axioms

Article

On the Iterative Multivalued ⊥-Preserving Mappings and an Application to Fractional Differential Equation

Muhammad Nazam [1,*], Sumit Chandok [2], Aftab Hussain [3,*] and Hamed H. Al Sulmi [3]

[1] Department of Mathematics, Allama Iqbal Open University, H-8, Islamabad 44000, Pakistan
[2] School of Mathematics, Thapar Institute of Engineering and Technology, Patiala 147004, Punjab, India
[3] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[*] Correspondence: muhammad.nazam@aiou.edu.pk (M.N.); aniassuirathka@kau.edu.sa (A.H.)

Abstract: In this paper, we introduce orthogonal multivalued contractions, which are based on the recently introduced notion of orthogonality in the metric spaces. We construct numerous fixed point theorems for these contractions. We show how these fixed point theorems aid in the generalization of a number of recently published findings. Additionally, we offer a theorem that establishes the existence of a fractional differential equation's solution.

Keywords: fixed point; (V, W)-orthogonal contractions; O-complete metric space

MSC: 47H10; 26A18; 26D20

check for updates

Citation: Nazam, M.; Chandok, S.; Hussain, A.; Al Sulmi, H.H. On the Iterative Multivalued ⊥-Preserving Mappings and an Application to Fractional Differential Equation. *Axioms* **2023**, *12*, 53. https://doi.org/10.3390/axioms12010053

Academic Editor: Clemente Cesarano

Received: 6 December 2022
Revised: 21 December 2022
Accepted: 24 December 2022
Published: 3 January 2023

1. Introduction and Preliminaries

The core of the metric fixed-point theory is the exploration of generalized contraction principles to add more applicable fixed-point theorems in the theory. The simplest and most applicable contraction principle is the Banach contraction principle. This contraction principle can be applied to show the existence of solutions to equations representing mathematical models. The contraction principle that appeared in [1] generalizes the Rakotch [2] contraction concept. Furthermore, Matkowski [3], Samet et al. [4], Karapinar et al. [5], and Pasicki [6] have all generalized the Boyd-Wong notion. The concept of F-contraction [7] is another notable generalization of the Banach contraction principle (BCP), and several research articles have been published in the previous decade (see [8–13], and references therein).

The role of fixed point theory in solving real-world problems has been described in many recently published papers. Recently, Turab et al. [14] proposed a generic stochastic functional equation that can be used to describe several psychological and learning theory experiments. The existence, uniqueness, and stability analysis of the suggested stochastic equation are examined by utilizing the notable fixed point theory tools. Khan et al. [15] proposed a fixed-point technique to investigate a system of fractional order differential equations. Rezapour et al. [16] proposed a labeling method for graph vertices, and then presented some existence results for solutions to a family of fractional boundary value problems (FBVPs) on the methyl propane graph by means of Krasnoselskii's and Scheafer's fixed point theorems.

The use of partial order, admissibility of a mapping, graph theory and binary relation are all being effectively utilized in metric fixed point theory. Recently, Gordji et al. [17] presented a special binary relation, termed the orthogonal relation, and presented several examples to clarify the concept of the orthogonal relation and, hence, orthogonal-set (see Ex 2.2 to Ex 2.11). Gordji et al. also presented a generalization of BCP in the *orthogonal metric space*. Later, Baghani et al. [8] generalized the study done in [17] by using the concept of F-contraction, while Nazam et al. [18] broadened the investigation conducted in [8].

On the other hand, Proinov [19] offered various fixed-point theorems that built on previous work in [1,3–7]. He introduced a generalized class of contractions by operating two functions $V, W : (0, \infty) \to (-\infty, \infty)$ on both sides of the Banach contraction and obtained several fixed point results. The class of contractions given in [19] encapsulate the contractions defined in [4,7,20,21].

In this paper, we extend some results of [19] to multivalued mappings subject to the class of orthogonal contractions. The class of orthogonal contractions generalizes ordered contractions, graphic contractions and α-admissible contractions. We demonstrate that every contraction is orthogonal but not vice versa. Along with several examples to validate the results, we also present an application for solving a fractional differential equation (FDE).

Let $\mathcal{U} \neq \emptyset$ and $\perp \subset \mathcal{U} \times \mathcal{U}$ satisfying the property (P),

$$(\mathrm{P}) : \exists\ \ell_0 \in \mathcal{U} : \text{ either } (\forall \tau \in \mathcal{U}; \ell_0 \perp \tau) \text{ or } (\forall \tau \in \mathcal{U}; \tau \perp \ell_0).$$

We call the pair $(\mathcal{U}, \perp)$ an orthogonal set (abbreviated as, O-set). The concept of orthogonality in an inner-product space is an example of $\perp$.

For the illustration of the orthogonal set, O-sequence, O-Cauchy and its examples, we suggest the reader read the articles [17,22].

Definition 1. *[17] The O-set $(\mathcal{U}, \perp)$ endowed with a metric d is called an O-metric space (in short, OMS) denoted by $(\mathcal{U}, \perp, d)$.*

Definition 2. *[17] Let $(\mathcal{U}, \perp, d)$ be an orthogonal metric space. A mapping $f : \mathcal{U} \to \mathcal{U}$ is said to be an orthogonal contraction if there exists $k \in [0, 1)$ such that*

$$d(fx, fy) \leq kd(x, y)\ \forall x, y \in \mathcal{U} \text{ with } x \perp y.$$

Terms such as continuity and orthogonal continuity, completeness and O-completeness, Banach contraction and orthogonal contraction have been explained in [10,13,17,22]. In the following, we give some comparisons between fundamental notions.

1. The continuity implies orthogonal continuity but the converse is not true. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(\ell) = [\ell],\ \forall \ell \in \mathbb{R}$ and the relation $\perp \subseteq \mathbb{R} \times \mathbb{R}$ is defined by

$$\ell \perp g \text{ if } \ell, g \in \left(i + \frac{1}{3}, i + \frac{2}{3}\right), i \in \mathbb{Z} \text{ or } \ell = 0.$$

   Then, f is $\perp$-continuous while f is discontinuous on $\mathbb{R}$.
2. The completeness of the metric space implies O-completeness, but the converse is not true. We know that $\mathcal{A} = [0, 1)$ with Euclidean metric d is not a complete metric space. If we define the relation $\perp \subseteq \mathcal{A} \times \mathcal{A}$ by

$$\ell \perp g \iff \ell \leq g \leq \frac{1}{2} \text{ or } \ell = 0,$$

   then $(\mathcal{A}, \perp, d)$ is an O-complete.
3. The Banach contraction implies orthogonal contraction but the converse is not true. Let $\mathcal{A} = [0, 10)$ with Euclidean metric d so that $(\mathcal{A}, d)$ is a metric space. If we define the relation $\perp \subseteq \mathcal{A} \times \mathcal{A}$ by

$$\ell \perp g \text{ if } \ell g \leq \ell \vee g,$$

   then $(\mathcal{A}, \perp, d)$ is an O-metric space. Define $f : \mathcal{A} \to \mathcal{A}$ by $f(\ell) = \frac{\ell}{2}$ (if $\ell \leq 2$) and $f(\ell) = 0$ (if $\ell > 2$). Since $d(f(3), f(2)) > kd(3, 2)$, f is not a contraction; rather, it is an orthogonal contraction.

Let
$P(\mathcal{U})$— set of non-empty subsets of $\mathcal{U}$.

$P_{cb}(\mathcal{U})$— set of all non-empty bounded and closed subsets of $\mathcal{U}$.
$K(\mathcal{U})$—set of non-empty compact subsets of $\mathcal{U}$.

If we let $E \in P_{cb}(\mathcal{U})$ and $g \in \mathcal{U}$, then $d(g, E) = \inf_{i \in E} d(g, i)$; d is a metric on $\mathcal{U}$. The mapping $H : P_{cb}(\mathcal{U}) \times P_{cb}(\mathcal{U}) \to [0, \infty)$ defined by

$$H(E_1, E_2) = \max\left\{ \sup_{r \in E_1} d(r, E_2), \sup_{w \in E_2} d(w, E_1) \right\} \text{ for all } E_1, E_2 \in P_{cb}(\mathcal{U}),$$

defines a metric on $P_{cb}(\mathcal{U})$. It is also known as the Pompieu-Hausdorff-metric. In the following, we define $\perp$-admissible mapping, $\perp$-preserving mapping and illustrate them with examples. Let $\Lambda = \{(x, y) \in \mathcal{U} \times \mathcal{U} : x \perp y\}$.

