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Boundary-Value Problem for Nonlinear Fractional Differential Equations of Variable Order with Finite Delay via Kuratowski Measure of Noncompactness

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Abstract: This paper is devoted to boundary-value problems for Riemann–Liouville-type fractional differential equations of variable order involving finite delays. The existence of solutions is first studied using a Darbo's fixed-point theorem and the Kuratowski measure of noncompactness. Secondly, the Ulam–Hyers stability criteria are examined. All of the results in this study are established with the help of generalized intervals and piecewise constant functions. We convert the Riemann–Liouville fractional variable-order problem to equivalent standard Riemann–Liouville problems of fractional-constant orders. Finally, two examples are constructed to illustrate the validity of the observed results.

Keywords: fractional differential equations of variable order; finite delay; boundary-value problem; fixed-point theorem; green function; Ulam–Hyers stability

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1. Introduction and Motivations

The concept of fractional calculus, whose origin goes back to 1695, is considered as one of the most important branches in mathematics. It has been shown that models with fractional derivatives may more accurately represent complex phenomena than integer-order models. Fractional integrals and derivatives have attracted the attention of the researchers due to their essential features such as long-term dependence properties and more degrees of freedom. As a result, in the last few decades we have witnessed the application of fractional calculus methods in modeling processes studied in computer sciences, physics, neuroscience, biology, medicine, engineering, etc. [1–7]. In view of their advantages, the Riemann–Liouville and Caputo types are the most applied fractional derivatives [3,5].

Additionally, various techniques have been introduced and applied to establish existence criteria for analytical, semi-analytical, and numerical solutions of fractional-order boundary-value problems. Different researchers applied fixed-point theorems [3], nondifferentiable traveling-wave techniques [8], the homotopy perturbation transform method and the Yang transform decomposition method [9], iteration transformation techniques [10], the natural transform method [11], measures of noncompactmess [12], almost sectorial operators [13], and some others.

On the other hand, the extended class of variable-order fractional derivatives have also been recently developed [14–17]. In fact, the generalizations performed by the fractional derivatives of a variable order offered great opportunities for applications and mathematical modeling approaches [18–20].

The main idea of variable-order fractional calculus is to substitute the constant fractional order μ with a function $\mu(.)$. Although this difference seems simple, a variable-order operator can explain and model several physical and natural phenomena [21,22]. The recent publications in the field confirm our understanding of the importance of this consideration [23–27].

Despite the proven potential in applications to describe the complicated behavior of real-world problems, the theory of variable-order delayed fractional differential equations is not well developed. Some numerical approaches to solve such differential equations have been developed in several articles. For example, in [28] a collocation numerical approach is applied with the aid of shifted Chebyshev polynomials to solve a multiterm variable-order fractional delay differential equation. The existence, uniqueness criteria, and stability results have been presented in [29] for linear systems with distributed delays and distributed-order fractional derivatives based on Caputo type single fractional derivatives with respect to a nonnegative density function. In [30], a numerical method based on the Lagrangian piece-wise interpolation is proposed to solve variable-order fractal-fractional time delay equations with power law, exponential decay, and Mittag–Leffler memories. The paper [31] applied a method based on the fundamental theorem of fractional calculus and the Lagrange polynomial interpolation to numerically solve a type of variable-order fractional delay differential equation.

However, as stated in [28], analytical solutions for variable-order delayed fractional differential equations are difficult to obtain since the kernel of the variable-order operators has a variable exponent. This explains the limited number of results related to the fundamental and qualitative results for the solutions of such equations. To the best of the authors' knowledge, the existence results are established only for a damped fractional subdiffusion equation with time delay with a variable-order fractional Caputo operator in a very resent publication [32] where the authors applied shifted Chebyshev polynomials to solve the presented problem by a matrix discretization technique. Similar results for delayed variable-order fractional differential equations involving Riemann–Liouville derivatives have not yet been reported in the existing literature. This is the main aim of our research.

In [33], the authors studied the existence of solutions for the following nonlinear fractional differential equations of constant order:

$$\begin{cases} D^{\mu}_{0+}\xi(s) = \varphi(s,\xi_s), & s \in \mathcal{N} := [0,N], \ \ \mu \in]0,1], \\ \xi(s) = \chi(s), & s \in (-\infty, 0], \end{cases}$$

where D^{μ}_{0+} is the standard Riemann–Liouville fractional derivative, $0 < N < +\infty$, φ and χ are well defined functions, and ξ_s is an element of $C((-\infty,0],\mathbb{R})$ defined by

$$\xi_s(\tau) := \xi(s+\tau), \ \tau \in (-\infty, 0]$$

for any function ξ defined on $(-\infty, N]$ and any $s \in \mathcal{N}$, $C((-\infty,0],\mathbb{R})$ is the class of all continuous functions from $(-\infty,0]$ to $\mathbb{R}$.

Since the paper [33] considers an infinite delay, the obtained existence results can be examined as a generalization of several existence results for delayed fractional differential equations with fractional constant-order derivatives. In fact, there have been some important existence results for such equations where different techniques have been applied [34–38]. However, as stated above, the corresponding results for delayed fractional variable-order boundary-value problems are very few.

Motivated by [15,23–27,33], in this paper we study the existence of solutions for the boundary-value problem of the nonlinear fractional differential equation of variable order with finite delay in the format

$$\begin{cases} D^{\mu(s)}_{0+}\xi(s) = \varphi(s,\xi_s), & s \in \mathcal{N} := [0,N], \\ \xi(s) = \chi(s), & s \in [-\gamma, 0], \gamma > 0, \end{cases} \tag{1}$$

where $1 < \mu(s) \leq 2$, $D_{0^+}^{\mu(s)}$ is the Riemann–Liouville fractional derivative of the variable-order $\mu(.)$, $\varphi : \mathcal{N} \times C([-\gamma, 0], \mathbb{R}) \to \mathbb{R}$. The initial function $\chi \in C([-\gamma, 0], \mathbb{R})$ and $\chi(0) = 0$, ξ_s in $C([-\gamma, 0], \mathbb{R})$ is given by

$$\xi_s(\tau) := \xi(s + \tau),\ \tau \in [-\gamma, 0],$$

for any function ξ defined on $[-\gamma, N]$ and any $s \in \mathcal{N}$.

Such problems have a great potential to model numerous real-world phenomena studied in science and engineering.

The main novelty of the paper is in the following five points: (1) a fractional boundary-value problem for delay differential equations in the variable-order Riemann–Liouville settings is introduced, which generalizes the fractional constant-order concepts; (2) new existence specifications of solutions are established; (3) we consider generalized subintervals by combining the existing notions in relation to the Kuratowski measure of noncompactness in the context of Darbo's fixed-point theorem; (4) we apply piecewise constant functions to convert the Riemann–Liouville fractional boundary-value problem of variable order (1) to standard Riemann–Liouville fractional constant-order boundary-value problems, which allows for the more accurate estimation of the solution operator and leads to a better exploration of the effect of the variable fractional order; and (5) the Ulam–Hyers stability behavior of the fractional variable-order problem is analyzed, and new stability criteria are proved.

The organization of the paper is as follows. Some definitions and preliminary results are presented in Section 2. In Section 3, the main existence criteria for solutions of the boundary-value problem of variable order (1) are established using Darbo's fixed-point theorem. Section 4 presents our main Ulam–Hyers stability results. Two illustrative examples are presented in Section 5 to complete the consistency of our findings. Finally, some conclusion notes and the future scope of this paper are given in Section 6.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts, which are used throughout this paper.

We denote by $C(\mathcal{N}, \mathbb{R})$ the space of real-valued continuous functions on $\mathcal{N}$ equipped with the supremum norm

$$\|\xi\|_{\mathcal{N}} = \sup\{|\xi(s)| : s \in \mathcal{N}\},$$

for any $\xi \in C(\mathcal{N}, \mathbb{R})$.

Definition 1 ([39,40])**.** *The left Riemann–Liouville fractional integral of variable-order $\mu(.)$, $\mu : [c, d] \to (0, +\infty)$, $-\infty < c < d < +\infty$, for a function $\xi(.)$, is defined by*

$$I_{c^+}^{\mu(s)} \xi(s) = \int_c^s \frac{(s - \tau)^{\mu(\tau)-1}}{\Gamma(\mu(\tau))} \xi(\tau) d\tau,\ \ s > c, \tag{2}$$

where the standard Gamma function is denoted by $\Gamma(.)$.

Definition 2 ([39,40])**.** *For $-\infty < c < d < +\infty$, we consider the mapping $\mu : [c, d] \to (m - 1, m)$, $m \in \mathbb{N}$. Then, the left Riemann–Liouville fractional derivative of variable-order $\mu(.)$ for a function ξ is defined by*

$$D_{c^+}^{\mu(s)} \xi(s) = \left(\frac{d}{ds}\right)^m I_{c^+}^{m-\mu(s)} \xi(s) = \left(\frac{d}{ds}\right)^m \int_c^s \frac{(s - \tau)^{m-\mu(\tau)-1}}{\Gamma(m - \mu(\tau))} \xi(\tau) d\tau,\ \ s > c. \tag{3}$$

Obviously, if the order $\mu(.)$ is a constant function, then the Riemann–Liouville fractional variable order derivative (3) and Riemann–Liouville fractional integral of variable-order (2) are reduced to the classical Riemann–Liouville fractional derivative and Riemann–Liouville fractional integral, respectively; see [3,5,14,39].

The following properties are some of the main ones of the fractional derivatives and integrals that we will use in our analysis.

