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Best Constant in Ulam Stability for the Third Order Linear Differential Operator with Constant Coefficients

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Abstract: The authors of the present paper previously proved the Ulam stability for the n-th-order linear differential operator with constant coefficients. They obtained its best Ulam constant for the case of distinct roots of the characteristic equation. However, a complete answer to the problem of the best Ulam constant was later obtained only for the second-order linear differential operator. This paper deals with the Ulam stability of the third-order linear differential operator with constant coefficients acting in a Banach space. The paper's main purpose is to obtain the best Ulam constant of this operator, thus completing the previous research in the field.

Keywords: linear differential operator; Ulam stability; best constant; Banach space

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1. Introduction

Let $\mathbb{K}$ be the field of real or complex numbers. Throughout this paper, $(X, \|\cdot\|)$ denotes a Banach space over the field $\mathbb{C}$ while $\mathcal{C}^n(\mathbb{R}, X)$ denotes the linear space of all n times differentiable functions with continuous n-th derivatives, defined on $\mathbb{R}$ with values in X.

Let A and B be two linear spaces over the field $\mathbb{K}$.

Definition 1. *The function $\rho_A : A \to [0, \infty]$ satisfying the following properties:*

(i) $\rho_A(x) = 0$ *if and only if* $x = 0$;

(ii) $\rho_A(\lambda x) = |\lambda|\rho_A(x)$ *for all* $x \in A, \lambda \in \mathbb{K}, \lambda \neq 0$,

is called a gauge *on* A.

For the function $f \in \mathcal{C}^n(\mathbb{R}, X)$, we define

$$\|f\|_\infty = \sup\{\|f(t)\| : t \in \mathbb{R}\}. \tag{1}$$

Then, $\|f\|_\infty$ is a gauge on $\mathcal{C}^n(\mathbb{R}, X)$. We suppose that both linear spaces $\mathcal{C}^n(\mathbb{R}, X)$ and $\mathcal{C}(\mathbb{R}, X)$ are endowed with the same gauge $\|\cdot\|_\infty$.

Let ρ_A, ρ_B be two gauges on the linear spaces A and B, respectively, and let $D : A \to B$ be a linear operator.

We denote by $\ker D = \{x \in A | Dx = 0\}$ the kernel of D and by $R(D) = \{Dx | x \in A\}$ the range of the operator D, respectively.

Definition 2. *We say that the operator D is Ulam-stable if there exists $K \geq 0$ such that, for every $\varepsilon > 0$ and every $x \in A$ with the property that $\rho_B(Dx) \leq \varepsilon$, there exists $z \in \ker D$ such that $\rho_A(x - z) \leq K\varepsilon$.*

The number K in the above definition is called an *Ulam constant* of the operator D. Further, we denote by K_D the infimum of all Ulam constants of the operator D. In general, the infimum of all Ulam constants of the operator D is not necessarily an Ulam constant of D (see [1]). However, for the case where K_D is also an Ulam constant of the operator D, we will call it *the best Ulam constant* of D.

The stability problem was initially raised by Ulam [2] in the fall of 1940 and partially answered a year later by Hyers [3], and it has developed ever since, growing as such into a vast area of research. Nowadays, Ulam stability follows various directions of research, from the stability of operators to the stability of different types of equations. For a complete approach to this topic, we refer the reader to [4,5].

For the sake of completeness, we will first present a brief historical background of the problem of finding the best Ulam constant of differential equations and operators. Consequently, we will mention here only some results in the field connected with the stability of operators that also serve the purpose of the present paper. As far as we know, the first Ulam stability result for differential equations was obtained by M.Obłoza in [6]. Hereafter, the topic was deeply investigated by many mathematicians. We can mention here T. Miura, S. Miyajima, S.E. Takahasi [7–9], and S. M Jung [10], who obtained stability results for various differential equations and partial differential equations. Representations of the best Ulam constants of linear and bounded operators acting on normed spaces were given in [1,11].

The study of Ulam stability has also been developed for the higher-order differential operators with constant coefficients. In [12,13], sharp estimates for the Ulam constant of the first-order and the higher-order linear differential operators were given. Later, the work was improved, and the best Ulam constant was obtained for the case of the first-order linear differential operator in [7]. Shortly after, in [14], A.R. Baias and D. Popa extended the study of Ulam stability to the case of the second-order linear differential operator with constant coefficients $D(y) = y'' + ay' + by, y \in \mathcal{C}^2(\mathbb{R}, X), a, b \in \mathbb{C}$ and obtained its best Ulam constant as

