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Quadrature Methods for Singular Integral Equations of Mellin Type Based on the Zeros of Classical Jacobi Polynomials

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Abstract: In this paper we formulate necessary conditions for the stability of certain quadrature methods for Mellin type singular integral equations on an interval. These methods are based on the zeros of classical Jacobi polynomials, not only on the Chebyshev nodes. The method is considered as an element of a special C^*-algebra such that the stability of this method can be reformulated as an invertibility problem of this element. At the end, the mentioned necessary conditions are invertibility properties of certain linear operators in Hilbert spaces. Moreover, for the proofs we need deep results on the zero distribution of the Jacobi polynomials.

Keywords: integral operator of Mellin type; collocation-quadrature method; stability

MSC: 65R20

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1. Introduction

The present paper is part of the efforts done during the last three decades to establish necessary and sufficient conditions for the stability of numerical methods for singular integral equations by using so called C^*-algebra techniques. The integral equations under consideration contain strong singular integral operators of Cauchy and Mellin type. In general, they are of the form

$$\begin{aligned} a(x)u(x) + \frac{b(x)}{\pi \mathbf{i}} \int_{-1}^{1} \frac{u(y)\,dy}{y-x} + c_-(x) \int_{-1}^{1} H_-\left(\frac{1+x}{1+y}\right) \frac{dy}{1+y} \\ + c_+(x) \int_{-1}^{1} H_+\left(\frac{1-x}{1-y}\right) \frac{dy}{1-y} + \int_{-1}^{1} K(x,y)u(y)\,dy = f(x)\,, \quad -1 < x < 1\,, \end{aligned} \tag{1}$$

where the functions $a, b, c_\pm : [-1,1] \longrightarrow \mathbb{C}$, $f : (-1,1) \longrightarrow \mathbb{C}$, $H_\pm : \mathbb{R}^+ \longrightarrow \mathbb{C}$ as well as $K : (-1,1) \times (-1,1) \longrightarrow \mathbb{C}$ are given and $u : (-1,1) \longrightarrow \mathbb{C}$ is looked for. As usual, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Moreover, by $\mathbb{R}^+ = \{t \in \mathbb{R} : t > 0\}$ we refer to the set of positive real numbers. The minimal conditions on the given functions are the piecewise continuity of the coefficient functions a, b, and $c_\pm$ as well as the continuity of the kernel functions $H_\pm(t)$ and $K(x,y)$. Moreover, the right-hand side f should belong to a Hilbert space $\mathbf{L}^2_{\alpha,\beta}$. Equation (1) is considered in this space $\mathbf{L}^2_{\alpha,\beta}$ and written shortly as

$$\mathcal{A}u := \left(a\mathcal{I} + b\mathcal{S} + c_-\mathcal{M}^-_{H_-} + c_+\mathcal{M}^+_{H_+} + \mathcal{K}\right)u = f\,. \tag{2}$$

For real numbers $\alpha, \beta > -1$, the Hilbert space $\mathbf{L}^2_{\alpha,\beta} := \mathbf{L}^2_{\alpha,\beta}(-1,1)$ is defined by the inner product

$$\langle f, g \rangle_{\alpha,\beta} = \int_{-1}^{1} f(x)\overline{g(x)} v^{\alpha,\beta}(x)\,dx\,,$$

where $v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ is a classical Jacobi weight. Hence the norm in $\mathbf{L}^2_{\alpha,\beta}$ is given by $\|f\|_{\alpha,\beta} = \sqrt{\langle f, f\rangle_{\alpha,\beta}}$. We call a function $a : [-1,1] \longrightarrow \mathbb{C}$ piecewise continuous if it is continuous at ± 1, the one-sided limits $a(x \pm 0)$ exist for all $x \in (-1,1)$, and at least one of them coincides with the function value $a(x)$. The set of these piecewise continuous functions is denoted by $\mathbf{PC} := \mathbf{PC}[-1,1]$.

For a continuous function $H : (0,\infty) \longrightarrow \mathbb{C}$, i.e., $H \in \mathbf{C}(\mathbb{R}^+)$, and a continuous function $K : (-1,1) \times (-1,1) \longrightarrow \mathbb{C}$ the Mellin-type operator $\mathcal{M}_H$ and the integral operator $\mathcal{K}$ are defined by

$$(\mathcal{M}_H^{\pm} u)(x) = \int_{-1}^{1} H\left(\frac{1 \mp x}{1 \mp y}\right) \frac{u(y)\,dy}{1 \mp y}, \quad x \in (-1,1), \tag{3}$$

and

$$(\mathcal{K}u)(x) = \int_{-1}^{1} K(x,y)u(y)\,dy, \quad x \in (-1,1), \tag{4}$$

respectively. Moreover, $\mathcal{S}$ denotes the Cauchy singular integral operator defined in the sense of a principal value integral as

$$(\mathcal{S}u)(x) = \frac{1}{\pi \mathbf{i}} \int_{-1}^{1} \frac{u(y)\,dy}{y-x}, \quad x \in (-1,1).$$

Furthermore, the operator of multiplication by a bounded function $a : [-1,1] \longrightarrow \mathbb{C}$ is defined by

$$a\mathcal{I} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}, \quad f \mapsto af, \quad a \in \mathbf{PC}.$$

Thus, $\mathcal{I} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$ itself denotes the identity operator. If $\mathcal{B} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$ is another operator, then we use the abbreviation $a\mathcal{B}$ for the product of the multiplication operator $a\mathcal{I}$ and the operator $\mathcal{B}$, i.e., $a\mathcal{B} := a\mathcal{I}\mathcal{B}$.

The numerical methods for the approximate solution of Equation (1), which are of interest here, are collocation and collocation-quadrature methods, which we will describe later on in more detail. Since we know that solutions of (1) usually contain singularities at the endpoints of the integration interval, in these methods we look for an approximate solution $u_n(x)$ to $u(x)$ of the form

$$u_n(x) = v^{\rho_0,\tau_0}(x)p_n(x) = (1-x)^{\rho_0}(1+x)^{\tau_0}p_n(x), \tag{5}$$

where $\rho_0, \tau_0 > -1$ are real numbers and $p_n(x)$ is a polynomial of degree less than n. In case of a collocation method we choose a sequence of collocation points

$$-1 < x_{nn} < x_{n-1,n} < \ldots < x_{1n} < 1, \quad k = 1, \ldots, n, \tag{6}$$

where $n \in \mathbb{N}$—the set of positive integers, and try to determine $u_n(x)$ with the help of the conditions

$$(\mathcal{A}u_n)(x_{jn}) = f_n(x_{jn}), \quad j = 1, \ldots, n, \tag{7}$$

where $f_n \in v^{\rho_0,\tau_0}\mathbf{P}_n$ is an approximation to f, $\mathbf{P}_n$ denotes the space of algebraic polynomials of degree less than n, and $v^{\rho_0,\tau_0}\mathbf{P}_n$ is considered as a subspace of $\mathbf{L}^2_{\alpha,\beta}$ (i.e., equipped with the norm of $\mathbf{L}^2_{\alpha,\beta}$).

To realize a so called collocation-quadrature method, in a first step we approximate the integral operators $\mathcal{M}_H^{\pm}$ and $\mathcal{K}$ with the help of a quadrature method of interpolation type

$$\int_{-1}^{1} g(x)\omega(x)\,dx \sim \sum_{k=1}^{n} \lambda_{kn} g(x_{kn}), \quad \lambda_{kn} = \int_{-1}^{1} \ell_{kn}(x)\omega(x)\,dx, \tag{8}$$

where

$$\ell_{kn}(x) = \prod_{j=1, j\neq k}^{n} \frac{x - x_{kn}}{x_{jn} - x_{kn}}, \quad k = 1, \ldots, n,$$

are the fundamental Lagrange interpolation polynomials with respect to the nodes x_{kn}. Thus, the Mellin type operator is approximated by

$$\left(\mathcal{M}^{\pm}_{n,H} u\right)(x) = \sum_{k=1}^{n} \frac{\lambda_{kn}}{\omega(x_{kn})} H\left(\frac{1 \mp x}{1 \mp x_{kn}}\right) \frac{u(x_{kn})}{1 \mp x_{kn}}$$

and the Fredholm integral operator $\mathcal{K}$ by

$$(\mathcal{K}_n u)(x) = \sum_{k=1}^{n} \frac{\lambda_{kn}}{\omega(x_{kn})} K(x, x_{kn}) u(x_{kn}).$$

In the second step, we again use the nodes x_{kn} as collocation points and try to determine $u_n(x)$ by solving

$$\left(a u_n + b\mathcal{S} u_n + c_- \mathcal{M}^{-}_{n,H_-} u_n + c_+ \mathcal{M}^{+}_{n,H_+} u_n + \mathcal{K}_n u_n\right)(x_{jn}) = f_n(x_{jn}), \quad j = 1, \ldots, n. \tag{9}$$

Note that, in the collocation-quadrature method (9), the quadrature rule (8) is not applied to $(\mathcal{S} u_n)(x)$.

Both the collocation method (7) and the collocation-quadrature method (9) can be written as an operator equation

$$\mathcal{A}_n u_n = f_n, \tag{10}$$

where $\mathcal{A}_n : v^{\rho_0,\tau_0}\mathbf{P}_n \longrightarrow v^{\rho_0,\tau_0}\mathbf{P}_n$ is a linear operator (cf. (44) and (45)). The definition of the stability of the method (10) or, in other words, of the stability of the sequence $(\mathcal{A}_n) = (\mathcal{A}_n)_{n=1}^{\infty}$ of the operators $\mathcal{A}_n$, includes the unique solvability of (10) for all sufficiently large n and the uniform boundedness of the inverse operators $\mathcal{A}_n^{-1} : v^{\rho_0,\tau_0}\mathbf{P}_n \longrightarrow v^{\rho_0,\tau_0}\mathbf{P}_n$ (see Definition 1).

Now, the application of C^*-algebra techniques is based on the idea to consider the sequence $(\mathcal{A}_n)$ as an element of a suitable C^*-algebra and to translate stability into invertibility modulo zero sequences of this element (see Section 4). To find necessary and sufficient conditions for the stability of the sequence $(\mathcal{A}_n)$ it is necessary to segue to certain C^*-subalgebras and quotient algebras (cf. Proposition 2). In Table 1 we give an overview on the efforts done in the literature during the last 25 years to equations of type (1) or (2), where we ignore the Fredholm integral operator $\mathcal{K}$.

Table 1. Cases already considered in the literature.

	Equation (1)	**Method**
[1]	$a, b \in \mathbf{C}[-1,1]$, $b(\pm 1) = 0$	$\rho_0, \tau_0 \in \left(-\frac{1}{4}, \frac{3}{4}\right)$, coll. with Jacobi nodes
[2–4]	$c_\pm \equiv 0$	$\rho_0 = \tau_0 = \frac{1}{2}$ resp. $\rho_0 = -\tau_0 = \frac{1}{2}$, coll. with Chebyshev nodes
[5,6]	$c_\pm \equiv 0$	$\rho_0, \tau_0 \in \left(-\frac{1}{4}, \frac{3}{4}\right)$, coll. with Chebyshev nodes
[7]	$c_\pm \not\equiv 0$	$\rho_0, \tau_0 \in \left(-\frac{1}{4}, \frac{3}{4}\right)$, coll. with Chebyshev nodes
[8–11]	$c_\pm \not\equiv 0$	$\rho_0 = -\tau_0 \in \left\{\pm\frac{1}{2}\right\}$, coll.-quadr. with Chebyshev nodes
[12–16]	$c_\pm \not\equiv 0$	numerical aspects, fast algorithms

The aim of the present paper is to extend the possible choices of ρ_0 and τ_0 in comparison to Table 1, where here we restrict to collocation-quadrature methods and the case $a \equiv 1$ and $b \equiv 0$, i.e., the Cauchy singular integral operator is absent in Equation (1). To reach this aim we use the zeros of classical Jacobi polynomials associated with weights $v^{\gamma,\delta}(x)$ as collocation and quadrature nodes, not only the zeros of Chebyshev polynomials. Unfortunately, for this general choice of collocation and quadrature nodes here we are only able to prove the necessity of the stability conditions. Their sufficiency will be the topic of a forthcoming paper.

There exists a series of papers (see, for example, refs. [17–21]) devoted to the application of the Nyström method to Fredholm integral equations of the form

$$(\mathcal{I} + \mathcal{K}_1 + \mathcal{K}_2)u = f \tag{11}$$

with non-compact integral operators $\mathcal{K}_1$, for which the operators $\mathcal{M}_H^{\pm}$ are respective examples. Thereby, Equation (11) is studied in spaces of continuous or weighted continuous functions. However, since the idea of proving stability and convergence of the Nyström method is essentially based on the concept of collectively compact operator sequences, which works only for compact operators $\mathcal{K}_1 + \mathcal{K}_2$, in the mentioned papers there is assumed that the norm of the operator $\mathcal{K}_1$ is less than 1 and that this is true uniformly also for the approximating operators $\mathcal{K}_{1n}$. Then, for $\mathcal{I} + \mathcal{K}_1$ and $\mathcal{I} + \mathcal{K}_{1n}$ one can use the Neumann series argument. In the present paper, we are not constrained to apply such a condition on the norm of an operator.

2. Preliminaries

2.1. Properties of Integral Operators with Mellin Kernels

Let us start with collecting some statements on integral operators of interest here and already proved in the literature.

Lemma 1 ([22], Proposition 3.13)**.** *Let $\beta \in (-1,1)$ and $H \in \mathbf{C}(\mathbb{R}^+)$. Moreover, we assume that there are real numbers p, q with $p < q$ such that $\frac{1+\beta}{2} \in (p,q)$ and such that*

$$\lim_{t\to+0} t^p k(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} t^q k(t) = 0\,.$$

Then, for all $\alpha \in (-1,1)$, the integral operator $\mathcal{M}_H^- : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$ is bounded.

Note that $\mathcal{M}_H^+ = \mathcal{R}\mathcal{M}_H^-\mathcal{R}$, where $\mathcal{R} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$ is given by $(\mathcal{R}f)(x) = f(-x)$.

Lemma 2 ([22], Lemma 3.8)**.** *Let $\alpha, \beta \in \mathbb{R}$. If the condition*

$$\int_{-1}^{1}\int_{-1}^{1} \left(\frac{1-x}{1-y}\right)^{\alpha} \left(\frac{1+x}{1+y}\right)^{\beta} |K(x,y)|^2\, dy\, dx < \infty$$

is fulfilled, then $\mathcal{K} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$ is a compact operator.

The following corollary is an immediate consequence of the previous lemma.

Corollary 1. *Let $\alpha, \beta \in (-1,1)$ and $\eta, \zeta, \psi, \chi \in \mathbb{R}$ such that*

$$\eta < \frac{1+\alpha}{2},\ \zeta < \frac{1+\beta}{2} \quad \text{and} \quad \psi < \frac{1-\alpha}{2},\ \chi < \frac{1-\beta}{2}\,.$$

If the function

$$(-1,1)\times(-1,1) \longrightarrow \mathbb{C}, \quad (x,y) \mapsto v^{\eta,\zeta}(x) K(x,y) v^{\psi,\chi}(y)$$

is continuous and bounded, then the operator $\mathcal{K} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$ is compact.

For $\rho \in \mathbb{R}$, we introduce the weighted $\mathbf{L}^2$-space $\mathbf{L}^2_\rho := \mathbf{L}^2_\rho(\mathbb{R}^+)$ defined by the norm

$$\|f\|_{\rho,2} := \left(\int_0^\infty |f(t)|^2 t^\rho \, dt \right)^{\frac{1}{2}}.$$

Lemma 3. *Let $H \in \mathbf{L}^2_{2s-1}$ and $f \in \mathbf{L}^2_{0,2s-1}$ for some $s \in \mathbb{R}$. Then $v^{0,s}(\mathcal{M}^-_H f)$ is a bounded function on $(-1,1)$.*

Proof. For $x \in (-1,1)$, we have

$$\begin{aligned}
&\left|(1+x)^s(\mathcal{M}^-_H f)(x)\right| \\
&\quad \le \int_{-1}^1 \left| H\left(\frac{1+x}{1+y}\right)\right| \frac{(1+x)^s}{(1+y)^{s+1}} \, (1+y)^s |f(y)| \, dy \\
&\quad \le \left(\int_{-1}^1 \left| H\left(\frac{1+x}{1+y}\right)\right|^2 \frac{(1+x)^{2s}}{(1+y)^{2s+1}} \, dy \right)^{\frac{1}{2}} \left(\int_{-1}^1 (1+y)^{2s-1} |f(y)|^2 \, dy \right)^{\frac{1}{2}} \\
&\quad = \left(\int_{\frac{1+x}{2}}^\infty |H(t)|^2 t^{2s-1} \, dt \right)^{\frac{1}{2}} \|f\|_{0,2s-1} \le \|H\|_{2s-1,2} \|f\|_{0,2s-1},
\end{aligned}$$

from which the assertion follows. □

For $n \in \mathbb{N}_0$ and $H \in \mathbf{C}^n(\mathbb{R}^+)$, we define the operators $\partial_n \mathcal{M}^-_H$ by

$$(\partial_n \mathcal{M}^-_H f)(x) = \int_{-1}^1 H^{(n)}\left(\frac{1+x}{1+y}\right) \frac{f(y)}{(1+y)^{n+1}} \, dy.$$

Let $\mathbf{R} = \mathbf{R}(-1,1)$ and $\mathbf{C} = \mathbf{C}(-1,1)$ denote the sets of all functions $f : (-1,1) \longrightarrow \mathbb{C}$ being bounded and Riemann integrable as well as continuous on each closed subinterval of $(-1,1)$, respectively. For $\mathbf{S} \in \{\mathbf{R}, \mathbf{C}\}$ and $\psi, \chi \in \mathbb{R}$ with $\psi, \chi \ge 0$, by $\widetilde{\mathbf{S}}^b_{\psi,\chi}$ we refer to the set of all functions $f \in \mathbf{S}$, for which the function $v^{\psi,\chi} f$ is bounded on $(-1,1)$. If we introduce the norm

$$\|f\|_{\psi,\chi,\infty} = \sup\{ v^{\psi,\chi}(x)|f(x)| : -1 < x < 1 \},$$

then $\left(\widetilde{\mathbf{S}}^b_{\psi,\chi}, \|.\|_{\psi,\chi,\infty}\right)$ becomes a Banach space. Moreover, by $\widetilde{\mathbf{S}}_{\psi,\chi}$ we denote the set of all functions $f \in \mathbf{S}(-1,1)$, for which the finite limits

$$\lim_{x \to 1-0} (1-x)^\psi f(x) \quad \text{and} \quad \lim_{x \to -1+0} (1+x)^\chi f(x) \tag{12}$$

exist, and by $\mathbf{S}_{\psi,\chi}$ the subspace of $\widetilde{\mathbf{S}}_{\psi,\chi}$ of those functions $f \in \mathbf{S}(-1,1)$, for which the limits in (12) are equal to zero if $\psi > 0$ or $\chi > 0$, respectively. The spaces $\widetilde{\mathbf{S}}_{\psi,\chi}$ and $\mathbf{S}_{\psi,\chi}$ are closed subspaces of $\widetilde{\mathbf{S}}^b_{\psi,\chi}$ and, consequently, also Banach spaces. Finally, for $\psi_0, \chi_0 > 0$, set

$$\mathbf{S}^0_{\psi_0,\chi_0} = \bigcup_{0 \le \psi < \psi_0, 0 \le \chi < \chi_0} \mathbf{S}_{\psi,\chi}. \tag{13}$$

Note that

$$\begin{aligned}
\mathbf{S}^0_{\psi_0,\chi_0} &= \bigcup_{0 \le \psi < \psi_0, 0 \le \chi < \chi_0} \mathbf{S}_{\psi,\chi} = \bigcup_{0 \le \psi < \psi_0, 0 \le \chi < \chi_0} \widetilde{\mathbf{S}}_{\psi,\chi} = \bigcup_{0 \le \psi < \psi_0, 0 \le \chi < \chi_0} \widetilde{\mathbf{S}}^b_{\psi,\chi} \\
&= \left\{ f \in \mathbf{S}(-1,1) : \exists \mathcal{C} > 0, \exists \varepsilon > 0 \text{ with } |f(x)| < \mathcal{C} v^{\varepsilon-\psi_0,\varepsilon-\chi_0}(x) \, \forall \, x \in (-1,1) \right\}.
\end{aligned} \tag{14}$$

Lemma 4. *Let $\alpha, \beta < 1$ and $\eta, \zeta, \psi, \chi \in \mathbb{R}$ such that*

$$0 \le \eta, \zeta \quad \text{and} \quad 0 \le \psi < \frac{1-\alpha}{2}, \quad 0 \le \chi < \frac{1-\beta}{2}.$$

Moreover, assume the map

$$[-1,1] \times [-1,1] \longrightarrow \mathbb{C}, \quad (x,y) \mapsto v^{\eta,\zeta}(x) K(x,y) v^{\psi,\chi}(y)$$

to be continuous. Then $\mathcal{K} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \widetilde{\mathbf{C}}_{\eta,\zeta}$ is a compact operator.

Proof. Let $u \in \mathbf{L}^2_{\alpha,\beta}$ and $\varepsilon > 0$. Then, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\begin{aligned}
&\left|(v^{\eta,\zeta}\mathcal{K}u)(x) - (v^{\eta,\zeta}\mathcal{K}u)(x')\right| \\
&\le \int_{-1}^{1} \left|v^{\eta,\zeta}(x)K(x,y) - v^{\eta,\zeta}(x')K(x',y)\right| |u(y)|\, dy \\
&\le \varepsilon \left(\int_{-1}^{1} v^{-2\psi-\alpha,-2\chi-\beta}(y)\, dy \right)^{\frac{1}{2}} \|u\|_{\alpha,\beta} \le \mathrm{const}\, \varepsilon \|u\|_{\alpha,\beta}
\end{aligned}$$

for all $x, x' \in [-1,1]$ with $|x - x'| < \delta$. Thus, the set

$$\left\{ v^{\eta,\zeta}\mathcal{K}u : u \in \mathbf{L}^2_{\alpha,\beta},\ \|u\|_{\alpha,\beta} \le 1 \right\}$$

is equicontinuous. In the same way one can show

$$\sup\left\{ \|\mathcal{K}u\|_{\eta,\zeta,\infty} : u \in \mathbf{L}^2_{\alpha,\beta},\ \|u\|_{\alpha,\beta} \le 1 \right\} \le \mathrm{const}\,.$$

Applying the Arzela-Ascoli theorem delivers the assertion. □

For $z \in \mathbb{C}$ and a measurable function $f : (0,\infty) \longrightarrow \mathbb{C}$, for which $t^{z-1}f(t)$ is integrable on each compact subinterval of $(0,\infty)$, the Mellin transform $\widehat{f}(z)$ is defined as

$$\widehat{f}(z) = \lim_{R \to \infty} \int_{R^{-1}}^{R} t^{z-1} f(t)\, dt, \tag{15}$$

if this limit exists. Moreover, for $\xi \in \mathbb{R}$ and $p < q$, let $\Gamma_\xi = \{z \in \mathbb{C} : \mathrm{Re}\, z = \xi\}$ and

$$\Gamma_{p,q} = \{z \in \mathbb{C} : p < \mathrm{Re}\, z < q\}, \quad \mathbf{C}_0(\Gamma_\xi) = \left\{ f \in \mathbf{C}(\Gamma_\xi) : \lim_{|\eta| \to \infty} f(\xi + \mathbf{i}\eta) = 0 \right\}.$$

Lemma 5. *Let $p, q \in \mathbb{R}$ with $p < q$ and $f \in \mathbf{L}^2_{2\xi-1} \cap \mathbf{C}(\mathbb{R}^+)$ for every $\xi \in (p,q)$. Then*

(a) *the Mellin transform $\widehat{f}$ belongs to the space $\mathbf{C}_0(\Gamma_\xi)$ for every $\xi \in (p,q)$ and is holomorphic in the strip $\Gamma_{p,q}$.*

Moreover, if $\widehat{f}(z)$ satisfies

$$\sup\left\{ (1+|z|)^{1+\tau} \left| \widehat{f}'(z) \right| : z \in \Gamma_{p_0,q_0} \right\} < \infty$$

for all closed intervals $[p_0,q_0] \subset (p,q)$ and some $\tau = \tau(p_0,q_0) > 0$, then

(b) $\lim_{t \to +0} t^{p+\varepsilon} f(t) = 0$ *and* $\lim_{t \to \infty} t^{q-\varepsilon} f(t) = 0$

for every $\varepsilon > 0$.

Proof. From $f \in \mathbf{L}^2_{2\xi-1}$, $p < \xi < q$, we can conclude that $f \in \mathbf{L}^1_{\xi-1}$, $p < \xi < q$ (see [23] (Lemma 3.4)). This implies $\widehat{f} \in \mathbf{C}_0(\Gamma_\xi)$, $p < \xi < q$ (cf. [23] (page 7)). By [22] (Lemma 2.14) we get that $\widehat{f}(z)$ is holomorphic in the strip $\Gamma_{p,q}$, and (a) is proved. For assertion (b) we have only to refer to [23] (Corollary 3.3). $\square$

For a function $H \in \mathbf{C}(\mathbb{R}^+)$ and a real number ξ , we formulate the following conditions:

(A_0) There exist real numbers p and q with $p < q$ such that $H \in \mathbf{L}^2_{2p-1}(\mathbb{R}^+) \cap \mathbf{L}^2_{2q-1}(\mathbb{R}^+)$, $\xi \in (p,q)$, and

$$\sup\left\{(1+|z|)^{1+\tau}\left|\widehat{H}'(z)\right| : z \in \Gamma_{p_0,q_0}\right\} < \infty$$

for all intervals $[p_0,q_0] \subset (p,q)$ and some $\tau = \tau(p_0,q_0) > 0$.

(A_1) There exist real numbers p and q with $p < q$ such that $H \in \mathbf{L}^2_{2p-1}(\mathbb{R}^+) \cap \mathbf{L}^2_{2q-1}(\mathbb{R}^+)$, $\xi \in (p,q)$, and

$$\sup\left\{(1+|z|)^{s+\tau}\left|\widehat{H}^{(s)}(z)\right| : z \in \Gamma_{p_0,q_0}\right\} < \infty, \quad s \in \{0,1\},$$

for all intervals $[p_0,q_0] \subset (p,q)$ and some $\tau = \tau(p_0,q_0) > 0$.

We set

$$\mathcal{A} := \mathcal{I} + c_-\mathcal{M}^-_{H_-} + c_+\mathcal{M}^+_{H_+}, \tag{16}$$

where $c_\pm \in \mathbf{L}^\infty(-1,1)$ and $H_\pm \in \mathbf{C}(\mathbb{R}^+)$. The following lemma is an application of [10] (Theorem 4.12) to the operator in (16).

Lemma 6. *Let $\alpha,\beta \in (-1,1)$, $c_\pm \in \mathbf{PC}$, and $H_\pm \in \mathbf{C}(\mathbb{R}^+)$. If the function $H = H_\pm$ satisfies condition (A_0) for $\xi = \xi_\pm$, where $\xi_+ = \frac{1+\alpha}{2}$ and $\xi_- = \frac{1+\beta}{2}$, then the integral operator $\mathcal{A} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$ defined in (16) is Fredholm if and only if the closed curve*

$$\Gamma_\mathcal{A} := \Gamma^-_\mathcal{A} \cup \Gamma^+_\mathcal{A}$$

does not contain the point 0 *, where*

$$\Gamma^\pm_\mathcal{A} := \left\{1 + c_\pm(\pm 1)\widehat{H}_\pm(\xi_\pm - \mathbf{i}t) : t \in \overline{\mathbb{R}}\right\}.$$

In this case, the Fredholm index of $\mathcal{A}$ is equal to the negative winding number of the curve $\Gamma_\mathcal{A}$, where the orientation of $\Gamma_\mathcal{A}$ is due to the above given parametrizations of $\Gamma^\pm_\mathcal{A}$.

Lemma 7 ([10], Proposition 6.1)**.** *Let $\alpha,\beta \in (-1,1)$, $a \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbf{C}(\mathbb{R}^+)$. If the function H satisfies condition (A_0) for $\xi = \frac{1+\beta}{2}$, then the homogeneous equations $(a\mathcal{I} + \mathcal{M}_H)u = 0$ in the space $\mathbf{L}^2_{\alpha,\beta}$ or $(a\mathcal{I} + \mathcal{M}_H) * v = 0$ in the space $\mathbf{L}^2_{-\alpha,-\beta}$ have only the trivial solution.*

2.2. Marcinkiewicz Inequalities

For $n \in \mathbb{N}_0$ and $\alpha,\beta > -1$, by $\widehat{P}^{\alpha,\beta}_n(x)$ we denote the monic orthogonal polynomial of degree n with respect to the weight $v^{\alpha,\beta}(x)$. Furthermore, let $x^{\alpha,\beta}_{kn}$, $k = 1,\cdots,n$, be the zeros of $\widehat{P}^{\alpha,\beta}_n(x)$. It is well known that these zeros are real, simple, and contained in $(-1,1)$. Therefore, we can write

$$x^{\alpha,\beta}_{kn} = \cos\theta^{\alpha,\beta}_{kn}, \quad \theta^{\alpha,\beta}_{kn} \in (0,\pi),$$

and order them as follows

$$-1 < x^{\alpha,\beta}_{nn} < x^{\alpha,\beta}_{n-1,n} < \cdots < x^{\alpha,\beta}_{1n} < 1, \quad \text{i.e.,} \quad 0 < \theta^{\alpha,\beta}_{1n} < \cdots < \theta^{\alpha,\beta}_{nn} < \pi.$$

We set $\theta^{\alpha,\beta}_{0n} = 0$ and $\theta^{\alpha,\beta}_{n+1,n} = \pi$ as well as $x^{\alpha,\beta}_{n+1,n} = -1$ and $x^{\alpha,\beta}_{0n} = 1$.