Lemma 1 ([3])**.** *Let $\varrho > 0$, $c \geq 0$, $\xi \in L^1(c,d)$, $D^{\varrho}_{c^+}\xi \in L^1(c,d)$. Then, the differential equation*

$$D^{\varrho}_{c^+}\xi = 0$$

has a solution

$$\xi(s) = \eta_1(s-c)^{\varrho-1} + \eta_2(s-c)^{\varrho-2} + \cdots + \eta_\ell(s-c)^{\varrho-\ell} + \cdots + \eta_m(s-c)^{\varrho-m},$$

where $m = [\varrho] + 1$, $\eta_\ell \in \mathbb{R}$, $\ell = 1,2,\ldots,m$.

Lemma 2 ([3])**.** *Let $\varrho > 0$, $c \geq 0$, $\xi \in L^1(c,d)$, $D^{\varrho}_{c^+}\xi \in L^1(c,d)$. Then,*

$$I^{\varrho}_{c^+}D^{\varrho}_{c^+}\xi(s) = \xi(s) + \eta_1(s-c)^{\varrho-1} + \eta_2(s-c)^{\varrho-2} + \cdots + \eta_\ell(s-c)^{\varrho-\ell} + \cdots + \eta_m(s-c)^{\varrho-m}, \tag{4}$$

where $m = [\varrho] + 1$, $\eta_\ell \in \mathbb{R}$, $\ell = 1,2,\ldots,m$.

Lemma 3 ([3])**.** *Let $\varrho > 0$, $c \geq 0$, $\xi \in L^1(c,d)$, $D^{\varrho}_{c^+}\xi \in L^1(c,d)$. Then,*

$$D^{\varrho}_{c^+}I^{\varrho}_{c^+}\xi(s) = \xi(s).$$

Lemma 4 ([3])**.** *Let $\varrho,\ \rho > 0$, $c \geq 0$, $\xi \in L^1(c,d)$. Then,*

$$I^{\varrho}_{c^+}I^{\rho}_{c^+}\xi(s) = I^{\rho}_{c^+}I^{\varrho}_{c^+}\xi(s) = I^{\varrho+\rho}_{c^+}\xi(s).$$

Remark 1 ([41,42])**.** *Generally, for two functions $\mu_1(s)$ and $\mu_2(s)$, the semigroup property does not hold, i.e.,*

$$I^{\mu_1(s)}_{c^+}I^{\mu_2(s)}_{c^+}\xi(s) \neq I^{\mu_1(s)+\mu_2(s)}_{c^+}\xi(s).$$

Definition 3 ([43])**.** *Let E be a Banach space and $\mathcal{P}_b(E)$ the family of bounded subsets of E. Then, $\zeta : \mathcal{P}_b(E) \to [0,+\infty[$ defined by*

$$\zeta(U) = inf\{\lambda > 0 :\ U \subseteq \cup_{k=1}^{n} B_k \text{ and } diam(B_k) < \lambda\}.$$

for every $U \in \mathcal{P}_b(E)$ is called the Kuratowski measure of noncompactness.

The Kuratowski measure of noncompactness satisfies the following properties:

Proposition 1 ([44,45])**.** *Let E be a Banach space. Then, for all bounded subsets $U,\ V$ of E, the following assertions hold:*

1. $\zeta(U) = 0 \Longleftrightarrow \overline{U}$ *is compact;*
2. $\zeta(\phi) = 0$;
3. $\zeta(U) = \zeta(\overline{U}) = \zeta(convU)$;
4. $(U \subset V) \Longrightarrow \zeta(U) \leq \zeta(V)$;
5. $\zeta(U+V) \leq \zeta(U) + \zeta(V)$;
6. $\zeta(\lambda U) = |\lambda|\zeta(U), \lambda \in \mathbb{R}$;
7. $\zeta(U \cup V) = max\{\zeta(U), \zeta(V)\}$;
8. $\zeta(U \cap V) \leq min\{\zeta(U), \zeta(V)\}$;
9. $\zeta(U + x_0) = \zeta(U)$ *for any* $x_0 \in E$.

Lemma 5 ([45])**.** *If the bounded set $U \subset C(\mathcal{N}, E)$ is equicontinuous, then* *(i)* *the function* $\zeta(U(s))$ *is continuous for* $s \in \mathcal{N}$, *and*

$$\zeta_{\mathcal{N}}(U) = \sup_{s\in\mathcal{N}} \zeta(U(s)).$$

(ii) $\zeta\left(\int_0^N \xi(s)ds : \xi \in U\right) \leq \int_0^N \zeta(U(s))ds,$

where

$$U(s) = \{\xi(s) : \ \xi \in U\}, \ s \in \mathcal{N}.$$

Remark 2. *For the definition and properties of equicontinuous sets, we refer to [45].*

Remark 3. *In the following, we shall use* ζ *and* $\zeta_{\mathcal{N}}$ *to denote the Kuratowski measures of noncompactness of sets in space* $\mathbb{R}$ *and space* $C(\mathcal{N}, \mathbb{R})$ *respectively.*

The following theorem will be needed.

Theorem 1 (Darbo's fixed-point theorem [43])**.** *Let* M *be a nonempty, bounded, convex, and closed subset of a Banach space* E *and* $T : M \longrightarrow M$ *is a continuous operator satisfying* $\zeta(TA) \leq L\zeta(A)$ *for any nonempty subset* A *of* M *and for some constant* $L \in [0, 1)$. *Then,* T *has at least one fixed point in* M.

Definition 4 ([46,47])**.** *Equation (1) is Ulam–Hyers is stable if there exists a real number* $c_\varphi > 0$ *such that for each* $\epsilon > 0$ *and any solution* $y \in C([-\gamma, N], \mathbb{R})$ *of the inequality*

$$\begin{cases} |D_{0^+}^{\mu(s)} y(s) - \varphi(s, y_s)| \leq \epsilon, & s \in \mathcal{N} := [0, N], \\ y(s) = \chi(s), & s \in [-\gamma, 0], \end{cases} \tag{5}$$

there exists a solution $\xi \in C([-\gamma, N], \mathbb{R})$ *of Equation (1) with*

$$|y(s) - \xi(s)| \leq c_\varphi \epsilon, \ s \in [-\gamma, N].$$

Remark 4. *A function* $y \in C([-\gamma, N], \mathbb{R})$ *is a solution of the inequality (5) if and only if a function* $h \in C([-\gamma, N], \mathbb{R})$ *(which depends on solution y) exists such that*

(i) $|h(s)| \leq \epsilon$, *for all* $s \in [-\gamma, N]$.

(ii) $D_{0^+}^{\mu(s)} y(s) = \varphi(s, y_s) + h(s)$ *for all* $s \in \mathcal{N}$.

Definition 5 ([15,48])**.** *Let* $I \subset \mathbb{R}$.

(a) *The interval I is called a generalized interval if it is either an interval or* $\{\rho_1\}$ *or* $\emptyset$.

(b) *A partition of I is a finite set* $\mathcal{P}$ *such that each x in I lies in exactly one of the generalized intervals E in* $\mathcal{P}$.

(c) *A function* $g : I \to \mathbb{R}$ *is called piecewise constant with respect to the partition* $\mathcal{P}$ *of I if for any* $E \in \mathcal{P}$, g *is constant on E.*

3. Existence Criteria

We will begin with the introduction of some main hypotheses:

(Hyp1) For an integer $n \in \mathbb{N}$, let the finite sequence of points $\{N_k\}_{k=0}^n$ be given such that $0 = N_0 < N_{k-1} < N_k < N_n = N$, $k = 2, \ldots, n-1$. Denote $\mathcal{N}_k := (N_{k-1}, N_k]$, $k = 1, 2, \ldots, n$ and consider the partition $\mathcal{P} = \{\mathcal{N}_k : 1 = 1, 2, \ldots, n\}$ of the interval $\mathcal{N}$. Let $\mu : \mathcal{N} \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$, represented as follows:

$$\mu(s) = \sum_{k=1}^{n} \mu_k I_k(s) = \begin{cases} \mu_1, & \text{if } s \in \mathcal{N}_1, \\ \mu_2, & \text{if } s \in \mathcal{N}_2, \\ . \\ . \\ . \\ \mu_n, & \text{if } s \in \mathcal{N}_n, \end{cases}$$

where $1 < \mu_k \leq 2$ are constants and I_k is an indicator of the interval $\mathcal{N}_k, k = 1, 2, \ldots, n$ defined by

$$I_k(s) = \begin{cases} 1, & \text{for } s \in \mathcal{N}_k, \\ 0, & \text{elsewhere.} \end{cases}$$

(Hyp2) Let $s^\sigma \varphi : \mathcal{N} \times C([-\gamma, 0], \mathbb{R}) \to \mathbb{R}$ be continuous $(0 < \sigma < 1)$. $K > 0$ exists, such that $s^\sigma |\varphi(s, y_s) - \varphi(s, z_s)| \leq K \|y_s - z_s\|_{[-\gamma,0]}$, for any $y,\ z \in C([-\gamma, N], \mathbb{R})$ and $s \in \mathcal{N}$.

The next definition of a solution of the problem (1) will be essential in this paper.

Definition 6. *Problem (1) has a solution, if there are functions $\xi_k, k = 1, 2, \ldots, n$, so that $\xi_k \in C([-\gamma, N_k], \mathbb{R})$ satisfying Equation (7) for $s \in [0, N_k]$, $\xi_k(s) = \chi(s)$ for $s \in [-\gamma, 0]$ and $\xi_k(0) = \xi_k(N_k) = 0$.*

In order to apply Darbo's fixed-point theorem and the Kuratowski measure of non-compactness, we will perform an essential analysis to the problem (1).

