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Article

Stability Analysis of Fractional-Order Predator-Prey System with Consuming Food Resource

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Abstract: The cardinal element of ecology is the predator-prey relationship. The population of interacting organisms is based on many factors such as food, water, space, and protection. A key component among these factors is food. The presence of food for the organisms shapes the structure of the habitat. The present study considers a predator and two types of prey. It is assumed that one prey species utilizes the same food resource as the predator, whereas the other prey species depends on a different food resource. The existence and uniqueness of the model are studied using the Lipschitz condition. The fixed points for the fractional-order model are sorted out, and the existence of the equilibrium points is discussed. The stability analysis of the model for the biologically important fixed points is provided. These include the coexistence fixed point and the prey-free (using the same food resources as the predator does) fixed point. A fractional-order scheme is implemented to support theoretical results for the stability of equilibrium points. The time series solution of the model is presented in the form of plots. Moreover, the impact of some mathematically and biologically important parameters is presented.

Keywords: ecology; fractional-order models; Lipschitz continuity; Caputo fractional derivative; stability

MSC: 34D20; 34D23; 65L07; 65L20

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1. Introduction

The interactive forces among predator and prey studies have remained a strikingly distinctive research topic in ecology and mathematical biology. The Lotka-Volterra system was a great initiative to put forth the dynamics of predator-prey studies [1]. In [2], the author has considered two categories of predator, juvenile and mature predator. He discussed the stability and bifurcation and showed the effects of the conversion of prey to a juvenile predator on the stability of the system. He also showed that the conversion parameter from juvenile to mature predator disturbs the system's stability. In [3], authors have studied the fractional-order singular Holling type-II predator-prey system. It is proved that the addition of the fractional order in a system of differential equations plays an important role in the system's stability. Zhang F. et al. evaluated a stage-structured model with two types of predators. They have discussed the effects of cannibalism on the model [4]. They have proved that cannibalism can reverse the stability of the system. In [5], authors have studied a fractional-order stage-structured model. The global stability of the system is investigated. Bifurcation analysis is also provided for the conversion coefficient of prey into an immature predator. The first SIR model was presented by Kermack and MacKendrick [6], which got much attention. The SI epidemic model, with its global stability and feedback control in a variable environment, was given by Li et al. [7]. Epidemiological models deal with the spread of infectious diseases among different species, which is evident from the facts mentioned in [8,9]. Such models deal with the realistic impact of the interaction between the predator-prey; however, the dynamics of the system would be highly influenced by the infectious disease [10,11]. The eco-epidemiological model defines the dimension of infection spread among predator and prey along with its control.

Taking into account the study [12], Hilker and Schmitz investigated the stability of predator-prey oscillations. The authors discussed the destabilizing effects of parasites and resource management. A detailed review of publications depicted that several studies were built on an eco-epidemiological model either by using prey [13–16] or predator [17–20] or using both populations [21–23]. The infectious disease could be the manifestation of a virus, bacteria, or any other natural or artificial calamity [24].

The functional response of a Holling type II predator in an eco-epidemiological model involving three species was studied in [11]. In this study, the authors investigated the effects of the recovery parameter of prey species on the system's dynamics and the optimization of net profit from the harvesting of predators. The generalization of ordinary differential equations is fractional-order differential equations with non-integer order, which have massive adaptations in biological and engineering sciences [25,26]. The main advantage of using fractional-order models is that they are non-local. This non-locality makes them flexible. These models are used for solving problems that have spatial or memory dependence. Since the current state of a system is dependent on its recent past state, the fractional-order derivative is considered a non-local operator [27].

Fractional calculus is fused into complex dynamic systems, revolutionizing the design theory and control performance of complex systems. Scientists have found that natural physical phenomena can be represented more accurately by fractional-order models than traditional full-order models [28]. Recently, some researchers have introduced fractional calculus into the predator-prey model to build a fractional predator-prey model. For example, the design and control of various ecological models [29–31].

Recently, fractional calculus has been successfully introduced into predator-prey models, and some interesting phenomena have been studied. In [32], the authors consider a fractionally ordered delayed predator-prey system with harvesting conditions. In [33], Mondal et al. found that solutions for fractional-order predator-prey systems slowly converge to their respective equilibrium points as the fractional order decreases. In [34], Chinnathambi and Rihan proposed that fractional ordering could enhance the stability of the prey-predator system and prevent the occurrence of oscillatory behavior. Fractional dynamics of lag-free predator-prey models have been documented [35,36]. In [37], the authors studied the population growth model under classical and non-classical operators. They used actual statistical data and proved that the fractional approach is better than the classical one.

The fractional-order differential equation imparts a memory effect in the model, making it more dynamic in behavior [38]. In [39], Zhao and Luo defined the general fractional derivative with memory effects dynamics. Bolton et al. [40] proved the accuracy of the fractional-order Gompertz growth model more than the integer-order Gompertz model with respect to the experimental dataset. Predator-prey relationships can be better described using fractional-order systems, as shown by the findings of [41,42], which conclude that the fractional-order differential equation can improve the modeling of biological phenomena. In [43], the authors depicted a fractional-order eco-epidemiological model with an infectious prey population. Some theoretical and numerical analyses supported the study. Many researchers have investigated the properties of fractional-order models [44]. The authors of [45] stress the need for text mining for detecting Financial Statement Fraud (FSF) incidents. The scientific community as a whole uses data mining to detect FSF most of the time. The study presented in [46] advocates for implementing the NIST Framework-based approach for properly managing cyber security in public sector enterprises within the context of the provision of digital services. With the help of a fractional linear regression equation, the author of [47] investigates the similarities and differences in carbohydrate metabolism and energy expenditure among different groups of humans who expend the same amount of oxygen during exercise. In [48], the author suggests a new way to solve second-order fractional differential equations using power series expansion. The current work [49] examined the stochastic dengue model's computational dynamics using the actual material. The main study area in these papers is stability and bifurcation analysis. According to [50,51], if these equations are defined on the real line, they transform into periodic signals. The proposed study focuses on the final state trajectory. All local paths with positive values are drawn to the limiting cycle (asymptotically stable limit cycle) [52].

As far as we know, no one has thought to use a fractional-order model in which multiple species share a single food source. We have developed a fractional order model with three species, one predator, and two prey. One prey and predator depend on the same resource, while the second prey depends on another resource. The setting of the paper is as follows:

Section 2 presents some fractional calculus results, which will help us understand the following sections. We have presented the concept of the Caputo fractional derivative along with Lipschitz continuity and stability conditions for the fractional model. Section 3 deals with the model formulation and description of the parameters involved. Section 4 discusses the model's existence, uniqueness, and non-negativity. Section 5 deals with the existing condition of all the equilibrium points of the model. Section 6 presents the stability of some important fixed points. Sections 7 and 8, respectively, deal with numerical simulations and concluding remarks.

2. Preliminaries

Some related concepts of fractional calculus are presented in this section, contributing to the study further.

