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Two Convergence Results for Inexact Infinite Products of Non-Expansive Mappings

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Abstract: We analyze the asymptotic behavior of infinite products of non-linear operators which take a non-empty, closed subset of a complete metric space into the space, taking into account summable computational errors. Our results can be applied in methods for solving convex feasibility and optimization problems.

Keywords: complete metric space; convergence analysis; inexact iteration; infinite product; non-expansive mapping

MSC: 47H09; 47H10; 54E35

Citation: Zaslavski, A.J. Two Convergence Results for Inexact Infinite Products of Non-Expansive Mappings. *Axioms* **2023**, *12*, 88. https://doi.org/10.3390/axioms12010088

Academic Editor: Behzad Djafari-Rouhani

Received: 2 December 2022
Revised: 6 January 2023
Accepted: 12 January 2023
Published: 14 January 2023

1. Introduction

The fixed-point theory of non-linear operators has been a rapidly growing area of research [1–19]. The starting point of this theory is Banach's classical result [20] on the existence of a unique fixed point for a strict contraction. Since that seminal paper, many developments have taken place in this field [18,19,21–30].

In our joint paper with D. Butnariu and S. Reich [3], it was established that if every sequence of iterates of a non-expansive operator converges, then this convergence property also takes place for every sequence of inexact iterates under the presence of summable errors. In our subsequent joint paper with D. Butnariu and S. Reich [31], this result was extended for inexact infinite products of non-expansive self-mappings of a complete metric space. Here, we analyze the convergence of inexact infinite products of non-expansive operators which take a non-empty, closed subset K of a complete metric space into the space, taking into account summable computational errors and obtaining a generalization of the result of [31] mentioned above. Namely, we show that for each pair of sequence of points $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ generated by our inexact infinite products which belong to the subset K, the distance between x_i and y_i tends to zero as $i \to \infty$.

2. Preliminaries

Suppose that (X, ρ) is a complete metric space equipped with a metric ρ. For an arbitrary element $\eta \in X$ and an arbitrary set $C \subset X$, put

$$\rho(\eta, C) = \inf\{\rho(\eta, \xi) : \ \xi \in C\}.$$

For any $\eta \in X$ and any $\gamma \in (0, \infty)$ put

$$B(\eta, \gamma) = \{\xi \in X : \ \rho(\eta, \xi) \le \gamma\}.$$

For any operator $S : X \to X$, let $S^0 y = y$ for every point $y \in X$.

In our joint paper with D. Butnariu and S. Reich [3], we investigated the influence of computational errors on the asymptotic behavior of iterates of non-expansive operators in complete metric spaces and established the following theorem (see also Theorem 2.72 of [16]).

Theorem 1. *Assume that $A : X \to X$ satisfies*

$$\rho(A\xi, A\eta) \leq \rho(\xi, \eta) \text{ every pair of points } \xi, \eta \in X,$$

$F(A)$ is the collection of all fixed points of the operator A and for every point $\xi \in X$, the sequence of iterates $\{A^n\xi\}_{n=1}^{\infty}$ converges (X, ρ).

Assume that $\{r_n\}_{n=0}^{\infty} \subset (0, \infty)$ satisfies

$$\sum_{n=0}^{\infty} r_n < \infty$$

and that a sequence of inexact iterates $\{x_n\}_{n=0}^{\infty} \subset X$ for every non-negative integer n satisfies

$$\rho(x_{n+1}, Ax_n) \leq r_n.$$

Then, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point of $F(A)$.

Theorem 1 has important applications and is an essential ingredient in the superiorization and perturbation resilience of algorithms [21–23,25,26]. The superiorization methodology works by analyzing the perturbation resilience of an iterative algorithm, and then applying proactively such perturbations in order to make the perturbed algorithm perform something useful in an addition to its original task. This methodology is illustrated by the next discussion.

