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On the Left Properness of the Model Category of Permutative Categories

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Abstract: In this paper, we introduce a notion of free cofibrations of permutative categories. We show that each cofibration of permutative categories is a retract of a free cofibration. The main goal of this paper is to show that the natural model category of permutative categories is a left proper model category.

Keywords: symmetric monoidal categories; free cofibrations; left proper model categories; homotopical algebra

1. Introduction

A *permutative* category is a symmetric monoidal category whose associativity and unit natural isomorphisms are identities. Permutative categories have generated significant interest in topology. An infinite loop space machine was constructed on permutative categories in [1]. A K-theory (multi-)functor from a *multicategory* of permutative categories into a symmetric monoidal category of symmetrical spectra, which preserves the multiplicative structure, was constructed in [2]. In [3], the K-theory of [2] was enhanced to a lax symmetric monoidal functor. It was shown in [4] that permutative categories model connective spectra.

Every symmetric monoidal category is equivalent (by a symmetric monoidal functor) to a permutative category. The category of symmetric monoidal categories **SMCAT** does NOT have a model category structure; however, its subcategory of permutative categories and strict symmetric monoidal functors **Perm** carries a model category structure. The category **Perm** is isomorphic to the category of algebras over the (categorical) Barrat–Eccles operad. Using this fact, the model category structure follows from [5] and ([6], Thm. 4.5). This model category structure is called the *natural* model category structure of permutative categories.

The main objective of this paper is to identify a class of cofibrations in the natural model category **Perm** called *free* cofibrations such that every cofibration in **Perm** is a retract of a free cofibration. A desirable property of free cofibrations is that cobase changes along a free cofibration preserve acyclic fibrations in the natural model category **Perm**. This property allows us to prove our main result that the natural model category **Perm** is *left proper*. Our primary motivation for proving the main result of this paper is the existence of left Bousfield localizations in combinatorial left-proper model categories ([7], Thm. 4.7). The main result of this paper has allowed us to construct two left Bousfield localizations of the natural model category **Perm** which are the model category of (permutative) compact closed categories $\mathbf{Perm}^{cc}$ [8] and the model category of (permutative) Picard groupoids $(\mathbf{Perm}, \mathcal{P}ic)$ [9].

Finally in Appendix A we present a construction of *Gabriel Factorization* of a unital symmetric monoidal functor between permutative categories. Our construction factors a unital symmetric monoidal functor into an essentially surjective strict symmetric monoidal functor followed by a fully faithful unital symmetric monoidal functor.

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Remark 1. *Some proofs in this paper assume standard results in homotopical algebra. We would like to provide the uninformed reader a list of references to standard textbooks on the subject: [10–12].*

2. Free Cofibrations in Perm

In this section, we define a class of maps called *free* cofibrations in the natural model category of permutative categories **Perm**. We show that a strict symmetric monoidal functor is a cofibration in **Perm** if and only if it is a retract of a free cofibration. A characterization of cofibrations in **Perm** was formulated purely in terms of object functions (which are monoid homomorphisms) of the underlying strict symmetric monoidal functor in [13]. In order to define free cofibrations, we will start by reviewing some basic notions of permutative categories:

Definition 1. *A symmetric monoidal category is called a* permutative *category or a strict symmetric monoidal category if it is strictly associative and strictly unital.*

Remark 2. *A permutative category is an internal category in the category of monoids.*

We recall that the forgetful functor $U : \mathbf{Perm} \to \mathbf{Cat}$ has a left adjoint $\mathcal{F} : \mathbf{Cat} \to \mathbf{Perm}$.

Definition 2. *A monoid M is called a* free *monoid if there exists a (dotted) lifting monoid homomorphism whenever we have the following (outer) commutative diagram of monoid homomorphisms:*

$$\begin{tikzcd} \ast \arrow[r] \arrow[d] & N \arrow[d, "p"] \\ M \arrow[r] \arrow[ur, dashed] & Q \end{tikzcd}$$

where p is a surjective monoid homomorphism, and $\ast$ is a zero *object in the category of monoids.*

Definition 3. *A* free cofibration *of permutative categories is a (strict symmetric monoidal) functor $i : A \to C$ whose object function is the inclusion $Ob(i) : Ob(A) \to (Ob(A) \vee M) = Ob(C)$, where M is a free monoid and the coproduct is taken in the category of monoids.*

The next proposition presents the desired characterization of cofibrations:

Proposition 1. *A strict symmetric monoidal functor $F : C \to D$ is a cofibration in* **Perm** *if and only if it is a retract of a free cofibration by a map that fixes C.*

Proof. Let us first assume that F is a retract of a free cofibration $i : E \to M$. We observe that the object function of a free cofibration has the left lifting property with respect to all surjective monoid homomorphisms; therefore, each free cofibration is a cofibration in **Perm**. A retract of a cofibration is again a cofibration. Thus, F is a cofibration in **Perm**.