Definition 3. *A mapping $f : \mathcal{U} \times \mathcal{U} \to [1, \infty)$ is said to be strictly $\perp$-admissible if $f(a, \theta) > 1$ for all $a, \theta \in \mathcal{U}$ with $a \perp \theta$ and $f(a, \theta) = 1$ otherwise.*

Example 1. *Let $\mathcal{U} = [0, 1)$ and define the relation $\perp \subset \mathcal{U} \times \mathcal{U}$ by*

$$a \perp \theta \text{ if } a\theta \in \{a, \theta\} \subset \mathcal{U}.$$

Then, $\mathcal{U}$ is an O-set. Define $f : \mathcal{U} \times \mathcal{U} \to [1, \infty)$ by

$$f(a, \theta) = \begin{cases} a + \frac{2}{1+\theta} & \text{if } a \perp \theta, \\ 1 & \text{otherwise}. \end{cases}$$

Then, f is $\perp$-admissible.

Definition 4. *Let $\mathcal{U}$ be a non-empty set. A set-valued mapping $L : \mathcal{U} \to P(\mathcal{U})$ satisfying the property (O) is called $\perp$-preserving.*

(O). For each $j \in \mathcal{U}$ and $l \in L(j)$ with $j \perp l$ or $l \perp j$, $\exists\, g \in L(l)$ with $l \perp g$ or $g \perp l$.

Example 2. *Let $\mathcal{U} = [0, 1)$ and define a relation $\perp \subset [0, 1) \times [0, 1)$ by*

$$g \perp h \text{ if } gh \in \{g, h\} \subset [0, 1).$$

Then, $\mathcal{U} := [0, 1)$ is an O-set. Now for a function $t : \mathcal{U} \times \mathcal{U} \to [1, \infty)$ defined by

$$t(g, h) = \begin{cases} g + \frac{2}{1+h} & \text{if } g \perp h, \\ 1 & \text{otherwise}. \end{cases}$$

Then, t is a $\perp$-admissible mapping. The mapping $r : \mathcal{U} \to P(\mathcal{U})$ defined by

$$r(g) = \begin{cases} \left[\frac{g}{15}, \frac{g+1}{7}\right] & \text{if } g \in \mathbb{Q} \cap \mathcal{U}, \\ \{0\} & \text{if } g \in \mathbb{Q}^c \cap \mathcal{U}, \end{cases}$$

is a $\perp$-preserving mapping.

The following facts have been stated in [19] and we carry them for our upcoming results.

Lemma 1. *Let $\{c_\alpha\} \subset (X, d)$ and it obeys the equation $\lim_{\alpha \to \infty} d(c_\alpha, c_{\alpha+1}) = 0$; then, there are subsequences $\{c_{\alpha_l}\}$, $\{c_{\beta_l}\}$ and $q > 0$ (whenever $\{c_\alpha\}$ is not Cauchy) following the equations:*

$$\lim_{l \to \infty} d(c_{\alpha_{l+1}}, c_{\beta_{l+1}}) = q + . \tag{1}$$

$$\lim_{l\to\infty} d(c_{\alpha_l}, c_{\beta_l}) = d(c_{\alpha_{l+1}}, c_{\beta_l}) = d(c_{\alpha_l}, c_{\beta_{l+1}}) = q. \tag{2}$$

The following result appeared in [23] and is very useful for our upcoming results.

Lemma 2. *Let (U, d) and $\ell > 1$, then, for all $w \in Q_1 \subseteq U$, there is a $g \in Q_2 \subseteq U$ following the inequality:*

$$d(w, g) \leq \ell H(Q_1, Q_2).$$

2. Multivalued $(V, W)_\perp$-Contractions

This section deals with the multivalued $(V, W)_\perp$-contractions. To guarantee the presence of fixed points of multivalued $(V, W)_\perp$-contractions, we study a number of constraints on the real valued nonlinear functions (V, W). The multivalued $(V, W)_\perp$-contraction is defined as follows.

Definition 5. *Let $(\mathcal{U}, \perp, d)$ be an OMS. A mapping $\mathcal{S} : \mathcal{U} \to P_{cb}(\mathcal{U})$ is called a multivalued $(V, W)_\perp$-contraction if there exists a strictly $\perp$-admissible function ν such that*

$$V(\nu(q, \ell) H(\mathcal{S}(q), \mathcal{S}(\ell))) \leq W(d(q, \ell)) \tag{3}$$

for all $q, \ell \in \Lambda$ with $H(\mathcal{S}(q), \mathcal{S}(\ell)) > 0$.

Remark 1. *The following observations indicate the generality of multivalued $(V, W)_\perp$ contraction for the specific definitions of the mappings V, W.*

1. *If $V(\ell) = \ell$ and $W(\ell) = \lambda\ell$, where $0 \leq \lambda < 1$, then $\mathcal{S}$ is an orthogonal Nadler contraction [23].*
2. *If $V(\ell) = \ell$, then $\mathcal{S}$ is an orthogonal multivalued Boyd-Wong contraction [1].*
3. *If V is lower semi-continuous and W is upper semi-continuous, then $\mathcal{S}$ is an orthogonal multivalued variant of the contraction defined in [24].*
4. *If $W(\ell) = F(V(\ell))$, then $\mathcal{S}$ is an orthogonal multivalued variant of the contraction defined in [21].*
5. *If $W(\ell) = \alpha(\ell)V(\ell)$ and $V(\ell) = \ell$, then $\mathcal{S}$ is an orthogonal variant of the contraction defined in [25].*
6. *If $W(\ell) = \lambda V(\ell)$, then $\mathcal{S}$ is an orthogonal multivalued variant of the contraction defined in [26].*
7. *If $W(\ell) = F(V(\ell))$ and $F(\ell) = \ell^\alpha$, then $\mathcal{S}$ is an orthogonal multivalued variant of the contraction defined in [20].*
8. *If $W(\ell) = V(\ell) - \tau$, then $\mathcal{S}$ is an orthogonal multivalued variant of the contraction defined in [7].*

Remark 2. It is noted that if $W(c) = V(c) - \tau$ for all $c \in (0, \infty)$, then the contractive condition (3) is a multivalued F-contraction [27]. If $W(c) = V(c) - \tau(c)$ for all $c \in (0, \infty)$, then it is a multivalued (τ, F_T)-contraction [10]. If we set $V(c) = \ln(c)$ for all c, then we have a Nadler contraction [23]. For, if the function $V : (0, \infty) \to (0, \infty)$ is non-decreasing and $p(j) \in (0, 1)$ for all $j \in (0, \infty)$ with $\limsup_{z\to\epsilon+} p(z) < 1$. Then, defining $W(z) = p(z)V(z)$ and $V(z) = z$ for all $z > 0$, we obtain the contraction defined in [28].

Let $\perp$RCOMS denote a $\perp$-regular complete orthogonal metric space.

The following theorem presents the first formula of this paper for the existence of fixed points.

Theorem 1. *Let $(\mathcal{U}, \perp, d)$ be a $\perp$RCOMS. Suppose that $\mathcal{S} : \mathcal{U} \to P_{cb}(\mathcal{U})$ is a $\perp$-preserving and satisfies (3). If $\perp$ is transitive and functions $V, W : (0, \infty) \to (-\infty, \infty)$ meet the following conditions:*

(i) there exists $c_1 \in \mathcal{S}(c_0)$ such that $c_1 \perp c_0$ or $c_0 \perp c_1$, for any $c_0 \in \mathcal{U}$,

(ii) V *is non-decreasing and* $W(c) < V(c)\ \forall\, c > 0$,
(iii) $\limsup_{c\to\gamma+} W(c) < V(\gamma+)\ (\forall\, \gamma > 0)$.

Then, there exists $c^* \in \mathcal{U}$ *such that* $c^* \in \mathcal{S}(c^*)$.