Using (3), we represent the equation of the problem (1) in the following form:

$$\frac{d^2}{ds^2} \int_0^s \frac{(s-\tau)^{1-\mu(\tau)}}{\Gamma(2-\mu(\tau))} \xi(\tau) d\tau = \varphi(s, \xi_s), \quad s \in \mathcal{N}. \tag{6}$$

According to **(Hyp1)**, we can represent Equation (6) on the interval $\mathcal{N}_k, k = 1, 2, \ldots,$ n as

$$\frac{d^2}{ds^2} \Big(\int_0^{N_1} \frac{(s-\tau)^{1-\mu_1}}{\Gamma(2-\mu_1)} \xi(\tau) d\tau + \ldots + \int_{N_{k-1}}^s \frac{(s-\tau)^{1-\mu_k}}{\Gamma(2-\mu_k)} \xi(\tau) d\tau \Big) = \varphi(s, \xi_s) \tag{7}$$

for $s \in \mathcal{N}_k$.

For $0 \leq s \leq N_{k-1}$, by taking $\xi(s) \equiv 0$, Equation (7) is reduced to

$$D^{\mu_k}_{N^+_{k-1}} \xi(s) = \varphi(s, \xi_s), \quad s \in \mathcal{N}_k.$$

Let us consider the following problem:

$$\begin{cases} D^{\mu_k}_{N^+_{k-1}} \xi(s) = \varphi(s, \xi_s), \quad s \in \mathcal{N}_k, \\ \xi(N_{k-1}) = 0, \qquad \xi(N_k) = 0, \\ \xi(s) = \chi_k(s),\ s \in [N_{k-1} - \gamma', N_{k-1}], \end{cases} \tag{8}$$

where $\gamma' = N_{k-1} + \gamma$ and

$$\chi_k(s) = \begin{cases} 0, & if\ s \in [0, N_{k-1}] \\ \chi(s), & if\ s \in [-\gamma, 0]. \end{cases}$$

The following auxiliary lemma will offer existence criteria for solutions for the problem (8).

Lemma 6. *The function $\xi \in C([-\gamma, N_k], \mathbb{R})$ is a solution of problem (8) if and only if ξ satisfies the integral equation*

$$\xi(s) = \begin{cases} -\int_{N_{k-1}}^{N_k} G_k(s,\tau)\varphi(\tau,\xi_\tau)d\tau, & if\ s \in \mathcal{N}_k, \\ \chi_k(s), & if\ s \in [-\gamma, N_{k-1}], \end{cases} \tag{9}$$

where $G_k(s,\tau)$ is a Green's function defined by

$$G_k(s,\tau) = \begin{cases} \frac{1}{\Gamma(\mu_k)}\Big[(N_k - N_{k-1})^{1-\mu_k}(s-N_{k-1})^{\mu_k-1}(N_k-\tau)^{\mu_k-1} - (s-\tau)^{\mu_k-1}\Big], \\ \qquad N_{k-1} \le \tau \le s \le N_k, \\ \frac{1}{\Gamma(\mu_k)}(N_k - N_{k-1})^{1-\mu_k}(s-N_{k-1})^{\mu_k-1}(N_k-\tau)^{\mu_k-1}, \\ \qquad N_{k-1} \le s \le \tau \le N_k, \end{cases}$$

$k = 1, 2, \ldots, n$.

Proof. Let $\xi \in C([-\gamma, N_k], \mathbb{R})$ be a solution of the problem (8). From (4), we have

$$\xi(s) = \eta_1(s-N_{k-1})^{\mu_k-1} + \eta_2(s-N_{k-1})^{\mu_k-2} + I^{\mu_k}_{N^+_{k-1}}\varphi(s,\xi_s), \quad s \in \mathcal{N}_k,\ k \in \{1,2,\ldots,n\}. \tag{10}$$

Using $\xi(N_{k-1}) = \xi(N_k) = 0$, we find that $\eta_2 = 0$ and

$$\eta_1 = -(N_k - N_{k-1})^{1-\mu_k}\, I^{\mu_k}_{N^+_{k-1}}\varphi(N_K, \xi_{N_k}).$$

By substituting the values of η_1 and η_2 in (10), we obtain

$$\xi(s) = -(N_k - N_{k-1})^{1-\mu_k}(s-N_{k-1})^{\mu_k-1}\, I^{\mu_k}_{N^+_{k-1}}\varphi(N_k,\xi_{N_k}) + I^{\mu_k}_{N^+_{k-1}}\varphi(s,\xi_s), \quad s \in \mathcal{N}_k.$$

Then, the solution of the problem (8) is given by

$$\begin{aligned} \xi(s) &= -(N_k - N_{k-1})^{1-\mu_k}(s-N_{k-1})^{\mu_k-1}\frac{1}{\Gamma(\mu_k)}\int_{N_{k-1}}^{N_k}(N_k-\tau)^{\mu_k-1}\varphi(\tau,\xi_\tau)d\tau \\ &+ \frac{1}{\Gamma(\mu_k)}\int_{N_{k-1}}^{s}(s-\tau)^{\mu_k-1}\varphi(\tau,\xi_\tau)d\tau \\ &= -\frac{1}{\Gamma(\mu_k)}\Big[\int_{N_{k-1}}^{s}\Big[(N_k-N_{k-1})^{1-\mu_k}(s-N_{k-1})^{\mu_k-1}(N_k-\tau)^{\mu_k-1} - (s-\tau)^{\mu_k-1}\Big]\varphi(\tau,\xi_\tau)d\tau \\ &+ \int_{s}^{N_k}(N_k-N_{k-1})^{1-\mu_k}(s-N_{k-1})^{\mu_k-1}(N_k-\tau)^{\mu_k-1}\varphi(\tau,\xi_\tau)d\tau\Big] \\ &= -\Big[\int_{N_{k-1}}^{s}G_k(s,\tau)\varphi(\tau,\xi_\tau)d\tau + \int_{s}^{N_k}G_k(s,\tau)\varphi(\tau,\xi_\tau)d\tau\Big] \end{aligned}$$

and the continuity of the Green function gives

$$\xi(s) = -\int_{N_{k-1}}^{N_k}G_k(s,\tau)\varphi(\tau,\xi_\tau)d\tau,\ s \in \mathcal{N}_k.$$

Conversely, let $\xi \in C([-\gamma, N_k], \mathbb{R})$ be a solution of integral Equation (9); then, by the continuity of function $S^\sigma\varphi$ and Lemma 3, we can easily obtain that ξ is the solution of the problem (8). □

Proposition 2 ([16]). *Let $0 < \sigma < 1$ and assume that $s^{\sigma}\varphi : \mathcal{N}_k \times C([-\gamma, 0], \mathbb{R}) \to \mathbb{R}$ is continuous, and $\mu : \mathcal{N}_k \to (1, 2]$ satisfies **(Hyp1)**. Then, the Green's function of problem (8) satisfies the following properties:*

(1) $G_k(s, \tau) \geq 0$ *for all* $N_{k-1} \leq s, \tau \leq N_k$,

(2) $\max\limits_{s \in \mathcal{N}_k} G_k(s, \tau) = G_k(\tau, \tau), \tau \in \mathcal{N}_k$,

(3) $G_i(s, s)$ *has a unique maximum given by*

$$\max_{\tau \in \mathcal{N}_k} G_k(\tau, \tau) = \frac{1}{\Gamma(\mu_k)} \left(\frac{N_k - N_{k-1}}{4} \right)^{\mu_k - 1},$$

where $k = 1, 2, \ldots, n$.

We will now establish the existence results for the Riemann–Liouville constant-order fractional problem (8). Our first result is based on Darbo's fixed-point theorem.

Theorem 2. *Suppose that both **(Hyp1)** and **(Hyp2)** hold, and*

$$\frac{K\left(N_k^{1-\sigma} - N_{k-1}^{1-\sigma}\right)\left(N_k - N_{k-1}\right)^{\mu_k - 1}}{4^{\mu_k - 1}(1 - \sigma)\Gamma(\mu_k)} < 1. \tag{11}$$

Then, the Riemann–Liouville constant-order fractional problem (8) possesses at least one solution on $C([-\gamma, N_k], \mathbb{R})$.

Proof. Consider the operator

$$\mathcal{L} : C([-\gamma, N_k], \mathbb{R}) \to C([-\gamma, N_k], \mathbb{R}),$$

defined by

$$(\mathcal{L}\xi)(s) = \begin{cases} \chi_k(s), \ s \in [-\gamma, N_{k-1}], \\ -\int_{N_{k-1}}^{N_k} G_k(s, \tau)\varphi(\tau, \xi_\tau)d\tau, \ s \in \mathcal{N}_k. \end{cases}$$

Let $v(.) : [-\gamma, N_k] \to \mathbb{R}$ be a function defined by

$$v(s) = \begin{cases} 0, \ if \ s \in \mathcal{N}_k, \\ \chi_k(s), \ if \ s \in [-\gamma, N_{k-1}]. \end{cases}$$

For each $z \in C([N_{k-1}, N_k], \mathbb{R})$, with $z(N_{k-1}) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(s) = \begin{cases} z(s), \ if \ s \in \mathcal{N}_k, \\ 0, \ if \ s \in [-\gamma, N_{k-1}]. \end{cases}$$

If $\xi(.)$ satisfies the integral equation

$$\xi(s) = -\int_{N_{k-1}}^{N_k} G_k(s, \tau)\varphi(\tau, \xi_\tau)d\tau,$$

then we can decompose $\xi(.)$ as $\xi(s) = z(s) + v(s), \ N_{k-1} \leq s \leq N_k$, which implies $\xi_s = \bar{z}_s + v_s$ for every $N_{k-1} \leq s \leq N_k$, and the function $z(.)$ satisfies

$$z(s) = -\int_{N_{k-1}}^{N_k} G_k(s, \tau)\varphi(\tau, z_\tau + v_\tau)d\tau.$$

Set

$$C_{N_{k-1}} = \{z \in C([N_{k-1}, N_k], \mathbb{R}) : \ z(N_{k-1}) = 0\}$$

and let $\|.\|_{N_k}$ be the norm in $C_{N_{k-1}}$ defined by

$$\|z\|_{N_k} = \sup_{s\in\mathcal{N}_k} |z(s)|,\ z \in C_{N_{k-1}}.$$

Thus, $C_{N_{k-1}}$ is a Banach space with the norm $\|.\|_{N_k}$. Let the operator $\mathcal{P} : C_{N_{k-1}} \to C_{N_{k-1}}$ be defined by

$$(\mathcal{P}z)(s) = -\int_{N_{k-1}}^{N_k} G_k(s,\tau)\varphi(\tau,\bar{z}_\tau + v_\tau)d\tau,\ s \in \mathcal{N}_k. \tag{12}$$

It follows from the properties of fractional integrals and from the continuity of function $s^\sigma\varphi$ that the operator $\mathcal{P} : C_{N_{k-1}} \to C_{N_{k-1}}$ in (12) is well defined.