Definition 1 [5]. *The Caputo fractional derivative is given as under*

$$\mathcal{D}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\varphi)^{n-\alpha-1} f^{(n)}(\phi) d\phi, \qquad n-1<\alpha<n, \qquad n \in N.$$

where Γ *represents the gamma function.*

Definition 2 [53]. *A real-valued function* $f : \mathcal{R} \to \mathcal{R}$ *is known as Lipschitz continuous, if* $\forall y_1,\ y_2 \epsilon\ \mathcal{R}$, $\exists$ *a positive constant k such that the following holds*

$$|f(y_1) - f(y_2)| \leq k|y_1 - y_2|.$$

Theorem 1 [54]. *For the fractional-order system* $\mathcal{D}^{\alpha} x = f(x)$ *with the following initial condition* $x(0) = x_0$, $x \in R^{\alpha}$, $0 < \alpha < 1$.

Then equilibrium point x_* is locally asymptotically stable if $J|_{x=x_*}$ has all the eigenvalues such that $|Arg(\lambda_i)| > \alpha\frac{\pi}{2}$, where J is the Jacobian matrix.

3. The Model

In recent years, fractional calculus has caught the attention of many researchers due to its widespread and diverse applications in different fields of study, such as engineering and science. Many applications are found in statistics, dynamics, electrochemistry, electromagnetism, signal processing, mathematical biology, optimization, and control theory. Mathematical models communicating real-life problems can be formulated well by fractional-order differential equations. We present a version of the predator-prey fractional-order model. We consider three species, a predator and two prey, considering the competition between the predator and a prey species for the food resource in the habitat. Moreover, it is assumed that the other prey species depend on some other food. Suppose $U(t)$ represents the density of the predator, $V_1(t)$ represents the species of prey depending on the same resources as the predator does where $V_2(t)$ indicates the species depending on a different resource. The parameters are described in the following Table 1.

Table 1. Breif description of variables and parameters.

Parameters	Physical Meaning
U	Density of Predator
$V_1(t)$	Density of prey using the same resources as the predator
$V_2(t)$	Density of Prey using different resources
a	Rate of consumption of common food by the predator
b	Rate of consumption of common food by prey
c	Rate of consumption of non-common food by prey
β_1, β_2	Death rate for prey due to predation
f	Common food resources
g	Non-common food resource for prey
η	Natural death rate of predator
μ_1	Natural death rate of first prey
μ_2	Natural death rate of the second prey

The following differential equations represent the model

$$D^{\alpha}\ (U\ (t)) = U\left(\frac{af}{aU + bV_1} + \beta_1 V_1 + \beta_2 V_2 - \eta\right) \tag{1}$$

$$D^{\alpha}\ (V_I\ (t)) = V_1\left(\frac{bf}{aU + bV_1} - \beta_1 U - \mu_1\right) \tag{2}$$

$$D^{\alpha}\ (V_2\ (t)) = V_2(cg - \beta_2 U - \mu_2). \tag{3}$$

Here, $\mathcal{D}^{\alpha}$ denotes Caputo derivative, which is taken with respect to time.

$$\mathcal{D}^{\alpha} f(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t (t-\varphi)^{n-\mu-1} f^n(\phi)\, d\phi, n-1 < \alpha < n, n \in N. \tag{4}$$

These models can be used for studying predator-prey relationships, in which predator and prey use the same food. Moreover, the predator has two prey available in the environment, and one uses the same food described earlier. It is a fact that most predators depend on more than one prey. For instance, an owl hunts a hawk and weasel. It is important to note that weasels and owls have common food in the form of the shrew.

4. Analysis of the Model

The mathematical analysis of the proposed model is presented here. Here, we use the Lipschitz criterion to demonstrate the existence and distinction of the model.

4.1. Existence and Uniqueness

We define the region $\Theta \times (0, t]$, where $\Theta = \left\{(U, V_1,\ V_2) \in \mathcal{R}^3 :\ \max((U, V_1,\ V_2) \leq \varphi\right\}$.

Theorem 2. *There exists a unique solution $\tau = (U, V_1, V_2)$ for the model with initial conditions $\chi_0 = (U_0, V_{10}, V_{20})\ \forall\ t \geq 0$.*

Proof of Theorem 2. We define a mapping $F(\chi) = (F_1(\chi), F_2(\chi), F_3(\chi))$ for χ, χ_0 belonging to Θ. Here,

$$F_1(\chi) = U\left(\frac{af}{aU + bV_1} + \beta_1 V_1 + \beta_2 V_2 - \eta\right) \tag{5}$$

$$F_2(\chi) = V_1\left(\frac{bf}{aU + bV_1} - \beta_1 U - \mu_1\right) \tag{6}$$

$$F_3(\chi) = V_2(cg - \beta_2 U - \mu_2). \tag{7}$$

We use the Lipschitz condition to prove the uniqueness of the solution for the model.

$$||F(\chi) - F(\overline{\chi})||$$

$$
\begin{aligned}
&= |F_1(\chi) - F_1(\overline{\chi})| + |F_2(\chi) - F_2(\overline{\chi})| + |F_3(\chi) - F_3(\overline{\chi})| \\
&= \left| U(\frac{af}{aU + bV_1} + \beta_1 V_1 + \beta_2 V_2 - \eta) - \overline{U}(\frac{af}{a\overline{U} + b\overline{V}_1} + \beta_1 \overline{V}_1 + \beta_2 \overline{V}_2 - \eta) \right| \\
&+ \left| V_1(\frac{bf}{aU + bV_1} - \beta_1 U - \mu_1) - \overline{V}_1(\frac{bf}{a\overline{U} + b\overline{V}_1} - \beta_1 \overline{U} - \mu_1) \right| \\
&+ \left| V_2(cg - \beta_2 U - \mu_2) - \overline{V}_2(cg - \beta_2 \overline{U} - \mu_2) \right| \\
&\leq abf\varphi(|V_1(t) - \overline{V}_1(t) + |U(t) - \overline{U}(t)|) + \beta_1\varphi(|U(t) - \overline{U}(t)| + |V_1(t) - \overline{V}_1(t)|) + \\
&\beta_2\varphi(|U(t) - \overline{U}(t)| + |V_2(t) - \overline{V}_2(t)|) + \eta|U(t) - \overline{U}(t)| + a^2 f\varphi|V_1(t) - \overline{V}_1(t)|) \\
&+ |U(t) - \overline{U}(t)| + \beta_2\varphi(|U(t) - \overline{U}(t)| + \beta_1\varphi(|U(t) - \overline{U}(t)| + |V_1(t) - \overline{V}_1(t)|) + \mu_1 + |V_1(t) \\
&- \overline{V}_1(t) + cg|V_2(t) - \overline{V}_2(t)| + \mu_2|V_2(t) - \overline{V}_2(t)| + \beta_2\varphi(|U(t) - \overline{U}(t)| + |V_2(t) - \overline{V}_2(t)|) \\
&= (abf\varphi + a^2 f\varphi + 2\beta_1\varphi + 2\beta_2\varphi + \eta)|U(t) - \overline{U}(t)| + (abf\varphi + a^2 f\varphi + 2\beta_1\varphi + \mu_1) \\
&|V_1(t) - \overline{V}_1(t)|(2\beta_2\varphi + cg + \mu_2)|V_2(t) - \overline{V}_2(t)| \\
&\leq M||\chi - \overline{\chi}||.
\end{aligned}
$$

where,

$$M = \max\{abf\varphi + a^2 f\varphi + 2\beta_1\varphi + 2\beta_2\varphi + \eta, abf\varphi + a^2 f\varphi + 2\beta_1\varphi + \mu_1, 2\beta_2\varphi + cg + \mu_2\}.$$