Conversely, let us assume that F is a cofibration in **Perm**. We have the following (outer) commutative diagram in the category of monoids

$$\begin{tikzcd} Ob(C) \arrow[r, "i"] \arrow[d, "Ob(F)"'] & Ob(C) \vee \mathcal{F}_m(Ob(D)) \arrow[d, "p"] \\ Ob(D) \arrow[r, equal] \arrow[ur, dashed, "L"] & Ob(D) \end{tikzcd}$$

where $\mathcal{F}_m(Ob(D))$ is the free monoid generated by the set $Ob(D)$, i is the inclusion into the coproduct, and $p = Ob(F) \vee \epsilon$. The summand $\epsilon : \mathcal{F}_m(Ob(D)) \to Ob(D)$ is the counit of the reflection:

$$\mathcal{F}_m : \mathbf{Set} \rightleftarrows \mathbf{Mon} : U$$

Since the right vertical homomorphism of monoids is surjective and F is a cofibration by assumption, there therefore exists a (dotted) lifting homomorphism L which makes the whole diagram commutative. Thus, $Ob(F)$ is a retract of the inclusion i in the category of monoids. We will construct a strict symmetric monoidal functor $I : C \to E$ whose object function is the inclusion i and show that F is a retract of I. We begin by constructing the category E:

The object set of E is $Ob(C) \vee F(Ob(D))$. The morphism monoid of E is defined by the following pullback square in the category of monoids:

$$\begin{tikzcd} Mor(E) \arrow[r, "p_1"] \arrow[d, "p_2"'] & Mor(D) \arrow[d, "(s_D,t_D)"] \\ (Ob(C) \vee F(Ob(D))) \times (Ob(C) \vee F(Ob(D))) \arrow[r, "p \times p"'] & Ob(D) \times Ob(D) \end{tikzcd} \tag{1}$$

We will denote the projection map p_2 in the above Cartesian square by (s_E, t_E). This pair will be source and target maps for the proposed category E. The projection map p_1 in the above Cartesian diagram restricts to a map between the set of composable arrows in E and D:

$$p_1^c : Mor(E) \underset{s_E = t_E}{\times} Mor(E) \to Mor(D) \underset{s_D = t_D}{\times} Mor(D).$$

Now, we observe the composite $(- \underset{D}{\circ} -) \circ p_1^c$ factors through $Mor(E)$ as follows:

$$\begin{tikzcd} Mor(E) \underset{s_E = t_E}{\times} Mor(E) \arrow[r, "- \underset{E}{\circ} -"] \arrow[d, "p_1^c"'] & Mor(E) \arrow[d, "p_1"] \\ Mor(D) \underset{s_D = t_D}{\times} Mor(D) \arrow[r, "- \underset{D}{\circ} -"'] & Mor(D) \end{tikzcd} \tag{2}$$

The map $- \underset{E}{\circ} -$ in the above commutative diagram provides the composition of category E. Finally, we define the symmetry natural transformation of E as follows:

$$\gamma^E_{z_1, z_2} := \gamma^D_{p(z_1), p(z_2)} \tag{3}$$

for each pair of objects $z_1, z_2 \in Ob(E)$. This defines a permutative category $(E, - \underset{E}{\boxtimes} -, \gamma^E)$, where the tensor product is uniquely determined by the monoid structures on $Ob(E)$ and $Mor(E)$.

The commutative diagrams (1) and (2) and the definition of the symmetry natural transformation (3) together imply that there is a strict symmetric monoidal functor $P : E \to D$ whose object homomorphism is p and morphism homomorphism is p_1. Further, P is surjective on objects and also fully faithful. This implies that P is an acyclic fibration in the natural model category **Perm**.

Now, we construct the free cofibration $I : C \to E$ mentioned above. The object homomorphism of I is the inclusion $i : Ob(C) \to Ob(C) \vee F(Ob(D))$. The morphism homomorphism of I is defined as follows:

$$Mor(I) := Mor(F).$$

In other words, $I(f) = F(f)$ for each morphism $f \in Mor(C)$. Now, we have the following (outer) commutative diagram in **Perm**:

$$\begin{tikzcd} C \arrow[r, "I"] \arrow[d, "F"'] & E \arrow[d, "P"] \\ D \arrow[ur, dashed, "L"] \arrow[r, equal] & D \end{tikzcd}$$

Since F is a cofibration and P is an acyclic fibration in the natural model category **Perm**, there exists a (dotted) lifting arrow L which makes the entire diagram commutative. This implies that F is a retract of the free cofibration I in the natural model category **Perm**. □

3. Left Properness of the Natural Model Category Perm

In this section we show that the natural model category of permutative categories **Perm** is left proper. We recall that a model category is left proper if the cobase change of a weak-equivalence along a cofibration is again a weak-equivalence. We will first show that the cobase change of a weak-equivalence along a free cofibration is a weak-equivalence. Using this intermediate result, we will prove the left properness of **Perm**.