Then, it is enough to show that the operator $\mathcal{P}$ has a fixed point z that will guarantee that the operator $\mathcal{L}$ has a fixed point $\xi = \bar{z} + v$, and in consequence, this fixed point will correspond to a solution of the problem (8). Indeed,

$$\begin{aligned}
\xi(s) &= \bar{z}(s) + v(s) \\
&= \begin{cases} z(s), & if\ s \in \mathcal{N}_k, \\ \chi_k(s), & if\ s \in [-\gamma, N_{k-1}] \end{cases} \\
&= \begin{cases} -\int_{N_{k-1}}^{N_k} G_k(s,\tau)\varphi(\tau,\bar{z}_\tau + v_\tau)d\tau, & if\ s \in \mathcal{N}_k, \\ \chi_k(s), & if\ s \in [-\gamma, N_{k-1}] \end{cases} \\
&= \begin{cases} -\int_{N_{k-1}}^{N_k} G_k(s,\tau)\varphi(\tau,\xi_\tau)d\tau, & if\ s \in \mathcal{N}_k, \\ \chi_k(s), & if\ s \in [-\gamma, N_{k-1}] \end{cases} \\
&= (\mathcal{L}\xi)(s).
\end{aligned}$$

Let

$$R_k \geq \frac{\frac{(K\|\chi\|_{[-\gamma,0]}+\varphi^\star)(N_k-N_{k-1})^{\mu_k-1}\left(N_k^{1-\sigma}-N_{k-1}^{1-\sigma}\right)}{4^{\mu_k-1}\Gamma(\mu_k)(1-\sigma)}}{1-\frac{K\left(N_k^{1-\sigma}-N_{k-1}^{1-\sigma}\right)\left(N_k-N_{k-1}\right)^{\mu_k-1}}{4^{\mu_k-1}(1-\sigma)\Gamma(\mu_k)}}$$

with $\varphi^\star = \sup_{s\in\mathcal{N}} s^\sigma|\varphi(s,0)|$, and consider the following set:

$$B_{R_k} = \{z \in C_{N_{k-1}},\ \|z\|_{N_k} \leq R_k\}.$$

Clearly, B_{R_k} is nonempty, convex, bounded, and closed.

For $z \in B_{R_k}$ and $s \in \mathcal{N}_k$, we have

$$\begin{aligned}
\|\bar{z}_s\|_{[-\gamma',0]} &= \sup_{-N_{k-1}-\gamma\leq\theta\leq 0} |\bar{z}_s(\theta)| \\
&= \sup_{-N_{k-1}-\gamma\leq\theta\leq 0} |\bar{z}(s+\theta)| \\
&\leq \sup_{-\gamma\leq\tau\leq N_k} |\bar{z}(\tau)| \\
&= \sup_{\tau\in\mathcal{N}_k} |z(\tau)| = \|z\|_{N_k}
\end{aligned}$$

and

$$\begin{aligned}
\|v_s\|_{[-\gamma',0]} &= \sup_{-N_{k-1}-\gamma\le\theta\le 0} |v_s(\theta)| \\
&= \sup_{-N_{k-1}-\gamma\le\theta\le 0} |v(s+\theta)| \\
&\le \sup_{-\gamma\le\tau\le N_k} |v(\tau)| \\
&= \sup_{-\gamma\le\tau\le 0} |v(\tau)| = \sup_{-\gamma\le\tau\le 0} |\chi(\tau)| = \|\chi\|_{[-\gamma,0]}.
\end{aligned}$$

We shall show that $\mathcal{P}$ satisfies Theorem 1 in five steps.
Step 1: $P(B_{R_k}) \subseteq (B_{R_k})$.
For $z \in B_{R_k}$, by Proposition 2 and **(Hyp2)**, we obtain

$$\begin{aligned}
|\mathcal{P}z(s)| &= \Big| \int_{N_{k-1}}^{N_k} G_k(s,\tau)\varphi(\tau,\bar{z}_\tau+v_\tau)d\tau \Big| \\
&\le \int_{N_{k-1}}^{N_k} G_k(s,\tau)|\varphi(\tau,\bar{z}_\tau+v_\tau)|d\tau \\
&\le \frac{1}{\Gamma(\mu_k)}\Big(\frac{N_k-N_{k-1}}{4}\Big)^{\mu_k-1}\int_{N_{k-1}}^{N_k} |\varphi(\tau,\bar{z}_\tau+v_\tau)|d\tau \\
&\le \frac{1}{\Gamma(\mu_k)}\Big(\frac{N_k-N_{k-1}}{4}\Big)^{\mu_k-1}\int_{N_{k-1}}^{N_k} \tau^{-\sigma}\tau^{\sigma}\Big|\varphi(\tau,\bar{z}_\tau+v_\tau)-f(\tau,0)\Big|d\tau \\
&+ \frac{1}{\Gamma(\mu_k)}\Big(\frac{N_k-N_{k-1}}{4}\Big)^{\mu_k-1}\int_{N_{k-1}}^{N_k} \tau^{-\sigma}\tau^{\sigma}|\varphi(\tau,0)|d\tau \\
&\le \frac{1}{\Gamma(\mu_k)}\Big(\frac{N_k-N_{k-1}}{4}\Big)^{\mu_k-1}\int_{N_{k-1}}^{N_k} \tau^{-\sigma}(K\|\bar{z}_\tau+v_\tau\|_{[-\gamma',0]})d\tau \\
&+ \frac{\varphi^\star(N_k-N_{k-1})^{\mu_k-1}}{\Gamma(\mu_k)4^{\mu_k-1}}\int_{N_{k-1}}^{N_k} \tau^{-\sigma}d\tau \\
&\le \frac{K}{\Gamma(\mu_k)}\Big(\frac{N_k-N_{k-1}}{4}\Big)^{\mu_k-1}\int_{N_{k-1}}^{N_k} (\|\bar{z}_\tau\|_{[-\gamma',0]}+\|v_\tau\|_{[-\gamma',0]})\tau^{-\sigma}d\tau \\
&+ \frac{\varphi^\star\Big(N_k-N_{k-1}\Big)^{\mu_k-1}\Big(N_k^{1-\sigma}-N_{k-1}^{1-\sigma}\Big)}{4^{\mu_k-1}\Gamma(\mu_k)(1-\sigma)} \\
&\le \frac{K}{\Gamma(\mu_k)}\Big(\frac{N_k-N_{k-1}}{4}\Big)^{\mu_k-1}(\|z\|_{N_k}+\|\chi\|_{[-\gamma,0]})\int_{N_{k-1}}^{N_k} \tau^{-\sigma}d\tau \\
&+ \frac{\varphi^\star\Big(N_k-N_{k-1}\Big)^{\mu_k-1}\Big(N_k^{1-\sigma}-N_{k-1}^{1-\sigma}\Big)}{4^{\mu_k-1}\Gamma(\mu_k)(1-\sigma)} \\
&\le \frac{K}{\Gamma(\mu_k)}\Big(\frac{N_k-N_{k-1}}{4}\Big)^{\mu_k-1}R_k\Big(\frac{N_k^{1-\sigma}-N_{k-1}^{1-\sigma}}{1-\sigma}\Big) \\
&+ \frac{K}{\Gamma(\mu_k)}\Big(\frac{N_k-N_{k-1}}{4}\Big)^{\mu_k-1}\|\chi\|_{[-\gamma,0]}\Big(\frac{N_k^{1-\sigma}-N_{k-1}^{1-\sigma}}{1-\sigma}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{\varphi^{\star}\Big(N_k - N_{k-1}\Big)^{\mu_k-1}\Big(N_k^{1-\sigma} - N_{k-1}^{1-\sigma}\Big)}{4^{\mu_k-1}\Gamma(\mu_k)(1-\sigma)} \\
&\leq \quad \frac{K\Big(N_k^{1-\sigma} - N_{k-1}^{1-\sigma}\Big)\Big(N_k - N_{k-1}\Big)^{\mu_k-1}}{4^{\mu_k-1}(1-\sigma)\Gamma(\mu_k)} R_k \\
&+ \quad \frac{\Big(N_k - N_{k-1}\Big)^{\mu_k-1}\Big(N_k^{1-\sigma} - N_{k-1}^{1-\sigma}\Big)}{4^{\mu_k-1}\Gamma(\mu_k)(1-\sigma)}\Big(K\|\chi\|_{[-\gamma,0]} + \varphi^{\star}\Big) \\
&\leq \quad R_k,
\end{aligned}
$$

which means that $\mathcal{P}(B_{R_k}) \subseteq B_{R_k}$.

Step 2: $\mathcal{P}$ is continuous.