The monic Jacobi polynomials $\widehat{P}^{\alpha,\beta}_n(x)$ satisfy the three-term recurrence relation (cf. [24] (Chapter V, (2.7),(2.29))

$$\widehat{P}^{\alpha,\beta}_{n+1}(x) = (x-\alpha_n)\widehat{P}^{\alpha,\beta}_n(x) - \beta_n^2\widehat{P}^{\alpha,\beta}_{n-1}(x), \quad n \in \mathbb{N}_0, \tag{17}$$

where $\widehat{P}_{-1}^{\alpha,\beta}(x) \equiv 0$, $\widehat{P}_{0}^{\alpha,\beta}(x) \equiv 1$, and

$$\alpha_n = \alpha_n^{\alpha,\beta} = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \quad n \in \mathbb{N},$$

$$\beta_n = \beta_n^{\alpha,\beta} = \sqrt{\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}}, \quad n \in \mathbb{N} \setminus \{1\},$$

and

$$\alpha_0 = \alpha_0^{\alpha,\beta} = \frac{\alpha+\beta}{2+\alpha+\beta}, \quad \beta_1 = \beta_1^{\alpha,\beta} = \sqrt{\frac{4(1+\alpha)(1+\beta)}{(2+\alpha+\beta)^2(3+\alpha+\beta)}}.$$

By $p_n^{\alpha,\beta}(x)$ we refer to the normalized (with respect to the inner product $\langle .,. \rangle_{\alpha,\beta}$) polynomials with positive leading coefficient. If we set

$$\beta_0 = \beta_0^{\alpha,\beta} := \sqrt{\int_{-1}^{1} v^{\alpha,\beta}(x)\,dx} = \sqrt{\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}}, \tag{18}$$

then (see [24] (Chapter I, (4.10))

$$p_n^{\alpha,\beta}(x) = \gamma_n^{\alpha,\beta}\widehat{P}_n^{\alpha,\beta}(x) \quad \text{with} \quad \gamma_n^{\alpha,\beta} = (\beta_0\beta_1\beta_2\cdots\beta_n)^{-1} \tag{19}$$

and from (17) we get

$$\beta_{n+1}p_{n+1}^{\alpha,\beta}(x) = (x-\alpha_n)p_n^{\alpha,\beta}(x) - \beta_n p_{n-1}^{\alpha,\beta}(x), \quad n \in \mathbb{N}_0, \tag{20}$$

where $p_{-1}^{\alpha,\beta}(x) \equiv 0$ and $p_0^{\alpha,\beta}(x) \equiv \beta_0^{-1}$. In view of

$$\int_{-1}^{1} (1-x^2)p_{n-1}^{\alpha+1,\beta+1}(x)x^k v^{\alpha,\beta}(x)\,dx = 0, \quad k = 0,1,\ldots,n-2,$$

we have

$$(1-x^2)p_{n-1}^{\alpha+1,\beta+1}(x) = A_n p_{n-1}^{\alpha,\beta}(x) + B_n p_n^{\alpha,\beta}(x) + C_n p_{n+1}^{\alpha,\beta}(x) \tag{21}$$

with certain real numbers A_n, B_n, and C_n. Together with (20) this yields

$$\left[1 - \left(x_{nk}^{\alpha,\beta}\right)^2\right] p_{n-1}^{\alpha+1,\beta+1}(x_{kn}^{\alpha,\beta}) = \left(A_n - \frac{C_n\beta_n}{\beta_{n+1}}\right) p_{n-1}^{\alpha,\beta}(x_{kn}^{\alpha,\beta}). \tag{22}$$

Let $\beta_n = \beta_n^{\alpha,\beta}$ and $\delta_n = \beta_n^{\alpha+1,\beta+1}$. With the help of the relations (18) and $\Gamma(z+1) = z\Gamma(z)$ we get

$$\frac{\delta_0}{\beta_0} = 2\sqrt{\frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)}}.$$

Furthermore,

$$\frac{\delta_k}{\beta_{k+1}} = \sqrt{\frac{k(k+\alpha+\beta+2)}{(k+1)(k+\alpha+\beta+1)}}, \quad k = 1,2,\ldots$$

Using the orthogonality properties of $p_n^{\alpha,\beta}(x)$ and (19) we obtain, for $n > 1$,

$$\begin{aligned}
A_n &= A_n(\alpha,\beta) \\
&= \int_{-1}^{1}(1-x^2)p_{n-1}^{\alpha+1,\beta+1}(x)p_{n-1}^{\alpha,\beta}(x)v^{\alpha,\beta}(x)\,dx \\
&= \gamma_{n-1}^{\alpha,\beta}\int_{-1}^{1}p_{n-1}^{\alpha+1,\beta+1}(x)x^{n-1}v^{\alpha+1,\beta+1}(x)\,dx \\
&= \frac{\gamma_{n-1}^{\alpha,\beta}}{\gamma_{n-1}^{\alpha+1,\beta+1}} = \frac{\delta_0\delta_1\cdots\delta_{n-1}}{\beta_0\beta_1\cdots\beta_{n-1}} \\
&= \begin{cases} \dfrac{\delta_0\delta_1}{\beta_0\beta_1} = \dfrac{2}{4+\alpha+\beta}\sqrt{\dfrac{(2+\alpha)(2+\beta)(2+\alpha+\beta+1)}{(4+\alpha+\beta-1)(4+\alpha+\beta+1)}} & : \quad n=2 \\ \dfrac{\delta_0}{\beta_0\beta_1}\cdot\dfrac{\delta_1\cdots\delta_{n-2}}{\beta_2\cdots\beta_{n-1}} & \\ \quad = \sqrt{2+\alpha+\beta}\sqrt{\dfrac{n+\alpha+\beta}{(2+\alpha+\beta)(n-1)}}\cdot & \\ \qquad \cdot\sqrt{\dfrac{4(n-1)(n+\alpha)(n+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}} & : \quad n>2 \end{cases} \qquad (23) \\
&= \frac{2}{2n+\alpha+\beta}\sqrt{\frac{(n+\alpha)(n+\beta)(n+\alpha+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta+1)}},
\end{aligned}$$

and, for $n = 1$,

$$A_1 = A_1(\alpha,\beta) = \frac{\delta_0}{\beta_0} = 2\sqrt{\frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)(\alpha+\beta+3)}}, \qquad (24)$$

as well as

$$\begin{aligned}
C_n &= \int_{-1}^{1}(1-x^2)p_{n-1}^{\alpha+1,\beta+1}(x)p_{n+1}^{\alpha,\beta}(x)v^{\alpha,\beta}(x)\,dx \\
&= -\gamma_{n-1}^{\alpha+1,\beta+1}\int_{-1}^{1}x^{n+1}p_{n+1}^{\alpha,\beta}(x)v^{\alpha,\beta}(x)\,dx \\
&= -\frac{\gamma_{n-1}^{\alpha+1,\beta+1}}{\gamma_{n+1}^{\alpha,\beta}} = -\frac{\beta_0\beta_1\cdots\beta_{n+1}}{\delta_0\delta_1\cdots\delta_{n-1}} = -\frac{\beta_n\beta_{n+1}}{A_n}, \quad n\in\mathbb{N}.
\end{aligned}$$

Hence, relation (22) can be written in the form

$$p_{n-1}^{\alpha,\beta}(x_{kn}^{\alpha,\beta}) = c_n\left[1-\left(x_{kn}^{\alpha,\beta}\right)^2\right]p_{n-1}^{\alpha+1,\beta+1}(x_{kn}^{\alpha,\beta}), \qquad (25)$$

where

$$\begin{aligned}
c_n &= \frac{1}{A_n+\dfrac{\beta_n^2}{A_n}} = \frac{A_n}{A_n^2+\beta_n^2} \\
&= \frac{2n+\alpha+\beta}{2}\sqrt{\frac{(n+\alpha+\beta+1)(2n+\alpha+\beta-1)}{(n+\alpha)(n+\beta)(n+\alpha+\beta)(2n+\alpha+\beta+1)}}, \quad n>1,
\end{aligned}$$

and

$$c_1 = \frac{\alpha+\beta+2}{2}\sqrt{\frac{(\alpha+\beta+2)}{(\alpha+1)(\beta+1)(\alpha+\beta+3)}}\,.$$

In what follows, by $\mathcal{C}$ we will denote a positive constant, which can assume different values at different places, and we will write $\mathcal{C} \neq \mathcal{C}(x,n,\dots)$ to indicate that $\mathcal{C}$ does not depend on the parameters $x, n, \dots$ If $A = A(x,n,\dots)$ and $B = B(x,n,\dots)$ are two positive functions depending on certain variables $x, n, \dots$, then we will write $A \sim_{x,n,\dots} B$ if there is a constant $\mathcal{C} \neq \mathcal{C}(x,n,\dots) > 0$ such that

$$\mathcal{C}^{-1}B(x,n,\dots) \le A(x,n,\dots) \le \mathcal{C}\, B(x,n,\dots) \quad \forall\, x,n,\dots$$

Let $\lambda_n^{\alpha,\beta}(x)$ stand for the n-th Christoffel function with respect to the weight $v^{\alpha,\beta}(x)$, i.e.,

$$\lambda_n^{\alpha,\beta}(x) = \left(\sum_{j=0}^{n-1}\left[p_j^{\alpha,\beta}(x)\right]^2\right)^{-1}, \quad x \in [-1,1]\,.$$

It is well-known that

$$\lambda_n^{\alpha,\beta}(x_{kn}^{\alpha,\beta}) = \int_{-1}^{1} \ell_{kn}^{\alpha,\beta}(x) v^{\alpha,\beta}(x)\, dx\,,$$

where $\ell_{kn}^{\alpha,\beta}(x)$ are the nth fundamental Lagrange interpolation polynomials

$$\ell_{kn}^{\alpha,\beta}(x) = \frac{p_n^{\alpha,\beta}(x)}{(x - x_{kn}^{\alpha,\beta})(p_n^{\alpha,\beta})'(x_{kn}^{\alpha,\beta})} = \prod_{j=1, j\neq k}^{n} \frac{x - x_{jn}^{\alpha,\beta}}{x_{kn}^{\alpha,\beta} - x_{jn}^{\alpha,\beta}}$$

with respect to the nodes $x_{kn}^{\alpha,\beta}$, $k = 1,\dots,n$. For $x \in [-1,1]$ and $n \in \mathbb{N}$, let $\varphi(x) := \sqrt{1-x^2}$, $x \in [-1,1]$, $\Delta_n(x) = \dfrac{\varphi(x)}{n} + \dfrac{1}{n^2}$, and $v_n^{\alpha,\beta}(x) := \left(\sqrt{1-x}+\dfrac{1}{n}\right)^{2\alpha}\left(\sqrt{1+x}+\dfrac{1}{n}\right)^{2\beta}$. Then (cf. [25] (Theorem 5))

$$\frac{\lambda_n^{\alpha,\beta}(x)}{\Delta_n(x) v_n^{\alpha,\beta}(x)} \sim_{x,n} 1\,. \tag{26}$$

Since Jacobi weights are so-called doubling weights (see, for example, [26] (Section 3.2.1, Exercise 3.2.4), we also have (see [27] (Theorem 1))

$$\frac{x_{kn}^{\alpha,\beta} - x_{k+1,n}^{\alpha,\beta}}{\Delta_n(t)} \sim_{t,k,n} 1\,, \quad k = 0,1,\dots,n\,,\ t \in \left[x_{kn}^{\alpha,\beta}, x_{k+1,n}^{\alpha,\beta}\right],\ n \in \mathbb{N}\,, \tag{27}$$

which can equivalently be written as (see [28] (Theorem 3.2) and cf. [26] (Exercise 3.2.25))

$$\theta_{k+1,n}^{\alpha,\beta} - \theta_{kn}^{\alpha,\beta} \sim_{k,n} \frac{1}{n}\,, \quad k = 0,\dots,n\,,\ n \in \mathbb{N}\,. \tag{28}$$

Note that, due to (28),

$$\theta_{kn}^{\alpha,\beta} = \sum_{j=1}^{k}\left(\theta_{jn}^{\alpha,\beta} - \theta_{j-1,n}^{\alpha,\beta}\right) \sim_{k,n} \frac{k}{n}\,, \qquad k = 1,\dots,n+1\,, \tag{29}$$

and

$$\pi - \theta_{kn}^{\alpha,\beta} = \sum_{j=k}^{n}\left(\theta_{j+1,n}^{\alpha,\beta} - \theta_{j,n}^{\alpha,\beta}\right) \sim_{k,n} \frac{n-k+1}{n}\,, \qquad k = 0,\dots,n\,. \tag{30}$$

Hence, for $k=1,\ldots,n$,

$$\sqrt{1-x_{kn}^{\alpha,\beta}}=\sqrt{2}\sin\frac{\theta_{kn}^{\alpha,\beta}}{2}\sim_{k,n}\theta_{k,n}^{\alpha,\beta}\sim_{k,n}\frac{k}{n} \tag{31}$$

and

$$\sqrt{1+x_{kn}^{\alpha,\beta}}=\sqrt{2}\cos\frac{\theta_{kn}^{\alpha,\beta}}{2}=\sqrt{2}\sin\frac{\pi-\theta_{kn}^{\alpha,\beta}}{2}\sim_{k,n}\pi-\theta_{k,n}^{\alpha,\beta}\sim_{k,n}\frac{n-k+1}{n} \tag{32}$$

as well as

$$\frac{1}{n}\le\frac{k(n-k+1)}{n^2}\sim_{k,n}\sqrt{1-(x_{kn}^{\alpha,\beta})^2}. \tag{33}$$

which implies, due to (28),

$$x_{kn}^{\alpha,\beta}-x_{k+1,n}^{\alpha,\beta}\sim_{t,k,n}\frac{\varphi(t)}{n},\quad k=1,\ldots,n-1,\ t\in\left[x_{k+1,n}^{\alpha,\beta},x_{kn}^{\alpha,\beta}\right]. \tag{34}$$

Lemma 8. *For $n\in\mathbb{N}$, $k=1,\cdots,n$, and $\alpha,\beta,\gamma,\delta>-1$, we have*

$$\lambda_n^{\alpha,\beta}(x_{kn}^{\gamma,\delta})\sim_{k,n}\frac{v^{\alpha+\frac{1}{2},\beta+\frac{1}{2}}(x_{kn}^{\gamma,\delta})}{n}. \tag{35}$$

Proof. In view of (26) we have

$$\lambda_n^{\alpha,\beta}(x_{kn}^{\gamma,\delta})\sim_{k,n}\left(\frac{\sqrt{1-(x_{kn}^{\gamma,\delta})^2}}{n}+\frac{1}{n^2}\right)\left(\sqrt{1-x_{kn}^{\gamma,\delta}}+\frac{1}{n}\right)^{2\alpha}\left(\sqrt{1+x_{kn}^{\gamma,\delta}}+\frac{1}{n}\right)^{2\beta},$$

and it remains to take into account

$$\sqrt{1\mp x_{kn}^{\gamma,\delta}}<\sqrt{1\mp x_{kn}^{\gamma,\delta}}+\frac{1}{n}\overset{(31),(32)}{\le}\mathcal{C}\sqrt{1\mp x_{kn}^{\gamma,\delta}}$$

and

$$\sqrt{1-(x_{kn}^{\gamma,\delta})^2}<\sqrt{1-(x_{kn}^{\gamma,\delta})^2}+\frac{1}{n}\overset{(33)}{\le}\mathcal{C}\sqrt{1-(x_{kn}^{\gamma,\delta})^2}.$$

□

Lemma 9 ([29], Theorem 2.6)**.** *Let $2\gamma_0+\alpha>-1$, $2\delta_0+\beta>-1$, and consider a system*

$$-1=x_{nn}<x_{n-1,n}<\cdots<x_{1n}=1$$

of nodes $x_{kn}=\cos\vartheta_{kn}$, $\vartheta_{kn}\in[0,\pi]$, satisfying $\vartheta_{kn}-\vartheta_{k-1,n}\sim_{k,n}\frac{1}{n}$ for $k=2,\ldots,n$ and $n\in\mathbb{N}$. Moreover, let m be a fixed positive integer. Then there exists a positive constant $\mathcal{C}\ne\mathcal{C}(n,Q)$, such that

$$\sum_{k=1}^{n}\lambda_n^{\alpha,\beta}(x_{kn})\left|v_n^{\gamma_0,\delta_0}(x_{kn})Q(x_{kn})\right|^2\le\mathcal{C}\left\|v^{\gamma_0,\delta_0}Q\right\|_{\alpha,\beta}^2 \tag{36}$$

holds true for all $Q\in\mathbf{P}_{mn}$, $n\in\mathbb{N}$, where $\mathbf{P}_n$ denotes the set of all algebraic polynomials of degree less than n.

Corollary 2. *Assume $2\gamma_0+\alpha>-1$, $2\delta_0+\beta>-1$, and consider a system*

$$-1<x_{nn}<x_{n-1,n}<\cdots<x_{1n}<1$$

of nodes $x_{kn}=\cos\vartheta_{kn}$, $\theta_{kn}\in[0,\pi]$, satisfying $\vartheta_{kn}-\vartheta_{k-1,n}\sim_{k,n}\frac{1}{n}$ for $k=1,\cdots,n+1$ and $n\in\mathbb{N}$, where $\vartheta_{n+1,n}=\pi$ and $\vartheta_{0n}=0$. Moreover, let m be a fixed positive integer. Then there exists a positive constant $\mathcal{C}\ne\mathcal{C}(n,Q)$, such that (36) is satisfied for all $Q\in\mathbf{P}_{mn}$, $n\in\mathbb{N}$.

Proof. Obviously

$$v^{\alpha,\beta}_{n+2}(x) \sim_{x,n} v^{\alpha,\beta}_n(x) \quad \text{and} \quad \Delta_{n+2}(x) \sim_{x,n} \Delta_n(x)\,, \quad x \in [-1,1]\,,\ n \in \mathbb{N}\,.$$

Using (26) we get

$$\lambda^{\alpha,\beta}_{n+2}(x) \sim_{x,n} \lambda^{\alpha,\beta}_n(x), \quad \forall\, x \in [-1,1].$$

Now, we consider the following system of nodes

$$\widetilde{x}_{n+2,n+2} := -1, \quad \widetilde{x}_{1,n+2} := 1, \quad \widetilde{x}_{k+1,n+2} = x_{kn}, \quad k = 1, \cdots, n.$$

If we apply Lemma 9 to this system, we immediately arrive at our assertion. □

In the particular case $x_{kn} = x^{\gamma,\delta}_{kn}$ and $\gamma_0 = \delta_0 = 0$, from (35) and Corollary 2 we get

$$\frac{1}{n}\sum_{k=1}^{n} v^{\alpha+\frac{1}{2},\beta+\frac{1}{2}}(x^{\gamma,\delta}_{kn})\left|Q(x^{\gamma,\delta}_{kn})\right|^2 \le \mathcal{C}\|Q\|^2_{\alpha,\beta}\,, \quad n \in \mathbb{N}\,, \quad \alpha,\beta,\gamma,\delta > -1\,, \tag{37}$$

for all $Q \in \mathbf{P}_{mn}$ and with $\mathcal{C} \ne \mathcal{C}(n,Q)$.

Lemma 10 ([29], Theorem 2.7)**.** *If $\alpha,\beta,\gamma,\delta > -1$ and $-\frac{1}{2} < \alpha-\gamma, \beta-\delta < \frac{3}{2}$, then there exists a positive constant $\mathcal{C} \ne \mathcal{C}(n,Q)$ such that*

$$\mathcal{C}\|Q\|^2_{\alpha,\beta} \le \sum_{k=1}^{n} \lambda^{\alpha,\beta}_n\left(x^{\gamma,\delta}_{kn}\right)\left|Q\left(x^{\gamma,\delta}_{kn}\right)\right|^2, \quad Q \in \mathbf{P}_n\,,\ n \in \mathbb{N}\,. \tag{38}$$

Corollary 3. *If $\alpha,\beta,\gamma,\delta > -1$ and*
(a) $-\frac{1}{2} < \alpha-\gamma, \beta-\delta < \frac{3}{2}$,
then there exists a positive constant $\mathcal{C} \ne \mathcal{C}(n,Q)$, such that

$$\mathcal{C}^{-1}\|Q\|^2_{\alpha,\beta} \le \frac{1}{n}\sum_{k=1}^{n} v^{\alpha+\frac{1}{2},\beta+\frac{1}{2}}(x^{\gamma,\delta}_{kn})\,\left|Q(x^{\gamma,\delta}_{kn})\right|^2 \le \mathcal{C}\|Q\|^2_{\alpha,\beta}$$

for all $Q \in \mathbf{P}_n$, $n \in \mathbb{N}$, where the second inequality holds true without condition (a).

Proof. Relation (37) and Lemma 10 together with (35) deliver the assertion. □

2.3. *The Algebra* alg $\mathcal{T}(\mathbf{PC})$

By alg $\mathcal{T}(\mathbf{PC})$ we denote the smallest C^*-subalgebra of the algebra $\mathcal{L}(\ell^2)$ of all linear and bounded operators on the Hilbert space ℓ^2 generated by the Toeplitz matrices

$$\mathbf{T}(g) = \left[\ \widehat{g}_{j-k}\ \right]_{j,k=0}^{\infty}$$

with piecewise continuous generating functions

$$g(t) := \sum_{\ell\in\mathbb{Z}} \widehat{g}_\ell t^\ell$$

defined on the unit circle $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$ and continuous on $\mathbb{T} \setminus \{\pm 1\}$.

Of course, alg $\mathcal{T}(\mathbf{PC})$ is a C^*-subalgebra of the C^*-algebra $\mathcal{L}_T(\ell^2) \subset \mathcal{L}(\ell^2)$ generated by all Toeplitz matrices $\mathbf{T}(f)$ with piecewise continuous generating function $f : \mathbb{T} \longrightarrow \mathbb{C}$. It is well known (see Chapter 16 in [30]) that there exists an isometrical isomorphism smb from the quotient algebra $\mathcal{L}_T(\ell^2)/\mathcal{K}(\ell^2)$ ($\mathcal{K}(\ell^2)$—the ideal in $\mathcal{L}(\ell^2)$ of compact operators) onto the algebra $(\mathbf{C}(\mathbb{M}), \|.\|_\infty)$ of all complex valued and continuous functions on the compact space $\mathbb{M} = \mathbb{T} \times [0,1]$, where the topology on $\mathbb{M}$ is defined by the neighborhoods (cf. Theorem 16.1 in [30])

$$U_{\varepsilon,\delta}(e^{\mathbf{i}\eta_0},0) := \Big\{(e^{\mathbf{i}\eta},\lambda) : \eta_0-\delta<\eta<\eta_0,\, 0\le\lambda\le 1\Big\} \cup \Big\{(e^{\mathbf{i}\eta_0},\lambda) : 0\le\lambda<\varepsilon\Big\},$$

$$U_{\varepsilon,\delta}(e^{\mathbf{i}\eta_0},1) := \Big\{(e^{\mathbf{i}\eta},\lambda) : \eta_0<\eta<\eta_0+\delta,\, 0\le\lambda\le 1\Big\} \cup \Big\{(e^{\mathbf{i}\eta_0},\lambda) : \varepsilon\le\lambda\le 1\Big\},$$

$$U_{\delta_1,\delta_2}(e^{\mathbf{i}\eta_0},\lambda_0) := \Big\{(e^{\mathbf{i}\eta_0},\lambda) : \lambda_0-\delta_1<\lambda<\lambda_0+\delta_2\Big\}$$

with $0<\delta_1<\lambda_0<1-\delta_2<1$, $0<\delta<2\pi$, and $0<\varepsilon<1$.

Proposition 1 ([30], Theorem 16.2, [31], Theorem 4.97)**.** *The mapping* smb *has the following properties:*

(a) $\mathbf{R} \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$ *is Fredholm if and only if* $\operatorname{smb}_{\mathbf{R}}(t,\lambda) \neq 0$ *for every* $(t,\lambda) \in \mathbb{T}\times[0,1]$, *where* $\operatorname{smb}_{\mathbf{R}} := \operatorname{smb}(\mathbf{R})$.

(b) *If* $\mathbf{R} \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$ *is Fredholm, then the index is equal to the negative winding number of the closed curve*

$$\begin{aligned}\Gamma_{\mathbf{R}} : = &\{\operatorname{smb}_{\mathbf{R}}(e^{\mathbf{i}s},0) :\ 0<s<\pi\} \cup \{\operatorname{smb}_{\mathbf{R}}(-1,s) :\ 0\le s\le 1\}\\ &\cup \{\operatorname{smb}_{\mathbf{R}}(-e^{\mathbf{i}s},0) :\ 0<s<\pi\} \cup \{\operatorname{smb}_{\mathbf{R}}(1,s) :\ 0\le s\le 1\},\end{aligned} \tag{39}$$

where the orientation of $\Gamma_{\mathbf{R}}$ *is due to the above given parametrization.*

Lemma 11 ([9], Lemma 2.10)**.** *Let* $H \in \mathbf{C}(\mathbb{R}^+)$. *If* H *fulfils condition* (A_1) *for* $\xi = \frac{1}{2}$, *then, for every* $s>0$, *the matrix*

$$\mathbf{M} := \left[H\left(\frac{j+s}{k+s}\right)\frac{1}{k+s}\right]_{j,k=0}^{\infty}$$

defines an operator $\mathbf{M} \in \mathcal{L}(\ell^2)$, *which belongs to the algebra* $\operatorname{alg}\mathcal{T}(\mathbf{PC})$, *and its symbol is given by*

$$\operatorname{smb}_{\mathbf{M}}(t,\lambda) = \begin{cases} \widehat{H}\left(\frac{1}{2}+\frac{\mathbf{i}}{2\pi}\log\frac{\lambda}{1-\lambda}\right) & : \quad t=1,\\ 0 & : \quad t\in\mathbb{T}\setminus\{1\}.\end{cases}$$

3. The Collocation-Quadrature Method

We consider the integral Equation (cf. (16))

$$\mathcal{A}u = (\mathcal{I} + c_-\mathcal{M}_{H_-}^- + c_+\mathcal{M}_{H_+}^+ + \mathcal{K})u = f, \tag{40}$$

where $c_\pm \in \mathbf{PC}$, $f \in \mathbf{L}^2_{\alpha,\beta}$, $H_\pm \in \mathbf{C}(\mathbb{R}^+)$, and the kernel function $K(x,y)$ of the integral operator $\mathcal{K}$ (cf. (4)) is supposed to be continuous on $(-1,1)\times(-1,1)$. In order to get approximate solutions, we use a polynomial collocation-quadrature method. To introduce that method, we need some further notations. Let $n\in\mathbb{N}$ and $\gamma,\delta,\rho,\tau>-1$ be real numbers. For $u : (-1,1) \longrightarrow \mathbb{C}$, the Lagrange interpolation operator $\mathcal{L}_n^{\gamma,\delta}$ is defined by

$$\mathcal{L}_n^{\gamma,\delta}u = \sum_{j=1}^{n} u(x_{jn}^{\gamma,\delta})\ell_{jn}^{\gamma,\delta},$$

$$\ell_{jn}^{\gamma,\delta}(x) = \frac{p_n^{\gamma,\delta}(x)}{(x-x_{jn}^{\gamma,\delta})(p_n^{\gamma,\delta})'(x_{jn}^{\gamma,\delta})} = \prod_{k=1,k\neq j}^{n}\frac{x-x_{kn}^{\gamma,\delta}}{x_{jn}^{\gamma,\delta}-x_{kn}^{\gamma,\delta}}.$$