Proof. By (i), for an arbitrary $c_0 \in \mathcal{U}$, there exists $c_1 \in \mathcal{S}(c_0)$ such that $c_0 \perp c_1$ or $c_1 \perp c_0$. Since the mapping $\mathcal{S}$ is $\perp$-preserving, there exists $c_2 \in \mathcal{S}(c_1)$ such that $c_1 \perp c_2$ or $c_2 \perp c_1$ and, thus, $c_3 \in \mathcal{S}(c_2)$ such that $c_2 \perp c_3$ or $c_3 \perp c_2$. In general, there exists $c_{n+1} \in \mathcal{S}(c_n)$ such that $c_n \perp c_{n+1}$ or $c_{n+1} \perp c_n$ for all $n \geq 0$. Hence, $\nu(c_n, c_{n+1}) > 1$ for all $n \geq 0$. If $c_n \in \mathcal{S}(c_n)$ (for some $n \geq 0$), then c_n is a fixed-point of $\mathcal{S}$. We assume that $c_n \notin \mathcal{S}(c_n)$ $(\forall\, n \geq 0)$. Then, $H(\mathcal{S}c_{n-1}, \mathcal{S}c_n) > 0$. So $\nu(c_n, c_{n+1}) > 1$ and $\mathcal{S}(c_n), \mathcal{S}(c_{n+1}) \in P_{cb}(\mathcal{U})(\forall\, n \geq 0)$. Hence, there exists $c_n \neq c_{n+1} \in \mathcal{S}(c_n)$ such that $d(c_n, c_{n+1}) \leq \nu(c_{n-1}, c_n)H(\mathcal{S}(c_{n-1}), \mathcal{S}(c_n))(\forall\, n \geq 1)$ (see Lemma 2). Since the function V is increasing, by (3), we have

$$V(d(c_n, c_{n+1})) \leq V(\nu(c_{n-1}, c_n)H(\mathcal{S}(c_{n-1}), \mathcal{S}(c_n))) \leq W(d(c_{n-1}, c_n)).$$

Since $W(c) < V(c)$ $(\forall\, c > 0)$, we have

$$V(d(c_n, c_{n+1})) \leq W(d(c_{n-1}, c_n)) < V(d(c_{n-1}, c_n)). \tag{4}$$

The monotonicity of the function V implies $d(c_n, c_{n+1}) < d(c_{n-1}, c_n)$ $(\forall n \geq 1)$ and, thus, the sequence $\{d(c_{n-1}, c_n)\}$ is monotone. Let $\delta \geq 0$ satisfy $\lim_{n\to\infty} d(c_{n-1}, c_n) = \delta+$. If $\delta > 0$, by (4), we have

$$V(\delta+) = \lim_{n\to\infty} V(d(c_n, c_{n+1})) \leq \lim_{n\to\infty} \sup W(d(c_{n-1}, c_n)) \leq \lim_{c\to\delta+} \sup W(c).$$

This is a contradiction to (iii). Thus, $\delta = 0$ and hence the mapping $\mathcal{S}$ is asymptotically-regular.

Now, we show that $\{c_n\}$ is a Cauchy sequence. Contrarily, suppose that the sequence $\{c_n\}$ is not Cauchy. By Lemma 1, there exist two subsequences $\{c_{n_k}\}$, $\{c_{m_k}\}$ of $\{c_n\}$ and $\epsilon > 0$ such that the equations (1) and (2) hold. By (1), we get that $d(c_{n_k+1}, c_{m_k+1}) > \epsilon$. Since $c_n \perp c_{n+1}$ $(\forall n \geq 0)$, by transitivity of $\perp$, we have $c_{n_k} \perp c_{m_k}$ and, hence, $\nu(c_{n_k}, c_{m_k}) > 1$ $(\forall k \geq 1)$. Setting $q = c_{n_k}$ and $\ell = c_{m_k}$ in (3), we have

$$V(d(c_{n_k+1}, c_{m_k+1})) \leq V(\nu(c_{n_k}, c_{m_k})H(\mathcal{S}c_{n_k}, \mathcal{S}c_{m_k})) \leq W(d(c_{n_k}, c_{m_k})),\ \text{for any } k \geq 1.$$

For if $a_k = d(c_{n_k+1}, c_{m_k+1})$ and $b_k = d(c_{n_k}, c_{m_k})$, we have

$$V(a_k) \leq W(b_k),\ \text{for any } k \geq 1. \tag{5}$$

By (1) and (2), we have $\lim_{k\to\infty} a_k = \epsilon+$ and $\lim_{k\to\infty} b_k = \epsilon$. By (9), we obtain

$$\lim \inf_{c\to\epsilon+} V(c) \leq \lim \inf_{k\to\infty} V(a_k) \leq \lim \sup_{k\to\infty} W(b_k) \leq \lim \sup_{c\to\epsilon} W(c). \tag{6}$$

But (6) contradicts (iii), thus, $\{c_n\}$ is a Cauchy sequence in $\mathcal{U}$. Since $(\mathcal{U}, \perp, d)$ is a complete OMS, $\lim_{n\to\infty} c_n = c^*$ for some $c^* \in \mathcal{U}$. Since the space $(\mathcal{U}, \perp, d)$ is $\perp$-regular, we have $c_n \perp c^*$ or $c^* \perp c_n$ such that $\nu(c_n, c^*) > 1$. We need to show that $d(c^*, \mathcal{S}(c^*)) = 0$ and contrarily suppose that $d(c^*, \mathcal{S}(c^*)) > 0$. Then, there exists $n_1 \in \mathbb{N}$ such that $d(c_n, \mathcal{S}(c^*)) > 0$ for all $n \geq n_1$. By (3)

$$V(d(c_{n+1}, \mathcal{S}(c^*))) \leq V(\nu(c_n, c^*)H(\mathcal{S}(c_n), \mathcal{S}(c^*))) \leq W(d(c_n, c^*)) < V(d(c_n, c^*)). \tag{7}$$

By monotonicity of V, we obtain that $d(c_{n+1}, \mathcal{S}(c^*)) < d(c_n, c^*)$. Taking the limit as $n \to \infty$ in (7), we have $d(c^*, \mathcal{S}(c^*)) < 0$, which is a contradiction. Thus, $d(c^*, \mathcal{S}(c^*)) = 0$. Since $\mathcal{S}(c^*)$ is closed, $c^* \in \mathcal{S}(c^*)$. □

The following theorem states another set of terms and conditions ensuring the existence of fixed points of multivalued $(V, W)_\perp$-contractions.

Theorem 2. *Let $(\mathcal{U}, \perp, d)$ be a $\perp$RCOMS with transitive $\perp$. Suppose that $\mathcal{S} : \mathcal{U} \to P_{cb}(\mathcal{U})$ is a $\perp$-preserving and satisfies (3) and the functions $V, W : (0, \infty) \to (-\infty, \infty)$ meet the following conditions:*

(i) *there exists $c_1 \in \mathcal{S}(c_0)$ such that $c_0 \perp c_1$ or $c_1 \perp c_0$, $c_0 \in \mathcal{U}$,*
(ii) *V is non-decreasing and $W(y) < V(y)$ for any $y > 0$,*
(iii) $\inf_{c>\epsilon} V(c) > -\infty$,
(iv) *for the strictly-decreasing sequences $\{V(c_n)\}$ and $\{W(c_n)\}$, if $\lim_{n\to\infty} V(c_n) = \lim_{n\to\infty} W(c_n) = L$, then $\lim_{n\to\infty} c_n = 0$,*
(v) $\limsup_{c\to\epsilon} W(c) < \liminf_{c\to\epsilon+} V(c)$ *for any $\epsilon > 0$,*
(vi) $\limsup_{c\to\epsilon_1} W(c) < \liminf_{c\to\epsilon} V(c)$ *for any $\epsilon, \epsilon_1 > 0$.*

Then, $\mathcal{S}$ admits at least one fixed-point in $\mathcal{U}$.

Proof. By (i), for $c_0 \in \mathcal{U}$, there exists $c_1 \in \mathcal{S}(c_0)$ such that $c_0 \perp c_1$ or $c_1 \perp c_0$. Since T is a $\perp$-preserving mapping, there exists $c_2 \in \mathcal{S}(c_1)$ such that $c_1 \perp c_2$ or $c_2 \perp c_1$ and then $c_3 \in \mathcal{S}(c_2)$ such that $c_2 \perp c_3$ or $c_3 \perp c_2$. In general, there exists $c_{n+1} \in \mathcal{S}(c_n)$ such that $c_n \perp c_{n+1}$ or $c_{n+1} \perp c_n$ ($\forall\, n \geq 0$). Hence, $\nu(c_n, c_{n+1}) > 1$ for all $n \geq 0$. If $c_n \in \mathcal{S}(c_n)$ then c_n is a fixed-point of $\mathcal{S}$ ($\forall\, n \geq 0$). If $c_n \notin \mathcal{S}(c_n)$ ($\forall n \geq 0$), then $H(\mathcal{S}c_{n-1}, \mathcal{S}c_n) > 0$. Since $\nu(c_n, c_{n+1}) > 1$ and $\mathcal{S}(c_n), \mathcal{S}(c_{n+1}) \in P_{cb}(\mathcal{U})$, $n \geq 0$, by Lemma 2, there exists $c_{n+1} \in \mathcal{S}(c_n)$ ($c_n \neq c_{n+1}$) such that $d(c_n, c_{n+1}) \leq \nu(c_{n-1}, c_n) H(\mathcal{S}(c_{n-1}), \mathcal{S}(c_n))$ for all $n \geq 1$. By monotonicity of V and (3), we have

$$V(d(c_n, c_{n+1})) \leq V(\nu(c_{n-1}, c_n) H(\mathcal{S}(c_{n-1}), \mathcal{S}(c_n))) \leq W(d(c_{n-1}, c_n)) < V(d(c_{n-1}, c_n)). \tag{8}$$

By (8) we get that $\{V(d(c_{n-1}, c_n))\}$ is a strictly decreasing-sequence.