We presume that the sequence (z_n) converges to z in $C_{N_{k-1}}$ and $s \in \mathcal{N}_k$. Then,

$$
\begin{aligned}
|\mathcal{P}(z_n)(s) - (\mathcal{P}z)(s)| \quad &\leq \quad \int_{N_{k-1}}^{N_k} G_k(s,\tau)\Big|\varphi(\tau, \overline{z_n}_\tau + v_\tau) - \varphi(\tau, \overline{z}_\tau + v_\tau)\Big| d\tau \\
&\leq \quad \frac{1}{\Gamma(\mu_k)}\Big(\frac{N_k - N_{k-1}}{4}\Big)^{\mu_k-1}\int_{N_{k-1}}^{N_k}\Big|\varphi(\tau, \overline{z_n}_\tau + v_\tau) - \varphi(\tau, \overline{z}_\tau + v_\tau)\Big| d\tau \\
&\leq \quad \frac{1}{\Gamma(\mu_k)}\Big(\frac{N_k - N_{k-1}}{4}\Big)^{\mu_k-1}\int_{N_{k-1}}^{N_k} \tau^{-\sigma} K\|\overline{z_n}_\tau - \overline{z}_\tau\|_{[-\gamma',0]} d\tau \\
&\leq \quad \frac{1}{\Gamma(\mu_k)}\Big(\frac{N_k - N_{k-1}}{4}\Big)^{\mu_k-1}(K\|z_n - z\|_{N_k})\int_{N_{k-1}}^{N_k} \tau^{-\sigma} d\tau \\
&\leq \quad \frac{K\Big(N_k^{1-\sigma} - T_{k-1}^{1-\sigma}\Big)\Big(N_k - N_{k-1}\Big)^{\mu_k-1}}{4^{\mu_k-1}(1-\sigma)\Gamma(\mu_k)}\|z_n - z\|_{N_k}.
\end{aligned}
$$

Hence, we obtain

$$
\|(\mathcal{P}z_n) - (\mathcal{P}z)\|_{N_k} \to 0 \text{ as } n \to \infty.
$$

Then, the operator $\mathcal{P}$ is a continuous on $C_{N_{k-1}}$.

Step 3: $\mathcal{P}(B_{R_k})$ is bounded set in $C_{N_{k-1}}$.

As in Step 1, we have $\mathcal{P}(B_{R_k}) \subset B_{R_k}$. This implies that $\mathcal{P}(B_{R_i})$ is bounded set in $C_{T_{i-1}}$.

Step 4: $\mathcal{P}(B_{R_k})$ is equicontinous set in $C_{N_{k-1}}$.

For arbitrary $s_1, s_2 \in \mathcal{N}_k$, with $s_1 < s_2$, let $z \in B_{R_k}$. Estimate

$$
\begin{aligned}
|\mathcal{P}(z)(t_2) \quad &- \quad (\mathcal{P}z)(t_1)| = \Big|\int_{N_{k-1}}^{N_k} G_k(s_2,\tau)\varphi(\tau, \overline{z}_\tau + v_\tau) d\tau - \int_{N_{k-1}}^{N_k} G_k(s_1,\tau)\varphi(\tau, \overline{z}_\tau + v_\tau) d\tau\Big| \\
&\leq \quad \int_{N_{k-1}}^{N_k}\Big|\Big(G_k(s_2,\tau) - G_k(s_1,\tau)\Big)\varphi(\tau, \overline{z}_\tau + v_\tau)\Big| d\tau \\
&\leq \quad \int_{N_{k-1}}^{N_k}\Big|G_k(s_2,\tau) - G_k(s_1,\tau)\Big||\varphi(\tau, \overline{z}_\tau + v_\tau)| d\tau \\
&\leq \quad \int_{N_{k-1}}^{N_k}\Big|G_k(s_2,\tau) - G_k(s_1,\tau)\Big|\tau^{-\sigma}\Big(\tau^{\sigma}\Big|\varphi(\tau, \overline{z}_\tau + v_\tau) - \varphi(\tau, 0)\Big| + \tau^{\sigma}|\varphi(\tau, 0)|\Big) d\tau
\end{aligned}
$$

$$\leq \int_{N_{k-1}}^{N_k} \Big|G_k(s_2,\tau) - G_k(s_1,\tau)\Big| \Big[\tau^{-\sigma}(K\|\overline{z}_\tau + v_\tau\|_{[-\gamma',0]}) + \tau^{-\sigma}\varphi^\star\Big] ds$$

$$\leq \int_{N_{k-1}}^{N_k} \Big|G_k(s_2,\tau) - G_k(s_1,\tau)\Big| \Big[\tau^{-\sigma}K(\|\overline{z}_\tau\|_{[-\gamma',0]} + \|v_\tau\|_{[-\gamma',0]}) + \tau^{-\sigma}\varphi^\star\Big] d\tau$$

$$\begin{aligned}
&\leq \int_{N_{k-1}}^{N_k} \Big|G_k(s_2,\tau) - G_k(s_1,\tau)\Big| \Big[\tau^{-\sigma}K(\|z\|_{N_k} + \|\chi\|_{[-\gamma,0]}) + \varphi^\star\Big] d\tau \\
&\leq KN_{k-1}^{-\sigma}(R + \|\chi\|_{[-\gamma,0]}) \int_{N_{k-1}}^{N_k} \Big|G_k(s_2,\tau) - G_k(s_1,\tau)\Big| d\tau \\
&+ \varphi^\star N_{k-1}^{-\sigma} \int_{N_{k-1}}^{N_k} \Big|G_k(s_2,\tau) - G_k(s_1,\tau)\Big| d\tau.
\end{aligned}$$

Hence, $|\mathcal{P}(z)(s_2) - (\mathcal{P}z)(s_1)| \to 0$ as $|s_2 - s_1| \to 0$. This implies that $\mathcal{P}(B_{R_k})$ is equicontinuous.

Note that [49] the inequality

$$\zeta\Big(s^\delta\varphi(s,B_1)\Big) \leq K\zeta_{[-\gamma,0]}(B_1)$$

is equivalent to **(Hyp2)** for each $B_1 \subset C([-\gamma,0],\mathbb{R})$ and $s \in \mathcal{N}$, where B_1 is bounded.

Step 5: $\mathcal{P}$ is L-set contraction.

For $U \subset B_{R_k}$, $s \in \mathcal{N}_k$, we obtain

$$\begin{aligned}
\zeta(\mathcal{P}(U)(s)) &= \zeta(\Big\{(\mathcal{P}z)(s), z \in U\Big\}) \\
&= \zeta(\Big\{-\textstyle\int_{N_{k-1}}^{N_k} G_k(s,\tau)\varphi(\tau,\overline{z}_\tau + v_\tau)d\tau, z \in U\Big\}) \\
&\leq \int_{N_{k-1}}^{N_k} G_k(s,\tau)\zeta(\Big\{\varphi(\tau,\overline{z}_\tau + v_\tau), z \in U\Big\}) \\
&\leq \int_{N_{k-1}}^{N_k} G_k(s,\tau)\tau^{-\sigma}\zeta(\Big\{\tau^\sigma\varphi(\tau,\overline{z}_\tau + v_\tau), z \in U\Big\}).
\end{aligned}$$

Remark 3 indicates that

$$\begin{aligned}
\zeta(\mathcal{P}(U)(s)) &\leq \int_{N_{k-1}}^{N_k} G_k(s,\tau)\tau^{-\sigma}[K(\zeta_{[-\gamma',0]}\{\overline{z}_\tau + v_\tau, z \in U\})]d\tau \\
&\leq \int_{N_{k-1}}^{N_k} G_k(s,\tau)\tau^{-\sigma}[K\zeta_{[-\gamma',0]}(\{\overline{z}_\tau, z \in U\} + v_\tau)]d\tau \\
&\leq \int_{N_{k-1}}^{N_k} G_k(s,\tau)\tau^{-\sigma}K[\zeta_{[-\gamma',0]}(\{\overline{z}_\tau, z \in U\})]d\tau \\
&\leq \int_{N_{k-1}}^{N_k} G_k(s,\tau)\tau^{-\sigma}K \sup_{-\gamma' \leq \theta \leq 0} \zeta(\{\overline{z}_\tau(\theta), z \in U\}d\tau \\
&\leq \int_{N_{k-1}}^{N_k} G_k(s,\tau)\tau^{-\sigma}K \sup_{-\gamma' \leq \theta \leq 0} \zeta(\{\overline{z}(\tau+\theta), z \in U\})d\tau \\
&\leq \int_{N_{k-1}}^{N_k} G_k(s,\tau)\tau^{-\sigma}K \sup_{-r \leq t \leq N_k} \zeta(\{\overline{z}(t), z \in U\})d\tau \\
&= \int_{N_{k-1}}^{N_k} G_k(s,\tau)\tau^{-\sigma}K \sup_{N_{k-1} \leq t \leq N_k} \zeta(\{\overline{z}(t), z \in U\} \cup \{0\})d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{N_{k-1}}^{N_k} G_i(s,\tau)\tau^{-\sigma} K \sup_{N_{k-1}\leq t\leq N_k} \zeta(\{\overline{z}(t), z\in U\})d\tau \\
&\leq \int_{N_{k-1}}^{N_k} G_k(s,\tau)\tau^{-\sigma} K \sup_{N_{k-1}\leq t\leq N_k} \zeta(\{z(t), z\in U\})d\tau \\
&\leq \int_{N_{k-1}}^{N_k} G_k(s,\tau)\tau^{-\sigma} K \sup_{t\in\mathcal{N}_k} \zeta(U(t))d\tau \\
&\leq \frac{1}{\Gamma(\mu_k)}\left(\frac{N_k - N_{k-1}}{4}\right)^{\mu_k-1}\left[K\zeta_{\mathcal{N}_k}(U)\int_{\mathcal{N}_{k-1}}^{N_k}\tau^{-\sigma}ds\right], \\
&\leq \frac{K\left(N_k^{1-\delta} - N_{k-1}^{1-\delta}\right)\left(N_k - N_{k-1}\right)^{\mu_k-1}}{4^{\mu_k-1}(1-\sigma)\Gamma(\mu_k)}\zeta_{\mathcal{N}_k}(U).
\end{aligned}
$$

Therefore,

$$
\zeta_{\mathcal{N}_k}(\mathcal{P}U) \leq \frac{K\left(N_k^{1-\sigma} - N_{k-1}^{1-\sigma}\right)\left(N_k - N_{k-1}\right)^{\mu_k-1}}{4^{\mu_k-1}(1-\sigma)\Gamma(\mu_k)}\zeta_{\mathcal{N}_k}(U).
$$

Consequently by (11), we deduce that $\mathcal{P}$ is a L-set contraction, where

$$
L := \frac{K\left(N_k^{1-\sigma} - N_{k-1}^{1-\sigma}\right)\left(N_k - N_{k-1}\right)^{\mu_k-1}}{4^{\mu_k-1}(1-\sigma)\Gamma(\mu_k)}.
$$

Therefore, since all conditions of Theorem 1 are fulfilled we deduce that $\mathcal{P}$ has a fixed point $z_k \in B_{R_k}$.