To the integral operators $\mathcal{M}_H^\pm$ and $\mathcal{K}$, we associate the quadrature operators

$$\mathcal{M}_{n,H}^\pm : \mathbf{C}(-1,1) \longrightarrow \mathbf{C}(-1,1)\,, \quad u \mapsto \sum_{k=1}^n \lambda_{kn}^{\gamma,\delta} H\left(\frac{1 \mp \cdot}{1 \mp x_{kn}^{\gamma,\delta}}\right) \frac{(v^{-\gamma,-\delta}u)(x_{kn}^{\gamma,\delta})}{1 \mp x_{kn}^{\gamma,\delta}}$$

and

$$\mathcal{K}_n : \mathbf{C}(-1,1) \longrightarrow \mathbf{C}(-1,1)\,, \quad u \mapsto \sum_{k=1}^n \lambda_{kn}^{\gamma,\delta} K(\cdot, x_{kn}^{\gamma,\delta})(v^{-\gamma,-\delta}u)(x_{kn}^{\gamma,\delta})\,, \tag{41}$$

respectively. For certain $\rho, \tau > -1$, the collocation-quadrature method seeks for approximations $u_n \in \mathbf{L}_{\alpha,\beta}^2$ of the form

$$u_n = \vartheta p_n := v^{\frac{\rho-\alpha}{2}, \frac{\tau-\beta}{2}} p_n, \quad p_n \in \mathbf{P}_n$$

to the exact solution of (40) by solving

$$\Big((\mathcal{I} + c_- \mathcal{M}_{n,H_-}^- + c_+ \mathcal{M}_{n,H_+}^+ + \mathcal{K}_n)u_n\Big)(x_{kn}^{\gamma,\delta}) = f_n(x_{kn}^{\gamma,\delta}), \quad k = 1, \ldots, n, \tag{42}$$

where $\mathbf{P}_n$ stands for the set of all algebraic polynomials of degree less than n and the functions $f_n : (-1,1) \longrightarrow \mathbb{C}$ are continuous and satisfy $\vartheta^{-1} f_n \in \mathbf{P}_n$ as well as

$$\lim_{n \longrightarrow \infty} \|f - f_n\|_{\alpha,\beta} = 0. \tag{43}$$

We set

$$\widetilde{p}_n := \vartheta p_n^{\rho,\tau}, \quad n = 0, 1, 2, \ldots$$

Note that $(\widetilde{p}_n)_{n=0}^\infty$ forms a complete orthonormal system in $\mathbf{L}_{\alpha,\beta}^2$. Using the weighted fundamental Lagrange interpolation polynomials

$$\widetilde{\ell}_{kn}^{\gamma,\delta}(x) := \frac{\vartheta(x)\ell_{kn}^{\gamma,\delta}(x)}{\vartheta(x_{kn}^{\gamma,\delta})}\,, \quad k = 1, 2, \ldots, n\,,$$

we can write u_n as

$$u_n = \sum_{j=0}^{n-1} \alpha_{jn} \widetilde{p}_j = \sum_{k=1}^n \xi_{kn} \widetilde{\ell}_{kn}^{\gamma,\delta}\,.$$

If we introduce the Fourier projections

$$\mathcal{L}_n : \mathbf{L}_{\alpha,\beta}^2 \longrightarrow \mathbf{L}_{\alpha,\beta}^2, \quad u \mapsto \sum_{j=0}^{n-1} \langle u, \widetilde{p}_j \rangle_{\alpha,\beta} \widetilde{p}_j$$

and the weighted Lagrange interpolation operator

$$\widetilde{\mathcal{L}}_n^{\gamma,\delta} := \vartheta \mathcal{L}_n^{\gamma,\delta} \vartheta^{-1} \mathcal{I}\,,$$

then the collocation system (42) can be written as

$$\mathcal{A}_n u_n = f_n \tag{44}$$

with

$$\mathcal{A}_n = \widetilde{\mathcal{L}}_n^{\gamma,\delta}\Big(\mathcal{I} + c_- \mathcal{M}_{n,H_-}^- + c_+ \mathcal{M}_{n,H_+}^+ + \mathcal{K}_n\Big)\mathcal{L}_n\,. \tag{45}$$

Note also that, with the introduced notations, the assertion of Corollary 3 remains true for $Q \in \text{im } \mathcal{L}_n$, i.e.,

$$\mathcal{C}^{-1}\|Q\|^2_{\alpha,\beta} \le \frac{1}{n}\sum_{k=1}^{n} v^{\alpha+\frac{1}{2},\beta+\frac{1}{2}}(x^{\gamma,\delta}_{kn})\left|Q(x^{\gamma,\delta}_{kn})\right|^2 \le \mathcal{C}\|Q\|^2_{\alpha,\beta}, \qquad Q \in \text{im } \mathcal{L}_n,\ n \in \mathbb{N}, \tag{46}$$

if $-\frac{1}{2} < \rho - \gamma, \tau - \delta < \frac{3}{2}$. Indeed, for $Q = \vartheta P \in \text{im } \mathcal{L}_n$, we have

$$\mathcal{C}^{-1}\|Q\|^2_{\alpha,\beta} = \mathcal{C}^{-1}\|P\|^2_{\rho,\tau} \le \frac{\pi}{n}\sum_{k=1}^{n} v^{\rho+\frac{1}{2},\tau+\frac{1}{2}}(x^{\gamma,\delta}_{kn})\left|P(x^{\gamma,\delta}_{kn})\right|^2 \le \mathcal{C}\|P\|^2_{\rho,\tau} = \mathcal{C}\|Q\|^2_{\alpha,\beta}$$

and

$$v^{\rho+\frac{1}{2},\tau+\frac{1}{2}}(x^{\gamma,\delta}_{kn})\left|P(x^{\gamma,\delta}_{kn})\right|^2 = v^{\alpha+\frac{1}{2},\beta+\frac{1}{2}}(x^{\gamma,\delta}_{kn})\left|Q(x^{\gamma,\delta}_{kn})\right|^2.$$

It is well known that, in the investigation of numerical methods for operator equations, the stability of the respective operator sequences plays an essential role.

Definition 1. *We call the sequence $(\mathcal{A}_n)$ in (44) stable (in $\mathbf{L}^2_{\alpha,\beta}$) if, for all sufficiently large n, the operators $\mathcal{A}_n : \text{im } \mathcal{L}_n \longrightarrow \text{im } \mathcal{L}_n$ are invertible and if the norms $\|\mathcal{A}_n^{-1}\mathcal{L}_n\|_{\mathcal{L}(\mathbf{L}^2_{\alpha,\beta})}$ are uniformly bounded.*

If the method is stable and if $\mathcal{A}_n\mathcal{L}_n$ converges strongly to $\mathcal{A} \in \mathcal{L}(\mathbf{L}^2_{\alpha,\beta})$, then the operator $\mathcal{A}$ is injective. If additionally the image of $\mathcal{A}$ equals $\mathbf{L}^2_{\alpha,\beta}$, then (43) implies the $\mathbf{L}^2_{\alpha,\beta}$-convergence of the solution u_n of (44) to the (unique) solution $u \in \mathbf{L}^2_{\alpha,\beta}$ of (40). This can be seen from the estimate

$$\begin{aligned}\|\mathcal{L}_n u - u_n\|_{\alpha,\beta} &= \left\|\mathcal{A}_n^{-1}\mathcal{L}_n(\mathcal{A}_n\mathcal{L}_n u - \mathcal{A}_n u_n)\right\|_{\alpha,\beta} \\ &\le \left\|\mathcal{A}_n^{-1}\mathcal{L}_n\right\|_{\mathcal{L}(\mathbf{L}^2_{\alpha,\beta})}\Big(\|\mathcal{A}_n\mathcal{L}_n u - \mathcal{A}u\|_{\alpha,\beta} + \|f - f_n\|_{\alpha,\beta}\Big).\end{aligned}$$

4. C^*-Algebra Framework

In order to investigate the stability of the collocation-quadrature method, we use specific C^*-algebra techniques. With the help of those tools, we are able to transform our stability problem into an invertibility problem in an appropriate C^*-algebra. The sequence $(\mathcal{A}_n)$ is considered as an element of such a C^*-algebra. To define that algebra, we need some operators and spaces.

Let $n \in \mathbb{N}$ and

$$\mathbf{X}_1 = \mathbf{L}^2_{\alpha,\beta}, \quad \mathbf{X}_2 = \mathbf{X}_3 = \ell^2, \quad \mathcal{L}^{(1)}_n = \mathcal{L}_n, \quad \mathcal{L}^{(2)}_n = \mathcal{L}^{(3)}_n = \mathbb{P}_n,$$

where

$$\mathbb{P}_n : \ell^2 \to \ell^2, \quad (\xi_j)_{j=0}^\infty \mapsto (\xi_0, \ldots, \xi_{n-1}, 0, \ldots).$$

Let the operators $\mathbb{F}_n \in \mathcal{L}(\text{im } \mathbb{P}_n)$ be given by

$$\mathbb{F}_n(\xi_0, \xi_1, \ldots, \xi_{n-1}, 0, \ldots) = (\mathbb{F}_n)^{-1}(\xi_0, \xi_1, \ldots, \xi_{n-1}, 0, \ldots) = (\xi_{n-1}, \xi_{n-2}, \ldots, \xi_0, 0, \ldots).$$

Moreover, for $t \in T := \{1, 2, 3\}$, we define $\mathcal{E}^{(t)}_n : \text{im } \mathcal{L}_n \to \text{im } \mathcal{L}^{(t)}_n$ by

$$\mathcal{E}^{(1)}_n := \mathcal{L}_n, \quad \mathcal{E}^{(2)}_n := \mathcal{V}_n, \quad \mathcal{E}^{(3)}_n := \mathbb{F}_n\mathcal{V}_n,$$

where

$$\mathcal{V}_n u := \left(n^{-\frac{1}{2}} v^{\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}}(x^{\gamma,\delta}_{1n})u(x^{\gamma,\delta}_{1n}), \ldots, n^{-\frac{1}{2}} v^{\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}}(x^{\gamma,\delta}_{nn})u(x^{\gamma,\delta}_{nn}), 0, \ldots\right).$$

All operators $\mathcal{E}_n^{(t)}$ are invertible with

$$(\mathcal{E}_n^{(1)})^{-1} = \mathcal{L}_n|_{\text{im } \mathcal{L}_n}\,,$$

$$(\mathcal{E}_n^{(2)})^{-1} = \sqrt{n}\sum_{k=1}^{n} \xi_{k-1} v^{-\frac{\alpha}{2}-\frac{1}{4},-\frac{\beta}{2}-\frac{1}{4}}(x_{kn}^{\gamma,\delta})\widetilde{\ell}_{kn}^{\gamma,\delta}\,,$$

$$(\mathcal{E}_n^{(3)})^{-1} = \sqrt{n}\sum_{k=1}^{n} \xi_{n-k} v^{-\frac{\alpha}{2}-\frac{1}{4},-\frac{\beta}{2}-\frac{1}{4}}(x_{kn}^{\gamma,\delta})\widetilde{\ell}_{kn}^{\gamma,\delta}\,.$$

Moreover,

$$\left\|\mathcal{E}_n^{(t)}u\right\|_{\ell^2}^2 = \frac{1}{n}\sum_{k=1}^{n} v^{\alpha+\frac{1}{2},\beta+\frac{1}{2}}(x_{kn}^{\gamma,\delta})\left|u(x_{kn}^{\gamma,\delta})\right|^2, \quad u\in \text{im } \mathcal{L}_n\,, \quad t=2,3\,.$$

Whence, in view of (46), the following lemma is proved.

Lemma 12. *The operators $\mathcal{E}_n^{(j)} : \text{im } \mathcal{L}_n \longrightarrow \text{im } \mathcal{L}_n^{(t)}$, $n\in\mathbb{N}$, $t=1,2,3$, are uniformly bounded together with their inverses if $-\frac{1}{2} < \rho-\gamma, \tau-\delta < \frac{3}{2}$.*

Lemma 13. *For $r,t\in T$ with $r\neq t$, the operators $\mathcal{E}_n^{r,t} := \mathcal{E}_n^{(r)}\left(\mathcal{E}_n^{(t)}\right)^{-1}\mathcal{L}_n^{(t)}$ converge together with their adjoints weakly to the zero operator if $-\frac{1}{2} < \rho-\gamma, \tau-\delta < \frac{3}{2}$.*

Proof. Since the operators $\mathcal{E}_n^{r,t}$ are uniformly bounded, it suffices to verify the convergence on a dense subset. At first, we consider the operator $\mathcal{E}_n^{2,1} = \mathcal{V}_n\mathcal{L}_n$. Let $k,m\in\mathbb{N}_0$ be arbitrary but fixed and $n>\max\{m,k\}$ as well as $e_k = (\delta_{j,k})_{j=0}^{\infty}\in\ell^2$, $k\in\mathbb{N}_0$. Using (31), we get

$$\left|\langle \mathcal{V}_n\mathcal{L}_n\widetilde{p}_m, e_k\rangle_{\ell^2}\right| = n^{-\frac{1}{2}}v^{\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}}(x_{k+1,n}^{\gamma,\delta})\left|\widetilde{p}_m(x_{k+1,n}^{\gamma,\delta})\right| \le C\,n^{-\rho-1}\,.$$

Taking into account $\rho>-1$, we conclude the weak convergence of $\mathcal{E}_n^{2,1}$ to the zero operator. Similarly, using (32) instead of (31), we get

$$\left|\langle \mathbb{F}_n\mathcal{V}_n\mathcal{L}_n\widetilde{p}_m, e_k\rangle_{\ell^2}\right| = n^{-\frac{1}{2}}v^{\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}}(x_{n-k,n}^{\gamma,\delta})\left|\widetilde{p}_m(x_{n-k,n}^{\gamma,\delta})\right| \le C\,n^{-\tau-1}\,.$$

Hence, $\mathcal{E}_n^{3,1}$ converges weakly to the zero operator. Fix $k\in\mathbb{N}_0$ and $\widetilde{p}(x) = \vartheta(x)(1-x^2)p(x)$ with $p(x)$ being a polynomial. Then, for all sufficiently large $n\in\mathbb{N}$,

$$\begin{aligned}
\left\langle \mathcal{V}_n^{-1}\mathbb{P}_n e_k, \widetilde{p}\right\rangle_{\alpha,\beta} &= n^{\frac{1}{2}}v^{-\frac{\alpha}{2}-\frac{1}{4},-\frac{\beta}{2}-\frac{1}{4}}(x_{k+1,n}^{\gamma,\delta})\langle\widetilde{\ell}_{k+1,n}^{\gamma,\delta}, \vartheta v^{1,1}p\rangle_{\alpha,\beta} \\
&= n^{\frac{1}{2}}v^{-\frac{\rho}{2}-\frac{1}{4},-\frac{\tau}{2}-\frac{1}{4}}(x_{k+1,n}^{\gamma,\delta})\langle\ell_{k+1,n}^{\gamma,\delta}, v^{\rho-\gamma,\tau-\delta}v^{1,1}p\rangle_{\gamma,\delta} \\
&= n^{\frac{1}{2}}v^{-\frac{\rho}{2}-\frac{1}{4},-\frac{\tau}{2}-\frac{1}{4}}(x_{k+1,n}^{\gamma,\delta})\langle\ell_{k+1,n}^{\gamma,\delta}, \mathcal{S}_n^{\gamma,\delta}v^{\rho-\gamma,\tau-\delta}v^{1,1}p\rangle_{\gamma,\delta} \\
&= n^{\frac{1}{2}}v^{-\frac{\rho}{2}-\frac{1}{4},-\frac{\tau}{2}-\frac{1}{4}}(x_{k+1,n}^{\gamma,\delta})\,\lambda_{k+1,n}^{\gamma,\delta}\left(\mathcal{S}_n^{\gamma,\delta}v^{\rho-\gamma,\tau-\delta}v^{1,1}p\right)(x_{k+1,n}^{\gamma,\delta})
\end{aligned}$$

(for the definition of $\mathcal{S}_n^{\gamma,\delta}$, see (51) below). Choose $\psi,\chi\in\mathbb{R}$ such that

$$\frac{\gamma}{2}-\frac{1}{4} < \psi < \min\left\{1+\gamma, \frac{\gamma}{2}+\frac{3}{4}, \gamma-\frac{\rho}{2}+\frac{1}{2}\right\}$$

and

$$\frac{\delta}{2}-\frac{1}{4} < \chi < \min\left\{1+\delta, \frac{\delta}{2}+\frac{3}{4}, \delta-\frac{\tau}{2}+\frac{1}{2}\right\},$$

which is possible since $\frac{\gamma}{2} - \frac{1}{4} < \gamma - \frac{\rho}{2} + \frac{1}{2}$ is equivalent to $\rho - \gamma < \frac{3}{2}$. In virtue of [32] (2.2) there is a constant $\mathcal{C} \neq \mathcal{C}(n,k,f)$ such that

$$\left|\left(\mathcal{S}_n^{\gamma,\delta} f\right)(x_{kn}^{\gamma,\delta})\right| v^{\psi,\chi}(x_{kn}^{\gamma,\delta}) \le \mathcal{C} \sup\{v^{\psi,\chi}(x)|f(x)| : -1 < x < 1\} \ln(n+1), \quad n \in \mathbb{N}.$$

Note that, due to (35),

$$\frac{\lambda_{kn}^{\gamma,\delta}}{v^{\gamma+\frac{1}{2},\delta+\frac{1}{2}}(x_{kn}^{\gamma,\delta})} = \frac{\lambda_n^{\gamma,\delta}(x_{kn}^{\gamma,\delta})}{v^{\gamma+\frac{1}{2},\delta+\frac{1}{2}}(x_{kn}^{\gamma,\delta})} \sim_{n,k} \frac{1}{n}, \quad k = 1, \ldots, n, \tag{47}$$

Consequently,

$$\begin{aligned}
&\left|\left\langle \mathcal{V}_n^{-1}\mathbb{P}_n e_k, \widetilde{p}\right\rangle_{\alpha,\beta}\right| \\
&\quad \le \mathcal{C}\, n^{-\frac{1}{2}} v^{\gamma-\frac{\rho}{2}+\frac{1}{4}-\psi,\delta-\frac{\tau}{2}+\frac{1}{4}-\chi}(x_{k+1,n}^{\gamma,\delta}) v^{\psi,\chi}(x_{k+1,n}^{\gamma,\delta}) \left|\left(\mathcal{S}_n^{\gamma,\delta} v^{\rho-\gamma,\tau-\delta} v^{1,1} p\right)(x_{k+1,n}^{\gamma,\delta})\right| \\
&\quad \overset{(31)}{\le} \mathcal{C}\, n^{\rho-2\gamma-1+2\psi} \ln(n+1) \longrightarrow 0
\end{aligned}$$

because of $2\psi < 2\gamma - \rho + 1$. Thus, $\mathcal{E}_n^{1,2}$ converges weakly to zero. Analogously, we get the same for $\mathcal{E}_n^{1,3}$.

Let $s, t \in \{2,3\}$ with $s \neq t$. The weak convergence of $\mathcal{E}_n^{s,t}$ follows by the relations $\mathcal{E}_n^{2,3} = \mathcal{V}_n(\mathbb{F}_n\mathcal{V}_n)^{-1}\mathbb{P}_n = \mathbb{F}_n = \mathbb{F}_n\mathcal{V}_n^{-1}\mathcal{V}_n\mathbb{P}_n = \mathcal{E}_n^{3,2}$ and $\langle \mathbb{F}_n e_k, e_m\rangle = \langle e_{n-1-k}, e_m\rangle = 0$ if $n > m + k + 1$. Finally, note that with $\mathcal{E}_n^{r,t}$ also $\left(\mathcal{E}_n^{r,t}\right)^*$ converges weakly to the zero operator, which follows immediately from the definition of the weak convergence. The lemma is proved. □

For all what follows we assume that $\rho - \gamma, \tau - \delta \in \left(-\frac{1}{2}, \frac{3}{2}\right)$. By $\mathfrak{F}$ we denote the set of all sequences $(\mathcal{A}_n) := (\mathcal{A}_n)_{n=1}^\infty$ of linear operators $\mathcal{A}_n : \operatorname{im} \mathcal{L}_n \to \operatorname{im} \mathcal{L}_n$ for which the strong limits

$$\mathcal{W}_t(\mathcal{A}_n) := \lim_{n\to\infty} \mathcal{E}_n^{(t)} \mathcal{A}_n \left(\mathcal{E}_n^{(t)}\right)^{-1} \mathcal{L}_n^{(t)} \quad \text{and} \quad (\mathcal{W}_t(\mathcal{A}_n))^* = \lim_{n\to\infty} \left(\mathcal{E}_n^{(t)} \mathcal{A}_n \left(\mathcal{E}_n^{(t)}\right)^{-1} \mathcal{L}_n^{(t)}\right)^*$$

exist for all $t \in T$. If we provide $\mathfrak{F}$ with the algebraic operations

$$\lambda_1(\mathcal{A}_n) + \lambda_2(\mathcal{B}_n) := (\lambda_1\mathcal{A}_n + \lambda_2\mathcal{B}_n),\ \lambda_1, \lambda_2 \in \mathbb{C},$$

$$(\mathcal{A}_n)(\mathcal{B}_n) := (\mathcal{A}_n\mathcal{B}_n), \quad (\mathcal{A}_n)^* = (\mathcal{A}_n^*),$$

and the supremum norm

$$\|(\mathcal{A}_n)\|_{\mathfrak{F}} := \sup_{n\ge1} \|\mathcal{A}_n\mathcal{L}_n\|_{\mathcal{L}(\mathbf{L}^2_{\alpha,\beta})},$$

one can easily check, that $\mathfrak{F}$ becomes a $\mathcal{C}^*$-algebra with the identity element $(\mathcal{L}_n)$.

Corollary 4. *Let $\mathcal{T}_t \in \mathcal{L}(\mathbf{X}_t)$, $t \in T$ be compact operators, i.e., $\mathcal{T}_t \in \mathcal{K}(\mathbf{X}_t)$. Then the sequences $\left(\mathcal{A}_n^{(t)}\right)$ with $\mathcal{A}_n^{(t)} = \left(\mathcal{E}_n^{(t)}\right)^{-1} \mathcal{L}_n^{(t)} \mathcal{T}_t \mathcal{E}_n^{(t)}$ belong to $\mathfrak{F}$, where*

$$\mathcal{E}_n^{(r)} \mathcal{A}_n^{(t)} \left(\mathcal{E}_n^{(r)}\right)^{-1} \mathcal{L}_n^{(r)} \longrightarrow 0 \quad \text{and} \quad \left(\mathcal{E}_n^{(r)} \mathcal{A}_n^{(t)} \left(\mathcal{E}_n^{(r)}\right)^{-1} \mathcal{L}_n^{(r)}\right)^* \longrightarrow 0, \quad n \longrightarrow \infty,$$

strongly in $\mathbf{X}_r$ for $r \in T, r \neq t$.

Proof. This is due to

$$\mathcal{E}_n^{(r)}\mathcal{A}_n^{(t)}\left(\mathcal{E}_n^{(r)}\right)^{-1}\mathcal{L}_n^{(r)} = \mathcal{E}_n^{r,t}\mathcal{T}_t\mathcal{E}_n^{t,r} \quad \text{and} \quad \left(\mathcal{E}_n^{(r)}\mathcal{A}_n^{(t)}\left(\mathcal{E}_n^{(r)}\right)^{-1}\mathcal{L}_n^{(r)}\right)^* = (\mathcal{E}_n^{t,r})^*\mathcal{T}_t^*(\mathcal{E}_n^{r,t})^*,$$

the compactness of $\mathcal{T}_t : \mathbf{X}_t \longrightarrow \mathbf{X}_t$ and the weak convergence of $\mathcal{E}_n^{r,t}$ as well as $\left(\mathcal{E}_n^{r,t}\right)^*$ to the zero operator if $r \neq t$ (see Corollary 4). □

Corollary 5. *The mappings $\mathcal{W}_t : \mathfrak{F} \longrightarrow \mathcal{L}(\mathbf{X}_t)$, $t \in T$, are unital *-homomorphisms with norm* 1.

Proof. Of course $\mathcal{W}_t : \mathfrak{F} \longrightarrow \mathcal{L}\left(\mathbf{X}^{(t)}\right)$, $t \in T$ are*- homomorphisms. The relation $\|\mathcal{W}_t\|_{\mathcal{L}(\mathbf{X}^{(t)})} = 1$ follows from the fact that *-homomorphism are bounded by 1 and that $\|\mathcal{W}_t(\mathcal{L}_n)\|_{\mathcal{L}(\mathbf{X}^{(t)})} = 1$. □

The convergence

$$\mathcal{E}_n^{(t)}\mathcal{A}_n^{(t)}\left(\mathcal{E}_n^{(t)}\right)^{-1}\mathcal{L}_n^{(t)} = \mathcal{L}_n^{(t)}\mathcal{T}_t\mathcal{L}_n^{(t)} \longrightarrow \mathcal{T}_t \quad \text{in} \quad \mathbf{X}_t \quad (\text{ even in } \mathcal{L}(\mathbf{X}_t))$$

together with Corollary 4 deliver that

$$\mathfrak{J} := \left\{ \left(\sum_{t\in T} \left(\mathcal{E}_n^{(t)}\right)^{-1}\mathcal{L}_n^{(t)}\mathcal{T}_t\mathcal{E}_n^{(t)} + \mathcal{C}_n \right) : (\mathcal{C}_n) \in \mathfrak{N},\ \mathcal{T}_t \in \mathcal{K}(\mathbf{X}_t),\ t \in T \right\}$$

is a subset of $\mathfrak{F}$, where $\mathfrak{N} \subset \mathfrak{F}$ is the two-sided closed ideal of $\mathfrak{F}$ of all sequences $(\mathcal{A}_n) \in \mathfrak{F}$ with $\|\mathcal{A}_n\mathcal{L}_n\|_{\mathcal{L}(\mathbf{L}^2_{\alpha,\beta})} \longrightarrow 0$.

Proposition 2 ([33,34], Theorem 10.33)**.** *The set $\mathfrak{J}$ forms a two-sided closed ideal in the C^*-algebra $\mathfrak{F}$. Moreover, a sequence $(\mathcal{A}_n) \in \mathfrak{F}$ is stable if and only if the operators $\mathcal{W}_t(\mathcal{A}_n) : \mathbf{X}_t \longrightarrow \mathbf{X}_t$, $t \in T$, and the coset $(\mathcal{A}_n) + \mathfrak{J} \in \mathfrak{F}/\mathfrak{J}$ are invertible.*

5. The Limit Operators of the Collocation-Quadrature Methods

In this section, under certain conditions, we prove that the sequence $(\mathcal{A}_n)$ of the collocation quadrature method, defined in (45), belongs to the algebra $\mathfrak{F}$ from the previous section. We do this by determine the limit operators $\mathcal{W}_t(\mathcal{A}_n)$, $t = 1,2,3$, and proving that also the sequences of the respective adjoint operators converge strongly.

The following Lemma is due to [35] (Satz III.2.1). Recall the definition of $\mathbf{R}^0_{\psi,\chi}$ in (13) and the equalities (14).

Lemma 14. *Let $\gamma, \delta > -1$.*

(a) *If $f \in \mathbf{R}^0_{1+\gamma,1+\delta}$, then $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \lambda_{kn}^{\gamma,\delta} f(x_{kn}^{\gamma,\delta}) = \int_{-1}^{1} f(x)v^{\gamma,\delta}(x)\,dx$.*

(b) *If $f \in \mathbf{R}^0_{\frac{1+\gamma}{2},\frac{1+\delta}{2}}$, then $\lim\limits_{n\to\infty} \left\|\mathcal{L}_n^{\gamma,\delta} f - f\right\|_{\gamma,\delta} = 0$.*

Lemma 15. *Let $\psi, \chi \in \mathbb{R}$, $\alpha, \beta, \gamma, \delta > -1$, and $f \in \mathbf{R}^0_{\frac{1+\alpha}{2},\frac{1+\beta}{2}}$, i.e.,*

$$|f(x)| \le C\,(1-x)^{\varepsilon-\frac{1+\alpha}{2}}\,(1+x)^{\varepsilon-\frac{1+\beta}{2}}, \quad x \in (-1,1)$$

for some $\varepsilon > 0$ (cf. (14)). Then we have

$$v^{\psi,\chi}\mathcal{L}_n^{\gamma,\delta}v^{-\psi,-\chi}f \longrightarrow f \quad \textit{in} \quad \mathbf{L}^2_{\alpha,\beta},$$

for $n \longrightarrow \infty$, *if*

$$-\frac{1}{2} < \alpha + 2\psi - \gamma < \frac{3}{2} \quad \text{and} \quad \alpha + 2\psi > -1\,,$$

as well as

$$-\frac{1}{2} < \beta + 2\chi - \delta < \frac{3}{2} \quad \text{and} \quad \beta + 2\chi > -1\,.$$

Proof. Let $\delta_1 > 0$ be arbitrary chosen and $n \in \mathbb{N}$. Since $\alpha + 2\psi, \beta + 2\chi > -1$ and $\left|v^{-\psi,-\chi}(x)f(x)\right|^2 v^{\alpha+2\psi,\beta+2\chi}(x) \le \mathcal{C}(1-x^2)^{2\varepsilon-1}$, we can choose a polynomial $p(x)$, such that $\|f - v^{\psi,\chi}p\|^2_{\alpha,\beta} = \|v^{-\psi,-\chi}f - p\|^2_{\alpha+2\psi,\beta+2\chi} < \delta_1$. For $n > \deg p$, we have

$$\left\|v^{\psi,\chi}\mathcal{L}_n^{\gamma,\delta}v^{-\psi,-\chi}f - f\right\|^2_{\alpha,\beta} \le 2\left(\left\|v^{\psi,\chi}\mathcal{L}_n^{\gamma,\delta}(v^{-\psi,-\chi}f - p)\right\|^2_{\alpha,\beta} + \|v^{\psi,\chi}p - f\|^2_{\alpha,\beta}\right).$$

By using Corollary 3 for $\alpha + 2\psi$ and $\beta + 2\chi$ instead of α and β, relation (35) for γ, δ instead of α, β, and Lemma 14, (a) we get

$$\begin{aligned}
\left\|v^{\psi,\chi}\mathcal{L}_n^{\gamma,\delta}(v^{-\psi,-\chi}f - p)\right\|^2_{\alpha,\beta} &= \left\|\mathcal{L}_n^{\gamma,\delta}(v^{-\psi,-\chi}f - p)\right\|^2_{\alpha+2\psi,\beta+2\chi} \\
&\le \frac{\mathcal{C}}{n}\sum_{k=1}^{n} v^{2\psi+\alpha+\frac{1}{2},2\chi+\beta+\frac{1}{2}}(x_{kn}^{\gamma,\delta})\left|(v^{-\psi,-\chi}f - p)(x_{kn}^{\gamma,\delta})\right|^2 \\
&\le \mathcal{C}\sum_{k=1}^{n} \lambda_{kn}^{\gamma,\delta} v^{2\psi+\alpha-\gamma,2\chi+\beta-\delta}(x_{kn}^{\gamma,\delta})\left|(v^{-\psi,-\chi}f - p)(x_{kn}^{\gamma,\delta})\right|^2 \\
&\longrightarrow \quad \mathcal{C}\int_{-1}^{1}\left|(v^{-\psi,-\chi}f - p)(x)\right|^2 v^{2\psi+\alpha,2\chi+\beta}(x)\,dx = \mathcal{C}\|f - v^{-\psi,-\chi}p\|^2_{\alpha,\beta}
\end{aligned}$$

for $n \longrightarrow \infty$ and some constant $\mathcal{C} \ne \mathcal{C}(n,f,p)$, where we took into account

$$v^{1+\gamma,1+\delta}(x)v^{2\psi+\alpha-\gamma,2\chi+\beta-\delta}(x)\left|v^{-\psi,-\chi}(x)f(x)\right|^2 \le \mathcal{C}(1-x^2)^{2\varepsilon}$$

and

$$v^{1+\gamma,1+\delta}(x)v^{2\psi+\alpha-\gamma,2\chi+\beta-\delta}(x)|p(x)|^2 \le \mathcal{C}v^{1+2\psi+\alpha,1+2\chi+\beta}(x) \le \mathcal{C}(1-x^2)^{\varepsilon_1}$$

with $\varepsilon_1 = \min\{1 + 2\psi + \alpha, 1 + 2\chi + \beta\} > 0$, which shows the applicability of Lemma 14, (a). Thus,

$$\limsup_{n\to\infty}\left\|v^{\psi,\chi}\mathcal{L}_n^{\gamma,\delta}v^{-\psi,-\chi}f - f\right\|^2_{\alpha,\beta} \le 2(\mathcal{C}+1)\delta_1\,.$$

Since this is true for all $\delta_1 > 0$, we get the assertion. □

The following corollary is an immediate consequence of the previous lemma and concerned with the case $\psi = \frac{\rho-\alpha}{2}$, $\chi = \frac{\tau-\beta}{2}$.