We have two cases:

Case 1. $\{V(d(c_{n-1}, c_n))\}$ is unbounded below.

By (iii), we have $\inf_{d(c_{n-1},c_n)>\epsilon} V(d(c_{n-1}, c_n)) > -\infty$. This implies that

$$\lim \inf_{d(c_{n-1},c_n)\to\epsilon+} V(d(c_{n-1}, c_n)) > -\infty.$$

Thus, $\lim_{n\to\infty} d(c_{n-1}, c_n) = 0$, otherwise, we have

$$\lim \inf_{d(c_{n-1},c_n)\to\epsilon+} V(d(c_{n-1}, c_n)) = -\infty.$$

This is a contradiction to the assumption (iii).

Case 2. $\{V(d(c_{n-1}, c_n))\}$ is bounded below.

The sequence is convergent and by (8), we have

$$\lim_{n\to\infty} \{W(d(c_{n-1}, c_n))\} = \lim_{n\to\infty} \{V(d(c_{n-1}, c_n))\}.$$

By (iv), we infer $\lim_{n\to\infty} d(c_{n-1}, c_n) = 0$.

Now, contrarily, if we let the sequence $\{c_n\}$ not be Cauchy, then by Lemma 1, there are subsequences $\{c_{n_k}\}$, $\{c_{m_k}\}$ of $\{c_n\}$ and $\epsilon > 0$ such that the Equations (1) and (2) hold. By (1), we get that $d(c_{n_k+1}, c_{m_k+1}) > \epsilon$. Since $c_n \perp c_{n+1}$ ($\forall n \geq 0$), by transitivity of $\perp$, we have $c_{n_k} \perp c_{m_k}$ and hence $\nu(c_{n_k}, c_{m_k}) > 1$ ($\forall k \geq 1$). Setting $q = c_{n_k}$ and $\ell = c_{m_k}$ in (3), we have

$$V(d(c_{n_k+1}, c_{m_k+1})) \leq V(\nu(c_{n_k}, c_{m_k}) H(\mathcal{S}c_{n_k}, \mathcal{S}c_{m_k})) \leq W(d(c_{n_k}, c_{m_k})), \text{ for any } k \geq 1.$$

For if $a_k = d(c_{n_k+1}, c_{m_k+1})$ and $b_k = d(c_{n_k}, c_{m_k})$, we have

$$V(a_k) \leq W(b_k), \text{ for any } k \geq 1. \tag{9}$$

By (1) and (2), we have $\lim_{k\to\infty} a_k = \epsilon+$ and $\lim_{k\to\infty} b_k = \epsilon$. By (9), we get that

$$\lim \inf_{c\to\epsilon+} V(c) \leq \lim \inf_{k\to\infty} V(a_k) \leq \lim \sup_{k\to\infty} W(b_k) \leq \lim \sup_{c\to\epsilon} W(c). \tag{10}$$

But (10) contradicts (v), thus, $\{c_n\}$ is a Cauchy sequence in $\mathcal{U}$. Since $\mathcal{U}$ is a complete OMS, the sequence $\{c_n\}$ converges to $c^* \in \mathcal{U}$.

We show that c^* is a fixed point of $\mathcal{S}$. There are two possibilities. (P1) If $d(c_{n+1}, \mathcal{S}c^*) = 0$ for a fixed n, then we have

$$d(c^*, \mathcal{S}c^*) \le d(c^*, c_{n+1}) + d(c_{n+1}, \mathcal{S}c^*) = d(c^*, c_{n+1}).$$

Taking the limit $n \to \infty$, we get $d(c^*, \mathcal{S}c^*) \le 0$. Thus, $d(c^*, \mathcal{S}(c^*)) = 0$. Since $\mathcal{S}(c^*)$ is closed, $c^* \in \mathcal{S}(c^*)$. (P2) If $d(c_{n+1}, \mathcal{S}c^*) > 0$ for all $n \ge 0$, then the $\perp$-regularity of the space $\mathcal{U}$ implies $c_n \perp c^*$ or $c^* \perp c_n$ and, thus, $\nu(c_n, c^*) > 1$. By the contractive condition (3), for all $n \ge 0$, we have

$$V(d(c_{n+1}, \mathcal{S}c^*)) \le V(\nu(c_n, c^*)H(\mathcal{S}c_n, \mathcal{S}c^*)) \le W(d(c_n, c^*)). \tag{11}$$

Set $a_n = d(c_{n+1}, \mathcal{S}c^*)$ and $b_n = d(c_n, c^*)$. Then, by (11), we have

$$V(a_n) \le W(b_n) \text{ for all } n \ge 0. \tag{12}$$

Suppose that $\epsilon = d(c^*, \mathcal{S}c^*)$ such that $\lim_{n\to\infty} a_n = \epsilon$ and $\lim_{n\to\infty} b_n = 0$. By (12), we have

$$\liminf_{c\to\epsilon} V(c) \le \liminf_{n\to\infty} V(a_n) \le \limsup_{n\to\infty} W(b_n) \le \liminf_{c\to 0} W(c). \tag{13}$$

If $\epsilon > 0$, then (13) contradicts (vi). Thus, we have $d(c^*, \mathcal{S}c^*) = 0$. Hence $c^* \in \mathcal{S}c^*$, that is, c^* is a fixed point of $\mathcal{S}$. $\square$

Remark 3. *If we replace $d(c, \ell)$ with $E(c, \ell)$ in the contractive condition (3), then according to Ćirić [29], Theorems 1 and 2 remain true.*

Uniqueness of the fixed point: The following three conditions are essential for the uniqueness of a fixed point of a multivalued mapping.

(U_1). For any multivalued mapping $\mathcal{M} : \mathcal{Q} \to P(\mathcal{Q})$, the set of fixed points of $\mathcal{M}$ ($F(\mathcal{M})$) is totally orthogonal (for any $w, e \in F(\mathcal{M})$ either $w \perp e$ or $e \perp w$).

(U_2). Let

$$\mathcal{Y}_{\mathcal{M}(\ell)}(q) = \{t \in \mathcal{M}(\ell) | d(q, t) = d(\mathcal{M}(\ell), q)\} \text{ for all } q \in \mathcal{Q}.$$

For any $\ell \in \mathcal{Q}$, $\exists\, \theta \in \mathcal{Y}_{\mathcal{M}(\ell)}(q)$ such that $\ell \perp \theta$.

(U_3). For all $i, b, \tau \in \mathcal{Q}$, $d(\tau, i) \ge d(b, i)$, whenever $i \perp b \perp \tau$.

Theorem 3. *Assume that, in addition to conditions stated in Theorem 1 (or Theorem 2), the conditions $(U_1) - (U_3)$ hold. Then, the mapping $\mathcal{M} : \mathcal{Q} \to P_{cb}(\mathcal{Q})$ admits a unique fixed point in $\mathcal{Q}$.*

Proof. Clearly the mapping $\mathcal{M}$ admits at least one fixed point in $\mathcal{Q}$ (by Theorem 1 (or Theorem 2)). Let w and e be two fixed points of $\mathcal{M}$, so that, $w \in \mathcal{M}(w)$ and $e \in \mathcal{M}(e)$. By (U_1), for any $w, e \in (F(\mathcal{M})$, either $w \perp e$ or $e \perp w$. In view of (U_2), $\exists g \in \mathcal{Y}_{\mathcal{M}(\ell)}(q)$ satisfying $\ell \perp g$ and $d(q, g) = d(\mathcal{M}(\ell), q)$. By (U_3), $q \perp \ell \perp g$, implies that $d(g, q) \ge d(\ell, q)$. Since $\ell \in \mathcal{M}(\ell)$ so that $d(g, q) \le d(\ell, q)$, hence, $d(\mathcal{M}(\ell), q) = d(q, g) = d(\ell, q)$. Now if $\ell \ne q$, then $d(\ell, q) > 0$. Moreover,

$$d(\ell, q) = d(\mathcal{M}(\ell), q) \le H(\mathcal{M}(\ell), \mathcal{M}(q)) < \nu(\ell, q)H(\mathcal{M}(\ell), \mathcal{M}(q)).$$

By (ii) stated in Theorem 1 and (3), we deduce that

$$V(d(\ell, q)) < V(\nu(\ell, q)H(\mathcal{M}(\ell), \mathcal{M}(q))) \le W(d(\ell, q)) < V(d(\ell, q)).$$

As V is an increasing mapping, we have $d(\ell, q) < d(\ell, q)$, a contradiction, thus, $\ell = q$. Hence, the multivalued mapping $\mathcal{M}$ has a unique fixed point. $\square$