Therefore, it is proved that $F(\chi)$ is a Lipschitz continuous function, hence the theorem. □

4.2. Non-Negativity of the Model

Here we show the non-negativity of the proposed model. It is easy to see that

$$\mathcal{D}^\alpha(\mathrm{U}(t))|_{U=0} = 0 \tag{8}$$

$$\mathcal{D}^\alpha(V_1(t))|_{V_1=0} = 0 \tag{9}$$

$$\mathcal{D}^\alpha(V_2(t))|_{V_2=0} = 0. \tag{10}$$

Equations (8)–(10) prove the non-negativity of the model.

5. Equilibrium Points of the Model

To calculate the model's equilibrium points, we have the following system of equations.

$$0 = U\left(\frac{af}{aU + bV_1} + \beta_1 V_1 + \beta_2 V_2 - \eta\right) \tag{11}$$

$$0 = V_1\left(\frac{bf}{aU + bV_1} - \beta_1 U - \mu_1\right) \tag{12}$$

$$0 = V_2(cg - \beta_2 U - \mu_2). \tag{13}$$

It is obvious from the above system of equations that (0, 0, 0) is an equilibrium point which has no importance ecologically. After some calculations, one can easily find the following equilibrium points.

1. Predator-free, one prey-free fixed point

$$\{U \to 0, V_1 \to \frac{f}{\mu_1}, V_2 \to 0\} \tag{14}$$

This particular equilibrium point represents the extinction of predator specie along with the prey using different resources in the habitat. One can observe that this fixed point always holds.

2. Prey (using another resource as a predator does) free fixed point

$$\{U \to -\frac{\mu_1}{\beta_1} + \frac{bf}{b\eta - a\mu_1}, V_1 \to \frac{\eta}{\beta_1} + \frac{af}{-b\eta + a\mu_1}, V_2 \to 0\}. \tag{15}$$

There is a second equilibrium point of the model, which guarantees the existence of both species using common food resources (predator and prey) when $b\eta > a\mu_1$ and $\eta(-b\eta + a\mu_1) > af\beta_1$.

3. Prey (using the same resource as a predator does) free fixed point

$$\{U \to \frac{cg - \mu_2}{\beta_2}, V_1 \to 0, V_2 \to \frac{\eta}{\beta_2} + \frac{f}{-cg + \mu_2}\} \tag{16}$$

The equilibrium point guarantees the extinction of prey using the same resources as a predator under the following condition $cg > \mu_2$ and $\eta(-cg + \mu_2) > f\beta_2$.

4. Preys free fixed point

$$\{U \to \frac{f}{\eta}, V_1 \to 0, V_2 \to 0\} \tag{17}$$

It is clear that prey-free points always exist.

5. Coexistence fixed point

$$\{U \to \frac{cg - \mu_2}{\beta_2}, V_1 \to \frac{\frac{f\beta_2^2}{\beta_2\mu_1 + \beta_1(cg - \mu_2)} + \frac{a(-cg+\mu_2)}{b}}{\beta_2}, V_2 \to \frac{\eta - \frac{a\mu_1}{b}}{\beta_2} - \frac{f\beta_1}{\beta_2\mu_1 + \beta_1(cg - \mu_2)}\} \tag{18}$$

This equilibrium point guarantees the existence of all species in the environment under the following conditions.

$$cg > \mu_2, \tag{19}$$

$$\frac{f\beta_2^2}{\beta_2\mu_1 + \beta_1(cg - \mu_2)} > \frac{a(-cg + \mu_2)}{b}, \tag{20}$$

$$\frac{\eta - \frac{a\mu_1}{b}}{\beta_2} > \frac{f\beta_1}{\beta_2\mu_1 + \beta_1(cg - \mu_2)}. \tag{21}$$

6. Stability of Equilibrium Points

This section deals with the stability of two fixed points of key interest: the prey (using the same food resources as the predator) free fixed point and the coexistence fixed point.

To deal with the stability of fixed points, we use the Routh Hurwitz criterion for fractional order. It is easy to observe that Jacobian of the system (1–3) is as under

$$J = \begin{bmatrix} -\eta - \frac{a^2fU}{(aU+bV_1)^2} + \frac{af}{aU+bV_1} + V_1\beta_1 + V_2\beta_2 & U\left(-\frac{abf}{(aU+bV_1)^2} + \beta_1\right) & U\beta_2 \\ V_1\left(-\frac{abf}{(aU+bV_1)^2} - \beta_1\right) & -\frac{b^2fV_1}{(aU+bV_1)^2} + \frac{bf}{aU+bV_1} - U\beta_1 - \mu_1 & 0 \\ -V_2\beta_2 & 0 & cg - U\beta_2 - \mu_2 \end{bmatrix}. \tag{22}$$

6.1. Stability of Coexistence Fixed Point

We have the following lemma for the coexistence fixed point.

Lemma 1. *The coexistence fixed point* (U^*, V_1^*, V_2^*) *is stable if* $A_2 > 0$*, where*

$$A_2 = -a_{11} - a_{22} \text{ and}$$

$$a_{11} = -\frac{a^2(\beta_2\mu_1 + \beta_1(cg - \mu_2))^2(cg - \mu_2)}{b^2 f \beta_2^3},$$

$$a_{22} = \frac{(\beta_2\mu_1 + \beta_1(cg - \mu_2))\left(a\beta_1(-cg + \mu_2)^2 - \beta_2(bf\beta_2 + a\mu_1(-cg + \mu_2))\right)}{bf\beta_2^3}.$$