Let $G : A \to B$ be an acyclic fibration in **Perm** and $i_A : A \to C$ be a free cofibration. Therefore, the object monoid of C can be written as a coproduct $Ob(A) \vee V$, where V is a free monoid. We observe that the following commutative square is co-Cartesian:

$$\begin{tikzcd} Ob(A) \arrow[r, "Ob(i_A)"] \arrow[d, "Ob(G)"'] & Ob(A) \vee V \arrow[d, "Ob(G)\vee id"] \\ Ob(B) \arrow[r, "i_B"'] & F(B) \vee V \end{tikzcd}$$

We will construct the following pushout square in **Perm**:

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "G"'] & C \arrow[d] \\ B \arrow[r] & B \underset{A}{\sqcup} C \end{tikzcd}$$

A strict symmetric monoidal functor $G : A \to B$ is an acyclic fibration in **Perm** if and only if there exists a unital symmetric monoidal section ([13], Cor. 3.5(3)) $S : B \to A$ such that $GS = id_D$ and a monoidal natural isomorphism $\epsilon_S : SG \cong id$. Let us fix such a section $S : B \to A$ and natural isomorphism ϵ_S.

Remark 3. *The above characterization of acyclic fibrations implies that $S : B \to A$ is a left-adjoint-right-inverse of $G : A \to B$. This means that $\epsilon_S : SG \cong id_A$ is a counit of an adjoint equivalence whose unit $\eta : GS = id_B$ is the identity natural transformation. This further implies that $G\epsilon_S \cdot \eta G = id_G$. In other words, for each $a \in A$, we have the following equality:*

$$G(\epsilon_S(a)) \circ \eta(G(a)) = id_{G(a)}.$$

Since it follows from ([13], Cor. 3.5(3)) that the unit natural transformation η is the identity, $G\epsilon_S = G$.

Remark 4. *Let b_1, b_2 be a pair of objects in B. Since ϵ_S is a monoidal natural transformation, we have the following commutative diagram:*

$$\begin{tikzcd} SG(S(b_1) \otimes S(b_2)) \arrow[r, "\epsilon_S(S(b_1), S(b_2))"] \arrow[d, "\lambda^S(b_1, b_2)"'] & S(b_1) \otimes S(b_2) \arrow[d, equal] \\ S(GS(b_1) \otimes GS(b_2)) \arrow[r, equal] & S(b_1) \otimes S(b_2) \end{tikzcd}$$

Thus, we have shown that

$$\lambda^S = \epsilon_S S.$$

This further implies that

$$G\lambda^S = G\epsilon_S S = GS = id_B.$$

The unital symmetric monoidal functor S gives us the following unital symmetric monoidal functor:

$$S \vee \mathcal{F}(C; V) : B \vee \mathcal{F}(C; V) \to A \vee \mathcal{F}(C; V),$$

where $\mathcal{F}(C; V)$ is the full permutative subcategory of C whose object set is the (free) monoid V, and the notation $S \vee \mathcal{F}(C; V)$ is an abbreviation for the coproduct of functors $S \vee id_{\mathcal{F}(C;V)}$. We observe that $S \vee \mathcal{F}(C; V)$ is a section of the strict symmetric monoidal functor $G \vee \mathcal{F}(C; V)$ i.e., $(G \vee \mathcal{F}(C; V)) \circ (S \vee \mathcal{F}(C; V)) = id$. Moreover, we obtain a monoidal natural isomorphism

$$\epsilon_S \vee \mathcal{F}(C; V) : (S \vee \mathcal{F}(C; V)) \circ (G \vee \mathcal{F}(C; V)) \cong id$$

Hence, the functor $G \vee \mathcal{F}(C; V)$ is an acyclic fibration in the natural model category **Perm** by ([13], Cor. 3.5(3)).

We observe the free cofibration i_A factors as follows:

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\iota_A"'] & & C \\ & A \vee \mathcal{F}(C; V) \arrow[ur, "i_{A,V}"'] & \end{tikzcd} \tag{4}$$

where $\iota_A : A \to A \vee \mathcal{F}(C; V)$ is the inclusion into the coproduct, and $i_{A,V} : A \vee \mathcal{F}(C; V) \to C$ is the unique map induced by the inclusions $i_A : A \to C$ and $i_V : \mathcal{F}(C, V) \to C$

Remark 5. *The following commutative square is a co-Cartesian:*

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "G"'] & A \vee \mathcal{F}(C; V) \arrow[d, "G \vee \mathcal{F}(C;V)"] \\ B \arrow[r, "\iota_B"'] & B \vee \mathcal{F}(C; V) \end{tikzcd} \tag{5}$$