Examples for the Explanation of Theory

Example 3. *Consider $X = \{0\} \cup]3,7]$ endowed with usual-metric*

$$d(q, \ell) = |q - \ell| \text{ for all } q, \ell \in X.$$

Define the relation $\perp \subset X^2$ by

$$a \perp b \text{ if and only if } a \wedge b = 0 \Rightarrow a \vee b \in]5,7].$$

Then, $\perp$ is an orthogonal relation and $(X, d, \perp)$ is a complete orthogonal metric space. Define $\mathcal{S} : X \to CB(X)$ by,

$$\mathcal{S}(q) = \begin{cases} \{5,7\}, & q \in]3,7], \\ \{4,6\}, & q = 0. \end{cases}$$

Let $A = \{5,7\}$ and $B = \{4,6\}$. Since

$$H(A, B) = \max\{d(A, B), d(B, A)\}, \tag{14}$$

$$\begin{aligned} d(A, B) &= \sup\{d(q, B) : q \in A\} = \inf\{d(q, \ell) : \ell \in B\} \\ d(B, A) &= \sup\{d(\ell, A) : \ell \in B\}. \end{aligned}$$

Consider,

$$\{d(q, B) : q \in A\} = \{d(5, B), d(7, B)\}, \tag{15}$$

where

$$\begin{aligned} d(5, B) &= \inf\{d(5,4), d(5,6)\} = \inf\{1,1\} = 1. \\ d(7, B) &= \inf\{d(7,4), d(7,6)\} = \inf\{3,1\} = 1. \end{aligned}$$

Thus, by (15), we get

$$d(A, B) = \sup\{d(q, B) : q \in A\} = \sup\{1,1\} = 1. \tag{16}$$

Consider

$$\{d(\ell, A) : \ell \in B\} = \{d(4, A), d(6, A)\}. \tag{17}$$

$$d(4, A) = \inf\{d(4,5), d(4,7)\} = \inf\{1,3\} = 1.$$

$$d(6, A) = \inf\{d(6,5), d(6,7)\} = \inf\{1,1\} = 1.$$

Thus, by (17), we get

$$d(B, A) = \sup\{d(\ell, A) : \ell \in B\} = \sup\{1,1\} = 1. \tag{18}$$

By virtue of Equations (16) and (18), Equation (14) implies that $H(A, B) = 1 > 0$. Define $\nu : X^2 \to [1, \infty)$ by

$$\nu(a, b) = \begin{cases} 2 & \text{if } a \perp b, \\ 1 & \text{otherwise}. \end{cases}$$

Then, ν is $\perp$-admissible.

Case 1: *If $\ell = 0$ and $q \in]5,7]$, then, $\ell \perp q$ and*

$$\begin{aligned} \frac{1}{294} - \frac{1}{2H(\mathcal{S}q,\mathcal{S}\ell)+1} &\leq \frac{49}{294} - \frac{1}{3} \\ = -\frac{1}{6} &< -\frac{1}{d(q,\ell)+1}. \end{aligned} \tag{19}$$

Case 2: *If $q = 0$ and $\ell \in]5,7]$, then, $\ell \perp q$ and*

$$\begin{aligned} \frac{1}{294} - \frac{1}{2H(\mathcal{S}q,\mathcal{S}\ell)+1} &\leq \frac{49}{294} - \frac{1}{3} \\ &= -\frac{1}{6} < -\frac{1}{d(q,\ell)+1}. \end{aligned} \tag{20}$$

By (19) and (20), we deduce that

$$\frac{1}{294} - \frac{1}{2H(\mathcal{S}q,\mathcal{S}\ell)+1} < -\frac{1}{d(q,\ell)+1},$$

for all $q, \ell \in X$ with $q \perp \ell$. Thus, by defining $V(t) = -\frac{1}{t+1}$ and $W(t) = V(t) - \tau$ for all $t \in (0,\infty)$ and $\tau = \frac{1}{294}$, we see that V and W satisfy conditions (ii) and (iii) of Theorem 1 and $\mathcal{S}$ is a multivalued $(V,W)_\perp$-contraction:

$$V(\nu(q,\ell)H(\mathcal{S}(q),\mathcal{S}(\ell))) \leq W(d(q,\ell)).$$

Here, we note that the fixed point of $\mathcal{S}$ is 7, because $7 \in \mathcal{S}(7)$.

Example 4. *Consider $X =]9,21]$ endowed with the usual metric:*

$$d(q,\ell) = |q-\ell| \text{ for all } q,\ell \in X.$$

Define the relation $\perp \subset X^2$ by

$$a \perp b \text{ if and only if } a \wedge b = 10 \Rightarrow a \vee b \in]17,21].$$

Then, $\perp$ is an orthogonal relation and $(X,d,\perp)$ is a complete orthogonal metric space. Define the mapping $\mathcal{S} : X \to CB(X)$ by,

$$\mathcal{S}(q) = \begin{cases} \{20,21\}, & q \in]9,21], \\ \{18,19\}, & q = 10. \end{cases}$$

Let $A = \{20,21\}$ and $B = \{18,19\}$. We know that

$$\begin{aligned} H(A,B) &= \max\{d(A,B), d(B,A)\};\ d(A,B) = \sup\{d(q,B) : q \in A\} \text{ and} \\ d(B,A) &= \sup\{d(\ell,A) : \ell \in B\}. \end{aligned} \tag{21}$$

Consider,

$$\{d(q,B) : q \in A\} = \{d(20,B), d(21,B)\}. \tag{22}$$

$$\begin{aligned} d(q,B) &= \inf\{d(q,\ell) : \ell \in B\} \\ d(20,B) &= \inf\{d(20,18), d(20,19)\} = \inf\{2,1\} = 1. \\ d(21,B) &= \inf\{d(21,18), d(21,19)\} = \inf\{3,2\} = 2. \end{aligned}$$

Thus, by (22), we get

$$d(A,B) = \sup\{d(q,B) : q \in A\} = \sup\{1,2\} = 2. \tag{23}$$

Similarly,

$$\{d(\ell, A) : y \in B\} = \{d(18, A), d(19, A)\}. \tag{24}$$

$d(18, A) = \inf\{d(18, 20), d(18, 21)\} = \inf\{2, 3\} = 2$, *and* $d(19, A) = \inf\{d(19, 20), d(19, 21)\} = \inf\{1, 2\} = 1$. *Thus, by (24), we get*

$$d(B, A) = \sup\{d(\ell, A) : \ell \in B\} = \sup\{2, 1\} = 2. \tag{25}$$

By Equations (23) and (25), the Equation (21) implies that $H(A, B) = 2 > 0$. *Define* $\nu : X^2 \to [1, \infty)$ *by*

$$\nu(a, b) = \begin{cases} 2 & \text{if } a \perp b, \\ 1 & \text{otherwise}. \end{cases}$$

Then, ν *is* $\perp$*-admissible.*

Case 1*: If* $q = 10$ *and* $\ell \in]17, 21]$*, then,* $q \perp \ell$ *and*

$$\begin{aligned} 1 + d(10, \ell) &\geq 8 \geq \frac{3}{2}(1 + 2H(\mathcal{S}10, \mathcal{S}\ell)) \\ \frac{3}{2}(1 + 2H(\mathcal{S}10, \mathcal{S}\ell)) &\leq 1 + d(10, \ell) \\ \ln(\frac{3}{2}(1 + 2H(\mathcal{S}10, \mathcal{S}\ell))) &\leq \ln(1 + d(10, \ell)) \\ \ln(\frac{3}{2}) + \ln(1 + 2H(\mathcal{S}10, \mathcal{S}\ell)) &\leq \ln(1 + d(10, \ell)). \end{aligned}$$

Case 2*: If* $\ell = 10$ *and* $q \in]17, 21]$*, then,* $q \perp \ell$ *and*

$$\begin{aligned} 1 + d(q, 10) &\geq 8 \geq \frac{3}{2}(1 + 2H(\mathcal{S}q, \mathcal{S}10)) \\ \frac{3}{2}(1 + 2H(\mathcal{S}q, \mathcal{S}10)) &\leq 1 + d(q, 10) \\ \ln(\frac{3}{2}(1 + 2H(\mathcal{S}q, \mathcal{S}10))) &\leq \ln(1 + d(q, 10)) \\ \ln(\frac{3}{2}) + \ln(1 + 2H(\mathcal{S}q, \mathcal{S}10)) &\leq \ln(1 + d(q, 10)). \end{aligned}$$

Thus, for all $q, \ell \in X$ *with* $q \perp \ell$*, Thus, by defining* $V(t) = \ln(t + 1)$ *and* $W(t) = V(t) - \tau$ *for all* $t \in (0, \infty)$ *and* $\tau = \ln \frac{3}{2}$*, we see that* V *and* W *satisfy conditions (ii)–(vi) of Theorem 2 and* $\mathcal{S}$ *is a multivalued* $(V, W)_\perp$*-contraction:*

$$V(\nu(q, \ell)H(\mathcal{S}(q), \mathcal{S}(\ell))) \leq W(d(q, \ell)).$$

We note that the fixed point of $\mathcal{S}$ *is* 20*, because* $20 \in \mathcal{S}(20)$*.*