Then, $\mathcal{L}$ has a fixed point; thus, the Riemann–Liouville constant-order fractional boundary-value problem (8) has at least one solution $\xi_k = \overline{z_k} + v \in C([-\gamma, N_k], \mathbb{R})$. □

Now, we will prove the existence result for the Riemann–Liouville fractional problem of variable order (1).

Theorem 3. *Let the hypotheses* **(Hyp1)**, **(Hyp2)** *and inequality (11) be satisfied for all* $k \in \{1, 2, \ldots, n\}$. *Then, the Riemann–Liouville fractional problem of variable order (1) possesses at least one solution in* $C([-\gamma, N], \mathbb{R})$.

Proof. For all $k \in \{1, 2, \ldots, n\}$ according to Theorem 2, the Riemann–Liouville constant-order fractional boundary-value problem (8) possesses at least one solution $\xi_k \in C([-\gamma, N_k], \mathbb{R})$. For any $k \in \{1, 2, \ldots, n\}$, we have

$$
\xi_1(s) = \overline{z_1}(s) + v(s) = \begin{cases} \chi(s), & s\in[-\gamma,0], \\ z_1(s), & s\in\mathcal{N}_1 \end{cases}
$$

and for any $k \in \{2, \ldots, n\}$

$$
\xi_k(s) = \overline{z_k}(s) + v(s) = \begin{cases} \chi(s), & s\in[-\gamma,0], \\ 0, & s\in[0,N_{k-1}], \\ z_k(s), & s\in\mathcal{N}_k. \end{cases}
$$

Thus, the function $\xi_k \in C([-\gamma, N_k], \mathbb{R})$ satisfies the integral Equation (7) for $s \in \mathcal{N}_k$ with $\xi_k(0) = 0$, $\xi_k(N_k) = z_k(N_k) = 0$ and $\xi_k(s) = \chi(s)$ for $s \in [-\gamma, 0]$.

Then, the function

$$\xi(s)=\begin{cases}\xi_1(s)=\begin{cases}\chi(s), & s\in[-\gamma,0],\\ z_1(s), & s\in\mathcal{N}_1,\end{cases}\\ \xi_2(s)=\begin{cases}\chi(s), & s\in[-\gamma,0],\\ 0, & s\in\mathcal{N}_1,\\ z_2(s), & s\in\mathcal{N}_2,\end{cases}\\ \cdot\\ \cdot\\ \cdot\\ \xi_n(s)=\begin{cases}\chi(s), & s\in[-\gamma,0],\\ 0, & s\in[0,N_{n-1}],\\ z_n(s), & s\in\mathcal{N}_n,\end{cases}\end{cases}.$$

gives the solution for the Riemann–Liouville fractional problem of variable order (1). □

Remark 5. *The existence results for fractional delay differential equations of constant order are well established [33–38], but very little research has been done on delay fractional variable-order systems because of the complex features of fractional variable-order derivatives [32]. Theorems 2 and 3 extend the existent results to boundary-value problems for variable-order fractional delay differential equations. The offered results are established by converting the Riemann–Liouville fractional boundary-value problem of variable order (1) to a standard Riemann–Liouville fractional boundary-value problem with constant-order fractional derivatives (8), and using piecewise constant functions, the Kuratowski measure of noncompactness in the context of Darbo's fixed-point theorem.*

Remark 6. *Our results also extend and generalize some recently published existence results on boundary-value problems for fractional variable-order differential equations without delays [15,23,24,26,27,50] to the delay case, considering that the delay terms in the models are more general and more relevant to the real-world applied problems.*

Remark 7. *Unlike the existing results in [32] for the delay fractional variable-order problem, in this study we consider the Riemann–Liouville variable-order fractional derivatives of order $\mu:\mathcal{N}\to(1,2]$ and apply Darbo's fixed-point theorem together with the Kuratowski measure of noncompactness. In fact, due to the superiority of this strategy, it is intensively applied to fractional variable-order problems [23,27]. In the further investigations of the proposed boundary-value problem, different approaches may be applied, and the corresponding comparisons can be made.*

We expect that the proposed results will motivate the researchers regarding further development of the topic.

4. Ulam–Hyers Stability

Existence criteria are necessary when we study the qualitative behavior of the solutions. In order to demonstrate the applicability of the proposed in Section 2 criteria, we will provide Ulam–Hyers stability results.

Theorem 4. *Assume that conditions **(Hyp1)**, **(Hyp2)** and (11) hold. Then, the Equation (1) is Ulam–Hyers stable.*

Proof. Let $\epsilon>0$ be arbitrary, and the function $y\in C([-\gamma,N],\mathbb{R})$ satisfies the following inequality:

$$\begin{cases}|D_{0^+}^{\mu(s)}y(s)-\varphi(s,y_s)|\le\epsilon, & s\in\mathcal{N}:=[0,N],\\ y(s)=\chi(s), & s\in[-\gamma,0].\end{cases}\tag{13}$$

We define the functions

$$y_1(s) = \begin{cases} y(s), & s \in [0, N_1], \\ \chi(s), & s \in [-\gamma, 0] \end{cases} \tag{14}$$

and for $k = 2, 3, \ldots, n$:

$$y_k(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0], \\ 0, & s \in [0, N_{k-1}], \\ y(s), & s \in \mathcal{N}_k. \end{cases} \tag{15}$$

For any $k \in \{1, 2, \ldots, n\}$ according to equality (7) for $s \in \mathcal{N}_k$, we obtain

$$D_{0^+}^{\mu(s)} y_k(s) = \frac{1}{\Gamma(2-\mu_k)} (\frac{d}{ds})^2 \int_{N_{k-1}}^{s} (s-\tau)^{1-\mu_k} y(\tau) d\tau.$$

Taking $I_{N_{k-1}^+}^{\mu_k}$ on both sides of (13), we obtain

$$\begin{aligned} \left| y(s) + \int_{N_{k-1}}^{N_k} G_k(s,\tau)\varphi(\tau, y_\tau) d\tau \right| &\leq \frac{\epsilon}{\Gamma(\mu_k)} \int_{N_{k-1}}^{s} (s-\tau)^{\mu_k - 1} d\tau \\ &\leq \epsilon \frac{\left(N_k - N_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k + 1)}. \end{aligned}$$

According to Theorem 3, the Riemann–Liouville fractional problem (1) of variable order has a solution $\xi \in C([-\gamma, N], \mathbb{R})$ defined by $\xi(s) = \xi_k(s)$for $s \in [0, N_k]$, $k = 1, 2, \ldots, n$, where

$$\xi_1(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0] \\ z_1(s), & s \in \mathcal{N}_1, \end{cases} \tag{16}$$

and for any $k \in \{2, \ldots, n\}$

$$\xi_k(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0], \\ 0, & s \in [0, N_{k-1}], \\ z_k(s), & s \in \mathcal{N}_k \end{cases} \tag{17}$$

and $\xi_k \in C([-\gamma, N_k], \mathbb{R})$ is a solution of the Riemann–Liouville constant-order fractional problem (8). According to Lemma 6, we have

$$\xi_k(s) = -\int_{N_{k-1}}^{N_k} G_k(s,\tau)\varphi(\tau, (\xi_k)_\tau) d\tau. \tag{18}$$