We observe that the object monoid of C is the same as the object monoid of $A \vee \mathcal{F}(C; V)$, namely the coproduct $(Ob(A)) \vee V$. This implies that for each $c \in Ob(C)$ there is the following isomorphism in C:

$$(i_{A,V} \circ (\epsilon_S \vee \mathcal{F}(C; V)))(c) : (S \vee \mathcal{F}(C; V)) \circ (G \vee \mathcal{F}(C; V))(c) \cong c,$$

Now, it follows from ([13], Prop. 2.7) that there exists a (uniquely defined) functor $S_C : C \to C$ and a natural isomorphism $\delta^C : id_C \cong S_C$. The functor S_C is defined on objects as follows:

$$S_C(c) := (S \vee \mathcal{F}(C; V)) \circ (G \vee \mathcal{F}(C; V))(c).$$

The following lemma now tells us that S_C is a unital symmetric monoidal functor, and δ^C is a monoidal natural isomorphism:

Lemma 1. *Given a unital oplax symmetric monoidal functor (F, λ_F) between two symmetric monoidal categories C and D, a functor $G : C \to D$ and a unital natural isomorphism $\alpha : F \cong G$, there is a unique natural isomorphism λ_G which enhances G to a unital oplax symmetric monoidal functor (G, λ_G) such that α is a monoidal natural isomorphism. If (F, λ_F) is unital symmetric monoidal, then so is (G, λ_G).*

Proof. We consider the following diagram:

$$
\begin{tikzcd}
C \times C \arrow[rr, "-\underset{C}{\otimes}-"] \arrow[d, bend right=60, "G\times G"'] \arrow[d, bend left=60, "F\times F"] \arrow[d, phantom, "\overset{\alpha\times\alpha}{\Leftarrow}"] & & C \arrow[d, bend right=60, "F"'] \arrow[d, bend left=60, "G"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\
D \times D \arrow[rr, "-\underset{D}{\otimes}-"'] & & D
\end{tikzcd}
$$

This diagram helps us define a composite natural isomorphism $\lambda_G : G \circ (- \underset{C}{\otimes} -) \Rightarrow (- \underset{D}{\otimes} -) \circ G \times G$ as follows:

$$\lambda_G := (id_{-\underset{D}{\otimes}-} \circ \alpha \times \alpha) \cdot \lambda_F \cdot (\alpha^{-1} \circ id_{-\underset{C}{\otimes}-}). \tag{6}$$

This composite natural isomorphism is the unique natural isomorphism which makes α a unital monoidal natural isomorphism. Now, we have to check that λ_G is a unital monoidal natural isomorphism with respect to the above definition. Clearly, λ_G is unital because both α and λ_F are unital natural isomorphisms. We first check the symmetry condition ([13], Defn. 2.4 OL. 2). This condition is satisfied because the following composite diagram commutes

$$
\begin{tikzcd}
G(c_1 \underset{C}{\otimes} c_2) \arrow[r, "\alpha^{-1}(c_1 \underset{C}{\otimes} c_2)"] \arrow[d, "G(\gamma^C(c_1,c_2))"'] & F(c_1 \underset{C}{\otimes} c_2) \arrow[r, "\lambda_F(c_1,c_2)"] \arrow[d, "F(\gamma^C(c_1,c_2))"] & F(c_1) \underset{D}{\otimes} F(c_2) \arrow[r, "\alpha(c_1) \underset{D}{\otimes} \alpha(c_2)"] \arrow[d, "\gamma^D(F(c_1),F(c_2))"] & G(c_1) \underset{D}{\otimes} G(c_2) \arrow[d, "\gamma^D(G(c_1),G(c_2))"] \\
G(c_2 \underset{C}{\otimes} c_1) \arrow[r, "\alpha^{-1}(c_2 \underset{C}{\otimes} c_1)"'] & F(c_2 \underset{C}{\otimes} c_1) \arrow[r, "\lambda_F(c_2,c_1)"'] & F(c_2) \underset{D}{\otimes} F(c_1) \arrow[r, "\alpha(c_2) \underset{D}{\otimes} \alpha(c_1)"'] & G(c_2) \underset{D}{\otimes} G(c_1)
\end{tikzcd}
$$