Corollary 6. *Let* $\alpha, \beta, \gamma, \delta, \rho, \tau > -1$. *If* $f \in \mathbf{R}^0_{\frac{1+\alpha}{2},\frac{1+\beta}{2}}$ *and* $\rho - \gamma, \tau - \delta \in \left(-\frac{1}{2}, \frac{3}{2}\right)$, *then* $\widetilde{\mathcal{L}}_n^{\gamma,\delta}f \longrightarrow f$ *in* $\mathbf{L}^2_{\alpha,\beta}$.

Lemma 16. *Let* $0 \le \psi < \frac{1+\alpha}{2}$, $0 \le \chi < \frac{1+\beta}{2}$, *and* $\alpha - \gamma, \beta - \delta \in \left(-\frac{1}{2}, \frac{3}{2}\right)$. *Moreover, let*

$$[-1,1] \times [-1,1] \longrightarrow \mathbb{C}, \quad (x,y) \mapsto f(x,y)v^{\psi,\chi}(y)$$

be a continuous function. Then

$$\lim_{n\to\infty} \sup\Big\{ \|\mathcal{L}_n^{\gamma,\delta} f(x,\cdot) - f(x,\cdot)\|_{\alpha,\beta} : -1 \le x \le 1 \Big\} = 0 .$$

Proof. Fix ψ_0, χ_0 such that $\psi < \psi_0 < \frac{1+\alpha}{2}$ and $\chi < \chi_0 < \frac{1+\beta}{2}$. By assumption $f^x \in \mathbf{C}_{\psi_0,\chi_0}$ for all $x \in [-1,1]$, where $f^x(y) := f(x,y)$. Moreover,

$$\lim_{x\to x_0} \|f^x - f^{x_0}\|_{\psi_0,\chi_0,\infty} = 0 \quad \text{for all} \quad x_0 \in [-1,1] .$$

Suppose the assertion of the lemma is not true. Then, there are an $\varepsilon > 0$ and a sequence $n_1 < n_2 < \cdots$ of natural numbers, satisfying

$$\sup\Big\{ \Big\|\mathcal{L}_{n_k}^{\gamma,\delta} f^x - f^x\Big\|_{\alpha,\beta} \,:\, -1 \le x \le 1 \Big\} \ge 2\varepsilon \quad \text{for all} \quad k \in \mathbb{N}.$$

Hence, for every $k \in \mathbb{N}$, there is an $x_k \in [-1,1]$ such that $\Big\|\mathcal{L}_{n_k}^{\gamma,\delta} f^{x_k} - f^{x_k}\Big\|_{\alpha,\beta} \ge \varepsilon$, and we can assume that $x_k \longrightarrow x^* \in [-1,1]$ for $k \longrightarrow \infty$. Due to our assumptions we have

$$M_0 := \sqrt{\int_{-1}^{1} (1-x)^{\alpha-2\psi_0}(1+x)^{\beta-2\chi_0}\,dx} < \infty$$

and, by Lemma 15 (choose $\psi = \chi = 0$ and use $\alpha-\gamma, \beta-\delta \in \left(-\frac{1}{2}, \frac{3}{2}\right)$) and the Banach-Steinhaus theorem,

$$M_1 := \sup\Big\{ \|\mathcal{L}_n^{\gamma,\delta}\|_{\mathbf{C}_{\psi_0,\chi_0} \to \mathbf{L}^2_{\alpha,\beta}} : n \in \mathbb{N} \Big\} < \infty.$$

Moreover, there is an $k_0 \in \mathbb{N}$ such that

$$\Big\|\mathcal{L}_{n_k}^{\gamma,\delta} f^{x^*} - f^{x^*}\Big\|_{\alpha,\beta} < \frac{\varepsilon}{3} \quad \text{and} \quad \Big\|f^{x_k} - f^{x^*}\Big\|_{\psi_0,\chi_0,\infty} < \frac{\varepsilon}{3\max\{M_0, M_1\}} \quad \text{for all} \quad k \ge k_0.$$

For $k \ge k_0$, we get the contradiction

$$\begin{aligned} \varepsilon &\le \Big\|\mathcal{L}_{n_k}^{\gamma,\delta} f^{x_k} - f^{x_k}\Big\|_{\alpha,\beta} \\ &\le \Big\|\mathcal{L}_{n_k}^{\gamma,\delta} (f^{x_k} - f^{x^*})\Big\|_{\alpha,\beta} + \Big\|\mathcal{L}_{n_k}^{\gamma,\delta} f^{x^*} - f^{x^*}\Big\|_{\alpha,\beta} + \Big\|f^{x^*} - f^{x_k}\Big\|_{\alpha,\beta} \\ &\le M_1 \Big\|f^{x_k} - f^{x^*}\Big\|_{\psi_0,\chi_0,\infty} + \frac{\varepsilon}{3} + M_0 \Big\|f^{x^*} - f^{x_k}\Big\|_{\psi_0,\chi_0,\infty} < \varepsilon , \end{aligned}$$

and the lemma is proved. □

Lemma 17. *Let $\alpha, \beta \in (-1,1)$ and $\eta, \zeta, \psi, \chi \in \mathbb{R}$ such that*

$$0 \le \eta < \frac{1+\alpha}{2} , \; 0 \le \zeta < \frac{1+\beta}{2} \quad \textit{and} \quad 0 \le \psi < \frac{1-\alpha}{2} , \; 0 \le \chi < \frac{1-\beta}{2} .$$

Moreover, let the map

$$[-1,1] \times [-1,1] \longrightarrow \mathbb{C}, \quad (x,y) \mapsto v^{\eta,\zeta}(x) K(x,y) v^{\psi,\chi}(y)$$

be continuous. If $\rho - \gamma, \tau - \delta \in \left(-\frac{1}{2}, \frac{3}{2}\right)$ *holds true, then*

$$\lim_{n\to\infty} \left\| \widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K} \mathcal{L}_n - \mathcal{L}_n \mathcal{K} \mathcal{L}_n \right\|_{\mathcal{L}\left(\mathbf{L}^2_{\alpha,\beta}\right)} = 0\,. \tag{48}$$

That means $\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K} \mathcal{L}_n\right) \in \mathfrak{J}$ *with*

$$\mathcal{W}_1\big(\widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K} \mathcal{L}_n\big) = \mathcal{K} \quad \textit{and} \quad \mathcal{W}_t\big(\widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K} \mathcal{L}_n\big) = \Theta, \quad t \in \{2,3\}. \tag{49}$$

Proof. By Corollary 6 we have $\widetilde{\mathcal{L}}_n^{\gamma,\delta} f \longrightarrow f$ in $\mathbf{L}^2_{\alpha,\beta}$ for all $f \in \widetilde{\mathbf{C}}_{\eta,\zeta}$. Together with the compactness of the operator $\mathcal{K} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \widetilde{\mathbf{C}}_{\eta,\zeta}$ (see Lemma 4) and the strong convergence $\mathcal{L}_n^* = \mathcal{L}_n \longrightarrow \mathcal{I}$ in $\mathbf{L}^2_{\alpha,\beta}$, this leads to the limit relation (48). Hence, $\widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K} \mathcal{L}_n = \mathcal{L}_n \mathcal{K} \mathcal{L}_n + \mathcal{C}_n$ with $(\mathcal{C}_n) \in \mathfrak{N}$ and consequently, by definition, $\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K} \mathcal{L}_n\right) \in \mathfrak{J}$. Corollary 4 yields (49). □

Define

$$\begin{aligned}\Omega = &\left\{(\omega_1,\omega_2) \in \mathbb{R}^2 : \omega_1,\omega_2 > -1,\, -\frac{1}{2} < \omega_1 - \omega_2 < \frac{1}{2},\, \omega_1 < 2\omega_2 + 1\right\} \\ &\cup \left\{(\omega_1,\omega_2) \in \mathbb{R}^2 : \omega_1,\omega_2 > -1,\, \frac{1}{2} < \omega_1 - \omega_2 < \frac{3}{2}\right\}.\end{aligned} \tag{50}$$

Furthermore, for $\gamma, \delta > -1$, by $\mathcal{S}_n^{\gamma,\delta}$ we refer to the Fourier operator given by

$$\mathcal{S}_n^{\gamma,\delta} f = \sum_{j=0}^{n-1} \langle f, p_j^{\gamma,\delta} \rangle_{\gamma,\delta} p_j^{\gamma,\delta}\,. \tag{51}$$

From [36] (Theorem 1) we infer the following lemma.

Lemma 18. *Let* $\alpha, \beta, \gamma, \delta, > -1$. *Then, there is a constant* $\mathcal{C} \neq \mathcal{C}(n, f)$ *such that, for all* $n \in \mathbb{N}$ *and* $f \in \mathbf{L}^2_{\alpha,\beta}$, *the inequality*

$$\left\| \mathcal{S}_n^{\gamma,\delta} f \right\|_{\alpha,\beta} \le \mathcal{C} \|f\|_{\alpha,\beta}$$

holds true if and only if

$$|\alpha - \gamma| < \min\left\{\frac{1}{2}, 1+\gamma\right\} \quad \textit{and} \quad |\beta - \delta| < \min\left\{\frac{1}{2}, 1+\delta\right\},$$

which is equivalent to

$$-\frac{1}{2} < \alpha - \gamma, \beta - \delta < \frac{1}{2}, \quad \alpha < 2\gamma + 1\,, \quad \textit{and} \quad \beta < 2\delta + 1\,. \tag{52}$$

Corollary 7. *Let* $a \in \mathbf{PC}$ *and* $\alpha, \beta, \gamma, \delta, \gamma_0, \delta_0 > -1$. *If*

$$-\frac{1}{2} < \alpha - \gamma, \beta - \delta < \frac{3}{2} \tag{53}$$

and

$$-\frac{1}{2} < \alpha - \gamma_0, \beta - \delta_0 < \frac{1}{2}, \quad \alpha < 2\gamma_0 + 1\,, \quad \beta < 2\delta_0 + 1 \tag{54}$$

are satisfied, then $\mathcal{L}_n^{\gamma,\delta} a \mathcal{S}_n^{\gamma_0,\delta_0} \longrightarrow a\mathcal{I}$ *strongly in* $\mathbf{L}^2_{\alpha,\beta}$.

Proof. The set $\mathbf{P}$ of all algebraic polynomials is dense in $\mathbf{L}^2_{\alpha,\beta}$. Moreover, for every $p \in \mathbf{P}$, we have $\mathcal{L}_n^{\gamma,\delta} a \mathcal{S}_n^{\gamma_0,\delta_0} p = \mathcal{L}_n^{\gamma,\delta} ap$ for all sufficiently large n and, by Lemma 15, $\mathcal{L}_n^{\gamma,\delta} ap \longrightarrow ap$ in $\mathbf{L}^2_{\alpha,\beta}$. Additionally, in view of Corollary 3 and Lemma 18, for $f \in \mathbf{L}^2_{\alpha,\beta}$ we can

$$\begin{aligned}
\left\| \mathcal{L}_n^{\gamma,\delta} a \mathcal{S}_n^{\gamma_0,\delta_0} u \right\|^2_{\alpha,\beta} &\le \frac{\mathcal{C}}{n} \sum_{k=1}^n v^{\alpha+\frac{1}{2},\beta+\frac{1}{2}}(x_{kn}^{\gamma,\delta}) \left| \left(a \mathcal{S}_n^{\gamma_0,\delta_0} f \right)(x_{kn}^{\gamma,\delta}) \right|^2 \\
&\le \|a\|^2_\infty \frac{\mathcal{C}}{n} \sum_{k=1}^n v^{\alpha+\frac{1}{2},\beta+\frac{1}{2}}(x_{kn}^{\gamma,\delta}) \left| \left(\mathcal{S}_n^{\gamma_0,\delta_0} f \right)(x_{kn}^{\gamma,\delta}) \right|^2 \\
&\le \mathcal{C} \|a\|^2_\infty \left\| \mathcal{S}_n^{\gamma_0,\delta_0} f \right\|^2_{\alpha,\beta} \le \mathcal{C} \|a\|^2_\infty \|u\|^2_{\alpha,\beta} .
\end{aligned}$$

Hence, the operators $\mathcal{L}_n^{\gamma,\delta} a \mathcal{S}_n^{\gamma_0,\delta_0} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$, $n \in \mathbb{N}$, are uniformly bounded. Now, the Banach-Steinhaus theorem gives the assertion. □

In the following Corollary we apply the previous Corollary to operators of the form

$$\vartheta v^{\gamma_0-\rho,\delta_0-\tau} \mathcal{L}_n^{\gamma,\delta} a \mathcal{S}_n^{\gamma_0,\delta_0} \vartheta^{-1} v^{\rho-\gamma_0,\tau-\delta_0} \mathcal{I} = v^{\gamma_0-\frac{\rho+\alpha}{2},\delta_0-\frac{\tau+\beta}{2}} \mathcal{L}_n^{\gamma,\delta} a \mathcal{S}_n^{\gamma_0,\delta_0} v^{\frac{\rho+\alpha}{2}-\gamma_0,\frac{\tau+\beta}{2}-\delta_0} \mathcal{I} .$$

Corollary 8. *Let $a \in \mathbf{PC}$ and $\alpha, \beta, \gamma, \delta, \gamma_0, \delta_0, \rho, \tau > -1$. Then,*

$$\vartheta v^{\gamma_0-\rho,\delta_0-\tau} \mathcal{L}_n^{\gamma,\delta} a \mathcal{S}_n^{\gamma_0,\delta_0} \vartheta^{-1} v^{\rho-\gamma_0,\tau-\delta_0} \mathcal{I} \longrightarrow a\mathcal{I}$$

strongly in $\mathbf{L}^2_{\alpha,\beta}$ if

$$-\frac{1}{2} < 2\gamma_0 - \rho - \gamma, 2\delta_0 - 2\tau - \delta < \frac{3}{2}, \tag{55}$$

and

$$-\frac{1}{2} < \rho - \gamma_0, \tau - \delta_0 < \frac{1}{2}, \quad \rho < 2\gamma_0 + 1, \quad \tau < 2\delta_0 + 1 . \tag{56}$$

Proof. The claimed strong convergence is equivalent to the strong convergence of the operators $\mathcal{L}_n^{\gamma,\delta} a \mathcal{S}_n^{\gamma_0,\delta_0}$ to $a\mathcal{I}$ in $\mathbf{L}^2_{2\gamma_0-\rho,2\delta_0-\tau}$. For $\alpha = 2\gamma_0 - \rho$ and $\beta = 2\delta_0 - \tau$, the conditions $\alpha, \beta > -1$, (53) and (54) are equivalent to (55) and (56). Hence, Corollary 7 is applicable. □

Lemma 19. *Let $\alpha, \beta \in (-1, 1)$ and $\eta, \zeta, \psi, \chi \in \mathbb{R}$ such that*

$$0 \le \eta < \frac{1+\alpha}{2}, \, 0 \le \zeta < \frac{1+\beta}{2} \quad \text{and} \quad 0 \le \psi < \frac{1-\alpha}{2}, \, 0 \le \chi < \frac{1-\beta}{2},$$

and such that the function

$$[-1,1] \times [-1,1] \longrightarrow \mathbb{C}, \quad (x,y) \mapsto v^{\eta,\zeta}(x) K(x,y) v^{\psi,\chi}(y)$$

is continuous. Let $(\rho,\gamma), (\tau,\delta) \in \Omega$. Then $\lim\limits_{n\to\infty} \left\| \widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K}_n \mathcal{L}_n - \mathcal{L}_n \mathcal{K} \mathcal{L}_n \right\|_{\mathcal{L}\left(\mathbf{L}^2_{\alpha,\beta}\right)} = 0$. That means (cf. Lemma 17) $\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K}_n \mathcal{L}_n \right) \in \mathfrak{J}$ with

$$\mathcal{W}_1\big(\widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K}_n \mathcal{L}_n\big) = \mathcal{K} \quad \text{and} \quad \mathcal{W}_t\big(\widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K}_n \mathcal{L}_n\big) = \Theta, \quad t \in \{2,3\}. \tag{57}$$

Proof. First of all, we notice that $\mathcal{K} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$ and $\mathcal{K} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \widetilde{\mathbf{C}}_{\eta,\zeta}$ are well-defined and compact (cf. Corollary 1 and Lemma 4). Choose $\eta_0 \in \left(\eta, \frac{1+\alpha}{2}\right)$ and $\zeta_0 \in \left(\zeta, \frac{1+\beta}{2}\right)$. By using Corollary 6 we get

$$\begin{aligned}
&\left\| \widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K}_n \mathcal{L}_n - \mathcal{L}_n \mathcal{K} \mathcal{L}_n \right\|_{\mathcal{L}\left(\mathbf{L}^2_{\alpha,\beta}\right)} \\
&\qquad \le \left\| \widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K}_n \mathcal{L}_n - \widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K} \mathcal{L}_n \right\|_{\mathcal{L}\left(\mathbf{L}^2_{\alpha,\beta}\right)} + \left\| \widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K} \mathcal{L}_n - \mathcal{L}_n \mathcal{K} \mathcal{L}_n \right\|_{\mathcal{L}\left(\mathbf{L}^2_{\alpha,\beta}\right)} \\
&\qquad \le \mathcal{C} \left\| \mathcal{K}_n \mathcal{L}_n - \mathcal{K} \mathcal{L}_n \right\|_{\mathcal{L}\left(\mathbf{L}^2_{\alpha,\beta}, \mathbf{C}_{\eta_0,\zeta_0}\right)} + \left\| \widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K} \mathcal{L}_n - \mathcal{L}_n \mathcal{K} \mathcal{L}_n \right\|_{\mathcal{L}\left(\mathbf{L}^2_{\alpha,\beta}\right)}.
\end{aligned}$$

Since, due to Lemma 17, $\left\| \widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{K} \mathcal{L}_n - \mathcal{L}_n \mathcal{K} \mathcal{L}_n \right\|_{\mathcal{L}\left(\mathbf{L}^2_{\alpha,\beta}\right)} \longrightarrow 0$ holds true, it suffices to verify the convergence

$$\left\| \mathcal{K}_n \mathcal{L}_n - \mathcal{K} \mathcal{L}_n \right\|_{\mathcal{L}\left(\mathbf{L}^2_{\alpha,\beta}, \mathbf{C}_{\eta_0,\zeta_0}\right)} \longrightarrow 0. \tag{58}$$

We define $r, s \in \{0, 1\}$ by

$$r = \begin{cases} 0 & : \quad -\frac{1}{2} < \rho - \gamma < \frac{1}{2}, \\ 1 & : \quad \frac{1}{2} < \rho - \gamma < \frac{3}{2}, \end{cases} \quad \text{and} \quad s = \begin{cases} 0 & : \quad -\frac{1}{2} < \tau - \delta < \frac{1}{2}, \\ 1 & : \quad \frac{1}{2} < \tau - \delta < \frac{3}{2}. \end{cases} \tag{59}$$

Recall that the application of the operator $\mathcal{K}_n$ to a function $u \in \mathbf{C}(-1, 1)$ can be written as (see (41))

$$(\mathcal{K}_n u)(x) = \int_{-1}^{1} \mathcal{L}_n^{\gamma,\delta} \left[v^{-\gamma,-\delta} K(x, \cdot) u \right](y)\, v^{\gamma,\delta}(y)\, dy.$$

We define

$$\begin{aligned}
\left(\widetilde{\mathcal{K}}_n u\right)(x) &= \int_{-1}^{1} \mathcal{L}_n^{\gamma,\delta} \left[v^{-\gamma,-\delta} K(x, \cdot) \vartheta v^{-r,-s} \right](y) \left(\vartheta^{-1} u\right)(y)\, v^{\gamma+r,\delta+s}(y)\, dy \\
&= \int_{-1}^{1} \mathcal{L}_n^{\gamma,\delta} \left[K(x, \cdot) v^{\frac{\rho-\alpha}{2} - \gamma - r, \frac{\tau-\beta}{2} - \delta - s} \right](y) \left(\vartheta^{-1} u\right)(y)\, v^{\gamma+r,\delta+s}(y)\, dy.
\end{aligned}$$

Let $u_n = \vartheta p_n \in \operatorname{im} \mathcal{L}_n$, i.e., $p_n \in \mathbf{P}_n$. Due to the algebraic accuracy of the Gaussian rule, in case of $r + s \le 1$ as well as in case of $\deg p_n < n - 1$ and $r = s = 1$, we have

$$\left(\widetilde{\mathcal{K}}_n u_n\right)(x) = \int_{-1}^{1} \mathcal{L}_n^{\gamma,\delta} \left[v^{-\gamma,-\delta} K(x, \cdot) \vartheta v^{-r,-s} \right](y)\, (v^{r,s} p_n)(y)\, v^{\gamma,\delta}(y)\, dy = (\mathcal{K}_n u_n)(x).$$

In case of $\deg p_n = n - 1$ and $r = s = 1$, we write $p_n = \varepsilon_n p_{n-1}^{\gamma+1,\delta+1} + p_{n-1}$ with $\deg p_{n-1} < n - 1$, i.e., $p_{n-1} = \mathcal{S}_{n-1}^{\gamma+1,\delta+1} p_n$ and $\varepsilon_n p_{n-1}^{\gamma+1,\delta+1} = \left(\mathcal{S}_n^{\gamma+1,\delta+1} - \mathcal{S}_{n-1}^{\gamma+1,\delta+1} \right) p_n$. We get, due to the previous considerations, $\widetilde{\mathcal{K}}_n u_n = \varepsilon_n \widetilde{\mathcal{K}}_n \vartheta p_{n-1}^{\gamma+1,\delta+1} + \mathcal{K}_n \vartheta p_{n-1}$. Moreover,

$$
\begin{aligned}
\left(\widetilde{\mathcal{K}}_n \vartheta p_{n-1}^{\gamma+1,\delta+1}\right)(x) &= \int_{-1}^{1} \mathcal{L}_n^{\gamma,\delta}\left[v^{-\gamma,-\delta}K(x,\cdot)\vartheta v^{-1,-1}\right](y)\,(1-y^2)p_{n-1}^{\gamma+1,\delta+1}(y)\,v^{\gamma,\delta}(y)\,dy \\
&= A_n(\gamma,\delta)\int_{-1}^{1} \mathcal{L}_n^{\gamma,\delta}\left[v^{-\gamma,-\delta}K(x,\cdot)\vartheta v^{-1,-1}\right](y)\,p_{n-1}^{\gamma,\delta}(y)\,v^{\gamma,\delta}(y)\,dy \\
&= A_n(\gamma,\delta)\sum_{k=1}^{n} \lambda_{kn}^{\gamma,\delta} v^{-\gamma,-\delta}(x_{kn}^{\gamma,\delta})K(x,x_{kn}^{\gamma,\delta})\vartheta(x_{kn}^{\gamma,\delta})\frac{p_{n-1}^{\gamma,\delta}(x_{kn}^{\gamma,\delta})}{1-(x_{kn}^{\gamma,\delta})^2} \\
&= A_n(\gamma,\delta)c_n(\gamma,\delta)\sum_{k=1}^{n} \lambda_{kn}^{\gamma,\delta} v^{-\gamma,-\delta}(x_{kn}^{\gamma,\delta})K(x,x_{kn}^{\gamma,\delta})\vartheta(x_{kn}^{\gamma,\delta})p_{n-1}^{\gamma+1,\delta+1}(x_{kn}^{\gamma,\delta}) \\
&= A_n(\gamma,\delta)c_n(\gamma,\delta)\left(\mathcal{K}_n \vartheta p_{n-1}^{\gamma+1,\delta+1}\right)(x) = \kappa_n\left(\mathcal{K}_n \vartheta p_{n-1}^{\gamma+1,\delta+1}\right)(x)
\end{aligned}
$$

with $\kappa_n = \frac{n+\gamma+\delta+1}{2n+\gamma+\delta+1}$, where we took into account relations (21) and (25) as well as the orthogonality properties of $p_n^{\gamma,\delta}(x)$ and $p_{n+1}^{\gamma,\delta}(x)$. Consequently, for $u \in \mathbf{L}^2_{\alpha,\beta}$, we have $\widetilde{\mathcal{K}}_n\widetilde{\mathcal{L}}_n u = \mathcal{K}_n\mathcal{L}_n u$, where

$$
\widetilde{\mathcal{L}}_n = \begin{cases} \mathcal{L}_n & : \quad r+s \le 1\,, \\ \vartheta\left[\frac{1}{\kappa_n}\left(\mathcal{S}_n^{\gamma+1,\delta+1} - \mathcal{S}_{n-1}^{\gamma+1,\delta+1}\right) + \mathcal{S}_{n-1}^{\gamma+1,\delta+1}\right]\vartheta^{-1}\mathcal{L}_n & : \quad r = s = 1\,. \end{cases}
$$

We show, that $\widetilde{\mathcal{L}}_n$ as well as $\widetilde{\mathcal{L}}_n^*$ converge strongly in $\mathbf{L}^2_{\alpha,\beta}$ to the identity operator. For the convergence of $\widetilde{\mathcal{L}}_n$, it suffices to show that, in case $r = s = 1$, $\vartheta\mathcal{S}_n^{\gamma+1,\delta+1}\vartheta^{-1}\mathcal{I} \longrightarrow \mathcal{I}$ strongly in $\mathbf{L}^2_{\alpha,\beta}$, which is equivalent to $\mathcal{S}_n^{\gamma+1,\delta+1} \longrightarrow \mathcal{I}$ strongly in $\mathbf{L}^2_{\rho,\tau}$, being again equivalent to

$$
\frac{1}{2} < \rho - \gamma < \frac{3}{2} \quad \text{and} \quad \frac{1}{2} < \tau - \delta < \frac{3}{2}\,, \tag{60}
$$

by Lemma 18. In the present situation, the last conditions are equivalent to the conditions which are satisfied in case $r = s = 1$. Since, for $f, g \in \mathbf{L}^2_{\alpha,\beta}$,

$$
\begin{aligned}
\left\langle \vartheta\mathcal{S}_n^{\gamma,\delta}\vartheta^{-1}f, g\right\rangle_{\alpha,\beta} &= \left\langle \mathcal{S}_n^{\gamma,\delta}\vartheta^{-1}f, \vartheta\, v^{\alpha-\gamma,\beta-\delta}g\right\rangle_{\gamma,\delta} \\
&= \left\langle f, \vartheta^{-1}v^{\gamma-\alpha,\delta-\beta}\mathcal{S}_n^{\gamma,\delta}\vartheta\, v^{\alpha-\gamma,\beta-\delta}g\right\rangle_{\alpha,\beta} \\
&= \left\langle f, v^{\gamma-\frac{\rho+\alpha}{2},\delta-\frac{\tau+\beta}{2}}\mathcal{S}_n^{\gamma,\delta}v^{\frac{\rho+\alpha}{2}-\gamma,\frac{\tau+\beta}{2}-\delta}g\right\rangle_{\alpha,\beta},
\end{aligned}
$$

we have to check if

$$
\left(\vartheta\mathcal{S}_n^{\gamma+1,\delta+1}\vartheta^{-1}\mathcal{I}\right)^* = v^{\gamma+1-\frac{\rho+\alpha}{2},\delta+1-\frac{\tau+\beta}{2}}\mathcal{S}_n^{\gamma+1,\delta+1}v^{\frac{\rho+\alpha}{2}-\gamma-1,\frac{\tau+\beta}{2}-\delta-1}\mathcal{I} \longrightarrow \mathcal{I} \quad \text{in} \quad \mathbf{L}^2_{\alpha,\beta}
$$

strongly, which is equivalent to $\mathcal{S}_n^{\gamma+1,\delta+1} \longrightarrow \mathcal{I}$ strongly in $\mathbf{L}^2_{2\gamma+2-\rho,2\delta+2-\tau}$. Due to Lemma 18, this is again equivalent to (60). As a consequence of these considerations we have that $\left\|\mathcal{K}\widetilde{\mathcal{L}}_n - \mathcal{K}\mathcal{L}_n\right\|_{\mathcal{L}\left(\mathbf{L}^2_{\alpha,\beta},\mathbf{C}_{\eta_0,\zeta_0}\right)} \longrightarrow 0$ if $n \longrightarrow \infty$. Hence, in order to prove (58) it suffices to show that

$$
\lim_{n\to\infty}\left\|\left(\widetilde{\mathcal{K}}_n - \mathcal{K}\right)\widetilde{\mathcal{L}}_n\right\|_{\mathcal{L}\left(\mathbf{L}^2_{\alpha,\beta},\mathbf{C}_{\eta_0,\zeta_0}\right)} = 0\,. \tag{61}
$$