$$K_D = \begin{cases} \frac{1}{|p-q|} \int_0^{\infty} |e^{-pv} - e^{-qv}| dv, & \text{if } \operatorname{Re} p > 0, \operatorname{Re} q > 0, p \neq q \\ \frac{1}{|p-q|} \int_{-\infty}^{0} |e^{-pv} - e^{-qv}| dv, & \text{if } \operatorname{Re} p < 0, \operatorname{Re} q < 0, p \neq q \\ \frac{1}{|p-q|} \left| \frac{1}{\operatorname{Re} p} - \frac{1}{\operatorname{Re} q} \right|, & \text{if } \operatorname{Re} p \cdot \operatorname{Re} q < 0 \\ \frac{1}{(\operatorname{Re} p)^2}, & \text{if } p = q, \end{cases}$$

where p and q are the characteristic roots of the equation; namely, $p = \frac{-a+\sqrt{a^2-4b}}{2}$ and $q = \frac{-a-\sqrt{a^2-4b}}{2}$. The stability of the second-order linear differential equations with variable coefficients was treated by M. Onitsuka in [15]. For linear differential equations with periodic coefficients, we can mention the stability results obtained in [16], while for second-order linear dynamic equations on time scales, we refer the reader to the paper by D.R. Anderson and M. Onitsuka [17]. The Ulam stability of some integro-differential equations was studied in [18,19].

In [8], it was proved that the n-order linear differential operator with constant coefficients is Ulam stable if and only if its characteristic equation has no pure imaginary roots. In this case, the Ulam constant is given by $\frac{1}{\prod_{k=1}^{n} |\operatorname{Re} r_k|}$, where r_k denote the roots of the characteristic equation. Important steps in finding the best Ulam constant of the same operator were made in [20], where the best Ulam constant was obtained only for the case of distinct roots in the characteristic equation.

The results in the next section streamline those given by [14] and extend them from the case of the second-order differential operator to the case of the third-order linear differential operator. In this paper, we first obtain a stability result for the third-order linear differential operator acting in a Banach space. However, the main purpose of the present paper is to give a complete answer to the problem of the best Ulam constant for this operator by obtaining an expression of the best Ulam constant in all possible cases. This research was motivated by the fact that the best Ulam constant of an equation or operator offers the best measure of the error between the approximate and the exact solution of the corresponding equation or operator.

2. Main Results

Let $a, b, c \in \mathbb{C}$ and let $D : \mathcal{C}^3(\mathbb{R}, X) \to \mathcal{C}(\mathbb{R}, X)$ be defined by

$$D(y) = y''' + ay'' + by' + cy, \quad y \in \mathcal{C}^3(\mathbb{R}, X). \tag{2}$$

If p, q, and r are the roots of the characteristic polynomial $P(z) = z^3 + az^2 + bz + c$, then, as is well known, the kernel of D takes one of the below forms, depending on the order of multiplicity of the roots of the characteristic equation:

$$\ker D = \{C_1 e^{px} + C_2 e^{qx} + C_3 e^{rx} | C_1, C_2, C_3 \in X\}, \tag{3}$$

for the case of distinct roots;

$$\ker D = \{(C_1 + C_2 x)e^{px} + C_3 e^{rx} | C_1, C_2, C_3 \in X\}, \tag{4}$$

for the case $p = q \neq r$;

$$\ker D = \{(C_1 + C_2 x + C_3 x^2)e^{px} | C_1, C_2, C_3 \in X\} \tag{5}$$

for the case $p = q = r$, respectively.

The operator D is surjective; so, for every $f \in \mathcal{C}(\mathbb{R}, X)$, one can find a particular solution of the equation $D(y) = f$, using, for example, the method of variation of constants. Next, we will determine the form of the particular solutions, taking into account the order of multiplicity of the roots of the characteristic equation.

For the case of **distinct roots** in the characteristic equation, the form of a particular solution to the equation $Dy = f$ is

$$y_p(x) = C_1(x)e^{px} + C_2(x)e^{qx} + C_3(x)e^{rx}, \quad x \in \mathbb{R},$$

where C_1, C_2, and C_3 are functions of class $\mathcal{C}^1(\mathbb{R}, X)$ that satisfy

$$\begin{pmatrix} e^{px} & e^{qx} & e^{rx} \\ pe^{px} & qe^{qx} & re^{rx} \\ p^2 e^{px} & q^2 e^{qx} & r^2 e^{rx} \end{pmatrix} \begin{pmatrix} C_1'(x) \\ C_2'(x) \\ C_3'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f(x) \end{pmatrix}, \quad x \in \mathbb{R}. \tag{6}$$

Consequently, we obtain

$$C_1'(x) = \frac{f(x)}{(r-p)(q-p)} e^{-px}, \quad C_2'(x) = \frac{f(x)}{(r-q)(p-q)} e^{-qx} \quad C_3'(x) = \frac{f(x)}{(r-q)(r-p)} e^{-rx}.$$