3. Consequences

It is noted that the Nadler fixed point theorem [30] is a particular case of Theorems 1 and 2 (let $V(c) = c$ and $W(c) = \lambda c$ for all $c > 0$ and $\lambda \in [0, 1)$). The multivalued version of the Wordowski Theorem can be derived by defining $V(c) = c$ for all $c > 0$ in Theorem 1. If we define $W(c) = V(c) - t$ $(t > 0)$ in Theorems 1 and 2, then we have an improvement of the results presented in [8,12,27,31] as follows:

Corollary 1. *Let* $(\mathcal{U}, \perp, d)$ *be a* $\perp$*RCOMS. Suppose that* $\mathcal{S} : \mathcal{U} \to P_{cb}(\mathcal{U})$ *is* $\perp$*-preserving and there exists a* $\perp$*-admissible function* ν *and* $t > 0$ *such that*

$$H(\mathcal{S}x, \mathcal{S}y) > 0 \text{ implies } t + V(\nu(x, y)H(\mathcal{S}x, \mathcal{S}y)) \leq V(d(x, y))) \text{ for all } x, y \in \mathcal{U}.$$

If there exists $c_1 \in \mathcal{S}(c_0)$ such that $c_0 \perp c_1$ or $c_1 \perp c_0$, $c_0 \in \mathcal{U}$ and $V : (0, \infty) \to \mathbb{R}$ is nondecreasing, then $\mathcal{S}$ admits a fixed point in $\mathcal{U}$.

Defining $W(c) = V(c) - \tau(c)$ for all $c \in (0, \infty)$ in Theorems 1 and 2, we have the following improvement of the result presented in [10].

Corollary 2. *Let $(\mathcal{U}, \perp, d)$ be a $\perp$RCOMS. Suppose that $\mathcal{S} : \mathcal{U} \to P_{cb}(\mathcal{U})$ is $\perp$-preserving and there exists a $\perp$-admissible function ν such that*

$$H(\mathcal{S}x, \mathcal{S}y) > 0 \text{ implies } \tau(d(x, y)) + V(\nu(x, y)H(\mathcal{S}x, \mathcal{S}y)) \leq V(d(x, y))) \text{ for all } x, y \in \mathcal{U},$$

$$\lim \inf_{c \to t+} \tau(c) > 0, \ \forall\, t \geq 0.$$

If there exists $c_1 \in \mathcal{S}(c_0)$ such that $c_0 \perp c_1$ or $c_1 \perp c_0$, $c_0 \in \mathcal{U}$ and $V : (0, \infty) \to \mathbb{R}$ is nondecreasing, then $\mathcal{S}$ admits a fixed point in $\mathcal{U}$.

Defining $W(c) = g(V(c))$ for all $c \in (0, \infty)$ in Theorem 1, we have the following improvement of Moradi's theorem [21].

Corollary 3. *Let $(\mathcal{U}, \perp, d)$ be a $\perp$RCOMS. Let $B \subset \mathbb{R}$ and $g : B \to [0, \infty)$ is an upper semi-continuous function satisfying $g(z) < z$ for all $z \in B$. Suppose that $\mathcal{S} : \mathcal{U} \to P_{cb}(\mathcal{U})$ is $\perp$-preserving and there exists a $\perp$-admissible function ν such that*

$$H(\mathcal{S}x, \mathcal{S}y) > 0 \text{ implies } V(\nu(x, y)H(\mathcal{S}x, \mathcal{S}y)) \leq g(V(d(x, y))) \text{ for all } x, y \in \mathcal{U}.$$

If there exists $c_1 \in \mathcal{S}(c_0)$ such that $c_0 \perp c_1$ or $c_1 \perp c_0$, $c_0 \in \mathcal{U}$ and $V : (0, \infty) \to B$ is nondecreasing, then $\mathcal{S}$ admits a fixed point in $\mathcal{U}$.

Defining $g(z) = z^{\omega}$ ($\omega \in (0, 1)$) in Corollary 3, we have the following conclusion.

Corollary 4. *Let $(\mathcal{U}, \perp, d)$ be a $\perp$RCOMS. Suppose that $\mathcal{S} : \mathcal{U} \to P_{cb}(\mathcal{U})$ is $\perp$-preserving and there exists a $\perp$-admissible function ν such that*

$$H(\mathcal{S}x, \mathcal{S}y) > 0 \text{ implies } V(\nu(x, y)H(\mathcal{S}x, \mathcal{S}y)) \leq (V(d(x, y)))^{\omega} \text{ for all } x, y \in \mathcal{U}$$

and $V : (0, \infty) \to (0, 1)$ is nondecreasing. If there exists $c_1 \in \mathcal{S}(c_0)$ such that $c_0 \perp c_1$ or $c_1 \perp c_0$, $c_0 \in \mathcal{U}$, then $\mathcal{S}$ admits a fixed point in $\mathcal{U}$.

Remark 4. *Corollary 4 shows the improvements of fixed point results presented in [20,32].*

If we define $W(y) = \lambda V(y)$ in Theorems 1 and 2, then we have the following improvement of the special case of Skof's fixed point theorem [26].

Corollary 5. *Let $(\mathcal{U}, \perp, d)$ be a $\perp$RCOMS. Suppose that $\mathcal{S} : \mathcal{U} \to P_{cb}(\mathcal{U})$ is $\perp$-preserving and there exists a $\perp$-admissible function ν such that*

$$H(\mathcal{S}x, \mathcal{S}y) > 0 \text{ implies } V(\nu(x, y)H(\mathcal{S}x, \mathcal{S}y)) \leq \lambda V(d(x, y)) \text{ for all } x, y \in \mathcal{U},$$

and V is a nondecreasing function that maps positive real numbers to positive real numbers and $\lambda \in (0, 1)$. If there exists $c_1 \in \mathcal{S}(c_0)$ such that $c_0 \perp c_1$ or $c_1 \perp c_0$, $c_0 \in \mathcal{U}$, then $\mathcal{S}$ has a unique fixed point in $\mathcal{U}$.

On the other hand, if V is a nondecreasing function that maps positive real numbers to positive real numbers and $\chi : (0, \infty) \to (0, 1)$ meets the condition $\lim \sup_{z \to \epsilon+} \chi(z) < 1$ for any $\epsilon > 0$, and $W(z) = \chi(z)V(z)$ and $V(z) = z$ for all $z > 0$ in Theorem 1, then we have an improvement of a theorem in [25].

Remark 5. *(R1). The $\perp$-admissibility of the mapping $\mathcal{S}$ can be dropped from all of the aforementioned results by replacing $P_{cb}(\mathcal{U})$ with $K(\mathcal{U})$ and the Lemma 2 is no more required.*

(R2). The condition:

$$V(\inf A) = \inf V(A) \text{ for all } A \subseteq (0,\infty) \text{ with } \inf A > 0,$$

can be used as an alternative of the $\perp$-admissibility of $\mathcal{S} : \mathcal{U} \to P_{cb}(\mathcal{U})$ in the above theorems.

The following theorem is a particular case of Theorem 1 and is useful for the upcoming result.

Theorem 4. *Let $\mathcal{S}$ be a $\perp$-preserving self-mapping defined on $\perp$RCOMS such that*

$$t + V(\nu(c,\ell)d(\mathcal{S}(c),\mathcal{S}(\ell))) \leq V(E(c,\ell)) \tag{26}$$

for all $c,\ell \in \Lambda$ with $d(\mathcal{S}(c),\mathcal{S}(\ell)) > 0$ and $V : (0,\infty) \to \mathbb{R}$ is nondecreasing and $t > 0$. If there exists $c_1 = \mathcal{S}(c_0)$ such that $c_0 \perp c_1$ or $c_1 \perp c_0$; $c_0 \in \mathcal{U}$, then $\mathcal{S}$ admits a fixed point.