Proof of Lemma 1. We calculate the Jacobian matrix for the model at $E_{co} = \left(U^*, V_1^*, V_2^*\right)$. Here,

$$U^* = \frac{cg - \mu_2}{\beta_2},$$

$$V_1^* = \frac{\frac{f\beta_2^2}{\beta_2\mu_1 + \beta_1(cg - \mu_2)} + \frac{a(-cg + \mu_2)}{b}}{\beta_2},$$

$$V_2^* = \frac{\frac{f\beta_2^2}{\beta_2\mu_1 + \beta_1(cg - \mu_2)} + \frac{a(-cg + \mu_2)}{b}}{\beta_2},$$

$$J(E_{\mathrm{co}}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix}.$$

where,

$$a_{11} = -\frac{a^2(\beta_2\mu_1 + \beta_1(cg - \mu_2))^2(cg - \mu_2)}{b^2 f \beta_2^3},$$

$$a_{12} = -\frac{(cg - \mu_2)(a\beta_2^2\mu_1^2 + a\beta_1^2(-cg + \mu_2)^2 - \beta_1\beta_2(bf\beta_2 + 2a\mu_1(-cg + \mu_2)))}{bf\beta_2^3},$$

$$a_{13} = cg - \mu_2,$$

$$a_{21} = \frac{(a\beta_2^2\mu_1^2 + \beta_1\beta_2(bf\beta_2 + 2a\mu_1(cg - \mu_2)) + a\beta_1^2(-cg + \mu_2)^2)(a\beta_1(-cg + \mu_2)^2 - \beta_2(bf\beta_2 + a\mu_1(-cg + \mu_2)))}{b^2 f \beta_2^3(\beta_2\mu_1 + \beta_1(cg - \mu_2))},$$

$$a_{22} = \frac{(\beta_2\mu_1 + \beta_1(cg - \mu_2))(a\beta_1(-cg + \mu_2)^2 - \beta_2(bf\beta_2 + a\mu_1(-cg + \mu_2)))}{bf\beta_2^3},$$

$$\mathrm{a}_{31} = -\eta + \frac{a\mu_1}{b} + \frac{f\beta_1\beta_2}{\beta_2\mu_1 + \beta_1(cg - \mu_2)}.$$

The characteristic polynomial can be written as under

$$P(\lambda) = \lambda^3 - \lambda^2(a_{11} + a_{22}) + a_{13}a_{22}a_{31} - \lambda(a_{12}a_{21} - a_{11}a_{22} + a_{13}a_{31}). \tag{23}$$

The solution of the above equation implies that we have the following eigenvalues

$$P(\lambda) = \lambda^3 + A_2\lambda^2 + A_1\lambda + A_0.$$

Here $A_0 = a_{13}a_{22}a_{31}$, $A_1 = (-a_{12}a_{21} + a_{11}a_{22} - a_{13}a_{31})$, $A_2 = -a_{11} - a_{22}$.
By taking $A_0 = A_1A_2$ we have the following eigenvalues

$$\lambda_1 = -A_2, \lambda_2 = -\mathrm{i}\sqrt{A_1}, \lambda_3 = \mathrm{i}\sqrt{A_1}.$$

We have $|Arg(\lambda_1)| = \pi > \alpha\frac{\pi}{2}$, provided $A_2 > 0$.
In addition, we have $|Arg(\lambda_2)| = \frac{\pi}{2} > \alpha\frac{\pi}{2}$ and $|Arg(\lambda_3)| = \frac{\pi}{2} > \alpha\frac{\pi}{2}$, hence the result.

□

6.2. Stability of Prey Free Equilibrium Point

For the stability of a prey-free (using common resources as predator) fixed point, we have the following lemma.

Lemma 2. *The prey-free (using common resources as predator) fixed point* $\left(U_{pf}, 0, V_{2pf}\right)$ *is asymptotically stable provided* $A_2 > 0$*. Where* $A_2 = -b_{11} - b_{22}$*, where* b_{11} *and* b_{22} *are defined below*

$$b_{11} = \frac{f\beta_2}{-cg + \mu_2}\,, \tag{24}$$

$$b_{22} = -\mu_1 + \frac{\beta_1(-cg + \mu_2)}{\beta_2} + \frac{bf\beta_2}{acg - a\mu_2}. \tag{25}$$

Proof of Lemma 2. To deal with the stability of fixed points, we use the Routh Hurwitz criterion. It is easy to observe that the Jacobian of the system (1–3) is given in (22). Now we calculate Jacobian at $\left(U_{pf}, 0, V_{2pf}\right)$, where

$$U_{pf} = \frac{cg - \mu_2}{\beta_2} \text{ and } V_{2pf} = \frac{\eta}{\beta_2} + \frac{f}{-cg + \mu_2}.$$

The Jacobian can be written as $J\left(E_{pf}\right) = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & 0 \\ b_{31} & 0 & 0 \end{bmatrix}$. where,

$$b_{11} = \frac{f\beta_2}{-cg + \mu_2},$$

$$b_{12} = \frac{(cg - \mu_2)(\beta_1 - \frac{bf\beta_2^2}{a(-cg+\mu_2)^2})}{\beta_2},$$

$$b_{13} = cg - \mu_2,$$

$$b_{22} = -\mu_1 + \frac{\beta_1(-cg + \mu_2)}{\beta_2} + \frac{bf\beta_2}{acg - a\mu_2},$$

$$b_{31} = -\eta + \frac{f\beta_2}{cg - \mu_2}.$$

The characteristic polynomial can be written as

$$P(\lambda) = \lambda^3 - \lambda^2(b_{11} + b_{22}) + b_{13}b_{22}b_{31} - \lambda(-b_{11}b_{22} + b_{13}b_{31})$$

$$P(\lambda) = \lambda^3 + A_2\lambda^2 + A_1\lambda + A_0.$$

Here, $A_0 = b_{13}b_{22}b_{31}$, $A_1 = b_{11}b_{22} - b_{13}b_{31}$ and $A_2 = -b_{11} - b_{22}$.
By taking $A_0 = A_1A_2$ we have the following eigenvalues

$$\lambda_1 = -A_2,\ \lambda_2 = -\mathrm{i}\sqrt{A_1} \text{ and } \lambda_3 = \mathrm{i}\sqrt{A_1}.$$

Now, $\left|Arg(\lambda_1) = \pi > \alpha\frac{\pi}{2}\right|$, provided $A_2 > 0$, $|Arg(\lambda_2)| = \frac{\pi}{2} > \alpha\frac{\pi}{2}$ and $|Arg(\lambda_3)| = \frac{\pi}{2} > \alpha\frac{\pi}{2}$, hence the result. □

7. Numerical Results

This section provides the simulation outcomes to support the theoretical results. We see the impact of fractional order "α" and other parameters on the stability of the coexistence of fixed points. We also provide the time series solution of the proposed model in this section.

Figure 1 presents the phase portraits of the fractional-order model (1–3) for different values of fractional parameter "α". The initial conditions taken are $U_S(0) = 0.40$, $U_I(0) = 0.20$, $V(0) = 0.20$, whereas the values of other parameters are $a = 0.9$, $\alpha_1 = 0.2$, $\beta = 0.2, m = 0.1, \alpha_2 = 0.9, \gamma = 0.91$, and $n = 0.3$. We have taken three values of fractional order as α = 0.4, 0.7, and 0.9.