For the sake of simplicity, we denote this by $V = (r-p)(r-q)(q-p)$; hence, a particular solution of the equation $D(y) = f$ is given by

$$y_p(x) = \frac{1}{V} \int_0^x \Big((r-q)e^{p(x-t)} - (r-p)e^{q(x-t)} + (q-p)e^{r(x-t)} \Big) f(t)dt, \quad x \in \mathbb{R}. \tag{7}$$

Analogously, for the case $p = q \neq r$, a particular solution is given by

$$y_p(x) = \frac{1}{(r-p)^2} \int_0^x \Big(((p-r)(x-t) - 1)e^{p(x-t)} + e^{r(x-t)} \Big) f(t)dt, \quad x \in \mathbb{R}, \tag{8}$$

while, for the case $p = q = r$, the form of the particular solution is

$$y_p(x) = \frac{1}{2}\int_0^x (t-x)^2 f(t) e^{p(x-t)} dt, \quad x \in \mathbb{R}. \tag{9}$$

The main results concerning the Ulam stability of the operator D are given in the next theorems.

Theorem 1. *Suppose that* p, q, *and* r *are* **distinct roots** *of the characteristic equation with* $\operatorname{Re} p \neq 0$, $\operatorname{Re} q \neq 0$, *and* $\operatorname{Re} r \neq 0$, *and let* $\varepsilon > 0$. *Thenm for every* $y \in \mathcal{C}^3(\mathbb{R}, X)$ *satisfying*

$$\|D(y)\|_\infty \leq \varepsilon \tag{10}$$

there exists a unique $y_0 \in \ker D$ *such that*

$$\|y - y_0\|_\infty \leq K\varepsilon \tag{11}$$

where

$$K = \begin{cases} \frac{1}{|V|}\int_0^\infty |(r-q)e^{-pu} + (p-r)e^{-qu} + (q-p)e^{-ru}|du, & \text{if } \operatorname{Re} p > 0, \operatorname{Re} q > 0, \operatorname{Re} r > 0; \\ \frac{1}{|V|}\left(\int_0^\infty |(r-q)e^{-pu} + (p-r)e^{-qu}|du + \int_0^\infty |(q-p)e^{ru}|du\right), & \text{if } \operatorname{Re} p > 0, \operatorname{Re} q > 0, \operatorname{Re} r < 0; \\ \frac{1}{|V|}\left(\int_0^\infty |(r-q)e^{-pu}|du + \int_0^\infty |(p-r)e^{qu} + (q-p)e^{ru}|du\right), & \text{if } \operatorname{Re} p > 0, \operatorname{Re} q < 0, \operatorname{Re} r < 0; \\ \frac{1}{|V|}\int_0^\infty |(r-q)e^{pu} + (p-r)e^{qu} + (q-p)e^{ru}|du, & \text{if } \operatorname{Re} p < 0, \operatorname{Re} q < 0, \operatorname{Re} r < 0, \end{cases} \tag{12}$$

Proof. Existence.

Suppose that $y \in \mathcal{C}^3(\mathbb{R}, X)$ satisfies Equation (10) and let $D(y) = f$. Then, $\|f\|_\infty \leq \varepsilon$ and

$$y(x) = C_1 e^{px} + C_2 e^{qx} + C_3 e^{rx} + y_p(x),\ x \in \mathbb{R},$$

for some $C_1, C_2\, C_3 \in X$, where $y_p(x)$ is a particular solution of the equation $D(y) = f$ given by Equation (7).

(i) Let $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$, and $\operatorname{Re} r > 0$. Define y_0 by the relation

$$y_0(x) = \widetilde{C_1} e^{px} + \widetilde{C_2} e^{qx} + \widetilde{C_3} e^{rx}, \quad x \in \mathbb{R},$$

where

$$\begin{aligned} \widetilde{C_1} &= C_1 + \frac{r-q}{V}\int_0^\infty f(t)e^{-pt}dt; \\ \widetilde{C_2} &= C_2 + \frac{p-r}{V}\int_0^\infty f(t)e^{-qt}dt; \\ \widetilde{C_3} &= C_3 + \frac{q-p}{V}\int_0^\infty f(t)e^{-rt}dt, \end{aligned}$$

The integrals in the definition of the constants $\widetilde{C_k}$, $1 \leq k \leq 3$ are convergent since $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$, $\operatorname{Re} r > 0$, and $\|f(t)\| \leq \varepsilon$, $\forall t \in \mathbb{R}$. Then,

$$y(x) - y_0(x) \quad = \quad \frac{-1}{V}\int_x^\infty ((r-q)e^{p(x-t)} + (p-r)e^{q(x-t)} + (q-p)e^{r(x-t)})f(t)dt.$$