Proof. Setting $W(y) = V(y) - t$ for all $y > 0$ in Theorem 1, we have the required result. □

Remark 6. *Define $E(c,\ell)$ as any one of the following. Then, Theorem 4 is applicable.*
(1) $\max\{d(c,\ell), d(c,\mathcal{S}(c)), d(\ell,\mathcal{S}(\ell))\}$.
(2) $\max\{d(c,\mathcal{S}(c)), d(\ell,\mathcal{S}(\ell))\}$.
(3) $\max\left\{d(c,\ell), \frac{d(c,\mathcal{S}(c))+d(\ell,\mathcal{S}(\ell))}{2}, \frac{d(\ell,\mathcal{S}(c))+d(c,\mathcal{S}(\ell))}{2}\right\}$.
(4) $a_1 d(c,\ell) + a_2(d(c,\mathcal{S}(c)) + d(\ell,\mathcal{S}(\ell))) + a_3(d(\ell,\mathcal{S}(c)) + d(c,\mathcal{S}(\ell)))$, $\sum_{i=1}^{3} a_i < 1$.
(5) $a_1 d(c,\ell) + a2d(c,\mathcal{S}(c)) + a_3 d(\ell,\mathcal{S}(\ell))$, $\sum_{i=1}^{3} a_i < 1$. *(6)* $d(c,\ell)$.

4. Application of Theorem 4 to FDE

Lacroix (1819) proposed and investigated several fractional differential properties. A number of new Caputo-Fabrizio derivative (CFD) models have recently been discovered and studied by authors in [33–35]. In this section, we will look at one of these models. The following notations are required.

Let $\mathcal{X} =: \mathcal{C}[0,1] = \{f | \nu : [0,1] \to (-\infty,\infty) \text{ and } f \text{ is continuous}\}$. The function $d : \mathcal{X} \times \mathcal{X} \to [0,\infty)$, defined by

$$d(f,g) = \|f-g\|_\infty = \max_{x\in[0,1]} |f(x) - g(x)|, \text{ for all } f,g \in \mathcal{C}[0,1],$$

is a metric on $\mathcal{X}$ and $(\mathcal{X},d)$ is a complete metric space. Define an orthogonal relation $\perp$ on $\mathcal{X}$ by

$$c \perp v \text{ iff } cv \geq 0, \text{ for all } c,v \in \mathcal{X}.$$

Then, $(\mathcal{X},\perp,d)$ is a complete OMS. Let $\nu : \mathcal{X} \times \mathcal{X} \to (1,\infty)$ be defined by

$$\nu(r,t) = e^{\|r+t\|_\infty} \text{ for all } r,t \in \mathcal{X} \text{ with } r \perp t.$$

Then, ν is a strictly $\perp$-admissible mapping. Let $K_1 : [0,1] \times (-\infty,\infty) \to (-\infty,\infty)$ be any mapping. We will apply Theorem 4 to show the existence of the solution to the following FDE:

$$^{CF}D^{\jmath} g(x) = K_1(x, g(x)); g \in \mathcal{X} \tag{27}$$
$$g(0) = 0,\ g'(1) = g'(0).$$

Here, $^{CF}D_0^{\jmath}$ denotes the Caputo-Fabrizio derivative (CFD) of order $\jmath$ defined by

$$^{CF}D_0^{\jmath} g(x) = \frac{1}{\Gamma(\zeta - \jmath)} \int_0^x (x-\theta)^{\zeta-\jmath-1} g(\theta))\, d\theta,$$

where

$$\zeta - 1 < \jmath < \zeta \text{ and } \zeta = [\jmath] + 1.$$

The integral operator is defined by

$$I^{\jmath} g(x) = \frac{1}{\Gamma(\jmath)} \int_0^x (x-\theta)^{\jmath-1} g(\theta)\, d\theta, \text{ with } \jmath > 0.$$

One of the transformations of (27) is as follow:

$$g(x) = \frac{1}{\Gamma(\jmath)} \int_0^x (x-\theta)^{\jmath-1} K_1(\theta, g(\theta))\, d\theta + \frac{2x}{\Gamma(\jmath)} \int_0^1 \int_0^\theta (\theta-u)^{\jmath-1} K_1(u, g(u))\, du d\theta.$$

Let

(I) $\exists\, \alpha > 0$ such that

$$|K_1(\theta, g(\theta)) - K_1(\theta, \ell(\theta))| \leq \frac{e^{-\alpha}\Gamma(\jmath+1)}{4M} |g(\theta) - \ell(\theta)| (M = \min\{d(g,\ell)| \quad g, \ell \in \mathcal{X}\}),$$

(II) for an arbitrary $g_0 \in \mathcal{X}$, we have

$$g_0(x) \leq \frac{1}{\Gamma(\jmath)} \int_0^x (x-\theta)^{\jmath-1} K_1(\theta, g_0(\theta))\, d\theta + \frac{2x}{\Gamma(\jmath)} \int_0^1 \int_0^\theta (\theta-u)^{\jmath-1} K_1(u, g_0(u))\, du d\theta.$$

Theorem 5. *If the conditions (I)–(II) stated above are satisfied, then the Equation (27) admits a solution in $\mathcal{X}$.*

Proof. Define the operator $\mathcal{S} : \mathcal{X} \to \mathcal{X}$, in line with the above information, by

$$\mathcal{S}(g)(x) = \frac{1}{\Gamma(\jmath)} \int_0^x (x-\theta)^{\jmath-1} K_1(\theta, g(\theta))\, d\theta + \frac{2x}{\Gamma(\jmath)} \int_0^1 \int_0^\theta (\theta-u)^{\jmath-1} K_1(u, g(u))\, du d\theta.$$

We note that whenever $g(x) \perp g(y)$ or $g(y) \perp g(x)$, $\mathcal{S}(g)(x) \perp \mathcal{S}(g)(y)$. By (II), there is an arbitrary function $g_0 \in \mathcal{X}$ such that $g_n = \mathcal{S}^n(g_0))$ with $g_n \perp g_{n+1}$ or $g_{n+1} \perp g_n$ $(\forall n \geq 0)$. We establish (26) of Theorem 4 in the next lines.

$$|\mathcal{S}(g)(x) - \mathcal{S}(\ell)(x)| = \left| \begin{array}{l} \frac{1}{\Gamma(\jmath)} \int_0^x (x-\theta)^{\jmath-1} K_1(\theta, g(\theta))\, d\theta \\ -\frac{1}{\Gamma(\jmath)} \int_0^x (x-\theta)^{\jmath-1} K_1(\theta, \ell(\theta))\, d\theta \\ +\frac{2x}{\Gamma(\jmath)} \int_0^1 \int_0^\theta (\theta-u)^{\jmath-1} K_1(u, g(u))\, du d\theta \\ -\frac{2x}{\Gamma(\jmath)} \int_0^1 \int_0^\theta (\theta-u)^{\jmath-1} K_1(u, \ell(u))\, du d\theta \end{array} \right| \text{ implies}$$

$$
\begin{aligned}
& |\mathcal{S}(g)(x) - \mathcal{S}(\ell)(x)| \\
\leq\ & \left| \int_0^x \left(\frac{1}{\Gamma(\jmath)}(x-\theta)^{\jmath-1} K_1(\theta, g(\theta)) - \frac{1}{\Gamma(\jmath)}(x-\theta)^{\jmath-1} K_1(\theta, \ell(\theta)) \right) d\theta \right| \\
+\ & \left| \int_0^1 \int_0^\theta \left(\frac{2}{\Gamma(\jmath)}(\theta-u)^{\jmath-1} K_1(\theta, g(\theta)) - \frac{2}{\Gamma(\jmath)}(\theta-u)^{\jmath-1} K_1(u, \ell(u)) \right) du d\theta \right| \\
\leq\ & \frac{1}{\Gamma(\jmath)} \frac{e^{-\alpha}\Gamma(\jmath+1)}{4M} \cdot \int_0^x (x-\theta)^{\jmath-1}(g(\theta) - \ell(\theta))\, d\theta \\
+\ & \frac{2}{\Gamma(\jmath)} \frac{e^{-\alpha}\Gamma(\jmath+1)}{4M} \cdot \int_0^1 \int_0^\theta (\theta-u)^{\jmath-1}(\ell(u) - g(u))\, du d\theta \\
\leq\ & \frac{1}{\Gamma(\jmath)} \frac{e^{-\alpha}\Gamma(\jmath+1)}{4M} \cdot d(g,\ell) \cdot \int_0^x (x-\theta)^{\jmath-1}\, d\theta \\
+\ & \frac{2}{\Gamma(\jmath)} \frac{e^{-\alpha}\Gamma(\jmath) \cdot \Gamma(\jmath+1)}{4M\Gamma(s) \cdot \Gamma(\jmath+1)} \cdot d(g,\ell) \cdot \int_0^1 \int_0^\theta (\theta-u)^{\jmath-1}\, du d\theta \\
\leq\ & \left(\frac{e^{-\alpha}\Gamma(\jmath) \cdot \Gamma(\jmath+1)}{4M\Gamma(\jmath) \cdot \Gamma(\jmath+1)} \right) \cdot d(g,\ell) + 2e^{-\alpha} B(\jmath+1, 1) \frac{\Gamma(\jmath) \cdot \Gamma(\jmath+1)}{4M\Gamma(\jmath) \cdot \Gamma(\jmath+1)} \cdot d(g,\ell) \\
\leq\ & \frac{e^{-\alpha}}{4M} d(g,\ell) + \frac{e^{-\alpha}}{2M} d(g,\ell) < \frac{e^{-\alpha}}{M} d(g,\ell),
\end{aligned}
$$