We have

$$\left|\left((\widetilde{\mathcal{K}}_n - \mathcal{K})\widetilde{\mathcal{L}}_n u\right)(x)\right|$$
$$\le \int_{-1}^{1} v^{\gamma+r-\frac{\rho}{2},\delta+s-\frac{\tau}{2}}(y)\left|\mathcal{L}_n^{\gamma,\delta}\left[K(x,\cdot)v^{\frac{\rho-\alpha}{2}-\gamma-r,\frac{\tau-\beta}{2}-\delta-s}\right](y) - K(x,y)v^{\frac{\rho-\alpha}{2}-\gamma-r,\frac{\tau-\beta}{2}-\delta-s}(y)\right|\cdot$$
$$\cdot v^{\frac{\alpha}{2},\frac{\beta}{2}}(y)\left|\left(\widetilde{\mathcal{L}}_n u\right)(y)\right| dy$$
$$\le \mathcal{C}\left\|\left(\mathcal{L}_n^{\gamma,\delta} - \mathcal{I}\right)K(x,\cdot)v^{\frac{\rho-\alpha}{2}-\gamma-r,\frac{\tau-\beta}{2}-\delta-s}\right\|_{2\gamma+2r-\rho,2\delta+2s-\tau}\|u\|_{\alpha,\beta}$$

Hence, in order to show relation (61) we can try to apply Lemma 16 for $\alpha_0 = 2\gamma + 2r - \rho$ and $\beta_0 = 2\delta + 2s - \tau$ instead of α and β, respectively, as well as for

$$f(x,y) = v^{\eta_0,\zeta_0}(x)K(x,y)v^{\frac{\rho-\alpha}{2}-\gamma-r,\frac{\tau-\beta}{2}-\delta-s}(y)\,.$$

Since $(\rho,\gamma) \in \Omega$ and

$$\alpha_0 = \rho + 2[r - (\rho-\gamma)] \ge \begin{cases} 2\gamma-\rho & : \quad r = 0\,, \\ \rho & : \quad r = 1,\ \frac{1}{2} < \rho-\gamma < 1\,, \\ \rho - 1 & : \quad r = 1,\ 1 \le \rho-\gamma < \frac{3}{2}\,, \end{cases}$$

we have $\alpha_0 > -1$ and, analogously, $\beta_0 > -1$. Moreover,

$$\alpha_0 - \gamma = \begin{cases} \gamma - \rho \in \left(-\frac{1}{2},\frac{1}{2}\right) & : \quad r = 0\,, \\ 2-(\rho-\gamma) \in \left(\frac{1}{2},\frac{3}{2}\right) & : \quad r = 1\,, \end{cases}$$

such that $\alpha_0 - \gamma$ and, analogously, $\beta_0 - \delta$ belong to the interval $\left(-\frac{1}{2},\frac{3}{2}\right)$. The conditions

$$\psi_0 := \psi + \gamma + r - \frac{\rho-\alpha}{2} < \frac{1+\alpha_0}{2} \quad \text{and} \quad \chi_0 := \chi + \delta + s - \frac{\tau-\beta}{2} < \frac{1+\beta_0}{2}$$

are equivalent to $\psi < \frac{1+\alpha}{2}$ and $\chi < \frac{1-\beta}{2}$. This all together implies that the function

$$f(x,y)v^{\psi_0,\chi_0}(y) = v^{\eta_0,\zeta_0}(x)K(x,y)v^{\psi_0+\frac{\rho-\alpha}{2}-\gamma-r,\chi_0+\frac{\tau-\beta}{2}-\delta-s}(y) = v^{\eta_0,\zeta_0}(x)K(x,y)v^{\psi,\chi}(y)$$

is continuous on $[-1,1]^2$ and that Lemma 16 is applicable to $f(x,y)$ with ψ_0 and χ_0 instead of ψ and χ. Hence,

$$\left\|\left(\widetilde{\mathcal{K}}_n - \mathcal{K}\right)\widetilde{\mathcal{L}}_n\right\|_{\mathcal{L}\left(\mathbf{L}^2_{\alpha,\beta},\mathbf{C}_{\eta_0,\zeta_0}\right)}$$
$$\le \sup\left\{\left\|\left(\mathcal{L}_n^{\gamma,\delta} - \mathcal{I}\right)v^{\eta_0,\zeta_0}(x)K(x,\cdot)v^{\frac{\rho-\alpha}{2}-\gamma-r,\frac{\tau-\beta}{2}-\delta-s}\right\|_{2\gamma+2r-\rho,2\delta+2s-\tau} : -1 \le x \le 1\right\}$$
$$\longrightarrow 0 \quad \text{if} \quad n \longrightarrow \infty\,.$$

For the proof of (57) it remains to refer to (49). □

Corollary 9. *Let $\alpha,\beta \in (-1,1)$ and $p,q \in \mathbb{R}$ with $p < \frac{1+\beta}{2} < q$, as well as $H \in \mathbf{C}(\mathbb{R}^+)$ such that*

$$\lim_{t\to 0} t^p H(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} t^q H(t) = 0\,.$$

Moreover, let $\xi \in \mathbb{R}$ with $-1 < \alpha + 2\xi < 1$ and $\chi_1, \chi_2 : [-1,1] \longrightarrow [0,1]$ be continuous functions which vanish in a neighbourhood of the point 1 and are identically 1 in a neighbourhood of the point -1. Then, for $(\rho,\gamma),(\tau,\delta) \in \Omega$, we have

$$\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta} v^{\xi,0}(\mathcal{M}_{n,H}^- - \chi_1\mathcal{M}_{n,H}^-\chi_2)v^{-\xi,0}\mathcal{L}_n\right) \in \mathfrak{J}\,,$$

where the limit operators are given by

$$\mathcal{W}_1\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta} v^{\xi,0}(\mathcal{M}_{n,H}^- - \chi_1\mathcal{M}_{n,H}^-\chi_2)v^{-\xi,0}\mathcal{L}_n\right) = v^{\xi,0}(\mathcal{M}_H^- - \chi_1\mathcal{M}_H^-\chi_2)v^{-\xi,0}\mathcal{I}$$

and

$$\mathcal{W}_t\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta} v^{\xi,0}(\mathcal{M}_{n,H}^- - \chi_1\mathcal{M}_{n,H}^-\chi_2)v^{-\xi,0}\mathcal{L}_n\right) = 0\,, \quad t \in \{2,3\}\,.$$

Proof. Due to our assumptions, the following functions are continuous on $[-1,1]^2$,

$$(x,y) \mapsto v^{\zeta+\xi,0}(x)v^{\chi-\xi,0}(y)H\left(\frac{1+x}{1+y}\right)\frac{1}{1+y}\,[1-\chi_1(x)]\,[1-\chi_2(y)]\,,$$

$$(x,y) \mapsto v^{\zeta+\xi,\eta}(x)v^{\chi-\xi,0}(y)H\left(\frac{1+x}{1+y}\right)\frac{1}{1+y}\,\chi_1(x)[1-\chi_2(y)]\,,$$

$$(x,y) \mapsto v^{\zeta+\xi,0}(x)v^{\chi-\xi,\psi}(y)H\left(\frac{1+x}{1+y}\right)\frac{1}{1+y}\,[1-\chi_1(x)]\chi_2(y)\,,$$

where $\zeta = \max\{-\xi,0\}$, $\chi = \max\{\xi,0\}$, and $\eta := \max\{p,0\}$, $\psi := \max\{1-q,0\}$. Since

$$\begin{aligned}&\widetilde{\mathcal{L}}_n^{\gamma,\delta} v^{\xi,0}(\mathcal{M}_{n,H}^- - \chi_1\mathcal{M}_{n,H}^-\chi_2)v^{-\xi,0}\mathcal{L}_n\\ &\quad= \widetilde{\mathcal{L}}_n^{\gamma,\delta} v^{\xi,0}(1-\chi_1)\mathcal{M}_{n,H}^-(1-\chi_2)v^{-\xi,0}\mathcal{L}_n\\ &\qquad+ \widetilde{\mathcal{L}}_n^{\gamma,\delta} v^{\xi,0}\chi_1\mathcal{M}_{n,H}^-(1-\chi_2)v^{-\xi,0}\mathcal{L}_n + \widetilde{\mathcal{L}}_n^{\gamma,\delta} v^{\xi,0}(1-\chi_1)\mathcal{M}_{n,H}^-\chi_2 v^{-\xi,0}\mathcal{L}_n\end{aligned}$$

and

$$0 \le \zeta < \frac{1+\alpha}{2}\,,\ 0 \le \eta < \frac{1+\beta}{2}\,, \quad \text{and} \quad 0 \le \chi < \frac{1-\alpha}{2}\,,\ 0 \le \psi < \frac{1-\beta}{2}\,,$$

we are able to apply Lemma 19. □

Analogously we can prove the following corollary.

Corollary 10. *Let $\alpha,\beta \in (-1,1)$ and $p,q \in \mathbb{R}$ with $p < \frac{1+\alpha}{2} < q$, as well as $H \in \mathbf{C}(\mathbb{R}^+)$ such that*

$$\lim_{t\to 0} t^p H(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} t^q H(t) = 0\,.$$

Moreover, let $\xi \in \mathbb{R}$ with $-1 < \beta + 2\xi < 1$ and $\chi_1, \chi_2 : [-1,1] \longrightarrow [0,1]$ be continuous functions which vanish in a neighbourhood of the point -1 and are identically 1 in a neighbourhood of the point 1. Then, for $(\rho,\gamma),(\tau,\delta) \in \Omega$, we have

$$\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta} v^{0,\xi}(\mathcal{M}_{n,H}^+ - \chi_1\mathcal{M}_{n,H}^+\chi_2)v^{0,-\xi}\mathcal{L}_n\right) \in \mathfrak{J}\,,$$

where the limit operators are given by

$$\mathcal{W}_1\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta} v^{0,\xi}(\mathcal{M}_{n,H}^+ - \chi_1\mathcal{M}_{n,H}^+\chi_2)v^{0,-\xi}\mathcal{L}_n\right) = v^{0,\xi}(\mathcal{M}_H^+ - \chi_1\mathcal{M}_H^+\chi_2)v^{0,-\xi}\mathcal{I}$$

and

$$\mathcal{W}_t\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta} v^{0,\xi}(\mathcal{M}_{n,H}^+ - \chi_1\mathcal{M}_{n,H}^+\chi_2)v^{0,-\xi}\mathcal{L}_n\right) = 0\,, \quad t \in \{2,3\}\,.$$

In what follows we identify an element $(\xi_0,\dots,\xi_{n-1},0,\dots)\in \operatorname{im}\mathbb{P}_n$ with the respective element $\left[\ \xi_k\ \right]_{k=0}^{n-1}\in\mathbb{C}^n$ and the linear operator $\mathbb{A}_n:\operatorname{im}\mathbb{P}_n\longrightarrow \operatorname{im}\mathbb{P}_n$ with its matrix representation $\mathbb{A}_n=\left[\ a_{jk}\ \right]_{j,k=1}^{n}\in\mathbb{C}^{n\times n}$, i.e.,

$$\mathbb{A}_n\left[\ \xi_k\ \right]_{k=0}^{n-1}=\left[\ \sum_{k=0}^{n-1}a_{j+1,k+1}\xi_k\ \right]_{j=0}^{n-1}.$$

For example, we have the representation

$$\begin{aligned}&\mathcal{V}_n\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{K}_n\mathcal{L}_n\mathcal{V}_n^{-1}\\&=\left[\ \lambda_{kn}^{\gamma,\delta}v^{-\gamma,-\delta}(x_{kn}^{\gamma,\delta})v^{\frac{\alpha}{2}+\frac14,\frac{\beta}{2}+\frac14}(x_{jn}^{\gamma,\delta})K(x_{jn}^{\gamma,\delta},x_{kn}^{\gamma,\delta})v^{-\frac{\alpha}{2}-\frac14,-\frac{\beta}{2}-\frac14}(x_{kn}^{\gamma,\delta})\ \right]_{j,k=1}^{n}.\end{aligned}\tag{62}$$

Let us formulate the following condition for a positive kernel function.

(B) For the function $H:\mathbb{R}^+\longrightarrow\mathbb{R}^+$, there are a positive constant c_M and a real number κ such that
 - (a) $0<t\le s$ implies $H(t)t^\kappa\le c_M H(s)s^\kappa$

   or
 - (b) $0<t\le s$ implies $H(t)t^\kappa\ge c_M H(s)s^\kappa$.

Corollary 11. *Let $\alpha,\beta\in(-1,1)$ and $H\in\mathbf{C}(\mathbb{R}^+)$ be a positive function, which satisfies condition (A_1) for $\xi=\frac{1+\beta}{2}$ and condition* (B)*. Then, for $\mathcal{A}_n=\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^{-}\mathcal{L}_n$ the sequences of operators $\mathcal{E}_n^{(t)}\mathcal{A}_n\left(\mathcal{E}_n^{(t)}\right)^{-1}\mathcal{L}_n^{(t)}:\mathbf{X}_t\longrightarrow\mathbf{X}_t$, $t\in\{1,2,3\}$, $n\in\mathbb{N}$, are uniformly bounded in case of $(\rho,\gamma),(\tau,\delta)\in\Omega$.*

Proof. Due to Lemma 12 it suffices to prove the assertion for $t=2$. Furthermore, in view of Corollary 9 in combination with Lemma 5,(b), we have only to prove the uniform boundedness of the operators $\mathbb{H}_n^\chi:\operatorname{im}\mathbb{P}_n\longrightarrow\operatorname{im}\mathbb{P}_n$ with (cf. also (62))

$$\begin{aligned}\mathbb{H}_n^\chi&:=\mathcal{V}_n\widetilde{\mathcal{L}}_n^{\gamma,\delta}\chi\mathcal{M}_{n,H}^{-}\chi\mathcal{L}_n(\mathcal{V}_n)^{-1}\\&=\left[\ \frac{\chi(x_{jn}^{\gamma,\delta})\chi(x_{kn}^{\gamma,\delta})\lambda_{kn}^{\gamma,\delta}v^{\frac{\alpha}{2}+\frac14,\frac{\beta}{2}+\frac14}(x_{jn}^{\gamma,\delta})}{v^{\gamma,\delta}(x_{kn}^{\gamma,\delta})}H\left(\frac{1+x_{jn}^{\gamma,\delta}}{1+x_{kn}^{\gamma,\delta}}\right)\frac{v^{-\frac{\alpha}{2}-\frac14,-\frac{\beta}{2}-\frac14}(x_{kn}^{\gamma,\delta})}{1+x_{kn}^{\gamma,\delta}}\ \right]_{j,k=1}^{n},\end{aligned}$$

where $\chi:[-1,1]\longrightarrow[0,1]$ is a continuous function, which vanishes in a neighbourhood of the point 1 and is identically 1 in a neighbourhood of the point -1. We use the notation $\mathbb{H}_n^\chi=\left[\ h_{jk}^{n,\chi}\ \right]_{j,k=1}^{n}$. Let us note that the entries $h_{jk}^{n,\chi}$ of the matrix $\mathbb{H}_n^\chi$ are positive and, in view of the choice of the function $\chi:[-1,1]\longrightarrow[0,1]$ and (47),

$$h_{jk}^{n,\chi}\le\mathcal{C}_\chi\frac{(1+x_{jn}^{\gamma,\delta})^{\frac{\beta}{2}+\frac14}}{n(1+x_{kn}^{\gamma,\delta})^{\frac{\beta}{2}+\frac34}}H\left(\frac{1+x_{jn}^{\gamma,\delta}}{1+x_{kn}^{\gamma,\delta}}\right)$$

with a constant $\mathcal{C}_\chi\ne\mathcal{C}_\chi(j,k,n)$. Due to (31), there is a constant $\mathcal{C}_1\ne\mathcal{C}_1(k,n)>0$, such that

$$\mathcal{C}_1^{-1}\left(\frac{n+1-k}{n}\right)^2\le 1+x_{kn}^{\gamma,\delta}\le\mathcal{C}_1\left(\frac{n+1-k}{n}\right)^2.\tag{63}$$

We conclude, by additionally using condition (B),

$$
\begin{aligned}
h_{jk}^{n,\chi} &\leq \frac{\mathcal{C}}{n}\,\frac{n}{n+1-k}\left(\frac{n+1-j}{n+1-k}\right)^{\beta+\frac{1}{2}}\left(\frac{1+x_{jn}^{\gamma,\delta}}{1+x_{kn}^{\gamma,\delta}}\right)^{-\kappa} H\left(\frac{1+x_{jn}^{\gamma,\delta}}{1+x_{kn}^{\gamma,\delta}}\right)\left(\frac{1+x_{jn}^{\gamma,\delta}}{1+x_{kn}^{\gamma,\delta}}\right)^{\kappa} \\
&\leq \frac{\mathcal{C}_a}{n+1-k}\left(\frac{n+1-j}{n+1-k}\right)^{\beta+\frac{1}{2}} H\left(\mathcal{C}_1^2\,\frac{(n+1-j)^2}{(n+1-k)^2}\right) =: h_{n+1-j,n+1-k}^{a}
\end{aligned}
\tag{64}
$$

if condition (B),(a) is in force, and

$$
h_{jk}^{n,\chi} \leq \frac{\mathcal{C}_b}{n+1-k}\left(\frac{n+1-j}{n+1-k}\right)^{\beta+\frac{1}{2}} H\left(\frac{1}{\mathcal{C}_1^2}\,\frac{(n+1-j)^2}{(n+1-k)^2}\right) =: h_{n+1-j,n+1-k}^{b} \tag{65}
$$

if condition (B),(b) is fulfilled, where $\mathcal{C}_d \neq \mathcal{C}_d(j,k,n)$, $d \in \{a,b\}$. As a consequence of these estimates we have

$$
\left\|\mathbb{H}_n^{\chi}\right\|_{\mathcal{L}(\ell^2)} \leq \left\|\mathbb{J}_n \mathbb{H}^d \mathbb{J}_n\right\|_{\mathcal{L}(\ell^2)}, \tag{66}
$$

where $\mathbb{H}^d = \left[\ h_{jk}^d\ \right]_{j,k=1}^{n}$, $d \in \{a,b\}$, and $\mathbb{J}_n = \left[\ \delta_{n-1-j,k}\ \right]_{j,k=0}^{n-1}$. For $d \in \{a,b\}$, consider the function $g_d : \mathbb{R}^+ \longrightarrow \mathbb{R}$ with

$$
g_d(x) = \begin{cases} c_M^a x^{\beta+\frac{1}{2}} H\left(\mathcal{C}_1^2\, x^2\right) & : \quad d = a\,, \\ c_M^b x^{\beta+\frac{1}{2}} H\left(\mathcal{C}_1^{-2} x^2\right) & : \quad d = b\,. \end{cases}
$$

Due to our assumptions, there exist $p,q \in \mathbb{R}$ with $p < q$ such that $H \in \mathbf{L}^2_{2p-1}(\mathbb{R}^+) \cap \mathbf{L}^2_{2q-1}(\mathbb{R}^+)$ and $\xi = \frac{1+\beta}{2} \in (p,q)$. From that, we derive

$$
g_d \in \mathbf{L}^2_{2(2p-\beta-\frac{1}{2})-1}(\mathbb{R}^+) \cap \mathbf{L}^2_{2(2q-\beta-\frac{1}{2})-1}(\mathbb{R}^+)
$$

and $\frac{1}{2} \in \left(2p-\beta-\frac{1}{2}, 2q-\beta-\frac{1}{2}\right)$. Moreover, the Mellin transforms of $g_a(x)$ and $g_b(x)$ are up to a multiplicative constant equal to $\mathcal{C}_1^{-z}\widehat{H}\left(\frac{z+\beta}{2}+\frac{1}{4}\right)$ and $\mathcal{C}_1^{z}\widehat{H}\left(\frac{z+\beta}{2}+\frac{1}{4}\right)$, respectively. Furthermore, $p < \frac{\frac{1}{2}+\beta}{2}+\frac{1}{4} < q$. Thus, g_d fulfils condition (A_1) for $\xi = \frac{1}{2}$. Because of $\mathbb{H}^d = \left[\frac{1}{k+1}\, g_d\left(\frac{j+1}{k+1}\right)\right]_{j,k=0}^{\infty}$ the uniform boundedness of the operators $\mathbb{H}_n^{\chi} : \ell^2 \longrightarrow \ell^2$ is a consequence of Lemma 11, inequality (66) and the fact that the norm of the operators $\mathbb{J}_n : \ell^2 \longrightarrow \ell^2$ is equal to 1. □

Analogously to the previous one, we can prove the following corollary.

Corollary 12. *Let $\alpha,\beta \in (-1,1)$ and $H \in \mathbf{C}(\mathbb{R}^+)$ be a positive function, which satisfies condition (A_1) for $\xi = \frac{1+\alpha}{2}$ and condition* (B)*. Then, for $\mathcal{A}_n = \widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^{+}\mathcal{L}_n$, the sequences of operators $\mathcal{E}_n^{(t)}\mathcal{A}_n\left(\mathcal{E}_n^{(t)}\right)^{-1}\mathcal{L}_n^{(t)} : \mathbf{X}_t \longrightarrow \mathbf{X}_t$, $t \in \{1,2,3\}$, $n \in \mathbb{N}$, are uniformly bounded in case of $(\rho,\gamma),(\tau,\delta) \in \Omega$.*

Remark 1. *Let $|\gamma| = |\delta| = \frac{1}{2}$. Then the assertions of Lemmas 11 and 12 remain true, if we only assume that $H \in \mathbf{C}(\mathbb{R}^+)$ satisfies condition (A_1) for $\sigma = \frac{1+\beta}{2}$ resp. $\xi = \frac{1+\alpha}{2}$. Thus, we do not need both the positivity of H and condition* (B)*.*

Proof. First of all, we notice that only the verification of (66) is necessary. Moreover, it is well known that

$$\theta_{jn}^{-\frac{1}{2},-\frac{1}{2}} = \frac{j-\frac{1}{2}}{n}\pi\,, \quad \theta_{jn}^{\frac{1}{2},\frac{1}{2}} = \frac{j}{n+1}\pi\,, \quad \theta_{jn}^{-\frac{1}{2},\frac{1}{2}} = \frac{j-\frac{1}{2}}{n+\frac{1}{2}}\pi\,, \quad \theta_{jn}^{\frac{1}{2},-\frac{1}{2}} = \frac{j}{n+\frac{1}{2}}\pi\,.$$

Hence, we can give up the usage of a cutting-off function χ, and the matrices $\mathbb{H}_n = \mathbb{H}_n^1$ have the form $\mathbb{H}_n = \mathbb{P}_n\mathbb{H}\mathbb{P}_n$ or $\mathbb{J}_n\mathbb{H}\mathbb{J}_n$ with

$$\mathbb{H} = \left[\ \frac{\sqrt{2}}{k}\left(\frac{j}{k}\right)^{\sigma+\frac{1}{2}} H\left(\frac{j^2}{k^2}\right)\ \right]_{j,k=1}^{\infty}$$

or

$$\mathbb{H} = \left[\ \frac{\sqrt{2}}{k-\frac{1}{2}}\left(\frac{j-\frac{1}{2}}{k-\frac{1}{2}}\right)^{\sigma+\frac{1}{2}} H\left(\frac{(j-\frac{1}{2})^2}{(k-\frac{1}{2})^2}\right)\ \right]_{j,k=1}^{\infty},$$

where $\sigma \in \{\alpha, \beta\}$. Finally, we have only to recall Lemma 11. □

Let $s > -1$ and J_s be the Bessel function of the first kind and of order s. We have

$$J_s(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{\left(\frac{x}{2}\right)^{2r+s}}{\Gamma(s+r+1)}\,, \quad x > 0\,.$$

It is well-known, that J_s has countable infinitely many positive simple zeros, which accumulate only at infinity. We denote these zeros in increasing order by $\psi_{s,k}$, $k = 1, 2, \ldots$ By using Legendre's duplication formula we get

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{and} \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x\,,$$

such that

$$\psi_{\frac{1}{2},k} = k\pi \quad \text{and} \quad \psi_{-\frac{1}{2},k} = (k-\tfrac{1}{2})\pi\,.$$

Lemma 20 ([37], Theorem 4.1)**.** *For $k \in \mathbb{N}$ fixed, the nodes $x_{kn}^{\alpha,\beta} = \cos\theta_{kn}^{\alpha,\beta}$ $(0 < \theta_{kn}^{\alpha,\beta} < \pi)$ of the Jacobi polynomial $p_n^{\alpha,\beta}(x)$ admit the representation*

$$\theta_{kn}^{\alpha,\beta} = \frac{\psi_{\alpha,k}}{c_n^{\alpha,\beta}} + \mathcal{O}\left(n^{-5}\right),$$

where

$$c_n^{\alpha,\beta} = \sqrt{\left(n + \frac{\alpha+\beta+1}{2}\right)^2 + \frac{1-\alpha^2-3\beta^2}{12}}\,.$$

The relation $p_n^{\alpha,\beta}(\cos\theta) = (-1)^n p_n^{\beta,\alpha}(\cos(\pi-\theta))$ yields $\theta_{kn}^{\alpha,\beta} = \pi - \theta_{n-k+1,n}^{\beta,\alpha}$. Hence,

$$x_{kn}^{\alpha,\beta} = -x_{n-k+1,n}^{\beta,\alpha}\,. \tag{67}$$

Corollary 13. *For $k \in \mathbb{N}$ fixed, we have*

$$\theta_{n-k+1,n}^{\alpha,\beta} = \pi - \frac{\psi_{\beta,k}}{c_n^{\beta,\alpha}} + \mathcal{O}\left(n^{-5}\right).$$

Lemma 21. *Let $j,k \in \mathbb{N}$, and ζ_1, ζ_2 be real numbers which satisfy $\zeta_1 + \zeta_2 = 1$. Moreover, let $H \in \mathbf{C}(\mathbb{R}^+)$. Then, for n tending to infinity, we have*

$$\frac{v^{\frac{\zeta_1}{2},0}(x_{jn}^{\alpha,\beta})v^{\frac{\zeta_2}{2},0}(x_{kn}^{\alpha,\beta})}{n(1-x_{kn}^{\alpha,\beta})} H\left(\frac{1-x_{jn}^{\alpha,\beta}}{1-x_{kn}^{\alpha,\beta}}\right) \longrightarrow \frac{\sqrt{2}(\psi_{\alpha,j})^{\zeta_1}}{(\psi_{\alpha,k})^{2-\zeta_2}} H\left(\left[\frac{\psi_{\alpha,j}}{\psi_{\alpha,k}}\right]^2\right)$$

and

$$\frac{v^{0,\frac{\zeta_1}{2}}(x_{n-j+1,n}^{\alpha,\beta})v^{0,\frac{\zeta_2}{2}}(x_{n-k+1,n}^{\tau_+,\tau_-})}{n(1+x_{n-k+1,n}^{\alpha,\beta})} H\left(\frac{1+x_{n-j+1,n}^{\alpha,\beta}}{1+x_{n-k+1,n}^{\alpha,\beta}}\right) \longrightarrow \frac{\sqrt{2}(\psi_{\beta,j})^{\zeta_1}}{(\psi_{\beta,k})^{2-\zeta_2}} H\left(\left[\frac{\psi_{\beta,j}}{\psi_{\beta,k}}\right]^2\right).$$