The condition ([13], Defn. 2.4 OL. 3) follows from the following equalities

$$
\begin{gathered}
\alpha_D(G(c_1), G(c_2), G(c_3)) \circ \lambda_G(c_1, c_2) \underset{D}{\otimes} id_{G(c_3)} \circ \lambda_G(c_1 \underset{C}{\otimes} c_2, c_3) = \\
(\alpha(c_1) \underset{D}{\otimes} \alpha(c_2)) \underset{D}{\otimes} \alpha(c_3) \circ \alpha_D(F(c_1), F(c_2), F(c_3)) \circ \lambda_F(c_1, c_2) \underset{D}{\otimes} id_{F(c_3)} \circ \\
\lambda_F(c_1 \underset{C}{\otimes} c_2, c_3) \circ \alpha^{-1}((c_1 \underset{C}{\otimes} c_2) \underset{C}{\otimes} c_3) = \\
(\alpha(c_1) \underset{D}{\otimes} \alpha(c_2)) \underset{D}{\otimes} \alpha(c_3) \circ id_{F(c_1)} \underset{D}{\otimes} \lambda_F(c_1, c_2) \circ \lambda_F(c_1, c_2 \underset{C}{\otimes} c_3) \circ F(\alpha_C(c_1, c_2, c_3)) \\
\circ \alpha^{-1}((c_1 \underset{C}{\otimes} c_2) \underset{C}{\otimes} c_3) = \\
id_{G(c_1)} \underset{D}{\otimes} \lambda_G(c_1, c_2) \circ \lambda_G(c_1, c_2 \underset{C}{\otimes} c_3) \circ G(\alpha_C(c_1, c_2, c_3)).
\end{gathered}
$$

If $F = (F, \lambda_F)$ is a symmetric monoidal functor, then so is $G = (G, \lambda_G)$ because (6) is a natural isomorphism. □

The section $S \vee \mathcal{F}(C;V)$ provides us with a unital symmetric monoidal functor $i_{A,V} \circ (S \vee \mathcal{F}(C;V)) : B \vee \mathcal{F}(C;V) \to C$ which we denote by $S^{\mathcal{F}}$. The unital symmetric monoidal functor $S^{\mathcal{F}}$ has the following Gabriel factorization:

$$\begin{tikzcd} B \vee \mathcal{F}(C;V) \arrow[rr, "S^{\mathcal{F}}"] \arrow[dr, "\Gamma_{S^{\mathcal{F}}}"'] & & C \\ & G(S^{\mathcal{F}}) \arrow[ur, "\Delta"'] & \end{tikzcd}$$

By Lemma A1, $(G(S^{\mathcal{F}}), -\boxdot -, \gamma)$ is a permutative category structure. In addition, by the same lemma, Γ is a strict symmetric monoidal functor.

Remark 6. *The following diagram of unital symmetric monoidal functors is commutative:*

$$\begin{tikzcd} B \vee \mathcal{F}(C;V) \arrow[r, "S^{\mathcal{F}}"] \arrow[d, "S\vee\mathcal{F}(C;V)"'] & C \\ A \vee \mathcal{F}(C;V) \arrow[r, "i_{A,V}"'] & C \arrow[u, "S_C"'] \end{tikzcd}$$

The above commutative diagram implies that for each object $z \in G(S^{\mathcal{F}})$, $\lambda^{S^{\mathcal{F}}}(z) = \lambda^{S_C}(z)$.

We claim that there exists a strict symmetric monoidal functor $P : C \to G(S^{\mathcal{F}})$ such that the following diagram, in **Perm**, is co-Cartesian:

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "G"'] & C \arrow[d, "P"] \\ B \arrow[r, "\Gamma"'] & G(S^{\mathcal{F}}) \end{tikzcd} \tag{7}$$

where $\Gamma = \Gamma_{S^{\mathcal{F}}} \circ \iota_B$. The object function of the functor P is the monoid homomorphism

$$Ob(G) \vee V : Ob(A) \vee V \to Ob(B) \vee V.$$

For any pair of objects $c_1, c_2 \in Ob(C)$, we observe the following equality:

$$G(S^{\mathcal{F}})(P(c_1), P(c_2)) = C(S_C(c_1), S_C(c_2)).$$

Now, we define the morphism function of P as follows:

$$P(f) := S_C(f),$$

where f is a morphism in C. The functoriality of P follows from that of S_C.

The object function of P is a monoid homomorphism; therefore, $P(c_1 \underset{C}{\otimes} c_2) = P(c_1) \boxdot P(c_2)$ for each pair of objects $c_1, c_2 \in Ob(C)$. The following commutative diagram shows that $P(f_1 \underset{C}{\otimes} f_2) = P(f_1) \boxdot P(f_2)$ for each pair of maps $(f_1, f_2) \in C(c_1, c_2) \times C(c_3, c_4)$:

$$\begin{tikzcd}
c_1 \underset{C}{\otimes} c_2 \arrow[r, "f_1 \underset{C}{\otimes} f_2"] \arrow[d, "\delta"', "\cong"] & c_3 \underset{C}{\otimes} c_4 \arrow[d, "\cong"', "\delta"] \\
S_C(c_1 \underset{C}{\otimes} c_2) \arrow[r, "P(f_1 \underset{C}{\otimes} f_2)"'] \arrow[d, equal] & S_C(c_3 \underset{C}{\otimes} c_4) \arrow[d, equal] \\
S^{\mathcal{F}}(P(c_1) \boxdot P(c_2)) \arrow[r, "P(f_1) \boxdot P(f_2)"'] \arrow[d, "\lambda_{S^{\mathcal{F}}}"'] & S^{\mathcal{F}}(P(c_3) \boxdot P(c_4)) \arrow[d, "\lambda_{S^{\mathcal{F}}}"] \\
S^{\mathcal{F}}(P(c_1)) \underset{C}{\otimes} S^{\mathcal{F}}(P(c_2)) \arrow[r, "P(f_1) \underset{C}{\otimes} P(f_2)"'] & S^{\mathcal{F}}(P(c_3)) \underset{C}{\otimes} S^{\mathcal{F}}(P(c_4))
\end{tikzcd}$$

Thus, we have defined a strict symmetric monoidal functor P which is fully faithful. Further, each object of $G(S^{\mathcal{F}})$ is isomorphic to one in the image of P. Thus, P is an equivalence of categories.

Proposition 2. *The commutative square (7) is co-Cartesian.*

Proof. In order to show that (7) is co-Cartesian, it is sufficient to show that the following commutative square is co-Cartesian in light of factorization (4) and Remark 5:

$$\begin{tikzcd}
A \vee \mathcal{F}(C;V) \arrow[r, "i_{A,V}"] \arrow[d, "G \vee \mathcal{F}(C;V)"'] & C \arrow[d, "P"] \\
B \vee \mathcal{F}(C;V) \arrow[r, "\Gamma_{S^{\mathcal{F}}}"'] & G(S^{\mathcal{F}})
\end{tikzcd}$$

We will show that whenever we have the following (outer) commutative diagram, there exists a unique dotted arrow L which makes the whole diagram commutative in **Perm**:

$$\begin{tikzcd}
A \vee \mathcal{F}(C;V) \arrow[r, "i_{A,V}"] \arrow[d, "G \vee \mathcal{F}(C;V)"'] & C \arrow[d, "P"] \arrow[ddr, bend left, "R"] & \\
B \vee \mathcal{F}(C;V) \arrow[r, "\Gamma_{S^{\mathcal{F}}}"'] \arrow[drr, bend right, "T"'] & G(S^{\mathcal{F}}) \arrow[dr, dashed, "L"] & \\
& & X
\end{tikzcd}$$

Since $Ob(\Gamma_{S^{\mathcal{F}}})$ is the identity, the object homomorphism $Ob(L)$ has to be the same as $Ob(T)$. In order to make the diagram commutative, we define $Ob(L) = Ob(T)$. The morphism function of L is defined as follows:

$$L_{z_1,z_2} := R_{S^{\mathcal{F}}(z_1),S^{\mathcal{F}}(z_2)} : G(S^{\mathcal{F}})(z_1,z_2) = C(S^{\mathcal{F}}(z_1), S^{\mathcal{F}}(z_2)) \to X(L(z_1), L(z_2))$$

for each pair of objects $z_1, z_2 \in Ob(G(S^{\mathcal{F}}))$. This defines a functor L which makes the diagram above commutative (in **Cat**). In order to verify that L is a strict symmetric monoidal functor, it is sufficient to show that for each pair of maps $f_1 : z_1 \to z_2$, $f_2 : z_3 \to z_4$ in $G(S^{\mathcal{F}})$,

$$L(f_1 \boxdot f_2) = L(f_1) \underset{X}{\otimes} L(f_2) = R(f_1) \underset{X}{\otimes} R(f_2). \tag{8}$$

We recall that the map $f_1 \boxdot f_2$ is defined by the following commutative diagram:

$$\begin{tikzcd} S^{\mathcal{F}}(z_1) \underset{C}{\otimes} S^{\mathcal{F}}(z_3) \arrow[r, "f_1 \underset{C}{\otimes} f_2"] & S^{\mathcal{F}}(z_2) \underset{C}{\otimes} S^{\mathcal{F}}(z_4) \\ S^{\mathcal{F}}(z_1 \underset{B}{\otimes} z_2) \arrow[u, "\lambda^{S^{\mathcal{F}}}"] \arrow[r, "f_1 \boxdot f_2"'] & S^{\mathcal{F}}(z_3 \underset{B}{\otimes} z_4) \arrow[u, "\lambda^{S^{\mathcal{F}}}"'] \end{tikzcd}$$