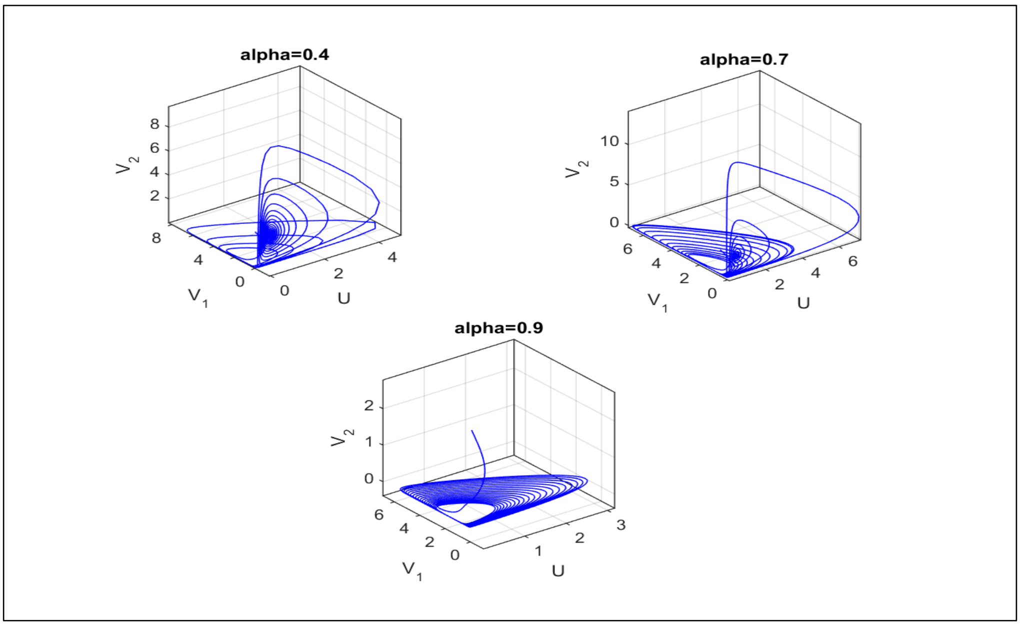

Figure 1. Phase Portraits different values of fractional parameter α.

Figure 2 demonstrates the phase portraits of the fractional-order system for three different values of "a" as 0.3, 0.4, and 0.5. The values of other parameters and ICs are $U(0) = 0.31$, $V_1(0) = 2.402$, $V_2(0) = 0.2003$, $\alpha = 0.5$, $b = 0.82$, $c = 0.21$, $f = 0.01$, $g = 0.55$, $\beta_1 = 0.2$, $\eta = 0.55$, $\mu_1 = 0.11$ and $\mu_2 = 0.05$.

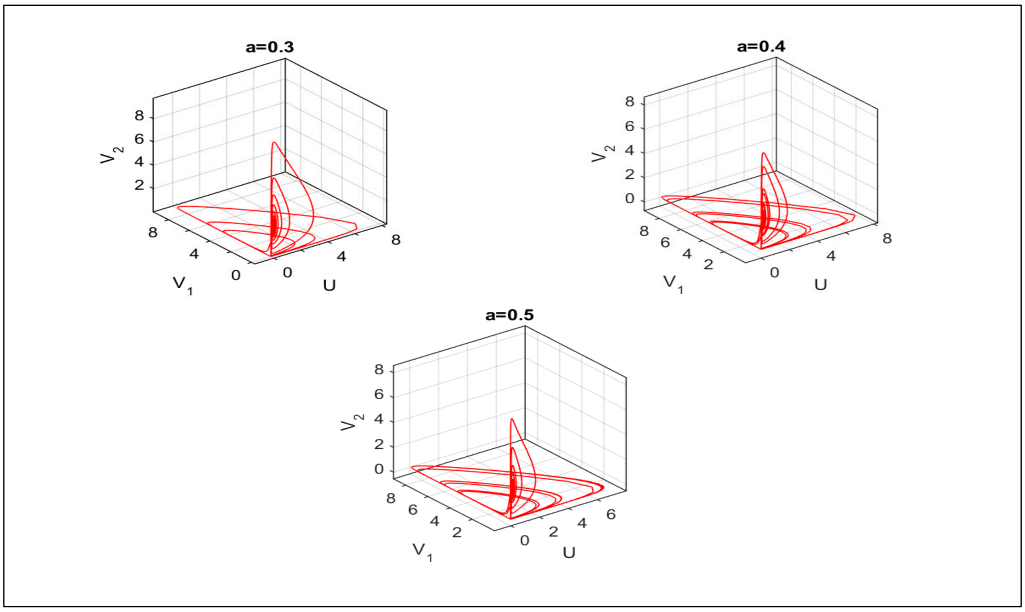

Figure 2. Phase portraits at different values of consumption rate of the common resource.

Figure 3 presents the phase portraits of the fractional-order model for different values of common food parameter "f" as 0.05, 0.07, and 0.09. The initial conditions and values of parameters are $U(0) = 0.3127$, $V_1(0) = 2.4016$, $V_2(0) = 0.2003$, $\alpha = 0.5$, $a = 0.50$, $b = 0.82$, $c = 0.21$, $\eta = 0.55$, $\mu_1 = 0.11$, $\mu_2 = 0.05$, $g = 0.55$, $\beta_1 = 0.2$, $\beta_2 = 0.20$. It is obvious that the system moves towards stability for higher resource parameter values.

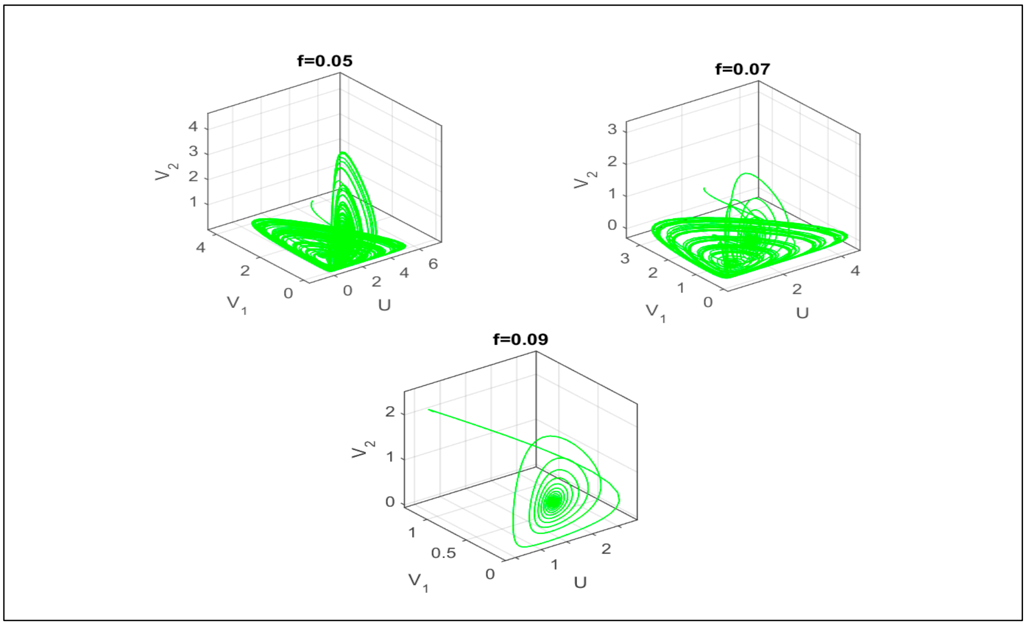

Figure 3. Phase portrait at different values of common resource *f*.

Figure 4 presents the time series solution of the model at different values of "*a*" 0.3, 0.4, and 0.5. The values of other parameters are the same as those in Figure 1.