Proof. The first relation is a consequence of

$$\begin{aligned}
&\frac{v^{\frac{\zeta_1}{2},0}(x_{jn}^{\alpha,\beta})v^{\frac{\zeta_2}{2},0}(x_{kn}^{\alpha,\beta})}{n(1-x_{kn}^{\alpha,\beta})} H\left(\frac{1-x_{jn}^{\alpha,\beta}}{1-x_{kn}^{\alpha,\beta}}\right) \\
&\quad = \sqrt{2}\,\frac{\sin^{\zeta_1}(\theta_{jn}^{\alpha,\beta}/2)}{(\theta_{jn}^{\alpha,\beta}/2)^{\zeta_1}}\,\frac{\sin^{\zeta_2}(\theta_{kn}^{\alpha,\beta}/2)}{(\theta_{kn}^{\alpha,\beta}/2)^{\zeta_2}}\,\frac{(\theta_{kn}^{\alpha,\beta}/2)^2}{\sin^2(\theta_{kn}^{\alpha,\beta}/2)}\,\frac{(n\theta_{jn}^{\alpha,\beta})^{\zeta_1}}{(n\theta_{kn}^{\alpha,\beta})^{2-\zeta_2}}\,\cdot \\
&\qquad \cdot H\left(\frac{\sin^2(\theta_{jn}^{\alpha,\beta}/2)}{(\theta_{jn}^{\alpha,\beta}/2)^2}\,\frac{(\theta_{kn}^{\alpha,\beta}/2)^2}{\sin^2(\theta_{kn}^{\alpha,\beta}/2)}\,\frac{(n\theta_{jn}^{\alpha,\beta})^2}{(n\theta_{kn}^{\alpha,\beta})^2}\right)
\end{aligned}$$

and Lemma 20. By applying (67) we get the second one. □

For an arbitrary $a \in \mathbf{PC}$, let us compute the sequence of the adjoints of the operators $\widetilde{\mathcal{L}}_n^{\gamma,\delta} a\mathcal{L}_n : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$. We define integers $r,s \in \{0,1\}$ as in (59) and obtain, for functions $f,g \in \mathbf{L}^2_{\alpha,\beta}$,

$$\begin{aligned}
\langle \widetilde{\mathcal{L}}_n^{\gamma,\delta} a\mathcal{L}_n f, g\rangle_{\alpha,\beta} &= \langle \mathcal{L}_n^{\gamma,\delta}\vartheta^{-1} a\mathcal{L}_n f, \vartheta^{-1} g\rangle_{\rho,\tau} \\
&= \langle \mathcal{L}_n^{\gamma,\delta}\vartheta^{-1} a\mathcal{L}_n f, v^{\rho-\gamma-r,\tau-\delta-s}\vartheta^{-1} g\rangle_{\gamma+r,\delta+s} \\
&= \langle \mathcal{L}_n^{\gamma,\delta}\vartheta^{-1} a\mathcal{L}_n f, \mathcal{S}_n^{\gamma+r,\delta+s} v^{\rho-\gamma-r,\tau-\delta-s}\vartheta^{-1} g\rangle_{\gamma+r,\delta+s} \\
&= \langle \mathcal{L}_n^{\gamma,\delta}\vartheta^{-1} a\mathcal{L}_n f, v^{r,s}\mathcal{S}_n^{\gamma+r,\delta+s} v^{\rho-\gamma-r,\tau-\delta-s}\vartheta^{-1} g\rangle_{\gamma,\delta}.
\end{aligned}$$

In case of $r+s \le 1$, we can proceed as follows using the algebraic accuracy of the Gaussian rule

$$\begin{aligned}
\langle \widetilde{\mathcal{L}}_n^{\gamma,\delta} a\mathcal{L}_n f, g\rangle_{\alpha,\beta} &= \sum_{k=1}^{n} \lambda_{kn}^{\gamma,\delta}\left(\vartheta^{-1} a\mathcal{L}_n f\right)(x_{kn}^{\gamma,\delta}) v^{r,s}(x_{kn}^{\gamma,\delta})\overline{\left(\mathcal{S}_n^{\gamma+r,\delta+s} v^{\rho-\gamma-r,\tau-\delta-s}\vartheta^{-1} g\right)(x_{kn}^{\gamma,\delta})} \\
&= \sum_{k=1}^{n} \lambda_{kn}^{\gamma,\delta}\left(\vartheta^{-1}\mathcal{L}_n f\right)(x_{kn}^{\gamma,\delta}) v^{r,s}(x_{kn}^{\gamma,\delta})\overline{\left(\bar{a}\,\mathcal{S}_n^{\gamma+r,\delta+s} v^{\rho-\gamma-r,\tau-\delta-s}\vartheta^{-1} g\right)(x_{kn}^{\gamma,\delta})} \\
&= \langle \vartheta^{-1}\mathcal{L}_n f, v^{r,s}\mathcal{L}_n^{\gamma,\delta}\bar{a}\,\mathcal{S}_n^{\gamma+r,\delta+s} v^{\rho-\gamma-r,\tau-\delta-s}\vartheta^{-1} g\rangle_{\gamma,\delta} \\
&= \langle \mathcal{L}_n f, \vartheta v^{\gamma+r-\rho,\delta+s-\tau}\mathcal{L}_n^{\gamma,\delta}\bar{a}\,\mathcal{S}_n^{\gamma+r,\delta+s} v^{\rho-\gamma-r,\tau-\delta-s}\vartheta^{-1} g\rangle_{\alpha,\beta} \\
&= \langle f, \mathcal{L}_n\vartheta v^{\gamma+r-\rho,\delta+s-\tau}\mathcal{L}_n^{\gamma,\delta}\bar{a}\,\mathcal{S}_n^{\gamma+r,\delta+s} v^{\rho-\gamma-r,\tau-\delta-s}\vartheta^{-1} g\rangle_{\alpha,\beta}.
\end{aligned}$$

Hence,

$$\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta} a\mathcal{L}_n\right)^* = \mathcal{L}_n \vartheta v^{\gamma+r-\rho,\delta+s-\tau} \mathcal{L}_n^{\gamma,\delta} \overline{a}\, \mathcal{S}_n^{\gamma+r,\delta+s} v^{\rho-\gamma-r,\tau-\delta-s} \vartheta^{-1} \mathcal{I}\,. \tag{68}$$

If $r = s = 1$ we use relations (21) and (25) as well as the orthogonality properties of the polynomials $p_n^{\gamma,\delta}(x)$ and $p_{n+1}^{\gamma,\delta}(x)$. With $g_0 = v^{\rho-\gamma-1,\tau-\delta-1}\vartheta^{-1}g$ we get

$$\begin{aligned}
\left\langle \widetilde{\mathcal{L}}_n^{\gamma,\delta} a\mathcal{L}_n f, g\right\rangle_{\alpha,\beta} &= \left\langle \mathcal{L}_n^{\gamma,\delta} \vartheta^{-1} a\mathcal{L}_n f, v^{1,1} \mathcal{S}_n^{\gamma+1,\delta+1} g_0\right\rangle_{\gamma,\delta} \\
&= \left\langle \mathcal{L}_n^{\gamma,\delta} \vartheta^{-1} a\mathcal{L}_n f, v^{1,1} \mathcal{S}_{n-1}^{\gamma+1,\delta+1} g_0\right\rangle_{\gamma,\delta} \\
&\qquad + \left\langle \mathcal{L}_n^{\gamma,\delta} \vartheta^{-1} a\mathcal{L}_n f, v^{1,1} \left\langle g_0, p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma+1,\delta+1} p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma,\delta} \\
&= \left\langle f, \mathcal{L}_n \vartheta v^{\gamma+1-\rho,\delta+1-\tau} \mathcal{L}_n^{\gamma,\delta} \overline{a}\, \mathcal{S}_{n-1}^{\gamma+1,\delta+1} v^{\rho-\gamma-1,\tau-\delta-1} \vartheta^{-1} g\right\rangle_{\alpha,\beta} \\
&\qquad + A_n(\gamma,\delta) \left\langle \mathcal{L}_n^{\gamma,\delta} \vartheta^{-1} a\mathcal{L}_n f, \left\langle g_0, p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma+1,\delta+1} p_{n-1}^{\gamma,\delta}\right\rangle_{\gamma,\delta}
\end{aligned}$$

and

$$\begin{aligned}
&A_n(\gamma,\delta) \left\langle \mathcal{L}_n^{\gamma,\delta} \vartheta^{-1} a\mathcal{L}_n f, \left\langle g_0, p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma+1,\delta+1} p_{n-1}^{\gamma,\delta}\right\rangle_{\gamma,\delta} \\
&\quad = A_n(\gamma,\delta) \sum_{k=1}^{n} \lambda_{kn}^{\gamma,\delta} \left(\vartheta^{-1} a\mathcal{L}_n f\right)(x_{kn}^{\gamma,\delta}) \overline{\left\langle g_0, p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma+1,\delta+1} p_{n-1}^{\gamma,\delta}(x_{kn}^{\gamma,\delta})} \\
&\quad = A_n(\gamma,\delta) c_n(\gamma,\delta) \sum_{k=1}^{n} \lambda_{kn}^{\gamma,\delta} \left(\vartheta^{-1} \mathcal{L}_n f\right)(x_{kn}^{\gamma,\delta}) \cdot \\
&\qquad\qquad \cdot \overline{\overline{a} \left\langle g_0, p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma+1,\delta+1} v^{1,1}(x_{kn}^{\gamma,\delta}) p_{n-1}^{\gamma+1,\delta+1}(x_{kn}^{\gamma,\delta})} \\
&\quad = A_n(\gamma,\delta) c_n(\gamma,\delta) \left\langle \vartheta^{-1} \mathcal{L}_n f, \mathcal{L}_n^{\gamma,\delta} \overline{a} v^{1,1} \left\langle g_0, p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma+1,\delta+1} p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma,\delta} \\
&\quad = \kappa_n \left\langle f, \mathcal{L}_n \vartheta v^{\gamma-\rho,\delta-\tau} \mathcal{L}_n^{\gamma,\delta} \overline{a} v^{1,1} \left(\mathcal{S}_n^{\gamma+1,\delta+1} - \mathcal{S}_{n-1}^{\gamma+1,\delta+1}\right) g_0\right\rangle_{\alpha,\beta} \\
&\quad = \kappa_n \left\langle f, \mathcal{L}_n \vartheta v^{\gamma-\rho,\delta-\tau} \mathcal{L}_n^{\gamma,\delta} \overline{a} v^{1,1} \left(\mathcal{S}_n^{\gamma+1,\delta+1} - \mathcal{S}_{n-1}^{\gamma+1,\delta+1}\right) v^{\rho-\gamma-1,\tau-\delta-1} \vartheta^{-1} g\right\rangle_{\alpha,\beta},
\end{aligned}$$

where $\kappa_n = \frac{n+\gamma+\delta+1}{2n+\gamma+\delta+1}$. Consequently, in case $r = s = 1$ we have

$$\begin{aligned}
\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta} a\mathcal{L}_n\right)^* &= \mathcal{L}_n \vartheta v^{\gamma+1-\rho,\delta+1-\tau} \mathcal{L}_n^{\gamma,\delta} \overline{a}\, \mathcal{S}_{n-1}^{\gamma+1,\delta+1} v^{\rho-\gamma-1,\tau-\delta-1} \vartheta^{-1} \mathcal{I} \\
&\quad + \kappa_n \mathcal{L}_n \vartheta v^{\gamma-\rho,\delta-\tau} \mathcal{L}_n^{\gamma,\delta} \overline{a} v^{1,1} \left(\mathcal{S}_n^{\gamma+1,\delta+1} - \mathcal{S}_{n-1}^{\gamma+1,\delta+1}\right) v^{\rho-\gamma-1,\tau-\delta-1} \vartheta^{-1} \mathcal{I}\,.
\end{aligned} \tag{69}$$

It is easy to see that $\widetilde{\mathcal{L}}_n^{\gamma,\delta} a\mathcal{L}_n$ converges strongly to $a\mathcal{I}$ in $\mathbf{L}^2_{\alpha,\beta}$ (see the beginning of the proof of Lemma 22 below). Let us discuss the convergence of the adjoint operators. At first, we consider the operators on the right hand side of (68). For $\gamma_0 = \gamma + r$ and $\delta_0 = \delta + s$, the conditions (55) and (56) are equivalent to $-\frac{1}{2} < \gamma - \rho + 2r\delta - \tau + 2s < \frac{3}{2}$ as well as $-\frac{1}{2} < \rho - \gamma - r, \tau - \delta - s < \frac{1}{2}$, $\rho < 2\gamma + 1$, $\tau < 2\delta + 1$, respectively. Hence, by definition (59) of the numbers $r, s \in \{0, 1\}$, we can apply Corollary 8 together with the strong convergence of $\mathcal{L}_n \longrightarrow \mathcal{I}$ in $\mathbf{L}^2_{\alpha,\beta}$ and get the strong convergence

$$\mathcal{L}_n \vartheta v^{\gamma+1-\rho,\delta+1-\tau} \mathcal{L}_n^{\gamma,\delta} \overline{a}\, \mathcal{S}_n^{\gamma+1,\delta+1} v^{\rho-\gamma-1,\tau-\delta-1} \vartheta^{-1} \mathcal{I} \longrightarrow \overline{a}\, \mathcal{I} \quad \text{in} \quad \mathbf{L}^2_{\alpha,\beta} \tag{70}$$

for $r,s\in\{0,1\}$. In particular, we have the strong convergence of $\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}a\mathcal{L}_n\right)^*$ for $r+s\le 1$. From (70) we also get the strong convergence of

$$\begin{aligned}&\mathcal{L}_n\vartheta v^{\gamma+1-\rho,\delta+1-\tau}\mathcal{L}_n^{\gamma,\delta}\overline{a}\,\mathcal{S}_{n-1}^{\gamma+1,\delta+1}v^{\rho-\gamma-1,\tau-\delta-1}\vartheta^{-1}\mathcal{I}\\ &\quad=\left(\mathcal{L}_n\vartheta v^{\gamma+1-\rho,\delta+1-\tau}\mathcal{L}_n^{\gamma,\delta}\overline{a}\,\mathcal{S}_n^{\gamma+1,\delta+1}v^{\rho-\gamma-1,\tau-\delta-1}\vartheta^{-1}\mathcal{I}\right)\cdot\\ &\qquad\cdot\left(\vartheta v^{\gamma+1-\rho,\delta+1-\tau}\mathcal{L}_{n-1}^{\gamma,\delta}\mathcal{S}_{n-1}^{\gamma+1,\delta+1}v^{\rho-\gamma-1,\tau-\delta-1}\vartheta^{-1}\mathcal{I}\right)\longrightarrow\overline{a}\,\mathcal{I}\,.\end{aligned}$$

Thus, due to formula (69), to prove the strong convergence of the operators $\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}a\mathcal{L}_n\right)^*$ in case $r=s=1$, it remains to show that the operators

$$\mathcal{C}_n:=\vartheta v^{\gamma-\rho,\delta-\tau}\mathcal{L}_n^{\gamma,\delta}\overline{a}v^{1,1}\left(\mathcal{S}_n^{\gamma+1,\delta+1}-\mathcal{S}_{n-1}^{\gamma+1,\delta+1}\right)v^{\rho-\gamma-1,\tau-\delta-1}\vartheta^{-1}\mathcal{I}$$

converge in $\mathbf{L}^2_{\alpha,\beta}$ strongly to the zero operator. Up to now we know that the operators $\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}a\mathcal{L}_n\right)^*$ and $\mathcal{L}_n\vartheta v^{\gamma+1-\rho,\delta+1-\tau}\mathcal{L}_n^{\gamma,\delta}\overline{a}\,\mathcal{S}_{n-1}^{\gamma+1,\delta+1}v^{\rho-\gamma-1,\tau-\delta-1}\vartheta^{-1}\mathcal{I}$ are uniformly bounded. Thus, in view of relation (69), also the operators $\mathcal{C}_n:\mathbf{L}^2_{\alpha,\beta}\longrightarrow\mathbf{L}^2_{\alpha,\beta}$ are uniformly bounded, and it suffices to show their convergence on a dense subset of $\mathbf{L}^2_{\alpha,\beta}$. Such a subset is the space $\left\{v^{\gamma+1-\rho,\delta+1-\tau}\vartheta P:P\in\mathbf{P}\right\}$ because of the relations

$$\left\|f-v^{\gamma+1-\rho,\delta+1-\tau}\vartheta P\right\|_{\alpha,\beta}=\left\|v^{\rho-\gamma-1,\tau-\delta-1}\vartheta^{-1}f-P\right\|_{2\gamma+2-\rho,2\delta+2-\tau}\quad\forall\,f\in\mathbf{L}^2_{\alpha,\beta}\,,$$

$2\gamma+2-\rho>-1\,,2\delta+2-\tau>-1$ (note that $\rho-\gamma,\tau-\delta<\frac{3}{2}$), and the density of $\mathbf{P}$ in $\mathbf{L}^2_{\alpha,\beta}$. These results can be used for the proof of the following lemma. Nevertheless, we will give a shorter proof of the strong convergence of the adjoint operators.

Lemma 22. *Let $a\in\mathbf{PC}$ and $(\rho,\gamma),(\tau,\delta)\in\Omega$. Then we have the strong convergences*

$$\widetilde{\mathcal{L}}_n^{\gamma,\delta}a\mathcal{L}_n\longrightarrow a\mathcal{I}\quad\textit{and}\quad\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}a\mathcal{L}_n\right)^*\longrightarrow\overline{a}\mathcal{I}$$

in the space $\mathbf{L}^2_{\alpha,\beta}$.

Proof. Since $\mathcal{L}_n=\vartheta\mathcal{S}_n^{\rho,\tau}\vartheta^{-1}\mathcal{I}$ and consequently $\widetilde{\mathcal{L}}_n^{\gamma,\delta}a\mathcal{L}_n=\vartheta\mathcal{L}_n^{\gamma,\delta}a\mathcal{S}_n^{\rho,\tau}\vartheta^{-1}\mathcal{I}$, The strong convergence of $\widetilde{\mathcal{L}}_n^{\gamma,\delta}a\mathcal{L}_n$ to $a\mathcal{I}$ in $\mathbf{L}^2_{\alpha,\beta}$ is equivalent to the strong convergence of $\mathcal{L}_n^{\gamma,\delta}a\mathcal{S}_n^{\rho,\tau}$ to $a\mathcal{I}$ in $\mathbf{L}^2_{\rho,\tau}$. To prove this, we can apply Corollary 7 for $\alpha=\gamma_0=\rho$ and $\beta=\delta_0=\tau$.

Let us turn to the convergence of the adjoint operators. We define integers $r,s\in\{0,1\}$ as in (59). Since we already know that these operators $\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}a\mathcal{L}_n\right)^*:\mathbf{L}^2_{\alpha,\beta}\longrightarrow\mathbf{L}^2_{\alpha,\beta}$ are uniformly bounded it suffices to prove their convergence on a dense subset of $\mathbf{L}^2_{\alpha,\beta}$. As such a subset we can take the set

$$\mathbf{X}_{rs}=\left\{\vartheta v^{\gamma+r-\rho,\delta+s-\tau}P:P\in\mathbf{P}\right\},\tag{71}$$

since $2\gamma+2r-\rho,2\delta+2s-\tau>-1$ for $(\rho,\gamma),(\tau,\delta)\in\Omega$ and $\mathbf{P}$ is dense in $\mathbf{L}^2_{2\gamma+2r-\rho,2\delta+2s-\tau}$ as well as

$$\left\|f-\vartheta v^{\gamma+r-\rho,\delta+s-\tau}P\right\|_{\alpha,\beta}=\left\|\vartheta^{-1}v^{\rho-\gamma-r,\tau-\delta-s}f-P\right\|_{2\gamma+2r-\rho,2\delta+2s-\tau}\quad\forall\,f\in\mathbf{L}^2_{\alpha,\beta}\,.$$

For $f \in \mathbf{L}^2_{\alpha,\beta}$, $g = \vartheta v^{\gamma+r-\rho,\delta+s-\tau} P$ with $P \in \mathbf{P}$, and all sufficiently large n, we obtain

$$\begin{aligned}
\left\langle \widetilde{\mathcal{L}}_n^{\gamma,\delta} a \mathcal{L}_n f, g \right\rangle_{\alpha,\beta} &= \left\langle \mathcal{L}_n^{\gamma,\delta} \vartheta^{-1} a \mathcal{L}_n f, \vartheta^{-1} g \right\rangle_{\rho,\tau} = \left\langle \mathcal{L}_n^{\gamma,\delta} \vartheta^{-1} a \mathcal{L}_n f, v^{\gamma+r-\rho,\delta+s-\tau} P \right\rangle_{\rho,\tau} \\
&= \left\langle \mathcal{L}_n^{\gamma,\delta} \vartheta^{-1} a \mathcal{L}_n f, v^{r,s} P \right\rangle_{\gamma,\delta} = \sum_{k=1}^{n} \lambda_{kn}^{\gamma,\delta} \left(\vartheta^{-1} a \mathcal{L}_n f \right)(x_{kn}^{\gamma,\delta}) v^{r,s}(x_{kn}^{\gamma,\delta}) \overline{P(x_{kn}^{\gamma,\delta})} \\
&= \sum_{k=1}^{n} \lambda_{kn}^{\gamma,\delta} \left(\vartheta^{-1} \mathcal{L}_n f \right)(x_{kn}^{\gamma,\delta}) v^{r,s}(x_{kn}^{\gamma,\delta}) \overline{(\overline{a} P)(x_{kn}^{\gamma,\delta})} = \left\langle v^{r,s} \vartheta^{-1} \mathcal{L}_n f, \mathcal{L}_n^{\gamma,\delta} \overline{a} P \right\rangle_{\gamma,\delta} \\
&= \left\langle \mathcal{L}_n f, \vartheta v^{\gamma+r-\rho,\delta+s-\tau} \mathcal{L}_n^{\gamma,\delta} \overline{a} P \right\rangle_{\alpha,\beta} = \left\langle f, \mathcal{L}_n \vartheta v^{\gamma+r-\rho,\delta+s-\tau} \mathcal{L}_n^{\gamma,\delta} \overline{a} P \right\rangle_{\alpha,\beta}.
\end{aligned}$$

It remains to show that $\vartheta v^{\gamma+r-\rho,\delta+s-\tau} \mathcal{L}_n^{\gamma,\delta} \overline{a} P$ converges to $\overline{a} g$ in $\mathbf{L}^2_{\alpha,\beta}$, which is equivalent to $\mathcal{L}_n^{\gamma,\delta} \overline{a} P \longrightarrow \overline{a} P$ in $\mathbf{L}^2_{2\gamma+2r-\rho,2\delta+2s-\tau}$. This follows from Lemma 15 by choosing the parameters $\alpha := 2\gamma + 2r - \rho$, $\beta := 2\delta + 2s - \tau$, as well as $\psi = \chi = 0$, and taking into account that the conditions

$$-\frac{1}{2} < \gamma + 2r - \rho, \delta + 2s - \tau < \frac{3}{2} \quad \text{and} \quad 2\gamma + 2r - \rho, 2\delta + 2s > -1$$

are satisfied for $(\rho, \gamma), (\tau, \delta) \in \Omega$ by the definition of $r, s \in \{0, 1\}$. □

For $H \in \mathbf{C}(\mathbb{R}^+)$, we define the integral operator

$$(\mathcal{N}_H^- f)(x) = \int_{-1}^{1} H\left(\frac{1+y}{1+x} \right) \frac{f(y)\, dy}{1+x}, \quad x \in (-1, 1).$$

We have the following relation

$$\mathcal{N}_H^- = \mathcal{M}_{H_1}^- \quad \text{with} \quad H_1(t) = H(t^{-1}) t^{-1}, \quad t \in \mathbb{R}^+. \tag{72}$$

Furthermore, one can easily show that the adjoint operator of $\mathcal{M}_H^- : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$ is equal to $\left(\mathcal{M}_H^- \right)^* = v^{-\alpha,-\beta} \mathcal{N}_{\overline{H}}^- v^{\alpha,\beta} \mathcal{I}$.

Lemma 23. *Let $\alpha, \beta \in (-1, 1)$, $p < \frac{1+\beta}{2} < q$ and $H \in \mathbf{C}^1(\mathbb{R}^+)$ satisfy $H \in \mathbf{L}^2_{2p-1} \cap \mathbf{L}^2_{2q-1}$ as well as $H' \in \mathbf{L}^2_{2p+1} \cap \mathbf{L}^2_{2q+1}$. If the operators $\widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{M}_{n,H}^- \mathcal{L}_n : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$ are uniformly bounded for some γ, δ with $(\rho, \gamma), (\tau, \delta) \in \Omega$, then we have the strong convergences*

$$\widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{M}_{n,H}^- \mathcal{L}_n \longrightarrow \mathcal{M}_H^- \quad \text{and} \quad \left(\widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{M}_{n,H}^- \mathcal{L}_n \right)^* \longrightarrow \left(\mathcal{M}_H^- \right)^* \quad \text{in} \quad \mathbf{L}^2_{\alpha,\beta}.$$

Proof. Due to the Banach-Steinhaus theorem, it suffices to show the convergence on a dense subset of $\mathbf{L}^2_{\alpha,\beta}$. We consider functions $g(x)$ of the form

$$g(x) = \vartheta(x)(1 - x^2)^2 p(x), \quad x \in (-1, 1), \tag{73}$$

where $p(x)$ is an arbitrary polynomial. By choosing $\chi \in \left(p, \frac{1+\beta}{2} \right) \cap \left(0, \frac{1+\beta}{2} \right)$ and taking Corollary 6 into account, we get $\sup \left\{ \left\| \widetilde{\mathcal{L}}_n^{\gamma,\delta} \right\|_{\mathcal{L}(\mathbf{C}_{0,\chi}, \mathbf{L}^2_{\alpha,\beta})} : n \in \mathbb{N} \right\} < \infty$. Consequently, since $g \in \operatorname{im} \mathcal{L}_n$,

$$\begin{aligned}
\left\| \widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{M}_{n,H}^- \mathcal{L}_n g - \mathcal{M}_H^- g \right\|_{\alpha,\beta} &= \left\| \widetilde{\mathcal{L}}_n^{\gamma,\delta} \left(\mathcal{M}_{n,H}^- g - \mathcal{M}_H^- g \right) + \widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{M}_H^- g - \mathcal{M}_H^- g \right\|_{\alpha,\beta} \\
&\leq C \left\| \mathcal{M}_{n,H}^- g - \mathcal{M}_H^- g \right\|_{0,\chi,\infty} + \left\| \widetilde{\mathcal{L}}_n^{\gamma,\delta} \mathcal{M}_H^- g - \mathcal{M}_H^- g \right\|_{\alpha,\beta}.
\end{aligned}$$

At first, we consider the term $\left\|\mathcal{M}^-_{n,H}g - \mathcal{M}^-_H g\right\|_{0,\chi,\infty}$. For $f : (-1,1) \longrightarrow \mathbb{C}$, we denote by $R_n^{\gamma,\delta}(f)$ the error

$$R_n^{\gamma,\delta}(f) = \int_{-1}^1 f(y)v^{\gamma,\delta}(y)\,dy - \sum_{k=1}^n \lambda_{kn}^{\gamma,\delta} f(x_{kn}^{\gamma,\delta}) .$$

From [38] ((5.1.35)) follows

$$\left|R_n^{\gamma,\delta}(f)\right| \le \frac{\mathcal{C}}{n}\int_{-1}^1 |f'(y)|v^{\gamma,\delta}(y)\varphi(y)\,dy, \tag{74}$$

where $\varphi(x) = \sqrt{1-x^2}$ and the constant does not depend on f and n. We have

$$(\mathcal{M}^-_{n,H}g)(x) - (\mathcal{M}_H g)(x) = R_n^{\gamma,\delta}(f_x), \quad x \in (-1,1)$$

with

$$f_x(y) = H\left(\frac{1+x}{1+y}\right)\frac{g_0(y)}{1+y} \quad \text{and} \quad g_0(y) = v^{-\gamma,-\delta}(y)g(y) .$$

With the help of Lemma 3, we get

$$\begin{aligned}
&\int_{-1}^1 |f_x'(y)|v^{\gamma,\delta}(y)\varphi(y)\,dy \le \int_{-1}^1 \left|H\left(\frac{1+x}{1+y}\right)\right|\frac{\varphi(y)|g(y)|}{(1+y)^2}\,dy \\
&\qquad + \int_{-1}^1 \left|H'\left(\frac{1+x}{1+y}\right)\right|\frac{(1+x)\varphi(y)|g(y)|}{(1+y)^3}\,dy + \int_{-1}^1 \left|H\left(\frac{1+x}{1+y}\right)\right|\frac{v^{\gamma,\delta}(y)\varphi(y)|g_0'(y)|}{1+y}\,dy \\
&\le \mathcal{C}\left\{\int_{-1}^1 \left|H\left(\frac{1+x}{1+y}\right)\right|\frac{dy}{1+y} + \int_{-1}^1 \left|H'\left(\frac{1+x}{1+y}\right)\right|\frac{1+x}{1+y}\frac{dy}{1+y}\right\} \\
&\le \mathcal{C}\,(1+x)^{-\chi}, \quad x \in (-1,1) .
\end{aligned}$$

Thus,

$$\frac{1}{n}\sup\left\{v^{0,\chi}(x)\int_{-1}^1 |f_x'(y)|v^{\gamma,\delta}(y)\sqrt{1-y^2}\,dy \,:\, -1 < x < 1\right\} \longrightarrow 0, \quad n \longrightarrow \infty,$$

and (74) delivers $\lim_{n\to\infty}\left\|\mathcal{M}^-_{n,H}g - \mathcal{M}^-_H g\right\|_{0,\chi,\infty} = 0$.