Now, letting $t - x = u$ in the above integral, we obtain

$$y(x) - y_0(x) = \frac{-1}{V}\int_0^\infty ((r-q)e^{-pu} + (p-r)e^{-qu} + (q-p)e^{-ru})f(x+u)du, \quad x \in \mathbb{R}.$$

Consequently,

$$\|y(x) - y_0(x)\| \leq \frac{\varepsilon}{|V|}\int_0^{\infty} |(r-q)e^{-pu} + (p-r)e^{-qu} + (q-p)e^{-ru}|du, \quad x \in \mathbb{R};$$

therefore,

$$\|y - y_0\|_\infty \leq K\varepsilon.$$

(ii) Let $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$, and $\operatorname{Re} r < 0$. Define y_0 by the relation

$$y_0(x) = \widetilde{C_1}e^{px} + \widetilde{C_2}e^{qx} + \widetilde{C_3}e^{rx}, \quad x \in \mathbb{R},$$

where

$$\begin{aligned}
\widetilde{C_1} &= C_1 + \frac{r-q}{V}\int_0^{\infty} f(t)e^{-pt}dt; \\
\widetilde{C_2} &= C_2 + \frac{p-r}{V}\int_0^{\infty} f(t)e^{-qt}dt; \\
\widetilde{C_3} &= C_3 - \frac{q-p}{V}\int_{-\infty}^{0} f(t)e^{-rt}dt,
\end{aligned}$$

Then,

$$y(x) - y_0(x) = \frac{-1}{V}\int_x^{\infty}((r-q)e^{p(x-t)} + (p-r)e^{q(x-t)})f(t)dt + \frac{q-p}{V}\int_{-\infty}^{x} e^{r(x-t)}f(t)dt.$$

Letting $t - x = u$ and, respectively, $t - x = -u$ in the above integrals, it follows that

$$y(x) - y_0(x) = \frac{-1}{V}\int_0^{\infty}((r-q)e^{-pu} + (p-r)e^{-qu})f(x+u)du + \frac{q-p}{V}\int_0^{\infty} e^{ru}f(x-u)du, \quad x \in \mathbb{R}.$$

Consequently,

$$\|y(x) - y_0(x)\| \leq \frac{\varepsilon}{|V|}\left(\int_0^{\infty} |(r-q)e^{-pu} + (p-r)e^{-qu}|du + |q-p|\int_0^{\infty} |e^{ru}|du\right), \quad x \in \mathbb{R};$$

therefore,

$$\|y - y_0\|_\infty \leq K\varepsilon.$$

(iii) Let $\operatorname{Re} p > 0$, $\operatorname{Re} q < 0$, and $\operatorname{Re} r < 0$. Define y_0 by the relation

$$y_0(x) = \widetilde{C_1}e^{px} + \widetilde{C_2}e^{qx} + \widetilde{C_3}e^{rx}, \quad x \in \mathbb{R},$$

where

$$\begin{aligned}
\widetilde{C_1} &= C_1 + \frac{r-q}{V}\int_0^{\infty} f(t)e^{-pt}dt; \\
\widetilde{C_2} &= C_2 - \frac{p-r}{V}\int_{-\infty}^{0} f(t)e^{-qt}dt; \\
\widetilde{C_3} &= C_3 - \frac{q-p}{V}\int_{-\infty}^{0} f(t)e^{-rt}dt,
\end{aligned}$$

Then,

$$y(x) - y_0(x) = -\frac{r-q}{V}\int_x^{\infty} e^{p(x-t)}f(t)dt + \frac{1}{V}\int_{-\infty}^{x}((p-r)e^{q(x-t)}) + (q-p)e^{r(x-t)})f(t)dt.$$

Letting $t - x = u$ and, respectively, $t - x = -u$ in the above integrals, it follows that

$$y(x) - y_0(x) = -\frac{r-q}{V}\int_0^\infty e^{-pu} f(x+u)du + \frac{1}{V}\int_0^\infty ((p-r)e^{-qu} + (q-p)e^{ru}) f(x-u)du, \quad x \in \mathbb{R}.$$

Consequently,

$$\|y(x) - y_0(x)\| \leq \frac{\varepsilon}{|V|}\left(|r-q|\int_0^\infty |e^{-pu}|du + \int_0^\infty |(p-r)e^{qu} + (q-p)e^{ru}|du\right), \quad x \in \mathbb{R};$$

therefore,

$$\|y - y_0\|_\infty \leq K\varepsilon.$$

(iv) Let $\operatorname{Re} p < 0$, and $\operatorname{Re} q < 0$, $\operatorname{Re} r < 0$. Define y_0 by the relation

$$y_0(x) = \widetilde{C_1}e^{px} + \widetilde{C_2}e^{qx} + \widetilde{C_3}e^{rx}, \quad x \in \mathbb{R},$$

where

$$\begin{aligned} \widetilde{C_1} &= C_1 - \frac{r-q}{V}\int_{-\infty}^0 f(t)e^{-pt}dt; \\ \widetilde{C_2} &= C_2 - \frac{p-r}{V}\int_{-\infty}^0 f(t)e^{-qt}dt; \\ \widetilde{C_3} &= C_3 - \frac{q-p}{V}\int_{-\infty}^0 f(t)e^{-rt}dt, \end{aligned}$$