The simplified form is given by

$$
Md(\mathcal{S}(g), \mathcal{S}(\ell)) \leq d(g,\ell) d(\mathcal{S}(g), \mathcal{S}(\ell)) \leq e^{-\alpha} d(g,\ell). \tag{28}
$$

Define the mapping $V(g(x)) = \ln(g(x))$ for all $g, \ell \in \mathcal{X}$. Then, the inequality (28) can be re-written as

$$
\alpha + V(d(g,\ell) d(\mathcal{S}(g), \mathcal{S}(\ell))) \leq V(d(g,\ell)).
$$

By Theorem 4, (27) admits a solution in. □

5. Conclusions

The multivalued contractions introduced in this paper encapsulate so many contractions, including Nadler, F, Boyd-Wong and Geraghty contractions. The results stated in this paper generalize and improve a number of results on the existence of fixed-points of the abovementioned contractions. The orthogonal relation is the useful generalization of binary relation. Fixed point methodology is used to investigate the presence of a solution to a fractional differential equation. Based on the recently developed concept of orthogonality in the metric spaces, we introduce orthogonal multivalued contractions in this study. For these contractions, we derive several fixed point theorems. We demonstrate how these fixed point theorems help to generalize several newly released findings. In addition, we provide a theorem that proves the existence of the solution to a fractional differential equation (FDE).

6. Future Work

The interested readers are suggested to try these results in vector-valued metric spaces or generalized metric spaces.

Author Contributions: Conceptualization, M.N.; methodology, M.N.; software, M.N.; validation, M.N., S.C., H.H.A.S. and A.H.; formal analysis, M.N.; investigation, M.N. and S.C.; resources, H.H.A.S.; writing—original draft preparation, M.N.; writing—review and editing, M.N., S.C. and A.H.; visualization, M.N.; supervision, M.N. and H.H.A.S.; project administration, A.H. and H.H.A.S.; funding acquisition, H.H.A.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no competing interest.

References

1. Boyd, D.W.; Wong, J.S.W. On nonlinear contractions. *Proc. Am. Math. Soc.* **1969**, *20*, 458–464. [CrossRef]
2. Rakotch, E. A note on contractive mappings. *Proc. Am. Math. Soc.* **1962**, *13*, 459–465. [CrossRef]
3. Matkowski, J. Integrable solutions of functional equations. *Diss. Math.* **1975**, *127*, 1–68.
4. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for (α, ψ)-contractive type mappings. *Nonlinear Anal.* **2012**, *75*, 2154–2165. [CrossRef]
5. Karapinar, E.; Samet, B. Generalized (α, Ψ)-contractive type mappings and related fixed point theorems with applications. *Abstr. Appl. Anal.* **2012**, *2012*, 793486. [CrossRef]
6. Pasick, L. The Boyd-Wong idea extended. *Fixed Point Theory Appl.* **2016**, *2016*, 63. [CrossRef]
7. Wardowski, D. Fixed point theory of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 94. [CrossRef]
8. Baghani, H.; Gordji, M.E.; Ramezani, M. Orthogonal sets: The axiom of choice and proof of a fixed point theorem. *J. Fixed Point Theory Appl.* **2016**, *18*, 465–477. [CrossRef]
9. Beg, I.; Mani, G.; Gnanaprakasam, A.J. Fixed point of orthogonal F-Suzuki contraction mapping on O-complete metric spaces with applications. *J. Funct. Spaces* **2021**, *2021*, 6692112. [CrossRef]
10. Chandok, S.; Sharma, R.K.; Radenović, S. Multivalued problems vian orthogonal contraction mappings with application to fractional differential equation. *J. Fixed Point Theory Appl.* **2021**, *23*, 14. [CrossRef]
11. Chandok, S.; Radenović, S. Existence of solution for orthogonal F-contraction mappings via Picard–Jungck sequences. *J. Anal.* **2022**, *30*, 677–690. [CrossRef]
12. Secelean, N.A. Weak F-contractions and some fixed point results. *Bull. Iranian Math. Soc.* **2016**, *42*, 779–798.
13. Sharma, R.K.; Chandok, S. Multivalued problems, orthogonal mappings, and fractional integro-differential equation. *J. Math.* **2020**, *2020*, 6615478. [CrossRef]
14. Turab, A.; Ali, W.; Park, C. A unified fixed point approach to study the existence and uniqueness of solutions to the generalized stochastic functional equation emerging in the psychological theory of learning. *Aims Math.* **2022**, *7*, 5291–5304. [CrossRef]
15. Khan, Z.A.; Ahmad, I.; Shah, K. Applications of Fixed Point Theory to Investigate a System of Fractional Order Differential Equations. *J. Funct. Spaces* **2021**, *2021*, 1399764. [CrossRef]
16. Rezapour, S.; Deressa, C.T.; Hussain, A.; Etemad, S.; George, R.; Ahmad, B. A Theoretical Analysis of a Fractional Multi-Dimensional System of Boundary Value Problems on the Methylpropane Graph via Fixed Point Technique. *Mathematics* **2022**, *10*, 568. [CrossRef]
17. Gordji, M.E.; Rameani, M.; Sen, M.D.L.; Cho, Y. On orthogonal sets and Banach fixed point theorem. *Fixed Point Theory* **2017**, *19*, 569–578. [CrossRef]
18. Nazam, M.; Park, C.; Arshad, M. Fixed point problems for generalized contractions with applications. *Adv. Differ. Equ.* **2021**, *2021*, 247. [CrossRef]
19. Proinov, P.D. Fixed point theorems for generalized contractive mappings in metric spaces. *J. Fixed Point Theory Appl.* **2020**, *22*, 21. [CrossRef]
20. Jleli, M.; Samet, B. A new generalization of the Banach contraction principle. *J. Inequal. Appl.* **2014**, *2014*, 38. [CrossRef]
21. Moradi, S. Fixed point of single-valued cyclic weak φF-contraction mappings. *Filomat* **2014**, *28*, 1747–1752. [CrossRef]
22. Nazam, M.; Aydi, H.; Hussain, A. Existence theorems for (Φ, Ψ)-orthogonal interpolative contractions and an application to fractional differential equations. *Optimization* **2022**, *72*, 1–31. [CrossRef]
23. Nadler, S.B. Multivalued contraction mappings. *Pacific J. Math.* **1969**, *30*, 475–488. [CrossRef]
24. Ran, A.C.M.; Reurings, M.C.B. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc.* **2003**, *132*, 1435–1443. [CrossRef]
25. Geraghty, M.A. On contractive mappings. *Proc. Am. Math. Soc.* **1973**, *40*, 604–608. [CrossRef]
26. Skof, F. Theoremi di punto fisso per applicazioni negli spazi metrici. *Atti. Acad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **1977**, *111*, 323–329.
27. Acar, O.; Altun, I. Multivalued F-contractive mappings with a graph and some fixed point results. *Publ. Math. Debr.* **2016**, *88*, 305–317. [CrossRef]
28. Afshari, H.; Aydi, H.; Karapinar, E. Some fixed point results for multivalued mappings in b-metric spaces. *East Asian Math. J.* **2016**, *32*, 319–332. [CrossRef]
29. Ćirić, L.B. Generalized contractions and fixed-point theorems. *Publ. Inst. Math.* **1971**, *26*, 19–26.
30. Banach, S. Sur les operations dans les ensembles abstraits et leurs applications aux equations integrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
31. Durmaz, G.; Altun, I. Fixed point results for α-admissible multivalued F-contractions. *Miskolc Math. Notes* **2016**, *17*, 187–199. [CrossRef]
32. Ahmad, J.; Al-Mazrooei, A.E.; Cho, Y.; Yang, Y. Fixed point results for generalized Θ-contractions. *J. Nonlinear Sci. Appl.* **2017**, *10*, 2350–2358. [CrossRef]

33. Alizadeh, S.; Baleanu, D.; Rezapour, S. Analyzing transient response of the parallel RCL circuit by using the Caputo-Fabrizio fractional derivative. *Adv. Differ. Equ.* **2020**, *2020*, 55. [CrossRef]
34. Baleanu, D.; Jajarmi, A.; Mohammadi, H.; Rezapour, S. A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative. *Chaos Solitons Fractals* **2020**, *134*, 109705. [CrossRef]
35. Baleanu, D.; Mohammadi, H.; Rezapour, S. Analysis of the model of HIV-1 infection of CD^{4+} T-cell with a new approach of fractional derivative. *Adv. Differ. Equ.* **2020**, *2020*, 71. [CrossRef]