Since R is a strict symmetric monoidal functor, $R(f_1 \underset{C}{\otimes} f_2) = R(f_1) \underset{X}{\otimes} R(f_2)$. Now, it sufficient to show that $R\lambda^{S^{\mathcal{F}}} = id$ in order to establish the equalities in (8). We observe that $\lambda^{S^{\mathcal{F}}} = i_{A,V}(\lambda^{S \vee \mathcal{F}(C;V)})$. Since $G \vee \mathcal{F}(C,V)$ is an acyclic fibration, it follows from Remark 4 that $G \vee \mathcal{F}(C,V)\lambda^{S \vee \mathcal{F}(C;V)} = id$. Since $T \circ G \vee \mathcal{F}(C,V) = R \circ i_{A,V}$, it follows that $R(\lambda^{S^{\mathcal{F}}}) = id$. The uniqueness of the object functor of L is obvious. The uniqueness of the morphism homomorphism of L can be easily checked. □

The main objective of this section is to show that the natural model category **Perm** is left proper. The next lemma serves as a first step in proving the main result. The lemma follows from the above discussion:

Lemma 2. *In the natural model category* **Perm**, *a pushout of a weak-equivalence along a free cofibration is a weak-equivalence.*

Proof. In light of the facts that each weak equivalence in a model category can be factored as an acyclic cofibration followed by an acyclic fibration and acyclic cofibrations are closed under cobase change, it is sufficient to see that the cobase change of an acyclic fibration is a weak-equivalence. This follows from the discussion above. □

Now, we state and prove the main result of this paper:

Theorem 1. *The natural model category of permutative categories* **Perm** *is a left proper model category.*

Proof. We will show that a pushout $P(F;q)$ of a weak equivalence $F : A \to D$ in **Perm** along a cofibration $q : A \to B$ in **Perm** is a weak-equivalence. We consider the following commutative diagram:

$$\begin{tikzcd} & & C \arrow[dr] \arrow[ddd, "P(F;l)"] & \\ A \arrow[urr, bend left, "r"] \arrow[r, "q"] \arrow[d, "F"'] & B \arrow[ur, "l"] \arrow[rr, equal] \arrow[d, "P(F;q)"] & & B \arrow[ddd, "P(F;q)"] \\ D \arrow[r] \arrow[drr, bend right] \arrow[ddrrr, bend right] & P \arrow[dr] & & \\ & & P_s \arrow[dr] & \\ & & & P \end{tikzcd}$$

Since F is a cofibration, by Proposition 1 there exists a free cofibration $r : A \to C$ such that F is a retract of r by a map that fixes A. The top left commutative square in the above diagram is co-Cartesian. The map $P(F;l)$ is a pushout of F along the free cofibration r and therefore a weak-equivalence by Lemma 2. Now, the result follows from the observation that the diagonal composite $P \to P_s \to P$ in the above diagram is the identity map and the commutativity of the above diagram. $\square$

4. Conclusions

In this paper, we identify a class of cofibrations between permutative categories which we call free cofibrations and show that every cofibration of permutative categories is a retract of a free cofibration. We go on to show that the natural model category of permutative categories is a left proper model category.

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Abbreviations

The following abbreviations are used in this manuscript:

Cat	The category of (small) categories and functors
Perm	The category of (small) permutative (or strict symmetric monoidal) categories and strict symmetric monoidal functors.
Mon	The category of monoids and monoid homomorphisms.
Set	The category of sets and functions
SMCat	The category of symmetric monoidal categories and symmetric monoidal functors.

Appendix A. Gabriel Factorization of Symmetric Monoidal Functors

In this appendix, we construct a *Gabriel Factorization* of a unital symmetric monoidal functor between permutative categories. Our construction factors a unital symmetric monoidal functor into an essentially surjective strict symmetric monoidal functor followed by a fully faithful unital symmetric monoidal functor.

Lemma A1. *Each unital symmetric monoidal functor $F : C \to D$ between permutative categories can be factored as follows:*

$$\begin{tikzcd} C \arrow[rr, "F"] \arrow[dr, "\Gamma_F"'] & & D \\ & G(F) \arrow[ur, "\Delta_F"'] & \end{tikzcd}$$

where Γ_F is a strict symmetric monoidal functor which is identity on objects, and Δ is fully faithful.

Proof. We begin by defining the permutative category $G(F)$. The object monoid of $G(F)$ is the same as $Ob(C)$. For a pair of objects $c_1, c_2 \in Ob(C)$, we define

$$G(F)(c_1, c_2) := C(F(c_1), F(c_2)).$$

The Gabriel factorization of the underlying functor of F gives us the following factorization in **Cat**:

$$\begin{tikzcd} C \arrow[rr, "F"] \arrow[dr, "\Gamma_F"'] & & D \\ & G(F) \arrow[ur, "\Delta_F"'] & \end{tikzcd}$$