Then,

$$y(x) - y_0(x) = \frac{1}{V}\int_{-\infty}^x ((r-q)e^{p(x-t)} + (p-r)e^{q(x-t)} + (q-p)e^{r(x-t)}) f(t)dt.$$

Now, letting $t - x = -u$ in the above integral, we obtain

$$y(x) - y_0(x) = \frac{1}{V}\int_0^\infty ((r-q)e^{pu} + (p-r)e^{qu} + (q-p)e^{ru}) f(x-u)du, \quad x \in \mathbb{R}.$$

Consequently,

$$\|y(x) - y_0(x)\| \leq \frac{\varepsilon}{|V|}\int_0^\infty |(r-q)e^{pu} + (p-r)e^{qu} + (q-p)e^{ru}|du, \quad x \in \mathbb{R};$$

therefore,

$$\|y - y_0\|_\infty \leq K\varepsilon.$$

Uniqueness. Suppose that, for some $y \in \mathcal{C}^3(\mathbb{R}, X)$ satisfying Equation (10), there exist $y_1, y_2 \in \ker D$ such that

$$\|y - y_j\|_\infty \leq K\varepsilon, \quad j = 1, 2.$$

Then,

$$\|y_1 - y_2\|_\infty \leq \|y_1 - y\|_\infty + \|y - y_2\|_\infty \leq 2K\varepsilon.$$

However, $y_1 - y_2 \in \ker D$; thus, $y_1 - y_2$ belongs to the set given by Equation (3). If $(C_1, C_2, C_3) \neq (0, 0, 0)$, then

$$\|y_1 - y_2\|_\infty = \sup_{x\in\mathbb{R}} \|y_1(x) - y_2(x)\| = +\infty,$$

which contradicts the boundedness of $y_1 - y_2$. We conclude that $C_k = 0$, $1 \leq k \leq 3$; therefore, $y_1 = y_2$. The theorem is proved. □

Theorem 2. *Suppose that $p = q \neq r$ with $\operatorname{Re} p \neq 0$ and $\operatorname{Re} r \neq 0$ and let $\varepsilon > 0$. Then, for every $y \in \mathcal{C}^3(\mathbb{R}, X)$ satisfying*

$$\|D(y)\|_\infty \le \varepsilon \tag{13}$$

there exists a unique $y_0 \in \ker D$ such that

$$\|y - y_0\|_\infty \le K\varepsilon, \tag{14}$$

where

$$K = \begin{cases} \dfrac{1}{|r-p|^2}\displaystyle\int_0^\infty \left(|((r-p)u-1)e^{-pu} + e^{-ru}|\right)du, & \text{if } \operatorname{Re} p > 0, \operatorname{Re} r > 0; \\ \dfrac{1}{|r-p|^2}\displaystyle\int_0^\infty |((r-p)u-1)e^{-pu}| + |e^{ru}|du, & \text{if } \operatorname{Re} p > 0, \operatorname{Re} r < 0; \\ \dfrac{1}{|r-p|^2}\displaystyle\int_0^\infty |((p-r)u-1)e^{pu} + e^{ru}|du, & \text{if } \operatorname{Re} p < 0, \operatorname{Re} r < 0. \end{cases} \tag{15}$$

Proof. Existence.

Let $y \in \mathcal{C}^3(\mathbb{R}, X)$ satisfying Equation (13) and let $D(y) = f$, with $\|f\|_\infty \le \varepsilon$. Then,

$$y(x) = (C_1 + C_2 x)e^{px} + C_3 e^{rx} + \frac{1}{(r-p)^2}\int_0^x \Big(((p-r)(x-t)-1)e^{p(x-t)} + e^{r(x-t)}\Big)f(t)dt,$$

$C_1, C_2, C_3 \in X$.

(i) Let $\operatorname{Re} p > 0$ and $\operatorname{Re} r > 0$. Define y_0 by the relation

$$y_0(x) = \widetilde{C_1}e^{px} + \widetilde{C_2}xe^{px} + \widetilde{C_3}e^{rx}, \quad x \in \mathbb{R},$$

where

$$\begin{aligned} \widetilde{C_1} &= C_1 + \frac{1}{(r-p)^2}\int_0^\infty ((r-p)t-1)f(t)e^{-pt}dt; \\ \widetilde{C_2} &= C_2 - \frac{1}{(r-p)^2}\int_0^\infty (r-p)f(t)e^{-pt}dt; \\ \widetilde{C_3} &= C_3 + \frac{1}{(r-p)^2}\int_0^\infty f(t)e^{-rt}dt. \end{aligned}$$

The proof of the convergence of the improper integrals is analogous to that given in Theorem 1.

