_{ji},B_{ji})_{m+1,\nu_i}, (1-\lambda+\eta;\rho) \end{matrix}\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\, q^{k\lambda + \frac{k(k-1)}{2}}\, [q^{-\eta};q]_k}{(q;q)_k\, (q^\lambda;q)_{k-\eta}} \\ &\quad \times I^{m,n+1}_{\mu_i+1,\nu_i+1,r}\left(zx^\rho;\, q \;\middle|\; \begin{matrix} (0;\rho), (a_j,A_j)_{1,n}, \cdots, (a_{ji},A_{ji})_{n+1,\mu_i} \\ (b_j,B_j)_{1,m}, \cdots, (b_{ji},B_{ji})_{m+1,\nu_i}, (k;\rho) \end{matrix}\right). \end{aligned} \tag{83}$$

Proof. If $\tau_i \to 1$ in (82), then the q-analogue of the one variable Aleph-function reduces to the q-analogue of the one variable I-function (see [40]). Then the identity (83) may follow from (82). $\square$

Corollary 8. *Let the restrictions in Section 2 and Corollary 7 with $r = 1$ be accordingly and suitably modified. Then*

$$\begin{aligned} &H^{m,n+1}_{\mu+1,\nu+1}\left(zx^{\rho};\, q \;\middle|\; \begin{matrix} (1-\lambda;\rho),\, (a_j, A_j)_{1,\mu} \\ (b_j, B_j)_{1,\nu},\, (1-\lambda+\eta;\rho) \end{matrix} \right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\, q^{k\lambda+\frac{k(k-1)}{2}}\, [q^{-\eta};q]_k}{(q;q)_k\, (q^{\lambda};q)_{k-\eta}}\, H^{m,n+1}_{\mu+1,\nu+1}\left(zx^{\rho};\, q \;\middle|\; \begin{matrix} (0;\rho),\, (a_j, A_j)_{1,\mu} \\ (b_j, B_j)_{1,\nu},\, (k;\rho) \end{matrix} \right). \end{aligned} \tag{84}$$

Proof. Setting $r = 1$ in Corollary 7, one finds that the q-analogue of the one variable I-function reduces to the q-analogue of the one variable H-function (see [41]). In this case, we get the Formula (84). $\square$

Corollary 9. *Let the restrictions in Section 2 and Corollary 8 with $(A_j)_{1,\mu} = (B_j)_{1,\nu} = 1$ be accordingly and suitably modified. Then*

$$\begin{aligned} &G^{m,n+1}_{\mu+1,\nu+1}\left(zx^{\rho};\, q \;\middle|\; \begin{matrix} (1-\lambda;\rho),\, (a_j)_{1,\mu} \\ (b_j)_{1,\nu},\, (1-\lambda+\eta;\rho) \end{matrix} \right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\, q^{k\lambda+\frac{k(k-1)}{2}}\, [q^{-\eta};q]_k}{(q;q)_k\, (q^{\lambda};q)_{k-\eta}}\, G^{m,n+1}_{\mu+1,\nu+1}\left(zx^{\rho};\, q \;\middle|\; \begin{matrix} (0;\rho),\, (a_j)_{1,\mu} \\ (b_j)_{1,\nu},\, (k;\rho) \end{matrix} \right). \end{aligned} \tag{85}$$

Proof. Take $(A_j)_{1,\mu} = (B_j)_{1,\nu} = 1$ in Corollary 8. Then the q-analogue of the H-function may reduce to the q-analogue of the Meijer's G-function. In this case, we may obtain the Formula (85). $\square$

6. Conclusions

The importance of our findings in this article may rest in their manifold generality. By specializing the various parameters as well as the variables in the q-extended several variable Aleph-function, we may obtain a large number of results, involving a remarkable variety of useful analogues of basic functions (or a product of such basic functions), which are expressible in terms of q-analogues of diverse special functions of one and several variables, including the q-extended I-function (see [20]), the q-extended H-function (see [40,41]), the q-extended Meijer's G-function (see [22]), the q-extended generalized hypergeometric function (see [4,5]), and Mac Robert's E-function. There are several ways to define Mac Robert's E-function in terms of the generalized hypergeometric function or the Meijer G-function (see [42] (Chapter V)). In this regard, Mac Robert's E-function may be q-extended by means of q-generalized hypergeometric function or q-Meijer G-function. Therefore, the formulae obtained in this research are of a fairly broad nature and may be helpful in a number of intriguing issues that have appeared in the literature of pure and applied mathematics and mathematical physics.

For further research, as well as some other properties and identities for the q-analogue of the several variable Aleph-function $\aleph(z_1, \cdots, z_\ell; q)$ in Definition 1 to be found, the results offered in this article are hoped and believed to find some applications, in particular, in quantum mechanics (see [4,43,44]). Furthermore, it may be interesting to make q-extensions of some of the results in the following articles [45,46]. Further, instead of $u(x) = x^{\lambda-1}$ in (57), choosing that $u(x)$ is another q-analogue of the several variable Aleph-function like $v(x)$, what does look like the resulting identity in Theorem 3?

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