Secondly, we deal with $\left\|\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}^-_H g - \mathcal{M}^-_H g\right\|_{\alpha,\beta}$. Again by Lemma 3 we can estimate

$$|(\mathcal{M}^-_H g)(x)| \le \mathcal{C}\int_{-1}^1 \left|H\left(\frac{1+x}{1+y}\right)\right|\frac{dy}{1+y} \le \mathcal{C}(1+x)^{-\chi}$$

and get $\mathcal{M}^-_H g \in \widetilde{\mathbf{C}}^b_{0,\chi}$, which implies, due to Corollary 6,

$$\lim_{n\to\infty}\left\|\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}^-_H g - \mathcal{M}^-_H g\right\|_{\alpha,\beta} = 0 . \tag{75}$$

Let us turn to the strong convergence of the adjoint operators. Let $f \in \mathbf{L}^2_{\alpha,\beta}$, define $r,s \in \{0,1\}$ as in (59), and take g from the dense subset $\mathbf{X}_{rs}$ defined in (71), what means $g = v^{\gamma+r-\rho,\delta+s-\tau}\vartheta P$ with $P \in \mathbf{P}$. We set

$$h(x,y) := H\left(\frac{1+x}{1+y}\right)\frac{1}{1+y}, \quad x,y \in (-1,1),$$

and get, for all sufficiently large n,

$$\begin{aligned}
&\left\langle \widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^{-}\mathcal{L}_n f, g\right\rangle_{\alpha,\beta} \\
&\quad = \left\langle \mathcal{L}_n^{\gamma,\delta}\vartheta^{-1}\mathcal{M}_{n,H}^{-}\mathcal{L}_n f, v^{r,s}P\right\rangle_{\gamma,\delta} \\
&\quad = \sum_{j=1}^{n}\lambda_{jn}^{\gamma,\delta}\vartheta^{-1}(x_{jn}^{\gamma,\delta})\Big(\mathcal{M}_{n,H}^{-}\mathcal{L}_n f\Big)(x_{jn}^{\gamma,\delta})v^{r,s}\overline{P(x_{jn}^{\gamma,\delta})} \\
&\quad = \sum_{j=1}^{n}\lambda_{jn}^{\gamma,\delta}\vartheta^{-1}(x_{jn}^{\gamma,\delta})\sum_{k=1}^{n}\lambda_{kn}^{\gamma,\delta}h(x_{jn}^{\gamma,\delta},x_{kn}^{\gamma,\delta})\Big(v^{-\gamma,-\delta}\mathcal{L}_n f\Big)(x_{kn}^{\gamma,\delta})\overline{(v^{r,s}P)(x_{jn}^{\gamma,\delta})} \\
&\quad = \sum_{k=1}^{n}\lambda_{kn}^{\gamma,\delta}\Big(v^{-\gamma,-\delta}\mathcal{L}_n f\Big)(x_{kn}^{\gamma,\delta})\overline{\sum_{j=1}^{n}\lambda_{jn}^{\gamma,\delta}\overline{h(x_{jn}^{\gamma,\delta},x_{kn}^{\gamma,\delta})}(v^{r,s}\vartheta^{-1}P)(x_{jn}^{\gamma,\delta})} \\
&\quad = \sum_{k=1}^{n}\lambda_{kn}^{\gamma,\delta}\Big(v^{-\gamma,-\delta}\mathcal{L}_n f\Big)(x_{kn}^{\gamma,\delta})\overline{\sum_{j=1}^{n}\lambda_{jn}^{\gamma,\delta}\overline{h(x_{jn}^{\gamma,\delta},x_{kn}^{\gamma,\delta})}(v^{-\gamma,-\delta}v^{\alpha,\beta}g)(x_{jn}^{\gamma,\delta})} \\
&\quad = \sum_{k=1}^{n}\lambda_{kn}^{\gamma,\delta}\Big(v^{-\gamma,-\delta}\mathcal{L}_n f\Big)(x_{kn}^{\gamma,\delta})\overline{\Big(\mathcal{N}_{n,\overline{H}}^{-}v^{\alpha,\beta}g\Big)(x_{jn}^{\gamma,\delta})} \\
&\quad = \sum_{k=1}^{n}\lambda_{kn}^{\gamma,\delta}\Big(v^{r,s}\vartheta^{-1}\mathcal{L}_n f\Big)(x_{kn}^{\gamma,\delta})\overline{\Big(v^{-\gamma-r,-\delta-s}\vartheta\mathcal{N}_{n,\overline{H}}^{-}v^{\alpha,\beta}g\Big)(x_{kn}^{\gamma,\delta})}
\end{aligned}$$

where

$$\mathcal{N}_{n,H}^{-}:\mathbf{C}(-1,1)\longrightarrow\mathbf{C}(-1,1),\quad u\mapsto\sum_{j=1}^{n}\lambda_{jn}^{\gamma,\delta}H\left(\frac{1+x_{jn}^{\gamma,\delta}}{1+\cdot}\right)\frac{(v^{-\gamma,-\delta}u)(x_{jn}^{\gamma,\delta})}{1+\cdot}.$$

In case of $r+s\le 1$, we use the algebraic accuracy of the Gaussian rule and obtain

$$\begin{aligned}
\left\langle \widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^{-}\mathcal{L}_n f, g\right\rangle_{\alpha,\beta} &= \left\langle v^{r,s}\vartheta^{-1}\mathcal{L}_n f, \mathcal{L}_n^{\gamma,\delta}v^{-\gamma-r,-\delta-s}\vartheta\mathcal{N}_{n,\overline{H}}^{-}v^{\alpha,\beta}g\right\rangle_{\gamma,\delta} \\
&= \left\langle \vartheta^{-2}\mathcal{L}_n f, v^{r,s}\vartheta\mathcal{L}_n^{\gamma,\delta}\vartheta^{-1}v^{\rho-\gamma-r,\tau-\delta-s}v^{-\alpha,-\beta}\mathcal{N}_{n,\overline{H}}^{-}v^{\alpha,\beta}g\right\rangle_{\gamma,\delta} \\
&= \left\langle f, \mathcal{L}_n v^{\gamma+r-\rho,\delta+s-\tau}\widetilde{\mathcal{L}}_n^{\gamma,\delta}v^{\rho-\gamma-r,\tau-\delta-s}v^{-\alpha,-\beta}\mathcal{N}_{n,\overline{H}}^{-}v^{\alpha,\beta}g\right\rangle_{\alpha,\beta}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Big(\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^{-}\mathcal{L}_n\Big)^{*}g &= \mathcal{L}_n v^{\gamma+r-\rho,\delta+s-\tau}\widetilde{\mathcal{L}}_n^{\gamma,\delta}v^{\rho-\gamma-r,\tau-\delta-s}v^{-\alpha,-\beta}\mathcal{N}_{n,\overline{H}}^{-}v^{\alpha,\beta}g \\
&= \mathcal{L}_n v^{\gamma+r-\frac{\rho+\alpha}{2},\delta+s-\frac{\tau+\beta}{2}}\mathcal{L}_n^{\gamma,\delta}v^{\frac{\rho+\alpha}{2}-\gamma-r,\frac{\tau+\beta}{2}-\delta-s}v^{-\alpha,-\beta}\mathcal{N}_{n,\overline{H}}^{-}v^{\alpha,\beta}g
\end{aligned} \tag{76}$$

if $r+s\le 1$. In case of $r=s=1$, we write

$$\begin{aligned}
\left\langle \widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^{-}\mathcal{L}_n f, g\right\rangle_{\alpha,\beta} &= \sum_{k=1}^{n}\lambda_{kn}^{\gamma,\delta}\Big(v^{1,1}\Big(\mathcal{S}_n^{\gamma+1,\delta+1}-\mathcal{S}_{n-1}^{\gamma+1,\delta+1}\Big)\vartheta^{-1}\mathcal{L}_n f\Big)(x_{kn}^{\gamma,\delta})\overline{N(x_{kn}^{\gamma,\delta})} \\
&\quad + \sum_{k=1}^{n}\lambda_{kn}^{\gamma,\delta}\Big(v^{1,1}\mathcal{S}_{n-1}^{\gamma+1,\delta+1}\vartheta^{-1}\mathcal{L}_n f\Big)(x_{kn}^{\gamma,\delta})\overline{N(x_{kn}^{\gamma,\delta})},
\end{aligned} \tag{77}$$

where we used the abbreviation $N(x) = \left(v^{-\gamma-1,-\delta-1}\vartheta\mathcal{N}^{-}_{n,\overline{H}}v^{\alpha,\beta}g\right)(x)$ and where, for the second sum in (77), we get

$$\begin{aligned}
&\sum_{k=1}^{n}\lambda_{kn}^{\gamma,\delta}\left(v^{1,1}\mathcal{S}_{n-1}^{\gamma+1,\delta+1}\vartheta^{-1}\mathcal{L}_nf\right)(x_{kn}^{\gamma,\delta})\,\overline{\left(v^{-\gamma-1,-\delta-1}\vartheta\mathcal{N}^{-}_{n,\overline{H}}v^{\alpha,\beta}g\right)(x_{kn}^{\gamma,\delta})}\\
&\quad=\left\langle \mathcal{S}_{n-1}^{\gamma+1,\delta+1}\vartheta^{-1}\mathcal{L}_nf,\mathcal{L}_n^{\gamma,\delta}v^{-\gamma-1,-\delta-1}\vartheta\mathcal{N}^{-}_{n,\overline{H}}v^{\alpha,\beta}g\right\rangle_{\gamma+1,\delta+1}\\
&\quad=\left\langle \vartheta^{-1}\mathcal{L}_nf,v^{1,1}\mathcal{S}_{n-1}^{\gamma+1,\delta+1}\mathcal{L}_n^{\gamma,\delta}v^{-\gamma-1,-\delta-1}\vartheta\mathcal{N}^{-}_{n,\overline{H}}v^{\alpha,\beta}g\right\rangle_{\gamma,\delta}\\
&\quad=\left\langle \vartheta^{-2}\mathcal{L}_nf,v^{1,1}\vartheta\mathcal{S}_{n-1}^{\gamma+1,\delta+1}\mathcal{L}_n^{\gamma,\delta}\vartheta^{-1}v^{\rho-\gamma-1,\tau-\delta-1}v^{-\alpha,-\beta}\mathcal{N}^{-}_{n,\overline{H}}v^{\alpha,\beta}g\right\rangle_{\gamma,\delta}\\
&\quad=\left\langle f,\mathcal{L}_nv^{\gamma+1-\rho,\delta+1-\tau}\vartheta\mathcal{S}_{n-1}^{\gamma+1,\delta+1}\mathcal{L}_n^{\gamma,\delta}\vartheta^{-1}v^{\rho-\gamma-1,\tau-\delta-1}v^{-\alpha,-\beta}\mathcal{N}^{-}_{n,\overline{H}}v^{\alpha,\beta}g\right\rangle_{\alpha,\beta}.
\end{aligned}\tag{78}$$

For the first sum in (77) we use the relations

$$\begin{aligned}
&\left(v^{1,1}\left(\mathcal{S}_n^{\gamma+1,\delta+1}-\mathcal{S}_{n-1}^{\gamma+1,\delta+1}\right)\vartheta^{-1}\mathcal{L}_nf\right)(x_{kn}^{\gamma,\delta})\\
&\qquad=\left\langle \vartheta^{-1}\mathcal{L}_nf,p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma+1,\delta+1}v^{1,1}(x_{kn}^{\gamma,\delta})p_{n-1}^{\gamma+1,\delta+1}(x_{kn}^{\gamma,\delta})\\
&\qquad\overset{(25)}{=}c_n^{-1}\left\langle \vartheta^{-1}\mathcal{L}_nf,p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma+1,\delta+1}p_{n-1}^{\gamma,\delta}(x_{kn}^{\gamma,\delta})
\end{aligned}$$

and conclude with the help of (21)

$$\begin{aligned}
&\sum_{k=1}^{n}\lambda_{kn}^{\gamma,\delta}\left(v^{1,1}\left(\mathcal{S}_n^{\gamma+1,\delta+1}-\mathcal{S}_{n-1}^{\gamma+1,\delta+1}\right)\vartheta^{-1}\mathcal{L}_nf\right)(x_{kn}^{\gamma,\delta})\,\overline{N(x_{kn}^{\gamma,\delta})}\\
&=c_n^{-1}\left\langle \vartheta^{-1}\mathcal{L}_nf,p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma+1,\delta+1}\sum_{k=1}^{n}\lambda_{kn}^{\gamma,\delta}p_{n-1}^{\gamma,\delta}(x_{kn}^{\gamma,\delta})\,\overline{N(x_{kn}^{\gamma,\delta})}\\
&=c_n^{-1}\left\langle \vartheta^{-1}\mathcal{L}_nf,p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma+1,\delta+1}\left\langle p_{n-1}^{\gamma,\delta},\mathcal{L}_n^{\gamma,\delta}N\right\rangle_{\gamma,\delta}\\
&=c_n^{-1}A_n^{-1}\left\langle \vartheta^{-1}\mathcal{L}_nf,p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma+1,\delta+1}\left\langle v^{1,1}p_{n-1}^{\gamma+1,\delta+1},\mathcal{L}_n^{\gamma,\delta}N\right\rangle_{\gamma,\delta}\\
&=\kappa_n^{-1}\left\langle \vartheta^{-1}\mathcal{L}_nf,p_{n-1}^{\gamma+1,\delta+1}\right\rangle_{\gamma+1,\delta+1}\left\langle p_{n-1}^{\gamma+1,\delta+1},\mathcal{L}_n^{\gamma,\delta}N\right\rangle_{\gamma+1,\delta+1}\\
&=\kappa_n^{-1}\left\langle \left(\mathcal{S}_n^{\gamma+1,\delta+1}-\mathcal{S}_{n-1}^{\gamma+1,\delta+1}\right)\vartheta^{-1}\mathcal{L}_nf,\mathcal{L}_n^{\gamma,\delta}N\right\rangle_{\gamma+1,\delta+1}\\
&=\kappa_n^{-1}\left\langle \vartheta^{-1}\mathcal{L}_nf,\left(\mathcal{S}_n^{\gamma+1,\delta+1}-\mathcal{S}_{n-1}^{\gamma+1,\delta+1}\right)\mathcal{L}_n^{\gamma,\delta}N\right\rangle_{\gamma+1,\delta+1},
\end{aligned}$$

which is equal to

$$\kappa_n^{-1}\left\langle f,\mathcal{L}_nv^{\gamma+1-\rho,\delta+1-\tau}\vartheta\left(\mathcal{S}_n^{\gamma+1,\delta+1}-\mathcal{S}_{n-1}^{\gamma+1,\delta+1}\right)\mathcal{L}_n^{\gamma,\delta}\vartheta^{-1}v^{\rho-\gamma-1,\tau-\delta-1}h\right\rangle_{\alpha,\beta}$$

with $h=v^{-\alpha,-\beta}\mathcal{N}^{-}_{n,\overline{H}}v^{\alpha,\beta}g$. That means, together with (76)–(78),

$$\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}^{-}_{n,H}\mathcal{L}_n\right)^{*}g=\mathcal{L}_nv^{\gamma+r-\frac{\rho+\alpha}{2},\delta+s-\frac{\tau+\beta}{2}}\widetilde{\mathcal{S}}_n\mathcal{L}_n^{\gamma,\delta}v^{\frac{\rho+\alpha}{2}-\gamma-r,\frac{\tau+\beta}{2}-\delta-s}v^{-\alpha,-\beta}\mathcal{N}^{-}_{n,\overline{H}}v^{\alpha,\beta}g,$$

where

$$\widetilde{\mathcal{S}}_n = \begin{cases} \mathcal{I} & : \quad r+s \le 1\,, \\ \kappa_n^{-1}\left(\mathcal{S}_n^{\gamma+1,\delta+1} - \mathcal{S}_{n-1}^{\gamma+1,\delta+1}\right) + \mathcal{S}_{n-1}^{\gamma+1,\delta+1} & : \quad r = s = 1\,. \end{cases}$$

If we apply Corollary 8 with $a \equiv 1$, $\gamma_0 = \gamma + r$, and $\delta_0 = \delta + s$, then we see that $v^{\gamma+r-\frac{\varrho+\alpha}{2},\delta+s-\frac{\tau+\beta}{2}}\widetilde{\mathcal{S}}_n v^{\frac{\varrho+\alpha}{2}-\gamma-r,\frac{\tau+\beta}{2}-\delta-s}\mathcal{I}$ converges in $\mathbf{L}^2_{\alpha,\beta}$ strongly to the identity operator. Hence, it remains to show the convergence

$$v^{\gamma+r-\frac{\varrho+\alpha}{2},\delta+s-\frac{\tau+\beta}{2}}\mathcal{L}_n^{\gamma,\delta} v^{\frac{\varrho+\alpha}{2}-\gamma-r,\frac{\tau+\beta}{2}-\delta-s} v^{-\alpha,-\beta}\mathcal{N}^-_{n,\overline{H}} v^{\alpha,\beta} g \longrightarrow v^{-\alpha,-\beta}\mathcal{N}^-_{\overline{H}} v^{\alpha,\beta} g \tag{79}$$

in $\mathbf{L}^2_{\alpha,\beta}$. For this, we can take g from the subset $\mathbf{X}^0_{rs} = \left\{\vartheta v^{\gamma+r-\rho,\delta+s-\tau}v^{2,2}P : P \in \mathbf{P}\right\}$ of $\mathbf{X}_{rs}$, which is also dense in $\mathbf{L}^2_{\alpha,\beta}$. At first we remark that, since the conditions of Lemma 15 are satisfied for $\psi = \gamma + r - \frac{\varrho+\alpha}{2}$ and $\chi = \delta + s - \frac{\tau+\beta}{2}$, we have, for all $f \in \mathbf{R}^0_{\frac{1+\alpha}{2},\frac{1+\beta}{2}}$,

$$v^{\gamma+r-\frac{\varrho+\alpha}{2},\delta+s-\frac{\tau+\beta}{2}}\mathcal{L}_n^{\gamma,\delta} v^{\frac{\varrho+\alpha}{2}-\gamma-r,\frac{\tau+\beta}{2}-\delta-s} f \longrightarrow f \quad \text{in} \quad \mathbf{L}^2_{\alpha,\beta}\,. \tag{80}$$

Choose $\psi_0 \in \left(1+\beta-q, \frac{1+\beta}{2}\right) \cap \left(0, \frac{1+\beta}{2}\right) \cap \left(\beta, \frac{1+\beta}{2}\right)$. With the help of (80) we can estimate

$$\begin{aligned} &\left\| v^{\gamma+r-\frac{\varrho+\alpha}{2},\delta+s-\frac{\tau+\beta}{2}}\mathcal{L}_n^{\gamma,\delta} v^{\frac{\varrho+\alpha}{2}-\gamma-r,\frac{\tau+\beta}{2}-\delta-s} v^{-\alpha,-\beta}\mathcal{N}^-_{n,\overline{H}} v^{\alpha,\beta} g - v^{-\alpha,-\beta}\mathcal{N}^-_{\overline{H}} v^{\alpha,\beta} g \right\|_{\alpha,\beta} \\ &\quad \le \mathcal{C}\left\| v^{-\alpha,-\beta}\mathcal{N}^-_{n,\overline{H}} v^{\alpha,\beta} g - v^{-\alpha,-\beta}\mathcal{N}^-_{\overline{H}} v^{\alpha,\beta} g \right\|_{\max\{0,\alpha\},\psi_0,\infty} \\ &\qquad + \left\| v^{\gamma+r-\frac{\varrho+\alpha}{2},\delta+s-\frac{\tau+\beta}{2}}\mathcal{L}_n^{\gamma,\delta} v^{\frac{\varrho+\alpha}{2}-\gamma-r,\frac{\tau+\beta}{2}-\delta-s} v^{-\alpha,-\beta}\mathcal{N}^-_{\overline{H}} v^{\alpha,\beta} g - v^{-\alpha,-\beta}\mathcal{N}^-_{\overline{H}} v^{\alpha,\beta} g \right\|_{\alpha,\beta}. \end{aligned}$$

We have

$$\left(\mathcal{N}^-_{n,\overline{H}} v^{\alpha,\beta} g\right)(x) - \left(\mathcal{N}^-_{\overline{H}} v^{\alpha,\beta} g\right)(x) = R_n^{\gamma,\delta}(f_x)\,, \quad x \in (-1,1)\,,$$

with

$$f_x(y) = \overline{H\left(\frac{1+y}{1+x}\right)}\,\frac{g_0(y)\,dy}{1+x} \quad \text{and} \quad g_0(y) = v^{\alpha-\gamma,\beta-\delta}(y)g(y) = v^{\frac{\alpha-\rho}{2}+r+2,\frac{\beta-\tau}{2}+s+2}P.$$

Set $H_2(t) = H'(t^{-1})t^{-2}$, $t \in \mathbb{R}_+$. Since $1 + \beta - q < \psi_0 < 1 + \beta - p$ holds true, we get $2p - 1 < 1 + 2\beta - 2\psi_0 < 2q - 1$. Hence, due to our assumptions $H \in \mathbf{L}^2_{2p-1} \cap \mathbf{L}^2_{2q-1}$ and $H' \in \mathbf{L}^2_{2p+1} \cap \mathbf{L}^2_{2q+1}$, we have $H \in \mathbf{L}^2_{2(\beta-\psi_0)+1}$ and $H' \in \mathbf{L}_{2(\beta-\psi_0)+3}$. Consequently, $H_1, H_2 \in \mathbf{L}^2_{2(\psi_0-\beta)-1}$, where H_1 is defined in (72). By

$$\frac{\alpha-\rho}{2} + \gamma + r + 2 + \frac{1}{2} = \frac{\gamma-\rho}{2} + \frac{\alpha+\gamma}{2} + r + \frac{5}{2} > -\frac{1}{2} - r - 1 + r + \frac{5}{2} = 1\,,$$

$$\frac{\beta-\tau}{2} + \delta + s + 2 + \frac{1}{2} = \frac{\delta-\tau}{2} + \frac{\beta+\delta}{2} + s + \frac{5}{2} > -\frac{1}{2} - s - 1 + s + \frac{5}{2} = 1\,,$$

and again using Lemma 3 we obtain

$$\begin{aligned}
&\int_{-1}^{1} |f_x'(y)| v^{\gamma,\delta}(y)\varphi(y)\,dy \\
&\le \int_{-1}^{1} \left| H'\left(\frac{1+y}{1+x}\right)\right| \frac{v^{\alpha,\beta}(y)\varphi(y)|g(y)|}{(1+x)^2}\,dy + \int_{-1}^{1} \left| H\left(\frac{1+y}{1+x}\right)\right| \frac{v^{\gamma,\delta}(y)\varphi(y)|g_0'(y)|}{1+x}\,dy \\
&\le \mathcal{C}\left[\int_{-1}^{1} \left| H'\left(\frac{1+y}{1+x}\right)\right| \frac{1+y}{1+x}\frac{dy}{1+x} + \int_{-1}^{1} \left| H\left(\frac{1+y}{1+x}\right)\right| \frac{dy}{1+x}\right] \\
&\le \mathcal{C}\left[\int_{-1}^{1} \left| H_2\left(\frac{1+x}{1+y}\right)\right| \frac{dy}{1+y} + \int_{-1}^{1} \left| H_1\left(\frac{1+x}{1+y}\right)\right| \frac{dy}{1+y}\right] \\
&\le \mathcal{C}\,(1+x)^{\beta-\psi_0}\,, \quad x \in (-1,1)\,.
\end{aligned}$$

Thus,

$$\frac{1}{n}\sup\left\{ v^{0,\psi_0-\beta}(x)\cdot \int_{-1}^{1} |f_x'(y)|\,dy \,:\, -1<x<1\right\} \longrightarrow 0\,, \quad n\longrightarrow\infty\,.$$

Relation (74) yields

$$\lim_{n\to\infty}\left\| v^{-\alpha,-\beta}\left(\mathcal{N}_{n,H}^{-} v^{\alpha,\beta} g - \mathcal{N}_{H}^{-} v^{\alpha,\beta} g\right)\right\|_{\max\{0,\alpha\},\psi,\infty} = 0\,.$$

As above we see that $v^{-\alpha,-\beta}\mathcal{N}_H^{-} v^{\alpha,\beta} g$ belongs to $\widetilde{\mathbf{C}}^{b}_{\max\{0,\alpha\},\psi_0}$, and (80) delivers

$$\lim_{n\to\infty}\left\| v^{\gamma+r-\frac{\rho+\alpha}{2},\delta+s-\frac{\tau+\beta}{2}}\mathcal{L}_n^{\gamma,\delta} v^{\frac{\rho+\alpha}{2}-\gamma-r,\frac{\tau+\beta}{2}-\delta-s} v^{-\alpha,-\beta}\mathcal{N}_H^{-} v^{\alpha,\beta} g - v^{-\alpha,-\beta}\mathcal{N}_H^{-} v^{\alpha,\beta} g\right\|_{\alpha,\beta} = 0\,.$$

The lemma is proved. □

The proof of the following lemma is analogous.

Lemma 24. *Let* $\alpha,\beta \in (-1,1)$, $p < \frac{1+\alpha}{2} < q$ *and* $H \in \mathbf{C}^1(\mathbb{R}^+)$ *satisfy* $H \in \mathbf{L}^2_{2p-1} \cap \mathbf{L}^2_{2q-1}$ *as well as* $H' \in \mathbf{L}^2_{2p+1} \cap \mathbf{L}^2_{2q+1}$. *If the operators* $\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^{+}\mathcal{L}_n : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$ *are uniformly bounded for some* γ,δ *with* $(\rho,\gamma),(\tau,\delta)\in\Omega$, *then we have the strong convergences*

$$\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^{+}\mathcal{L}_n \longrightarrow \mathcal{M}_H^{+} \quad \text{and} \quad \left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^{+}\mathcal{L}_n\right)^{*} \longrightarrow \left(\mathcal{M}_H^{+}\right)^{*} \quad \text{in} \quad \mathbf{L}^2_{\alpha,\beta}\,.$$

The following lemma is a version of the dominated convergence theorem and will be useful in proofs of the strong convergence of operator sequences in the Hilbert space ℓ^2.

Lemma 25. *Let* $\xi,\eta \in \ell^2$, $\xi^n = (\xi_j^n)_{j=0}^{\infty}$, $|\xi_j^n| \le |\eta_j|\ \forall\, j = 0,1,2,\ldots, \forall\, n \ge n_0$ *and*

$$\lim_{n\to\infty}\xi_j^n = \xi_j \forall\, j = 0,1,2,\ldots$$

Then $\lim\limits_{n\to\infty}\|\xi^n - \xi\|_{\ell^2} = 0$.

For what follows we set

$$b_{jk}^{n,\pm} := \frac{\varphi(x_{kn}^{\gamma,\delta})}{n}\,\frac{v^{\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}}(x_{jn}^{\gamma,\delta})}{v^{\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}}(x_{kn}^{\gamma,\delta})}\,H\left(\frac{1\mp x_{jn}^{\gamma,\delta}}{1\mp x_{kn}^{\gamma,\delta}}\right)\frac{1}{1\mp x_{kn}^{\gamma,\delta}}\,. \tag{81}$$

Moreover, we need the limit relations (see Lemma 20)

$$\lim_{n\to\infty} n^2(1-x_{kn}^{\gamma,\delta}) = \lim_{n\to\infty} \frac{\sin^2 \frac{\theta_{kn}^{\gamma,\delta}}{2}}{2\left(\frac{\theta_{kn}^{\gamma,\delta}}{2}\right)^2} n^2\left(\theta_{kn}^{\gamma,\delta}\right)^2 = \frac{(\psi_{\gamma,k})^2}{2} \tag{82}$$

and ([39] (15.3.11))

$$\lim_{n\to\infty} n^{\gamma+1}\sqrt{\lambda_{kn}^{\gamma,\delta}} = \frac{2^{\frac{\delta-\gamma+1}{2}}(\psi_{\gamma,k})^\gamma}{\left|J_\gamma'(\psi_{\gamma,k})\right|}, \tag{83}$$

which are true for all $\gamma,\delta > -1$ and fixed $k \in \mathbb{N}$.

Lemma 26. *Let the conditions of Corollary 11 be satisfied. Then the strong limits of the operators*

$$\mathcal{V}_n\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^-\mathcal{L}_n\mathcal{V}_n^{-1}\mathcal{P}_n : \ell^2 \longrightarrow \ell^2 \quad \textit{and} \quad \left(\mathcal{V}_n\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^-\mathcal{L}_n\mathcal{V}_n^{-1}\mathcal{P}_n\right)^* : \ell^2 \longrightarrow \ell^2$$

as well as

$$\mathbb{F}_n\mathcal{V}_n\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^-\mathcal{L}_n\mathcal{V}_n^{-1}\mathbb{F}_n\mathcal{P}_n : \ell^2 \longrightarrow \ell^2 \quad \textit{and} \quad \left(\mathbb{F}_n\mathcal{V}_n\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^-\mathcal{L}_n\mathcal{V}_n^{-1}\mathbb{F}_n\mathcal{P}_n\right)^* : \ell^2 \longrightarrow \ell^2$$

exist, where $\mathcal{W}_2\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^-\mathcal{L}_n\right) = 0$ *and*

$$\mathcal{W}_3\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H}^-\mathcal{L}_n\right) = \left[\ \left[\frac{\psi_{\delta,j}}{\psi_{\delta,k}}\right]^{\beta+\frac{1}{2}} \frac{4}{\psi_{\delta,k}\left|J_\delta'(\psi_{\delta,k})\right|^2} H\left(\left[\frac{\psi_{\delta,j}}{\psi_{\delta,k}}\right]^2\right)\ \right]_{j,k=1}^\infty. \tag{84}$$

Proof. Due to Corollary 11 and the uniform boundedness of the operators $\left(\mathcal{E}_n^{(t)}\right)^{\pm 1}$ (see Lemma 12), all sequences of operators under consideration here are uniformly bounded. Thus, in view of the Banach-Steinhaus theorem, it suffices to verify the convergence on the set $\{e_m = (\delta_{j,m})_{j=0}^\infty : m = 0,1,\dots\} \subset \ell^2$. Moreover, in view of Corollary 9 and Lemma 5 we can replace the operators $\mathcal{M}_{n,H}^-$ by $\chi\mathcal{M}_{n,H}^-\chi\mathcal{I}$, where $\chi : [-1,1] \longrightarrow [0,1]$ is a continuous function, which vanishes in a neighbourhood of the point 1 and is identically 1 in a neighbourhood of the point -1.