Then,

$$y(x) - y_0(x) = \frac{-e^{px}}{(r-p)^2}\int_x^\infty ((x-t)(p-r)-1)f(t)e^{-pt}dt + \frac{e^{rx}}{(r-p)^2}\int_x^\infty f(t)e^{-rt}dt.$$

Now, letting $t - x = u$ in the above integral, we obtain

$$y(x) - y_0(x) = \frac{-1}{(r-p)^2}\int_0^\infty (((r-p)u-1)e^{-pu} + e^{-ru})f(x+u)du, \quad x \in \mathbb{R}.$$

Consequently,

$$\begin{aligned} \|y(x) - y_0(x)\| &\le \frac{1}{|r-p|^2}\int_0^\infty \left(\left|((r-p)u-1)f(x+u)e^{-pu}\right| + \left|e^{-ru}f(x-u)\right|\right)du \\ &\le \frac{\varepsilon}{|r-p|^2}\int_0^\infty \left(\left|((r-p)u-1)e^{-pu}\right| + \left|e^{-ru}\right|\right)du, \quad x \in \mathbb{R}; \end{aligned}$$

therefore,

$$\|y - y_0\|_\infty \le K\varepsilon.$$

(ii) Let $\operatorname{Re} p > 0$ and $\operatorname{Re} r < 0$. The proof follows analogously, defining y_0 by the relation

$$y_0(x) = \widetilde{C_1}e^{px} + \widetilde{C_2}xe^{px} + \widetilde{C_3}e^{rx}, \quad x \in \mathbb{R},$$

with

$$\begin{aligned}
\widetilde{C_1} &= C_1 + \frac{1}{(r-p)^2}\int_0^\infty ((r-p)t-1)f(t)e^{-pt}dt; \\
\widetilde{C_2} &= C_2 - \frac{1}{(r-p)^2}\int_0^\infty (r-p)f(t)e^{-pt}dt; \\
\widetilde{C_3} &= C_3 - \frac{1}{(r-p)^2}\int_{-\infty}^0 f(t)e^{-rt}dt.
\end{aligned}$$

Then,

$$y(x) - y_0(x) = \frac{-e^{px}}{(r-p)^2}\int_x^\infty ((x-t)(p-r)-1)f(t)e^{-pt}dt + \frac{e^{rx}}{(r-p)^2}\int_{-\infty}^x f(t)e^{-rt}dt.$$

Letting $t - x = u$ and, respectively, $t - x = -u$ in the above integrals, it follows that

$$y(x) - y_0(x) = \frac{1}{(r-p)^2}\left(-\int_0^\infty ((r-p)u-1)f(x+u)e^{-pu}du + \int_0^\infty f(x-u)e^{ru}du\right), \quad x \in \mathbb{R}.$$

Consequently,

$$\begin{aligned}
\|y(x) - y_0(x)\| &\le \frac{1}{|r-p|^2}\int_0^\infty |((r-p)u-1)f(x+u)e^{-pu}|du + \int_0^\infty |f(x-u)e^{ru}|du \\
&\le \frac{\varepsilon}{|r-p|^2}\int_0^\infty (|((r-p)u-1)e^{-pu}| + |e^{ru}|)du, \quad x \in \mathbb{R};
\end{aligned}$$

therefore,

$$\|y - y_0\|_\infty \le K\varepsilon.$$

(iii) Let $\operatorname{Re} p < 0$ and $\operatorname{Re} r < 0$. Then, we define y_0 by the relation

$$y_0(x) = \widetilde{C_1}e^{px} + \widetilde{C_2}xe^{px} + \widetilde{C_3}e^{rx}, \quad x \in \mathbb{R},$$

with

$$\begin{aligned}
\widetilde{C_1} &= C_1 - \frac{1}{(r-p)^2}\int_{-\infty}^0 ((r-p)t-1)f(t)e^{-pt}dt; \\
\widetilde{C_2} &= C_2 + \frac{1}{(r-p)^2}\int_{-\infty}^0 (r-p)f(t)e^{-pt}dt; \\
\widetilde{C_3} &= C_3 - \frac{1}{(r-p)^2}\int_{-\infty}^0 f(t)e^{-rt}dt.
\end{aligned}$$

Therefore,

$$y(x) - y_0(x) = \frac{e^{px}}{(r-p)^2}\int_{-\infty}^x ((x-t)(p-r)-1)f(t)e^{-pt}dt + \frac{e^{rx}}{(r-p)^2}\int_{-\infty}^x f(t)e^{-rt}dt.$$