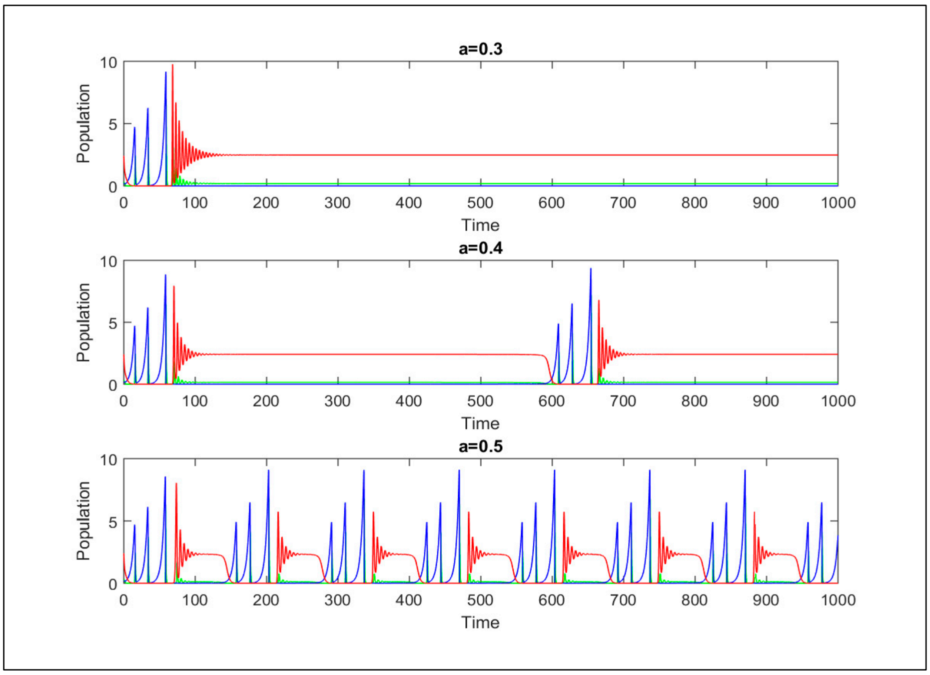

Figure 4. Time series solution at different values of consumption rate of common resource '*a*'.

Figure 5 depicts the time series behavior of the fractional-order model for the disease transmission fractional order "α". The values of the parameters are the same as those taken in Figure 2. It is clear from the plot that for smaller values of "α", the system reaches its stability at an earlier time.

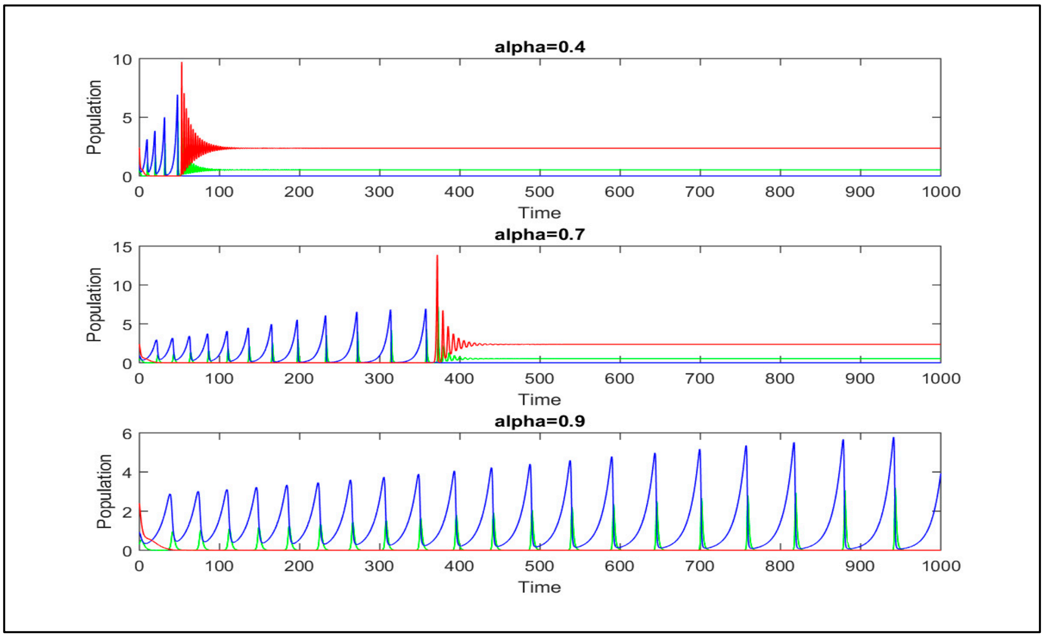

Figure 5. Time series solution at different values of α.

Figure 6 shows the time series solution for the fractional-order model for different values of "f". The values of the parameters are the same as in Figure 3. It is shown that higher values of "f" cause faster stability of the model, whereas low values of "f" take longer to move to the system's stability.

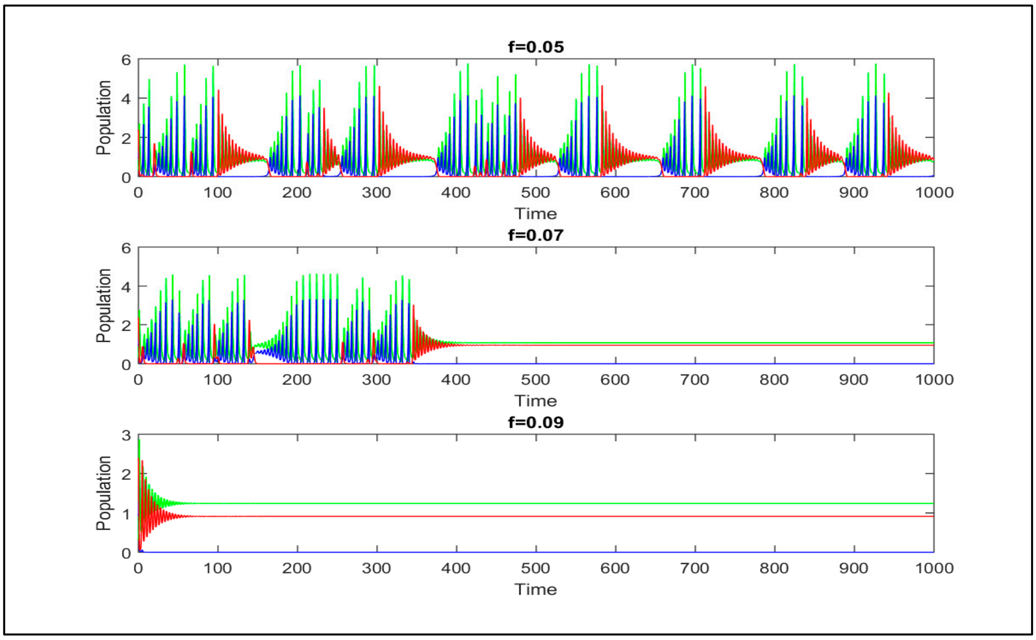

Figure 6. Time series solution at different values of f.