Assume that $(X, \|\cdot\|)$ is a Banach space equipped with the norm $\|\cdot\|$, $\rho(\xi, \eta) = \|\xi - \eta\|$ for all $\xi, \eta \in X$, an operator $A : X \to X$ satisfies

$$\|A(\xi) - A(\eta)\| \leq \|\xi - \eta\|, \ \xi, \eta \in X$$

and that for any point $\eta \in X$, the sequence $\{A^n\eta\}_{n=1}^{\infty}$ converges in the norm topology, $\xi_0 \in X$, $\{\alpha_t\}_{t=0}^{\infty} \subset (0, \infty)$ satisfies

$$\sum_{t=0}^{\infty} \alpha_t < \infty,$$

$\{u_t\}_{t=0}^{\infty} \subset X$ satisfies

$$\sup\{\|u_t\| : \ t = 0, 1, \ldots\} < \infty$$

and that for every non-negative integer $t \geq 0$,

$$\xi_{t+1} = A(\xi_t + \alpha_t u_t).$$

Theorem 1 implies that $\{\xi_k\}_{k=0}^{\infty}$ converges and its limit ξ satisfies $A(\xi) = \xi$. In this case, the mapping A is called bounded perturbations resilient [22].

Now, assume that $\xi_0 \in X$ and the summable sequence of positive numbers $\{\alpha_t\}_{t=0}^{\infty}$ are given. We construct a sequence of iterates $\{\xi_t\}_{t=1}^{\infty}$ determined by the equation above. Under an appropriate choice of $\{u_t\}_{t=0}^{\infty}$, the sequence of inexact iterates $\{\xi_t\}_{t=1}^{\infty}$ has some useful properties. Namely, the sequence $\{f(\xi_t)\}_{t=1}^{\infty}$ can be decreasing, where f is a given objective function.

In our joint paper with D. Butnariu and S. Reich [31], we extended Theorem 1 for inexact infinite products of non-expansive self-mappings of a complete metric space. In the present paper, we investigate the convergence of inexact infinite products of non-expansive mappings which take a non-empty, closed subset K of a complete metric space into the space and obtain a generalization of the result of the work [31]. Namely, we show that for each pair of sequence of points $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ generated by our inexact infinite products which belong to the subset K, the distance between x_i and y_i tends to zero as $i \to \infty$.

The most important and well-known application of the results obtained in [3,31] and here is the convex feasibility problem: to find a common point of a family of convex, closed subsets C_i, $i = 1, \dots, m$ of a Hilbert space. The convex feasibility problems arises in radiation planning and computer tomography. In order to solve this problem, one usually uses infinite products of projections on the sets C_i, $i = 1, \dots, m$ or more advanced dynamic string-averaging projection methods [18,19,25]. Our results, as well as the results of [3,31], explain stability effects arising in numerical experiments under the presence of small computational errors [21].

3. A Convergence Result in a Metric Space

Assume that K is a non-empty, closed set in a complete metric space (X, ρ) equipped with the metric ρ. Denote by $\mathcal{A}$ the collection of all operators $S : K \to X$ for which

$$\rho(S(\eta), S(\xi)) \le \rho(\eta, \xi),\ \eta, \xi \in K. \tag{1}$$

Assume that $\mathcal{R}$ is a collection of maps $T : \{1, 2, \dots, \} \to \mathcal{A}$ which have the following two properties:

(a) For every map $T \in \mathcal{R}$ and every natural number s the map $\tilde{T}(t) = T(t+s)$, $t \in \{1, 2, \dots\}$ belongs to $\mathcal{R}$;

(b) For any map $T \in \mathcal{R}$ and every pair $\{\xi_t\}_{t=0}^{\infty}, \{\eta_t\}_{t=0}^{\infty} \subset K$ for which

$$\xi_{t+1} = T(t+1)(\xi_t),\ \eta_{t+1} = T(t+1)(\eta_t),\ t = 0, 1, \dots$$

the equation

$$\lim_{t \to \infty} \rho(\xi_t, \eta_t) = 0$$

is true.

We will prove the following result.

Theorem 2. *Assume that $T \in \mathcal{R}$, $\Delta > 0$, $\{\Delta_i\}_{i=1}^{\infty} \subset (0, \infty)$ satisfies*

$$\sum_{i=1}^{\infty} \Delta_i < \infty \tag{2}$$

and that $\{x_t\}_{t=0}^{\infty}$, $\{y_t\}_{t=0}^{\infty} \subset K$ satisfy for every non-negative integer t,

$$\rho(x_{t+1}, T(t+1)(x_t)) \le \Delta_{t+1},\ \rho(y_{t+1}, T(t+1)(y_t)) \le \Delta_{t+1}, \tag{3}$$