Let $s \in \mathcal{N}_k, k \in \{1, 2, \ldots, n\}$.Then, by (15), (16), (17), and (18), we obtain

$$\begin{aligned} |y(s) - \xi(s)| &= |y(s) - \xi_k(s)| = |y_k(s) - \xi_k(s)| \\ &= |y_k(s) + \int_{N_{k-1}}^{N_k} G_k(s,\tau)\varphi(\tau, (\xi_k)_\tau) d\tau| \\ &\leq \left| y_k(s) + \int_{N_{k-1}}^{N_k} G_k(s,\tau)\varphi(\tau, (y_k)_\tau) d\tau \right| + \int_{N_{k-1}}^{N_k} G_k(s,\tau) \left| \varphi(\tau, (y_k)_\tau) - \varphi(\tau, (\xi_k)_\tau) \right| d\tau \\ &\leq \epsilon \frac{\left(N_k - N_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k + 1)} \\ &+ K \frac{1}{\Gamma(\mu_k)} \Big(\frac{N_k - N_{k-1}}{4} \Big)^{\mu_k - 1} \int_{N_{k-1}}^{N_k} \tau^{-\sigma} \|(y_k)_\tau - (\xi_k)_\tau\|_{[-\gamma', 0]} d\tau \end{aligned}$$

$$
\begin{aligned}
&\le \epsilon\frac{\left(N_k-N_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k+1)}\\
&+ K\frac{1}{\Gamma(\mu_k)}\left(\frac{N_k-N_{k-1}}{4}\right)^{\mu_k-1}\int_{N_{k-1}}^{N_k}\tau^{-\sigma}\sup_{-N_{k-1}-\gamma\le\theta\le0}|(y_k)_\tau(\theta)-(\xi_k)_\tau(\theta)|d\tau\\
&\le \epsilon\frac{\left(N_k-N_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k+1)}\\
&+ K\frac{1}{\Gamma(\mu_k)}\left(\frac{N_k-N_{k-1}}{4}\right)^{\mu_k-1}\int_{N_{k-1}}^{N_k}\tau^{-\sigma}\sup_{-N_{k-1}-\gamma\le\theta\le0}|y_k(\tau+\theta)-\xi_k(\tau+\theta)|d\tau\\
&\le \epsilon\frac{\left(N_k-N_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k+1)}\\
&+ K\frac{1}{\Gamma(\mu_k)}\left(\frac{N_k-N_{k-1}}{4}\right)^{\mu_k-1}\int_{N_{k-1}}^{N_k}\tau^{-\sigma}\sup_{-\gamma\le t\le T_k}|y_k(t)-x_k(t)|d\tau\\
&\le \epsilon\frac{\left(N_k-N_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k+1)}\\
&+ K\frac{1}{\Gamma(\mu_k)}\left(\frac{N_k-N_{k-1}}{4}\right)^{\mu_k-1}\int_{N_{k-1}}^{N_k}\tau^{-\sigma}\|y_k-\xi_k\|_{[-\gamma,N_k]}d\tau\\
&\le \epsilon\frac{\left(N_k-N_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k+1)}\\
&+ K\frac{1}{\Gamma(\mu_k)}\left(\frac{N_k-N_{k-1}}{4}\right)^{\mu_k-1}\|y_k-\xi_k\|_{[-\gamma,N_k]}\int_{N_{k-1}}^{N_k}\tau^{-\sigma}d\tau\\
&\le \epsilon\frac{\left(N_k-N_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k+1)}+\frac{K\left(N_k^{1-\sigma}-N_{k-1}^{1-\sigma}\right)\left(N_k-N_{k-1}\right)^{\mu_k-1}}{4^{\mu_k-1}(1-\sigma)\Gamma(\mu_k)}\|y_k-\xi_k\|_{[-\gamma,N_k]}\\
&\le \epsilon\frac{\left(N_k-N_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k+1)}+\nu\|y_k-\xi_k\|_{[-\gamma,N_k]},
\end{aligned}
$$

where

$$\nu=\max_{k=1,2,\dots,n}\frac{K\left(N_k^{1-\sigma}-N_{k-1}^{1-\sigma}\right)\left(N_k-N_{k-1}\right)^{\mu_k-1}}{4^{\mu_k-1}(1-\sigma)\Gamma(\mu_k)}.$$

Then,

$$\|y-\xi\|_{[-\gamma,N_k]}(1-\nu)\le\epsilon\frac{\left(N_k-N_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k+1)},$$

and so for $c_\varphi:=\frac{\left(N_k-N_{k-1}\right)^{\mu_k}}{(1-\nu)\Gamma(\mu_k+1)}$,

$$\|y-\xi\|_{[-\gamma,N_k]}\le c_\varphi\epsilon,$$

i.e.,

$$|y(s)-\xi(s)|\le c_\varphi\epsilon,\ \ s\in[-\gamma,N_k].$$

Then, by Definition 4, the Riemann–Liouville fractional problem (1) of variable order is Ulam–Hyers stable. □

Remark 8. *With the established result in this section, we contribute to the development of the Ulam–Hyers stability theory for fractional variable-order models. In fact, due to the great opportunities for applications, this stability notion has been studied by numerous authors [24,46,47,50]. In addition,* *the qualitative results offered by Theorem 4 demonstrate the opportunities for applications of the existence criteria proved in Theorems 2 and 3.*

5. Illustrative Examples

Example 1. *Let* $\gamma > 0$,

$$\mu(s) = \begin{cases} \frac{7}{5}, & s \in \mathcal{N}_1 := [0,1], \\ \frac{3}{2}, & s \in \mathcal{N}_2 :=]1,2] \end{cases} \tag{19}$$

and consider the following Riemann–Liouville fractional variable-order boundary-value problem:

$$\begin{cases} D_{0+}^{\mu(s)}\xi(s) = \frac{s^{-\frac{1}{2}}}{4e^s(1+\|\xi_s\|_{[-\gamma,0]})}, & s \in \mathcal{N} :=]0,2], \\ \xi(s) = \chi(s), & s \in [-\gamma,0]. \end{cases} \tag{20}$$

The choice of $\mu(s)$ *guarantee that* ***(Hyp1)*** *holds. Let*

$$\varphi(s,y_s) = \frac{s^{-\frac{1}{2}}}{4e^s(1+\|y_s\|_{[-\gamma,0]})}, \ (s,y_s) \in [0,2] \times C([-\gamma,0],\mathbb{R}).$$

For $y,z \in C([-\gamma,2],\mathbb{R})$ *and* $s \in \mathcal{N}$, *we have*

$$\begin{aligned} s^{\frac{1}{2}}|\varphi(s,y_s) - \varphi(s,z_s)| &= \left| \frac{1}{4e^s}\left(\frac{1}{1+\|y_s\|_{[-\gamma,0]}} - \frac{1}{1+\|z_s\|_{[-\gamma,0]}} \right) \right| \\ &\leq \frac{|\|y_s\|_{[-\gamma,0]} - \|z_s\|_{[-\gamma,0]}|}{4e^s(1+\|y_s\|_{[-\gamma,0]})(1+\|y_s\|_{[-\gamma,0]})} \\ &\leq \frac{1}{4e^s}(\|y_s - z_s\|_{[-\gamma,0]}) \\ &\leq \frac{1}{4}\|y_s - z_s\|_{[-\gamma,0]}. \end{aligned}$$

Hence, ***(Hyp2)*** *holds for* $\sigma = \frac{1}{2}$ *and* $K = \frac{1}{4}$.

By (19), according to (8) we consider the following two auxiliary problems for Riemann–Liouville fractional differential equations of constant orders:

$$\begin{cases} D_{0+}^{\frac{7}{5}}\xi(s) = \frac{s^{-\frac{1}{2}}}{4e^s(1+\|\xi_s\|_{[-\gamma,0]})}, & s \in \mathcal{N}_1, \\ \xi(0) = 0, \ \xi(1) = 0, \\ \xi(s) = \chi_1(s), & s \in [-\gamma,0] \end{cases} \tag{21}$$

and

$$\begin{cases} D_{0+}^{\frac{3}{2}}\xi(s) = \frac{s^{-\frac{1}{2}}}{4e^s(1+\|\xi_s\|_{[-\gamma,0]})}, & s \in \mathcal{N}_2, \\ \xi(1) = 0, \ \xi(2) = 0, \\ \xi(s) = \chi_2(s), & s \in [-\gamma,1], \end{cases} \tag{22}$$

where $\chi_1 = \chi$ *and*

$$\chi_2(s) = \begin{cases} 0, \ , & if \ s \in [0,1], \\ \chi(s), & if \ s \in [-\gamma,0]. \end{cases}$$

We will show also that condition (11) is satisfied for $k = 1$. *Indeed,*

$$\frac{K\left(N_1^{1-\sigma} - N_0^{1-\sigma}\right)\left(N_1 - N_0\right)^{\mu_1 - 1}}{4^{\mu_1-1}(1-\sigma)\Gamma(\mu_1)} = \frac{\frac{1}{4}\left(1^{1-\frac{1}{2}} - 0^{1-\frac{1}{2}}\right)\left(1-0\right)^{\frac{7}{5}-1}}{4^{\frac{7}{5}-1}(1-\frac{1}{2})\Gamma(\frac{7}{5})} \simeq 0.323663 < 1.$$

By Theorem 2, the problem (21) has a solution $\xi_1 \in C([-\gamma,1],\mathbb{R})$, where

$$\xi_1(s) \quad = \quad \begin{cases} \chi(s), & s \in [-\gamma,0], \\ z_1(s), & s \in \mathcal{N}_1. \end{cases}$$

We also have that

$$\frac{K\left(N_2^{1-\sigma} - N_1^{1-\sigma}\right)\left(N_2 - N_1\right)^{\mu_2-1}}{4^{\mu_2-1}(1-\sigma)\Gamma(\mu_2)} = \frac{\frac{1}{4}\left(2^{1-\frac{1}{2}} - 1^{1-\frac{1}{2}}\right)\left(2-1\right)^{\frac{3}{2}-1}}{4^{\frac{3}{2}-1}(1-\frac{1}{2})\Gamma(\frac{3}{2})}$$
$$\simeq 0.11684748 < 1.$$

Thus, (11) is fulfilled for $k = 2$. According to Theorem 2, the problem (22) possesses a solution $\xi_2 \in C([-\gamma,2],\mathbb{R})$, where

$$\xi_2(s) = \begin{cases} \chi(s), & s \in [-\gamma,0], \\ 0, & s \in [0,1], \\ z_2(s), & t \in \mathcal{N}_2. \end{cases}$$

Then, by Theorem 3, the problem (20) has a solution

$$\xi(s) = \begin{cases} \xi_1(s) = \begin{cases} \chi(s), & s \in [-\gamma,0], \\ z_1(s), & s \in \mathcal{N}_1, \end{cases} \\ \xi_2(s) = \begin{cases} \chi(s), & s \in [-\gamma,0], \\ 0, & s \in \mathcal{N}_1, \\ z_2(s), & s \in \mathcal{N}_2, \end{cases} \end{cases}$$

In addition, according to Theorem 4, problem (20) is Ulam–Hyers-stable.