Letting $x - t = u$ in the above integrals, it follows that

$$y(x) - y_0(x) = \frac{1}{(r-p)^2}\left(\int_0^\infty ((p-r)u-1)f(x-u)e^{pu}du + \int_0^\infty f(x-u)e^{ru}du\right), \quad x \in \mathbb{R}.$$

Consequently,

$$\begin{aligned} \|y(x) - y_0(x)\| &\leq \frac{1}{|r-p|^2} \int_0^\infty |((p-r)u - 1)e^{pu} + e^{ru}| f(x-u) du \\ &\leq \frac{\varepsilon}{|r-p|^2} \int_0^\infty |((p-r)u - 1)e^{pu} + e^{ru}| du, \quad x \in \mathbb{R}; \end{aligned}$$

therefore,

$$\|y - y_0\|_\infty \leq K\varepsilon.$$

Uniqueness. Suppose that, for some $y \in \mathcal{C}^3(\mathbb{R}, X)$ satisfying Equation (13), there exist $y_1, y_2 \in \ker D$ such that

$$\|y - y_j\|_\infty \leq K\varepsilon, \quad j = 1, 2.$$

Then,

$$\|y_1 - y_2\|_\infty \leq \|y_1 - y\|_\infty + \|y - y_2\|_\infty \leq 2K\varepsilon.$$

However, $y_1 - y_2 \in \ker D$; hence, there exist $C_1, C_2, C_3 \in X$, such that

$$y_1(x) - y_2(x) = (C_1 + C_2 x)e^{px} + C_3 e^{rx}, \quad x \in \mathbb{R}. \tag{16}$$

If $(C_1, C_2, C_3) \neq (0, 0, 0)$, then

$$\|y_1 - y_2\|_\infty = \sup_{x \in \mathbb{R}} \|y_1(x) - y_2(x)\| = +\infty,$$

which contradicts the boundedness of $y_1 - y_2$. We conclude that $C_k = 0$, $1 \leq k \leq 3$; therefore, $y_1 = y_2$. The theorem is proved. □

Theorem 3. *Suppose that $p = q = r$ with $\operatorname{Re} p \neq 0$ and let $\varepsilon > 0$. Then, for every $y \in \mathcal{C}^3(\mathbb{R}, X)$ satisfying*

$$\|D(y)\|_\infty \leq \varepsilon \tag{17}$$

there exists a unique $y_0 \in \ker D$ such that

$$\|y - y_0\|_\infty \leq \frac{\varepsilon}{|\operatorname{Re} p|^3}. \tag{18}$$

Proof. Existence. Let $y \in \mathcal{C}^3(\mathbb{R}, X)$ satisfying Equation (17) and let $D(y) = f$, with $\|f\|_\infty \leq \varepsilon$. Then,

$$y(x) = (C_1 + xC_2 + x^2 C_3)e^{px} + y_p(x),$$

$C_1, C_2, C_3 \in X$, where y_p is given by Equation (9).

(i) Let $\operatorname{Re} p > 0$. Define y_0 by the relation

$$y_0(x) = \widetilde{C_1}e^{px} + \widetilde{C_2}xe^{px} + \widetilde{C_3}x^2 e^{px}, \quad x \in \mathbb{R},$$

where

$$\begin{aligned} \widetilde{C_1} &= C_1 + \frac{1}{2}\int_0^\infty t^2 f(t)e^{-pt}dt; \\ \widetilde{C_2} &= C_2 - \int_0^\infty t f(t)e^{-pt}dt; \\ \widetilde{C_3} &= C_3 + \frac{1}{2}\int_0^\infty f(t)e^{-pt}dt. \end{aligned}$$

Then,

$$y(x) - y_0(x) \quad = \quad \frac{-e^{px}}{2} \int_x^{\infty} (t-x)^2 f(t) e^{-pt} dt.$$

Now, letting $t - x = u$ in the above integral, we obtain

$$y(x) - y_0(x) = \frac{-1}{2} \int_0^{\infty} u^2 f(x+u) e^{-pu} du, \quad x \in \mathbb{R}. \tag{19}$$

Consequently,

$$\|y(x) - y_0(x)\| \quad \leq \quad \frac{\varepsilon}{2} \int_0^{\infty} |u^2 e^{-pu}| du = \frac{\varepsilon}{2} \int_0^{\infty} u^2 e^{-u \operatorname{Re} p} du = \frac{\varepsilon}{|\operatorname{Re} p|^3}, \quad x \in \mathbb{R}.$$