8. Conclusions

In the present study, we have formulated and analyzed a dynamical fractional-order model focusing on the food resources used by the predator-prey population. Here we have taken into account a predator and two species of prey (one depending on the sources used by the predator). The fixed points of the fractional-order model were computed. We have proved that there exists a non-negative unique solution of the model. The existence of fixed points is discussed by providing existence conditions for all the points mentioned above separately. The biologically important equilibrium points, coexistence fixed points and prey-free fixed points, are studied in detail. We have proved that both equilibrium points are conditionally stable for the original system. The theoretical results are supported by providing simulations. The impact of fractional order parameter α, food resource parameter f and consumption rate of food a by the predator on the stability of said fixed points is studied. The time series solution of the model is provided for different values of the fractional order parameter, food resource, and consumption rate of food by the predator. It is shown that larger values of α cause the system's instability. For smaller values of a the system is stable, but as soon as the values of the consumption parameter are increased, the system stability is disturbed. We observed that the system switches stability after regular intervals. Similarly, for larger values of f the system moves towards stability. Following the completion of this study, it is possible to propose alternative applications for the current methods in addition to the current uses [55–58].

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References

1. Boccara, N. *Modeling Complex Systems*; Springer: Berlin/Heildelberg, Germany, 2010.
2. Khajanchi, S. Modeling the dynamics of stage-structure predator-prey system with Monod-Haldane type response function. *Appl. Math. Comput.* **2017**, *302*, 122–143. [CrossRef]
3. Nosrati, K.; Shafiee, M. Dynamic analysis of fractional-order singular Holling type-II predator-prey system. *Appl. Math. Comput.* **2017**, *313*, 159–179. [CrossRef]
4. Zhang, F.; Chen, Y.; Li, J. Dynamical analysis of a stage-structured predator-prey model with cannibalism. *Math. Biosci.* **2018**, *307*, 33–41. [CrossRef]
5. Moustafa, M.; Mohd, M.H.; Ismail, A.I.; Abdullah, F.A. Stage structure and refuge effects in the dynamical analysis of a frac-tional order Rosenzweig-MacArthur prey-predator model. *Prog. Fract. Differ. Appl.* **2019**, *5*, 1–16. [CrossRef]
6. Kermack, W.O.; McKendrick, A.G. A contribution to the mathematical theory of epidemics. *Proc. R. Soc. London Ser. A Math. Phys. Sci.* **1927**, *115*, 700–721. [CrossRef]
7. Li, H.-L.; Zhang, L.; Teng, Z.; Jiang, Y.-L.; Muhammadhaji, A. Global stability of an SI epidemic model with feedback controls in a patchy environment. *Appl. Math. Comput.* **2018**, *321*, 372–384. [CrossRef]
8. Anderson, R.M.; May, R.M. *Infectious Diseases of Humans: Dynamics and Control*; Oxford University Press: London, UK, 1992.
9. Stone, L.; Shulgin, B.; Agur, Z. Theoretical examination of the pulse vaccination policy in the SIR epidemic model. *Math. Comput. Model.* **2000**, *31*, 207–215. [CrossRef]
10. Mukherjee, D. Hopf bifurcation in an eco-epidemic model. *Appl. Math. Comput.* **2010**, *217*, 2118–2124. [CrossRef]
11. Juneja, N.; Agnihotri, K. Conservation of a predator species in SIS prey-predator system using optimal taxation policy. *Chaos Solitons Fractals* **2018**, *116*, 86–94. [CrossRef]
12. Hilker, F.M.; Schmitz, K. Disease-induced stabilization of predator-prey oscillations. *J. Theor. Biol.* **2008**, *255*, 299–306. [CrossRef]
13. Mortoja, G.; Panja, P.; Mondal, K. Dynamics of a predator-prey model with nonlinear incidence rate, Crowley-Martin type functional response and disease in prey population. *Ecol. Genet. Gen.* **2018**, *10*, 100035. [CrossRef]
14. Meng, X.-Y.; Qin, N.-N.; Huo, H.-F. Dynamics analysis of a predator-prey system with harvesting prey and disease in prey species. *J. Biol. Dyn.* **2018**, *12*, 342–374. [CrossRef] [PubMed]
15. Greenhalgh, D.; Haque, M. A predator-prey model with disease in the prey species only. *Math. Methods Appl. Sci.* **2006**, *30*, 911–929. [CrossRef]
16. Shaikh, A.A.; Das, H.; Sarwardi, S. Dynamics of an eco-epidemiological system with disease in competitive prey species. *J. Appl. Math. Comput.* **2019**, *62*, 525–545. [CrossRef]
17. Rana, S.; Samanta, S.; Bhattacharya, S. The interplay of Allee effect in an eco-epidemiological system with disease in predator population. *Bull. Calcutta Math. Soc.* **2016**, *108*, 103–122.