and

$$B(x_t, \Delta),\ B(y_t, \Delta) \subset K. \tag{4}$$

Then,

$$\lim_{t \to \infty} \rho(x_t, y_t) = 0.$$

4. Proof of Theorem 2

We may assume without loss of generality that

$$\Delta < 1.$$

Let

$$\epsilon \in (0, \Delta). \tag{5}$$

In view of Equation (2), there is an integer $n_0 \ge 1$ for which

$$\sum_{j=n_0}^{\infty} \Delta_j < \epsilon/9. \tag{6}$$

Set

$$\tilde{x}_{n_0} = x_{n_0} \tag{7}$$

and

$$\tilde{x}_{n_0+1} = T(n_0+1)(\tilde{x}_{n_0}). \tag{8}$$

By (3), (7) and (8),

$$\rho(\tilde{x}_{n_0+1}, x_{n_0+1}) = \rho(x_{n_0+1}, T(n_0+1)(x_{n_0})) \le \Delta_{n_0+1}. \tag{9}$$

Equations (4), (6) and (9) imply that

$$\tilde{x}_{n_0+1} \in K.$$

Therefore, we can define

$$\tilde{x}_{n_0+2} = T(n_0+2)(\tilde{x}_{n_0+1}).$$

By induction, we define iterates $\tilde{x}_j$ for all natural numbers $j > n_0$. If $j > n_0$ is an integer and $\tilde{x}_j \in K$ was defined, then we set

$$\tilde{x}_{j+1} = T(j+1)(\tilde{x}_j). \tag{10}$$

Assume that $m > n_0$ is an integer and that $\tilde{x}_i \in K$, $i = n_0, \dots, m$ are defined and that for each $i \in \{n_0+1, \dots, m\}$,

$$\rho(\tilde{x}_i, x_i) \le \sum_{j=n_0+1}^{i} \Delta_j. \tag{11}$$

(Clearly, by Equation (9), our assumption is true for $m = n_0 + 1$.) Equations (5), (6) and (11) imply that

$$\rho(x_m, \tilde{x}_m) \le \sum_{j=n_0+1}^{\infty} \Delta_j < \epsilon/8 < \Delta/4. \tag{12}$$

By Equations (4) and (12), we have

$$\tilde{x}_m \in K$$

and then

$$\tilde{x}_{m+1} = T(m+1)\tilde{x}_m$$

is defined.

Equations (1), (3) and (11) imply that

$$\begin{aligned} \rho(\tilde{x}_{m+1}, x_{m+1}) &\le \rho(T(m+1)(\tilde{x}_m), T(m+1)(x_m)) + \rho(T(m+1)(x_m), x_{m+1}) \\ &\le \rho(\tilde{x}_m, x_m) + \Delta_{m+1} \\ &\le \textstyle\sum_{j=n_0+1}^{m} \Delta_j + \Delta_{m+1} = \sum_{j=n_0+1}^{m+1} \Delta_j. \end{aligned} \tag{13}$$

In view of (13), Equation (11) is true for $i = m+1$. By (4)–(6) and (13),

$$\rho(\tilde{x}_{m+1}, x_{m+1}) < \epsilon/8 < \Delta/8.$$

and

$$\tilde{x}_{m+1} \in K.$$

Thus, the assumption which was made for m is true for $m+1$ as well. By induction, we showed that $\tilde{x}_i \in K$ is defined for all integers $i \ge n_0$ and (11) is true for all integers $i \ge n_0 + 1$. Set

$$\tilde{y}_{n_0} = y_{n_0}$$

and if an integer $i \ge n_0$ and $\tilde{y}_i \in K$ is defined, then set

$$\tilde{y}_{i+1} = T(i+1)(\tilde{y}_i).$$

Arguing as before, we can show that for any natural number $i \geq n_0$, $\tilde{y}_i \in K$ is defined and that

$$\rho(\tilde{y}_i, y_i) \leq \sum_{j=n_0+1}^{i} \Delta_j. \tag{14}$$

Properties (a) and (b) imply that

$$\lim_{i \to \infty} \rho(\tilde{x}_i, \tilde{y}_i) = 0 \tag{15}$$

By Equation (15), there is a natural number $n_1 \geq n_0$ such that for any natural number $i \geq n_1$, we have

$$\rho(\tilde{x}_i, \tilde{y}_i) \leq \epsilon/4. \tag{16}$$

Equations (8), (11), (14) and (16) imply that for any natural number $i \geq n_1$,

$$\rho(x_i, y_i) \leq \rho(x_i, \tilde{x}_i) + \rho(\tilde{x}_i, \tilde{y}_i) + \rho(\tilde{y}_i, y_i)$$

$$\leq 2 \sum_{j=n_0+1}^{i} \Delta_j + \epsilon/4 \leq \epsilon/8 + \epsilon/8 + \epsilon/4.$$

Theorem 2 is proved.