We will show that the functor Γ_F is strict symmetric monoidal, and Δ_F is unital symmetric monoidal. We define a symmetric monoidal structure on $G(F)$ which we denote by $(G(F), \boxdot, \gamma)$. For any pair of objects $c_1, c_2 \in Ob(G(F))$, we define $c_1 \boxdot c_2 := c_1 \underset{C}{\otimes} c_2$. For a pair of maps $f_1 : c_1 \to c_3$ and $f_2 : c_2 \to c_4$, we define $f_1 \boxdot f_2$ to be the following arrow:

$$\begin{tikzcd}
F(c_1 \underset{C}{\otimes} c_2) \arrow[r, "f_1 \boxdot f_2"] \arrow[d, "\lambda_F"', "\cong"] & F(c_3 \underset{C}{\otimes} c_4) \arrow[d, "\cong"', "\lambda_F"] \\
F(c_1) \underset{D}{\otimes} F(c_2) \arrow[r, "f_1 \underset{D}{\otimes} f_2"'] & F(c_3) \underset{D}{\otimes} F(c_4)
\end{tikzcd}$$

It is easy to establish that $- \boxdot -$ is a bifunctor. Let $f_3 : c_3 \to c_5$ and $f_4 : c_4 \to c_6$ be another pair of arrows in $G(F)$. Now, we consider the following commutative diagram:

$$\begin{tikzcd}
F(c_1 \underset{C}{\otimes} c_2) \arrow[r, "f_1 \boxdot f_2"] \arrow[d, "\lambda_F"', "\cong"] & F(c_3 \underset{C}{\otimes} c_4) \arrow[r, "f_3 \boxdot f_4"] \arrow[d, "\cong"', "\lambda_F"] & F(c_5 \underset{C}{\otimes} c_6) \arrow[d, "\cong"', "\lambda_F"] \\
F(c_1) \underset{D}{\otimes} F(c_2) \arrow[r, "f_1 \underset{D}{\otimes} f_2"'] & F(c_3) \underset{D}{\otimes} F(c_4) \arrow[r, "f_3 \underset{D}{\otimes} f_4"'] & F(c_5) \underset{D}{\otimes} F(c_6)
\end{tikzcd}$$

The above diagram tells us that:

$$(f_3 \boxdot f_4) \circ (f_1 \boxdot f_2) = (f_3 \circ f_1) \boxdot (f_4 \circ f_2)$$

because the composite map in the bottom row of the above diagram, namely $(f_3 \underset{D}{\otimes} f_4) \circ (f_1 \underset{D}{\otimes} f_2)$ is the same as $(f_3 \circ f_1) \underset{D}{\otimes} (f_4 \circ f_2)$. The tensor product $- \boxdot -$ on $G(F)$ is strictly associative because the object set of $G(F)$ is a monoid, and the tensor product of morphisms is associative because the tensor product of morphisms in $G(F)$ is inherited from that in D which is strictly associative. The symmetry natural transformation γ is defined on objects as follows:

$$\gamma_{c_1, c_2} := F(\gamma^C_{c_1, c_2}).$$

Let $f_1 : c_1 \to c_3$ and $f_2 : c_2 \to c_4$ be a pair of maps in $G(F)$. The following commutative diagram shows us that γ is a natural isomorphism:

$$\begin{tikzcd}
F(c_1 \underset{C}{\otimes} c_2) \arrow[r, "\lambda_F"'] \arrow[rrr, bend left, "\gamma_{c_1, c_2}"] \arrow[d, "f_1 \boxtimes f_2"'] & F(c_1) \underset{D}{\otimes} F(c_2) \arrow[r, "\gamma^D_{F(c_1), F(c_2)}"'] \arrow[d, "f_1 \underset{D}{\otimes} f_2"'] & F(c_3) \underset{D}{\otimes} F(c_4) \arrow[r, "\lambda_F"'] \arrow[d, "f_2 \underset{D}{\otimes} f_1"] & F(c_2 \underset{C}{\otimes} c_1) \arrow[d, "f_2 \boxtimes f_1"] \\
F(c_3 \underset{C}{\otimes} c_4) \arrow[r, "\lambda_F"] \arrow[rrr, bend right, "\gamma_{c_3, c_4}"'] & F(c_3) \underset{D}{\otimes} F(c_4) \arrow[r, "\gamma^D_{F(c_3), F(c_4)}"] & F(c_4) \underset{D}{\otimes} F(c_3) \arrow[r, "\lambda_F"] & F(c_4 \underset{C}{\otimes} c_3)
\end{tikzcd}$$

which shows that γ is a natural transformation. The following equalities verify the symmetry condition:

$$\gamma_{c_1, c_2} \circ \gamma_{c_2, c_1} = F(\gamma^C_{c_1, c_2}) \circ F(\gamma^C_{c_2, c_1}) = F(\gamma^C_{c_1, c_2} \circ \gamma^C_{c_2, c_1}) = id.$$

This defines a permutative category structure on the category $G(F)$. Using the definition of the symmetric monoidal structure on $G(F)$, one can easily check that Γ_F is a strict symmetric monoidal functor. $\square$

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