18. Juneja, N.; Agnihotri, K. Global stability of harvested prey-predator model with infection in predator species. In *Information and Decision Sciences*; Springer: Berlin/Heidelberg, Germany, 2018; pp. 559–568.
19. Pal, P.J.; Haque, M.; Mandal, P.K. Dynamics of a predator-prey model with disease in the predator. *Math. Methods Appl. Sci.* **2013**, *37*, 2429–2450. [CrossRef]
20. Bulai, I.M.; Hilker, F.M. Eco-epidemiological interactions with predator interference and infection. *Theor. Popul. Biol.* **2019**, *130*, 191–202. [CrossRef]
21. Agnihotri, K.; Juneja, N. An eco-epidemic model with disease in both prey and predator. *IJAEEE* **2015**, *4*, 50–54.
22. Hsieh, Y.-H.; Hsiao, C.-K. Predator-prey model with disease infection in both populations. *Math. Med. Biol. A J. IMA* **2008**, *25*, 247–266. [CrossRef]
23. Gao, X.; Pan, Q.; He, M.; Kang, Y. A predator-prey model with diseases in both prey and predator. *Phys. A Stat. Mech. Its Appl.* **2013**, *392*, 5898–5906. [CrossRef]
24. Almeida, R.; Cruz, B.; Martins, N.; Monteiro, T. An epidemiological MSEIR model described by the Caputo fractional deriva-tive. *Int. J. Dyn. Control* **2018**, *7*, 776–784. [CrossRef]
25. Heymans, N.; Podlubny, I. Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. *Rheol. Acta* **2005**, *45*, 765–771. [CrossRef]
26. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
27. Ansari, S.P.; Agrawal, S.K.; Das, S. Stability analysis of fractional-order generalized chaotic susceptible-infected-recovered epidemic model and its synchronization using active control method. *Pramana* **2014**, *84*, 23–32. [CrossRef]
28. Laskin, N. Fractional quantum mechanics. *Phys. Rev. E* **2000**, *62*, 3135–3145. [CrossRef]
29. Mandal, M.; Jana, S.; Nandi, S.K.; Kar, T.K. Modeling and of a fractional-order prey-predator system incorporating harvesting. *Model. Earth Syst. Environ.* **2021**, *7*, 1159–1176. [CrossRef]
30. Karakaya, H.; Ozturk, I.; Kartal, S.; Gurcan, F. Dynamical analysis of discretized Logistic model with Caputo-Fabrizio fractional derivative. *Comput. Ecol. Softw.* **2021**, *11*, 21–34.
31. Yousef, F.B.; Yousef, A.; Maji, C. Effects of fear in a fractional-order predator-prey system with predator density-dependent prey mortality. *Chaos Solitons Fractals* **2021**, *145*, 110711. [CrossRef]
32. Song, P.; Zhao, H.; Zhang, X. Dynamic analysis of a fractional order delayed predator-prey system with harvesting. *Theory Biosci.* **2016**, *135*, 59–72. [CrossRef]
33. Mondal, S.; Lahiri, A.; Bairagi, N. Analysis of a fractional order eco-epidemiological model with prey infection and type 2 functional response. *Math. Methods Appl. Sci.* **2017**, *40*, 6776–6789. [CrossRef]
34. Chinnathambi, R.; Rihan, F.A. Stability of fractional-order prey-predator system with time-delay and Monod-Haldane functional response. *Nonlinear Dyn.* **2018**, *92*, 1637–1648. [CrossRef]
35. Moustafa, M.; Mohd, M.H.; Ismail, A.I.; Abdullah, F.A. Dynamical analysis of a fractional-order Rosenzweig-MacArthur model incorporating a prey refuge. *Chaos Solitons Fractals* **2018**, *109*, 1–13. [CrossRef]
36. Xie, Y.; Lu, J.; Wang, Z. Stability analysis of a fractional-order diffused prey-predator model with prey refuges. *Phys. A Stat. Mech. Its Appl.* **2019**, *526*, 120773. [CrossRef]
37. Sania, Q.; Abdullah, Y.; Shaheen, A. Fractional numerical dynamics for the logistic population growth model under conformable Caputo: A case study with real observation. *Phys. Scr.* **2021**, *96*, 114002.
38. dos Santos, J.P.C.; Cardoso, L.C.; Monteiro, E.; Lemes, N.H.T. A Fractional-Order Epidemic Model for Bovine Babesiosis Disease and Tick Populations. *Abstr. Appl. Anal.* **2015**, *2015*, 729894. [CrossRef]
39. Zhao, D.; Luo, M. Representations of acting processes and memory effects: General fractional derivative and its application to theory of heat conduction with finite wave speeds. *Appl. Math. Comput.* **2018**, *346*, 531–544. [CrossRef]
40. Bolton, L.; Cloot, A.H.J.J.; Schoombie, S.W.; Slabbert, J.P. A proposed fractional-order Gompertz model and its application to tumour growth data. *Math. Med. Biol. A J. IMA* **2014**, *32*, 187–209. [CrossRef]
41. Li, H.-L.; Muhammadhaji, A.; Zhang, L.; Teng, Z. Stability analysis of a fractional-order predator-prey model incorporating a constant prey refuge and feedback control. *Adv. Differ. Equ.* **2018**, *2018*, 325. [CrossRef]
42. Ahmed, E.; El-Sayed, A.; El-Saka, H.A. Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models. *J. Math. Anal. Appl.* **2007**, *325*, 542–553. [CrossRef]
43. Nugraheni, K.; Trisilowati, T.; Suryanto, A. Dynamics of a Fractional Order Eco-Epidemiological Model. *J. Trop. Life Sci.* **2017**, *7*, 243–250. [CrossRef]
44. Delavari, H.; Baleanu, D.; Sadati, J. Stability analysis of Caputo fractional-order nonlinear systems revisited. *Nonlinear Dyn.* **2011**, *67*, 2433–2439. [CrossRef]
45. Yadav, A.K.S.; Sora, M. An optimized deep neural network-based financial statement fraud detection in text mining. *3ciencias* **2021**, *10*, 77–105. [CrossRef]
46. Delgado, M.F.; Esenarro, D.; Regalado, F.F.J.; Reátegui, M.D. Methodology based on the NIST cybersecurity framework as a proposal for cybersecurity management in government organizations. *3c TIC Cuad. Desarro. Apl. A Las TIC* **2021**, *10*, 123–141. [CrossRef]
47. Gao, J.; Alotaibi, F.S.; Ismail, R.I. The Model of Sugar Metabolism and Exercise Energy Expenditure Based on Fractional Linear Regression Equation. *Appl. Math. Nonlinear Sci.* **2021**, *7*, 123–132. [CrossRef]

48. Liu, C. Precision algorithms in second-order fractional differential equations. *Appl. Math. Nonlinear Sci.* **2021**, *7*, 155–164. [CrossRef]
49. Shatanawi, W.; Raza, A.; Arif, M.S.; Rafiq, M.; Bibi, M.; Mohsin, M. Essential features preserving dynamics of stochastic Dengue model. *Comput. Model. Eng. Sci.* **2021**, *126*, 201–215. [CrossRef]
50. Area, I.; Losada, J.; Nieto, J.J. On Fractional Derivatives and Primitives of Periodic Functions. *Abstr. Appl. Anal.* **2014**, *2014*, 392598. [CrossRef]
51. Shatanawi, W.; Raza, A.; Arif, M.S.; Abodayeh, K.; Rafiq, M.; Bibi, M. Design of nonstandard computational method for stochastic susceptible-infected-treated-recovered dynamics of coronavirus model. *Adv. Differ. Equ.* **2020**, *2020*, 505. [CrossRef]
52. Li, X.; Wu, R. Hopf bifurcation analysis of a new commensurate fractional-order hyperchaotic system. *Nonlinear Dyn.* **2014**, *78*, 279–288. [CrossRef]
53. Heinonen, J. Lipschitz function. In *Lectures on Analysis on Metric Spaces*; Springer: New York, NY, USA, 2001.
54. Diethelm, K. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*; Springer: Berlin/Heidelberg, Germany, 2010; p. 547.
55. Arif, M.S.; Raza, A.; Rafiq, M.; Bibi, M.; Abbasi, J.N.; Nazeer, A.; Javed, U. Numerical Simulations for Stochastic Computer Virus Propagation Model. *Comput. Mater. Contin.* **2020**, *62*, 61–77. [CrossRef]
56. Bibi, M.; Nawaz, Y.; Arif, M.S.; Abbasi, J.N.; Javed, U.; Nazeer, A. A finite difference method and effective modification of gradient descent optimization algorithm for MHD fluid flow over a linearly stretching surface. *Comput. Mater. Contin.* **2020**, *62*, 657–677.
57. Arif, S.M.; Biwi, M.; Jahangir, A. Solution of algebraic lyapunov equation on positive-definite hermitian matrices by using extended Hamiltonian algorithm. *Comput. Mater. Contin.* **2018**, *54*, 181–195.
58. Raza, A.; Rafiq, M.; Baleanu, D.; Arif, M.S. Numerical simulations for stochastic meme epidemic model. *Adv. Differ. Equ.* **2020**, *2020*, 1–16. [CrossRef]