5. A Weak Convergence Result

Assume that K is a non-empty, closed set in a Banach space $(E, \|\cdot\|)$ equipped with the norm $\|\cdot\|$ with a dual space $(E^*, \|\cdot\|_*)$. For each $\xi, \eta \in E$, put $\rho(\xi, \eta) = \|\xi - \eta\|$. Denote by $\mathcal{A}$ the collection of all maps $S : K \to E$, for which

$$\|S(\eta) - A(\xi)\| \leq \|\eta - \xi\|, \ \eta, \xi \in K. \tag{17}$$

Assume that $\mathcal{R}$ is a collection of maps $T : \{1, 2, \ldots, \infty\} \to \mathcal{A}$ which have the following two properties:

(a) For every map $T \in \mathcal{R}$ and every natural number s, the map $\tilde{T}(t) = T(t+s)$, $t \in \{1, 2, \ldots\}$ belongs to $\mathcal{R}$;

(b) For any map $T \in \mathcal{R}$ and each $\{x_t\}_{t=0}^{\infty}, \{y_t\}_{t=0}^{\infty} \subset K$ which satisfies

$$x_{t+1} = T(t+1)(x_t), \ y_{t+1} = T(t+1)(y_t), \ t = 0, 1, \ldots,$$

the sequence $\{x_t - y_t\}_{t=0}^{\infty}$ converges weakly in X to the zero.

We will prove the following result.

Theorem 3. *Assume that $T \in \mathcal{R}$, $\Delta > 0$, $\{\Delta_j\}_{j=1}^{\infty} \subset (0, \infty)$ satisfies*

$$\sum_{j=1}^{\infty} \Delta_j < \infty \tag{18}$$

and that $\{x_t\}_{t=0}^{\infty}$, $\{y_t\}_{t=0}^{\infty} \subset K$ satisfy for every non-negative integer t,

$$\|x_{t+1} - T(t+1)(x_t)\| \leq \Delta_{t+1}, \ \|y_{t+1} - T(t+1)(y_t)\| \leq \Delta_{t+1}, \tag{19}$$

and

$$B(x_t, \Delta), \ B(y_t, \Delta) \subset K. \tag{20}$$

Then the sequence $\{x_t - y_t\}_{t=0}^{\infty}$ converges weakly in X to the zero.

6. Proof of Theorem 3

We may assume without loss of generality that

$$\Delta < 1.$$

Let $f \in E^*$ satisfy

$$\|f\|_* \leq 1, \; \epsilon \in (0, \Delta). \tag{21}$$

In order to prove the theorem, it is sufficient to show that

$$\lim_{i\to\infty} f(y_i - x_i) = 0.$$

By (18), there is $n_0 \in \{1, 2, \dots\}$, for which

$$\sum_{i=n_0}^{\infty} \Delta_i < \epsilon/8. \tag{22}$$

Set

$$\tilde{x}_{n_0} = x_{n_0} \tag{23}$$

and

$$\tilde{x}_{n_0+1} = T(n_0+1)(\tilde{x}_{n_0}). \tag{24}$$

By (19)–(24),

$$\|\tilde{x}_{n_0+1} - x_{n_0+1}\| \leq \Delta_{n_0+1}, \; \tilde{x}_{n_0+1} \in K. \tag{25}$$