Example 2. *Let $\gamma > 0$,*

$$\mu(s) = \begin{cases} \frac{7}{5}, & s \in \mathcal{N}_1 := [0,1], \\ \frac{6}{5}, & s \in \mathcal{N}_2 :=]1,\frac{3}{2}], \\ \frac{3}{2}, & s \in \mathcal{N}_3 :=]\frac{3}{2},2]. \end{cases} \tag{23}$$

and consider the following Riemann–Liouville fractional variable-order boundary-value problem:

$$\begin{cases} D_{0^+}^{\mu(s)}\xi(s) = \frac{s^{-\frac{1}{3}}}{(e^{e^{\frac{s^3}{1+s}}}+6)(1+\|x_s\|_{[-\gamma,0]})}, & s \in \mathcal{N} :=]0,2], \\ \xi(s) = \chi(s), & s \in [-\gamma,0], \end{cases} \tag{24}$$

*The choice of $\mu(s)$ guarantees that **(Hyp1)** holds. Let*

$$\varphi(s,y_s) = \frac{s^{-\frac{1}{3}}}{(e^{e^{\frac{s^3}{1+s}}}+6)(1+\|y_s\|_{[-\gamma,0]})}, \ (s,y_s) \in [0,2] \times C([-\gamma,0],\mathbb{R}).$$

For $y,z \in C([-\gamma,2],\mathbb{R})$ and $s \in \mathcal{N}$, we have

$$\begin{aligned} s^{\frac{1}{3}}|\varphi(s,y_s) - \varphi(s,z_s)| \quad &= \quad \left|\frac{1}{(e^{e^{\frac{s^3}{1+s}}}+6)}\left(\frac{1}{1+\|y_s\|_{[-\gamma,0]}} - \frac{1}{1+\|z_s\|_{[-\gamma,0]}}\right)\right| \\ &\leq \quad \frac{|\|y_s\|_{[-\gamma,0]} - \|z_s\|_{[-\gamma,0]}|}{(e^{e^{\frac{s^3}{1+s}}}+6)(1+\|y_s\|_{[-\gamma,0]})(1+\|y_s\|_{[-\gamma,0]})} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{(e^{e^{\frac{s^3}{1+s}}}+6)}\|y_t - z_s\|_{[-\gamma,0]} \\
&\leq \frac{1}{e+6}\|y_s - z_s\|_{[-\gamma,0]}.
\end{aligned}$$

Hence, ***(Hyp2)*** *holds for* $\sigma = \frac{1}{3}$ *and* $K = \frac{1}{e+6}$.

By (23), according to (8) we consider three auxiliary problems for Riemann–Liouville fractional differential equations of constant order

$$\begin{cases} D_{0^+}^{\frac{7}{5}}\xi(s) = \frac{s^{-\frac{1}{3}}}{(e^{e^{\frac{s^3}{1+s}}}+6)(1+\|x_s\|_{[-\gamma,0]})}, & s \in \mathcal{N}_1, \\ \xi(0) = 0,\ \xi(1) = 0, \\ \xi(s) = \chi_1(s), & s \in [-\gamma, 0], \end{cases} \tag{25}$$

$$\begin{cases} D_{0^+}^{\frac{6}{5}}\xi(s) = \frac{s^{-\frac{1}{3}}}{(e^{e^{\frac{s^3}{1+s}}}+6)(1+\|x_s\|_{[-\gamma,0]})}, & s \in \mathcal{N}_2, \\ \xi(1) = 0,\ \xi(\frac{3}{2}) = 0, \\ \xi(s) = \chi_2(s), & s \in [-\gamma, 1], \end{cases} \tag{26}$$

and

$$\begin{cases} D_{0^+}^{\frac{3}{2}} = \frac{s^{-\frac{1}{3}}}{(e^{e^{\frac{s^3}{1+s}}}+6)(1+\|x_s\|_{[-\gamma,0]})}, & s \in \mathcal{N}_3, \\ \xi(\frac{3}{2}) = 0,\ x(2) = 0, \\ \xi(s) = \chi_2(s), & s \in [-\gamma, \frac{3}{2}], \end{cases} \tag{27}$$

where $\chi_1 = \chi$ *,*

$$\chi_2(s) = \begin{cases} 0, \ , \ if\ s \in [0,1], \\ \chi(s), \ if\ s \in [-\gamma, 0] \end{cases}$$

and

$$\chi_3(s) = \begin{cases} 0, \ , \ if\ s \in [0,\frac{3}{2}], \\ \chi(s), \ if\ s \in [-\gamma, 0]. \end{cases}$$

We will also show that condition (11) is satisfied for $k = 1$. *Indeed,*

$$\frac{K\Big(N_1^{1-\sigma} - N_0^{1-\sigma}\Big)\Big(N_1 - N_0\Big)^{\mu_1 - 1}}{4^{\mu_1-1}(1-\sigma)\Gamma(\mu_1)} = \frac{\frac{1}{e+6}\Big(1^{1-\frac{1}{3}} - 0^{1-\frac{1}{3}}\Big)\Big(1-0\Big)^{\frac{7}{5}-1}}{4^{\frac{7}{5}-1}(1-\frac{1}{3})\Gamma(\frac{7}{5})} \simeq 0.11137 < 1.$$

By Theorem 2, the problem (25) has a solution $\xi_1 \in C([-\gamma, 1], \mathbb{R})$, *where*

$$\xi_1(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0] \\ z_1(s), & s \in \mathcal{N}_1. \end{cases}$$

We also have that

$$\frac{K\Big(N_2^{1-\sigma} - N_1^{1-\sigma}\Big)\Big(N_2 - N_1\Big)^{\mu_2 - 1}}{4^{\mu_2-1}(1-\sigma)\Gamma(\mu_2)} = \frac{\frac{1}{e+6}\Big(\frac{3}{2}^{1-\frac{1}{3}} - 1^{1-\frac{1}{3}}\Big)\Big(\frac{3}{2}-1\Big)^{\frac{6}{5}-1}}{4^{\frac{6}{5}-1}(1-\frac{1}{3})\Gamma(\frac{6}{5})}$$

$$\simeq 0.03837 < 1.$$

Thus, (11) is fulfilled for $k = 2$. *According to Theorem 2, the BVP (26) possesses a solution* $\xi_2 \in C([-\gamma, \frac{3}{2}], \mathbb{R})$, *where*

$$\xi_2(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0], \\ 0, & s \in [0, 1], \\ z_2(s), & s \in \mathcal{N}_2. \end{cases}$$

We also have that

$$\frac{K\left(N_3^{1-\sigma} - N_2^{1-\sigma}\right)\left(N_3 - N_2\right)^{\mu_3 - 1}}{4^{\mu_3 - 1}(1-\sigma)\Gamma(\mu_3)} = \frac{\frac{1}{e+6}\left(2^{1-\frac{1}{3}} - \frac{3}{2}^{1-\frac{1}{3}}\right)\left(2 - \frac{3}{2}\right)^{\frac{3}{2}-1}}{4^{\frac{3}{2}-1}(1-\frac{1}{3})\Gamma(\frac{3}{2})}$$
$$\simeq 0.01901 < 1.$$

Thus, (11) is fulfilled for $k = 3$. *According to Theorem 2, the BVP (27) possesses a solution* $\xi_3 \in C([-\gamma, 2], \mathbb{R})$,
where

$$\xi_3(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0], \\ 0, & s \in [0, \frac{3}{2}], \\ z_3(s), & s \in \mathcal{N}_3. \end{cases}$$

Then, by Theorem 3, problem (24) has a solution

$$\xi(s) = \begin{cases} \xi_1(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0], \\ z_1(s), & s \in \mathcal{N}_1, \end{cases} \\ \xi_2(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0], \\ 0, & s \in]0, N_1], \\ z_2(s), & s \in \mathcal{N}_2, \end{cases} \\ \xi_3(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0], \\ 0, & s \in]0, N_2], \\ z_3(s), & s \in \mathcal{N}_3. \end{cases} \end{cases}$$

In addition, according to Theorem 4, problem (24) *is Ulam–Hyers stable.*

Remark 9. *The constructed examples show the capability of the elaborated existence and stability results.*

6. Conclusions

This research introduces a boundary-value problem for a Riemann–Liouville nonlinear fractional differential equation of variable order with finite delay. The analytical solutions have been successfully investigated via three strategies: the Kuratowski measure of noncompactness, Darbo's fixed-point theorem, and the Ulam–Hyers stability concept. We established existence and stability criteria for the solutions of the problem under consideration. The presented new results generalize some existing results for the Riemann–Liouville delayed fractional differential equation of constant order considering the variable order of fractional derivatives. Two examples are given at the end to support and validate the potentiality of the obtained results. We expect that the proposed results will motivate the researchers in the further development of the topic. The established existence results are essential in the qualitative investigation of the introduced problem. Additionally, since the Riemann–Liouville delayed fractional differential equations of variable order are intensively applied in the mathematical modeling, our research is practically important. Hence, the application of our results to some Riemann–Liouville fractional-neural-network models of variable order with finite delay is an interesting topic for a future research. The obtained results can also be applied in the investigation of numerous qualitative properties of the solutions. In addition, it is possible to extend the proposed results to the impulsive case and study the effect of some impulsive controllers on the fundamental and qualitative behavior of the solutions.

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