Fix $k \in \mathbb{N}$. Regarding the proof of Corollary 11, for $n > k$, we have

$$\begin{aligned}
&\mathbb{F}_n\mathcal{V}_n\widetilde{\mathcal{L}}_n^{\gamma,\delta}\chi\mathcal{M}_{n,H}^-\chi\mathcal{L}_n\mathcal{V}_n^{-1}\mathbb{F}_n\mathcal{P}_n e_{k-1} \\
&= \left[\ h_{n+1-j,n+1-k}^{n,\chi}\ \right]_{j=1}^n \\
&= \left[\ \frac{\chi(x_{n+1-j,n}^{\gamma,\delta})n\,\lambda_{n+1-k,n}^{\gamma,\delta}}{v^{\gamma+\frac{1}{2},\delta+\frac{1}{2}}(x_{n+1-k,n}^{\gamma,\delta})} b_{n+1-j,n+1-k}^{n,-}\chi(x_{n+1-k,n}^{\gamma,\delta})\ \right]_{j=1}^n
\end{aligned}$$

with $b_{jk}^{n,-}$ defined in (81) and the entries $h_{jk}^{n,\chi}$ of the matrix $\mathbf{H}_n^\chi$ defined in the proof of Corollary 11. For fixed j, Lemma 21 yields $\left(\zeta_1 = \beta + \frac{1}{2}, \zeta_2 = -\beta + \frac{1}{2}\right)$

$$\chi(x_{n+1-j,n}^{\gamma,\delta})b_{n+1-j,n+1-k}^{n,-}\chi(x_{n+1-k,n}^{\gamma,\delta}) \longrightarrow \frac{2(\psi_{\delta,j})^{\beta+\frac{1}{2}}}{(\psi_{\delta,k})^{\beta+\frac{3}{2}}} H\left(\left[\frac{\psi_{\delta,j}}{\psi_{\delta,k}}\right]^2\right) \quad \text{if} \quad n \longrightarrow \infty,$$

since $\lim_{n\to\infty} x^{\gamma,\delta}_{n+1-j,n} = -1$. Moreover, taking into account (82) and (83) we get

$$\frac{n\,\lambda^{\gamma,\delta}_{n+1-k,n}}{v^{\gamma+\frac{1}{2},\delta+\frac{1}{2}}(x^{\gamma,\delta}_{n+1-k,n})} = \frac{\left(n^{\delta+1}\sqrt{\lambda^{\delta,\gamma}_{kn}}\right)^2}{\left(1+x^{\delta,\gamma}_{kn}\right)^{\gamma+\frac{1}{2}}\left[n^2(1-x^{\delta,\gamma}_{kn})\right]^{\delta+\frac{1}{2}}}$$

$$\longrightarrow \left[\frac{2^{\frac{\gamma-\delta+1}{2}}(\psi_{\delta,k})^{\delta}}{\left|J'_{\delta}(\psi_{\delta,k})\right|}\right]^2 \frac{1}{2^{\gamma+\frac{1}{2}}\left[\frac{(\psi_{\delta,k})^2}{2}\right]^{\delta+\frac{1}{2}}} = \frac{2}{\left|J'_{\delta}(\psi_{\delta,k})\right|^2\psi_{\delta,k}}$$

if $n \longrightarrow \infty$. Due to (64) and (65), respectively, we can estimate

$$h^{n,\chi}_{n+1-j,n+1-k} \le h^{d}_{jk}\,, \tag{85}$$

where $\left[\ h^d_{jk}\ \right]_{j=1}^{\infty} = \mathbf{H}^d e_{k-1} \in \ell^2$, since $\mathbf{H}^d \in \mathcal{L}(\ell^2)$ (cf. the end of the proof of Corollary 11). Hence, it remains to apply Lemma 25 with $\xi^n_j = h^{n,\chi}_{n-j,n+1-k}$ and $\eta_j = h^d_{j+1,k}$ for $k = 1,2,\ldots$ to get formula (84).

On the other hand, since $\lim_{n\to\infty} x^{\gamma,\delta}_{jn} = 1$ and $\chi(x) = 0$ in a neighbourhood of the point 1 , we have $\chi(x^{\gamma,\delta}_{jn}) = 0$ for all sufficiently large n . Hence,

$$\left(\mathcal{V}_n\mathcal{L}^{\gamma,\delta}_n\chi\mathcal{M}^{-}_{n,H}\chi\mathcal{V}_n^{-1}\mathbb{P}_n e_{k-1}\right)_j = h^{n,\chi}_{jk} \longrightarrow 0 \quad \text{for all} \quad j\in\mathbb{N}$$

if n tends to infinity. Moreover, again due to the choice of $\chi(x)$, we have $h^{n,\chi}_{jk} = 0$ for all $n > n_0 = n_0(k)$ and $j = 1,\ldots,n$. Consequently, if we set

$$M := \max\left\{h^{n,\chi}_{jk} : n = 1,\ldots,n_0\,,\ j = 1,\ldots,n\right\},$$

then $h^{n,\chi}_{jk} \le f_{j-1}$, $j \in \mathbb{N}$, where $f_j = M$, $j = 0,\ldots,n_0-1$ and $f_j = 0$, $j = n_0, n_0+1,\ldots$, such that $f = \left[\ f_j\ \right]_{j=0}^{\infty} \in \ell^2$. Thus, the application of Lemma 25 with $\xi^n_j = h^{n,\chi}_{j+1,k}$ and $\eta_j = f_j$ yields $\mathcal{W}_2\left(\widetilde{\mathcal{L}}^{\gamma,\delta}_n\mathcal{M}^{-}_{n,H}\mathcal{L}_n\right) = 0$. The proof of the strong convergence of the adjoint oprators follows the same ideas by using that $\left(\mathbf{H}^d\right)^*$ belongs to $\mathcal{L}(\ell^2)$ (see (85)). □

In the same way we can prove the following.

Lemma 27. *Let the conditions of Corollary 12 be in force. Then the strong limits of the operators*

$$\mathcal{V}_n\widetilde{\mathcal{L}}^{\gamma,\delta}_n\mathcal{M}^{+}_{n,H}\mathcal{L}_n\mathcal{V}_n^{-1}\mathcal{P}_n : \ell^2 \longrightarrow \ell^2 \quad \textit{and} \quad \left(\mathcal{V}_n\widetilde{\mathcal{L}}^{\gamma,\delta}_n\mathcal{M}^{+}_{n,H}\mathcal{L}_n\mathcal{V}_n^{-1}\mathcal{P}_n\right)^* : \ell^2 \longrightarrow \ell^2$$

as well as

$$\mathbb{F}_n\mathcal{V}_n\widetilde{\mathcal{L}}^{\gamma,\delta}_n\mathcal{M}^{+}_{n,H}\mathcal{L}_n\mathcal{V}_n^{-1}\mathbb{F}_n\mathcal{P}_n : \ell^2 \longrightarrow \ell^2 \quad \textit{and} \quad \left(\mathbb{F}_n\mathcal{V}_n\widetilde{\mathcal{L}}^{\gamma,\delta}_n\mathcal{M}^{+}_{n,H}\mathcal{L}_n\mathcal{V}_n^{-1}\mathbb{F}_n\mathcal{P}_n\right)^* : \ell^2 \longrightarrow \ell^2$$

exist, where $\mathcal{W}_3\left(\widetilde{\mathcal{L}}^{\gamma,\delta}_n\mathcal{M}^{+}_{n,H}\mathcal{L}_n\right) = 0$ *as well as*

$$\mathcal{W}_2\left(\widetilde{\mathcal{L}}^{\gamma,\delta}_n\mathcal{M}^{+}_{n,H}\mathcal{L}_n\right) = \left[\ \left[\frac{\psi_{\gamma,j}}{\psi_{\gamma,k}}\right]^{\alpha+\frac{1}{2}} \frac{4}{\psi_{\gamma,k}\left|J'_{\gamma}(\psi_{\gamma,k})\right|^2} H\left(\left[\frac{\psi_{\gamma,j}}{\psi_{\gamma,k}}\right]^2\right)\ \right]_{j,k=1}^{\infty}.$$

Lemma 28. *Let $a \in \mathbf{PC}$ and $(\rho,\gamma),(\tau,\delta) \in \Omega$. Then the strong limits of the operators*

$$\mathcal{V}_n \widetilde{\mathcal{L}}_n^{\gamma,\delta} a \mathcal{L}_n \mathcal{V}_n^{-1} \mathcal{P}_n : \ell^2 \longrightarrow \ell^2 \quad \text{and} \quad \left(\mathcal{V}_n \widetilde{\mathcal{L}}_n^{\gamma,\delta} a \mathcal{L}_n \mathcal{V}_n^{-1} \mathcal{P}_n\right)^* : \ell^2 \longrightarrow \ell^2$$

as well as

$$\mathbb{F}_n \mathcal{V}_n \widetilde{\mathcal{L}}_n^{\gamma,\delta} a \mathcal{L}_n \mathcal{V}_n^{-1} \mathbb{F}_n \mathcal{P}_n : \ell^2 \longrightarrow \ell^2 \quad \text{and} \quad \left(\mathbb{F}_n \mathcal{V}_n \widetilde{\mathcal{L}}_n^{\gamma,\delta} a \mathcal{L}_n \mathcal{V}_n^{-1} \mathbb{F}_n \mathcal{P}_n\right)^* : \ell^2 \longrightarrow \ell^2$$

exist, where

$$\mathcal{W}_2(\widetilde{\mathcal{L}}_n^{\gamma,\delta} a \mathcal{L}_n) = a(1)\mathbf{I} \quad \text{and} \quad \mathcal{W}_3(\widetilde{\mathcal{L}}_n^{\gamma,\delta} a \mathcal{L}_n) = a(-1)\mathbf{I}$$

as well as $\mathbf{I} : \ell^2 \longrightarrow \ell^2$ is the identity operator.

Proof. We are only going to show the first two convergences. The proof of the other convergences can be done in the same way. In view of Lemmas 12 and 22 it suffices to show the convergence on a dense subset of the space ℓ^2. Let $e_m = (\delta_{j,m})_{j=0}^{\infty}$, $m \in \mathbb{N}_0$, For $n > m+1$, we have

$$\mathcal{V}_n \widetilde{\mathcal{L}}_n^{\gamma,\delta} a \mathcal{L}_n \mathcal{V}_n^{-1} \mathcal{P}_n e_m = \left[\ a(x_{m+1,n}^{\gamma,\delta})\delta_{j,m+1}\ \right]_{j=1}^{n} = a(x_{m+1,n}^{\gamma,\delta}) e_m \,. \tag{86}$$

Since $a\big(x_{m+1,n}^{\gamma,\delta}\big) \longrightarrow a(1)$ as n tends to infinity, we obtain $\mathcal{W}_2(\widetilde{\mathcal{L}}_n^{\gamma,\delta} a \mathcal{L}_n) = a(1)\mathbf{I}$. Moreover, from (86) we infer $\mathcal{V}_n \widetilde{\mathcal{L}}_n^{\gamma,\delta} a \mathcal{L}_n \mathcal{V}_n^{-1} \mathcal{P}_n = \operatorname{diag}\left[\ a(x_{kn}^{\gamma,\delta})\ \right]_{k=1}^{n}$. Hence,

$$\left(\mathcal{V}_n \widetilde{\mathcal{L}}_n^{\gamma,\delta} a \mathcal{L}_n \mathcal{V}_n^{-1} \mathcal{P}_n\right)^* = \operatorname{diag}\left[\ \overline{a(x_{kn}^{\gamma,\delta})}\ \right]_{k=1}^{n},$$

and the strong convergence of these adjoint operators follows as before. □

6. The Stability Theorem

We recall that $\mathcal{A}_n = \widetilde{\mathcal{L}}_n^{\gamma,\delta}\Big(\mathcal{I} + c_- \mathcal{M}_{n,H_-}^- + c_+ \mathcal{M}_{n,H_+}^+ + \mathcal{K}_n\Big)\mathcal{L}_n$.

Theorem 1. *Let $\alpha,\beta \in (-1,1)$, $c_\pm \in \mathbf{PC}$, and $H = H_\pm \in \mathbf{C}^1(\mathbb{R}^+)$ be positive functions, satisfying condition (A_1) for $\xi = \xi_\pm$ and real numbers $p = p_\pm$, $q = q_\pm$, where $\xi_+ = \frac{1+\alpha}{2}$ and $\xi_- = \frac{1+\beta}{2}$, as well as $H_\pm' \in \mathbf{L}^2_{2p_\pm+1} \cap \mathbf{L}^2_{2q_\pm+1}$ and condition* (B)*, respectively. Moreover, let $K : [-1,1] \times [-1,1] \longrightarrow \mathbb{C}$ be a function, which fulfils the requirements of Lemma 19. Then $(\mathcal{A}_n)$ belongs to the algebra $\mathfrak{F}$ for all γ,δ with $(\rho,\gamma),(\tau,\delta) \in \Omega$. If this is the case, then for the sequence $(\mathcal{A}_n)$ to be stable it is necessary that the operators*

$$\mathcal{W}_1(\mathcal{A}_n) = \mathcal{I} + c_- \mathcal{M}_{H_-}^- + c_+ \mathcal{M}_{H_+}^+ + \mathcal{K} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$$

and

$$\mathcal{W}_2(\mathcal{A}_n) = \mathbf{I} + c_+(1)\left[\ \left[\frac{\psi_{\gamma,j}}{\psi_{\gamma,k}}\right]^{\alpha+\frac{1}{2}} \frac{4}{\psi_{\gamma,k}\left|J_\gamma'(\psi_{\gamma,k})\right|^2} H_+\left(\left[\frac{\psi_{\gamma,j}}{\psi_{\gamma,k}}\right]^2\right)\ \right]_{j,k=1}^{\infty} : \ell^2 \longrightarrow \ell^2,$$

$$\mathcal{W}_3(\mathcal{A}_n) = \mathbf{I} + c_-(-1)\left[\ \left[\frac{\psi_{\delta,j}}{\psi_{\delta,k}}\right]^{\beta+\frac{1}{2}} \frac{4}{\psi_{\delta,k}\left|J_\delta'(\psi_{\delta,k})\right|^2} H_-\left(\left[\frac{\psi_{\delta,j}}{\psi_{\delta,k}}\right]^2\right)\ \right]_{j,k=1}^{\infty} : \ell^2 \longrightarrow \ell^2.$$

are invertible.

Proof. At first let $\mathcal{K}$ be the zero operator. We notice that

$$\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}(\mathcal{I}+c_-\mathcal{M}_{n,H_-}^-+c_+\mathcal{M}_{n,H_+}^+)\mathcal{L}_n\right)$$
$$=(\mathcal{L}_n)+\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}c_-\mathcal{L}_n\right)\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H_-}^-\mathcal{L}_n\right)+\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}c_+\mathcal{L}_n\right)\left(\widetilde{\mathcal{L}}_n^{\gamma,\delta}\mathcal{M}_{n,H_+}^+\mathcal{L}_n\right).$$

From Corollaries 11 and 12, Lemmas 22–24 follows

$$\mathcal{A}_n \longrightarrow \mathcal{I}+c_-\mathcal{M}_{H_-}^-+c_+\mathcal{M}_{H_+}^+=\mathcal{W}_1(\mathcal{A}_n) \quad \text{and} \quad (\mathcal{A}_n)^* \longrightarrow (\mathcal{W}_1(\mathcal{A}_n))^*.$$

Lemmas 26–28 deliver the existence of

$$\mathcal{W}_2(\mathcal{A}_n)=\mathbf{I}+c_+(1)\left[\left[\frac{\psi_{\gamma,j}}{\psi_{\gamma,k}}\right]^{\alpha+\frac12}\frac{4}{\psi_{\gamma,k}\left|J_\gamma'(\psi_{\gamma,k})\right|^2}H_+\left(\left[\frac{\psi_{\gamma,j}}{\psi_{\gamma,k}}\right]^2\right)\right]_{j,k=1}^{\infty},$$

and

$$\mathcal{W}_3(\mathcal{A}_n)=\mathbf{I}+c_-(-1)\left[\left[\frac{\psi_{\delta,j}}{\psi_{\delta,k}}\right]^{\beta+\frac12}\frac{4}{\psi_{\delta,k}\left|J_\gamma'(\psi_{\delta,k})\right|^2}H_-\left(\left[\frac{\psi_{\delta,j}}{\psi_{\delta,k}}\right]^2\right)\right]_{j,k=1}^{\infty},$$

as well as the strong convergence of the sequences of the respective adjoint operators. Hence $(\mathcal{A}_n)\in\mathfrak{F}$. This allows us to apply Proposition 2, which immediately delivers the assertion. If the integral operator $\mathcal{K}$ does not vanish, then the assertion follows in combination with Lemma 19. □

In case of Chebyshev nodes, we can formulate the following theorem.

Theorem 2. *Let $\alpha,\beta\in(-1,1)$, $c_\pm\in\mathbf{PC}$, and $H_\pm\in\mathbf{C}^1(\mathbb{R}^+)$ be positive functions, satisfying condition (A_1) for $\xi=\xi_\pm$ and real numbers $p_\pm,q_\pm$, where $\xi_+=\frac{1+\alpha}{2}$, $\xi_-=\frac{1+\beta}{2}$, and let $H_\pm'\in\mathbf{L}^2_{2p_\pm+1}\cap\mathbf{L}^2_{2q_\pm+1}$. Moreover, assume that $K:[-1,1]\times[-1,1]\longrightarrow\mathbb{C}$ fulfils the requirements of Lemma 19. Then, $(\mathcal{A}_n)$ belongs to the algebra $\mathfrak{F}$ for all γ,δ with $|\gamma|=|\delta|=\frac12$ and $(\rho,\gamma),(\tau,\delta)\in\Omega$. In that case, for the sequence $(\mathcal{A}_n)$ to be stable it is necessary that*

(a) *the kernel of the operator $\mathcal{W}_1(\mathcal{A}_n)=\mathcal{I}+c_-\mathcal{M}_{H_-}^-+c_+\mathcal{M}_{H_+}^++\mathcal{K}:\mathbf{L}^2_{\alpha,\beta}\longrightarrow\mathbf{L}^2_{\alpha,\beta}$ is trivial,*

(b) *the curves*

$$\left\{1+c_-(-1)\widehat{H}_-(\xi_--\mathbf{i}t):t\in\overline{\mathbb{R}}\right\} \quad \text{and} \quad \left\{1+c_+(1)\widehat{H}_+(\xi_+-\mathbf{i}t):t\in\overline{\mathbb{R}}\right\}$$

do not contain the point 0 and their winding numbers are equal to zero,

(c) *the kernels of the operators*

$$\mathcal{W}_2(\mathcal{A}_n)=\mathbf{I}+c_+(1)\left[\left[\frac{\psi_{\gamma,j}}{\psi_{\gamma,k}}\right]^{\alpha+\frac12}\frac{2\pi}{\psi_{\gamma,k}}H_+\left(\left[\frac{\psi_{\gamma,j}}{\psi_{\gamma,k}}\right]^2\right)\right]_{j,k=1}^{\infty}:\ell^2\longrightarrow\ell^2,$$

$$\mathcal{W}_3(\mathcal{A}_n)=\mathbf{I}+c_-(-1)\left[\left[\frac{\psi_{\delta,j}}{\psi_{\delta,k}}\right]^{\beta+\frac12}\frac{2\pi}{\psi_{\delta,k}}H_-\left(\left[\frac{\psi_{\delta,j}}{\psi_{\delta,k}}\right]^2\right)\right]_{j,k=1}^{\infty}:\ell^2\longrightarrow\ell^2$$

are trivial.

If $\mathcal{K}$ is the zero operator and one of the functions $c_\pm$ vanishes, then condition (a) *is automatically a consequence of condition* (b)*, which due to Lemma 7.*

Proof. In view of Remark 1, in comparison with Theorem 1 we can omit the positivity of the Mellin kernel functions $H_\pm$ and condition (B), and we only have to show that the operators $\mathcal{W}_1(\mathcal{A}_n)$, $\mathcal{W}_2(\mathcal{A}_n)$, and $\mathcal{W}_3(\mathcal{A}_n)$ are invertible. Since $\mathcal{K} : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$ is compact (cf. Lemma 4), we can make use of Lemma 6. Thus, conditions (a) and (b) deliver the invertibility of the operator $\mathcal{W}_1(\mathcal{A}_n) : \mathbf{L}^2_{\alpha,\beta} \longrightarrow \mathbf{L}^2_{\alpha,\beta}$. It remains to check the invertibility of $\mathcal{W}_2(\mathcal{A}_n)$, $\mathcal{W}_3(\mathcal{A}_n) : \ell^2 \longrightarrow \ell^2$. Without loss of generality, we assume that $\gamma = \delta = \frac{1}{2}$. Then,

$$\mathcal{W}_2(\mathcal{A}_n) = \mathbf{I} + c_+(1)\left[\frac{2}{k}\left[\frac{j}{k}\right]^{\alpha+\frac{1}{2}} H_+\left(\left[\frac{j}{k}\right]^2\right) \right]_{j,k=1}^{\infty},$$

$$\mathcal{W}_3(\mathcal{A}_n) = \mathbf{I} + c_-(-1)\left[\frac{2}{k}\left[\frac{j}{k}\right]^{\beta+\frac{1}{2}} H_-\left(\left[\frac{j}{k}\right]^2\right) \right]_{j,k=1}^{\infty}.$$

We consider the function $g_\pm : \mathbb{R}^+ \longrightarrow \mathbb{R}$ with

$$g_d(x) = \begin{cases} x^{\alpha+\frac{1}{2}} H_+\left(x^2\right) & : \ d = +, \\ x^{\beta+\frac{1}{2}} H_-\left(x^2\right) & : \ d = -. \end{cases}$$

Due to our assumption we have $H_\pm \in \mathbf{L}^2_{2p_\pm-1}(\mathbb{R}^+) \cap \mathbf{L}^2_{2q_\pm-1}(\mathbb{R}^+)$ and $\xi_\pm \in (p_\pm, q_\pm)$. From that, we derive

$$g_+ \in \mathbf{L}^2_{2(2p_+-\alpha-\frac{1}{2})-1}(\mathbb{R}^+) \cap \mathbf{L}^2_{2(2q_+-\alpha-\frac{1}{2})-1} \quad \text{and} \quad \frac{1}{2} \in \left(2p_+ - \alpha - \frac{1}{2}, 2q_+ - \alpha - \frac{1}{2}\right)$$

as well as

$$g_- \in \mathbf{L}^2_{2(2p_--\beta-\frac{1}{2})-1}(\mathbb{R}^+) \cap \mathbf{L}^2_{2(2q_--\beta-\frac{1}{2})-1} \quad \text{and} \quad \frac{1}{2} \in \left(2p_- - \beta - \frac{1}{2}, 2q_- - \beta - \frac{1}{2}\right).$$

Moreover, the Mellin transforms of $g_+(x)$ and $g_-(x)$ are equal to

$$\widehat{H}_+\left(\frac{z+\alpha}{2} + \frac{1}{4}\right) \quad \text{and} \quad \widehat{H}_-\left(\frac{z+\beta}{2} + \frac{1}{4}\right),$$

respectively, and $p_+ < \frac{\frac{1}{2}+\alpha}{2} + \frac{1}{4} < q_+$ and $p_- < \frac{\frac{1}{2}+\beta}{2} + \frac{1}{4} < q_-$. Thus, $g_\pm$ fulfils condition (A_1) for $\xi = \frac{1}{2}$. This allows us to apply Lemma 11. Consequently, $\mathcal{W}_2(\mathcal{A}_n)$, $\mathcal{W}_3(\mathcal{A}_n) \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$ with

$$\Gamma_{\mathcal{W}_2(\mathcal{A}_n)} = \left\{1 + c_+(1)\widehat{H}_+(\xi_+ + \mathbf{i}t) : t \in \overline{\mathbb{R}}\right\}$$

and

$$\Gamma_{\mathcal{W}_3(\mathcal{A}_n)} = \left\{1 + c_-(-1)\widehat{H}_-(\xi_- + \mathbf{i}t) : t \in \overline{\mathbb{R}}\right\}.$$

In view of condition (b), the curves $\Gamma_{\mathcal{W}_2(\mathcal{A}_n)}$, $\Gamma_{\mathcal{W}_3(\mathcal{A}_n)}$ do not contain the point 0 and their winding number is zero. From Proposition 1 we derive that $\mathcal{W}_2(\mathcal{A}_n)$ and $\mathcal{W}_3(\mathcal{A}_n)$ are Fredholm operators with vanishing index. Thus, condition (c) delivers the invertibility of those operators. □

7. Final Remarks

Finally, let us discuss the progress we have made in the present paper for possible representations of endpoint singularities of the approximate solution (cf. (5))

$$u_n(x) = (1-x)^{\rho_0}(1+x)^{\tau_0} p_n(x) \tag{87}$$

in comparison with the paper [7]. Recall that for our method we can choose the parameters $\gamma, \delta > -1$ for the nodes $x_{jn}^{\gamma,\delta}$, the parameters $\alpha, \beta \in (-1,1)$ for the space $\mathbf{L}^2_{\alpha,\beta}$, and the parameters $\rho, \tau > -1$ for the orthonormal system $\left(v^{\rho_0,\tau_0} p_n^{\rho,\tau}\right)_{n=0}^{\infty}$ in $\mathbf{L}^2_{\alpha,\beta}$, where

$$\rho_0 = \frac{\rho - \alpha}{2} \quad \text{and} \quad \tau_0 = \frac{\tau - \beta}{2}.$$

In [7], the case $\gamma = \delta = \rho = \tau = \frac{1}{2}$ together with $\alpha, \beta \in (-1,1)$ is considered. That means that the range for ρ_0 and τ_0 is given by the interval $\left(-\frac{1}{4}, \frac{3}{4}\right)$. If, in the present paper, we choose $\gamma = \delta = \frac{1}{2}$, then for the choice of ρ and τ we have to fulfil the condition

$$\left(\rho, \frac{1}{2}\right), \left(\tau, \frac{1}{2}\right) \in \Omega, \tag{88}$$

where (cf. (50))

$$\Omega = \left\{(\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_1, \omega_2 > -1,\, -\frac{1}{2} < \omega_1 - \omega_2 < \frac{1}{2},\, \omega_1 < 2\omega_2 + 1\right\}$$
$$\cup \left\{(\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_1, \omega_2 > -1,\, \frac{1}{2} < \omega_1 - \omega_2 < \frac{3}{2}\right\}.$$

Condition (88) is equivalent to $\rho, \tau \in (0,1) \cup (1,2)$, such that, for ρ_0 and τ_0 we have the possible ranges

$$\rho_0 \in \left\{\frac{\rho - \alpha}{2} : \rho \in (0,1) \cup (1,2), \alpha \in (-1,1)\right\}$$
$$= \bigcup_{\alpha \in (-1,1)} \left[\left(-\frac{\alpha}{2}, \frac{1-\alpha}{2}\right) \cup \left(\frac{1-\alpha}{2}, \frac{2-\alpha}{2}\right)\right] = \left(-\frac{1}{2}, \frac{3}{2}\right)$$

and

$$\tau_0 \in \left\{\frac{\tau - \beta}{2} : \tau \in (0,1) \cup (1,2), \beta \in (-1,1)\right\}$$
$$= \bigcup_{\beta \in (-1,1)} \left[\left(-\frac{\beta}{2}, \frac{1-\beta}{2}\right) \cup \left(\frac{1-\beta}{2}, \frac{2-\beta}{2}\right)\right] = \left(-\frac{1}{2}, \frac{3}{2}\right).$$

These possible ranges for ρ_0 and τ_0 can be extended, if we do not fix γ and δ. We see that, for every $\rho > -1$ and $\tau > -1$, there exist $\gamma > -1$ and $\delta > -1$ such that $(\rho, \gamma) \in \Omega$ and $(\tau, \delta) \in \Omega$. Consequently, for every $\rho_0 > -1$ and $\tau_0 > -1$, we can choose parameters $\rho, \tau > -1$ and $\gamma, \delta > -1$ such that the respective collocation-quadrature method (42) looks for approximate solutions of the form (87) with a polynomial $p_n(x)$.

Another distinction between [7] and the present investigations is that in [7] the collocation method is studied, the implementation of which is much more expansive (cf. [12,16]) than the collocation-quadrature method considered here.

Of course, the advantage of the results in [7] is that there also the sufficiency of the stability conditions is proved and that in (1) also the case $b(x) \not\equiv 0$ is considered. These problems will be studied for the collocation-quadrature methods considered here in forthcoming papers.

Finally, we can conclude that we were able to prove necessary conditions for the stability of, in comparison with the existing literature, a wider class of collocation-quadrature methods based on the zeros of classical Jacobi polynomials. In this way we can enlarge the range of endpoint singularities of the solutions of singular integral equations of Mellin type, which we can represent in the respective approximate solutions. The questions on the sufficiency of the formulated stability conditions and on the extension of the results presented here to Cauchy singular integral equations remain open for further studies.

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