By induction, we define $\tilde{x}_t \in K$ for every natural number $t > n_0$. If $i > n_0$ is an integer and $\tilde{x}_i \in K$ was defined, then we set

$$\tilde{x}_{i+1} = T(i+1)(\tilde{x}_i). \tag{26}$$

Assume that $m > n_0$ is an integer and that $\tilde{x}_i \in K$; $i = n_0, \dots, m$ are defined using (26) and for each $i \in \{n_0+1, \dots, m\}$,

$$\|\tilde{x}_i - x_i\| \leq \sum_{j=n_0+1}^{i} \Delta_j. \tag{27}$$

(It should be mentioned that by (25) our assumption is valid for $m = n_0 + 1$.) By (27), we have

$$\|x_m - \tilde{x}_m\| \leq \sum_{j=n_0+1}^{m} \Delta_j. \tag{28}$$

Set

$$\tilde{x}_{m+1} = T(m+1)(\tilde{x}_m).$$

Equations (17), (19), (28) and (29) imply that

$$\begin{aligned} \|\tilde{x}_{m+1} - x_{m+1}\| &\leq \|T(m+1)(\tilde{x}_m) - T(m+1)(x_m)\| + \|T(m+1)(x_m) - x_{m+1}\| \\ &\leq \|\tilde{x}_m - x_m\| + \Delta_{m+1} \\ &\leq \textstyle\sum_{j=n_0+1}^{m} \Delta_j + \Delta_{m+1} = \sum_{j=n_0+1}^{m+1} \Delta_j. \end{aligned} \tag{29}$$

In view of (29), Equation (27) is true for $i = m + 1$. By (20)–(22) and (29),

$$\|\tilde{x}_{m+1} - x_{m+1}\| < \epsilon/8 < \Delta/8.$$

and

$$\tilde{x}_{m+1} \in K.$$

Thus, the assumption which was made for m is true for $m+1$ as well. By induction, we showed that $\tilde{x}_i \in K$ is defined for all integers $i \geq n_0$ by (26) and (27) holds for all integers $i \geq n_0 + 1$. Set

$$\tilde{y}_{n_0} = y_{n_0} \tag{30}$$

and if an integer $i \geq n_0$ and $\tilde{y}_i \in K$ is defined, then set

$$\tilde{y}_{i+1} = T(i+1)(\tilde{y}_i). \tag{31}$$

Arguing as before, we can show that for every integer $i \geq n_0 + 1$, $\tilde{y}_i \in K$ is defined and that

$$\|\tilde{y}_i - y_i\| \leq \sum_{j=n_0+1}^{i} \Delta_j. \tag{32}$$

Properties (a) and (b) and Equations (23), (26), (30) and (31) imply that

$$\tilde{x}_i - \tilde{y}_i \to 0 \text{ weakly in } E \text{ as } i \to \infty. \tag{33}$$

In order to complete the proof of our result, it is sufficient to show that the inequality

$$|f(x_i - y_i)| < \epsilon$$

is true for all sufficiently large natural numbers $i \geq 0$. By (33),

$$\lim_{i \to \infty} f(\tilde{y}_i - \tilde{x}_i) = 0.$$

Thus, there is a natural number $n_1 > n_0$ such that for every natural number $i \geq n_1$,

$$|f(\tilde{x}_i - \tilde{y}_i)| \leq \epsilon/8. \tag{34}$$

Following Equations (22), (27), (32) and (34), for every natural number $i \geq n_1$,

$$\begin{aligned}
|f(x_i - y_i)| &\leq |f(x_i - \tilde{x}_i)| + |f(\tilde{x}_i - \tilde{y}_i)| + |f(\tilde{y}_i - y_i)| \\
&\leq \|f\|_* \|x_i - \tilde{x}_i\| + \epsilon/8 + \|f\|_* \|y_i - \tilde{y}_i\| \\
&\leq 2 \sum_{j=n_0+1}^{\infty} \Delta_j + \epsilon/8 < \epsilon.
\end{aligned}$$

Theorem 3 is proved.

7. Conclusions

We analyze the asymptotic behavior of infinite products of non-linear operators which take a non-empty, closed subset K of a complete metric space into the space, taking into account summable computational errors and obtaining a generalization of the result of [31]. More precisely, we show that for each pair of sequence of points $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ generated by our inexact infinite products which belong to the subset K, the distance between x_i and y_i tends to zero as $i \to \infty$. The most important and well-known application of the results obtained in [3,31] and here is the convex feasibility problem: to find a common point of a family of convex, closed subsets C_i, $i = 1, \ldots, m$ of a Hilbert space. The convex feasibility problems arises in radiation planning and computer tomography. In order to solve this problem, one usually uses infinite products of projections on the sets C_i, $i = 1, \ldots, m$ or more advanced dynamic string-averaging projection methods [18,19,25]. Our results as well as the results of [3,31] explain stability effects arising in numerical experiments under the presence of small computational errors [21].

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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