(ii) Let $\operatorname{Re} p < 0$. Define y_0 by the relation

$$y_0(x) = \widetilde{C_1} e^{px} + \widetilde{C_2} x e^{px} + \widetilde{C_3} e^{rx}, \quad x \in \mathbb{R},$$

where

$$\begin{aligned} \widetilde{C_1} &= C_1 + \frac{1}{2} \int_{-\infty}^{0} t^2 f(t) e^{-pt} dt; \\ \widetilde{C_2} &= C_2 - \int_{-\infty}^{0} t f(t) e^{-pt} dt; \\ \widetilde{C_3} &= C_3 + \frac{1}{2} \int_{-\infty}^{0} f(t) e^{-pt} dt. \end{aligned}$$

Then,

$$y(x) - y_0(x) \quad = \quad \frac{e^{px}}{2} \int_x^{\infty} (t-x)^2 f(t) e^{-pt} dt.$$

Now, letting $x - t = u$ in the above integral, we obtain

$$y(x) - y_0(x) = \frac{1}{2} \int_0^{\infty} u^2 f(x-u) e^{pu} du, \quad x \in \mathbb{R}.$$

Consequently,

$$\|y(x) - y_0(x)\| \quad \leq \quad \frac{\varepsilon}{2} \int_0^{\infty} u^2 |e^{pu}| du = \frac{\varepsilon}{|\operatorname{Re} p|^3}, \quad x \in \mathbb{R}.$$

The existence is proved.

Uniqueness. Suppose that, for some $y \in \mathcal{C}^3(\mathbb{R}, X)$ satisfying Equation (17), there exist $y_1, y_2 \in \ker D$ such that

$$\|y - y_j\|_\infty \leq K\varepsilon, \quad j = 1, 2.$$

Then,

$$\|y_1 - y_2\|_\infty \leq \|y_1 - y\|_\infty + \|y - y_2\|_\infty \leq 2K\varepsilon.$$

However, $y_1 - y_2 \in \ker D$; hence, there exist $C_1, C_2, C_3 \in X$ such that

$$y_1(x) - y_2(x) = (C_1 + C_2 x + C_3 x^2) e^{px}, \quad x \in \mathbb{R}. \tag{20}$$

If $(C_1, C_2, C_3) \neq (0, 0, 0)$, then

$$\|y_1 - y_2\|_\infty = \sup_{x \in \mathbb{R}} \|y_1(x) - y_2(x)\| = +\infty,$$

which contradicts the boundedness of $y_1 - y_2$. We conclude that $C_k = 0$, $1 \le k \le 3$; therefore, $y_1 = y_2$, and the result holds. $\square$

Theorem 4. *Suppose that p, q, and r are* **distinct roots** *of the characteristic equation with $\operatorname{Re} p \neq 0$, $\operatorname{Re} q \neq 0$, and $\operatorname{Re} r \neq 0$. Then, the best Ulam constant of D is given by*

$$K_D = \begin{cases} \frac{1}{|V|}\int_0^\infty |(r-q)e^{-pu} + (p-r)e^{-qu} + (q-p)e^{-ru}|du, & \text{if } \operatorname{Re} p > 0, \operatorname{Re} q > 0, \operatorname{Re} r > 0, \\ \frac{1}{|V|}\int_0^\infty |(r-q)e^{pu} + (p-r)e^{qu} + (q-p)e^{ru}|du, & \text{if } \operatorname{Re} p < 0, \operatorname{Re} q < 0, \operatorname{Re} r < 0, \\ \frac{1}{|V|}\left(\int_0^\infty |(r-q)e^{-pu} + (p-r)e^{-qu}|du + \int_0^\infty |(q-p)e^{ru}|du\right), & \text{if } \operatorname{Re} p > 0, \operatorname{Re} q > 0, \operatorname{Re} r < 0; \\ \frac{1}{|V|}\left(\int_0^\infty |(r-q)e^{-pu}|du + \int_0^\infty |(p-r)e^{qu} + (q-p)e^{ru}|du\right), & \text{if } \operatorname{Re} p > 0, \operatorname{Re} q < 0, \operatorname{Re} r < 0. \end{cases} \tag{21}$$

Proof. Suppose that D admits an Ulam constant $K < K_D$.

(i) First, let $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$, and $\operatorname{Re} r > 0$. Then,

$$K_D = \frac{1}{|V|}\int_0^\infty |(r-q)e^{-pu} + (p-r)e^{-qu} + (q-p)e^{-ru}|du.$$

Let $h(x) = (r-q)e^{-px} + (p-r)e^{-qx} + (q-p)e^{-rx}$, $x \in \mathbb{R}$. Take $s \in X$, $\|s\| = 1$, and $\theta > 0$ as arbitrarily chosen and consider $f : \mathbb{R} \to X$ defined by

$$f(x) = \frac{\overline{h(x)}}{|h(x)| + \theta e^{-x}} s, \quad x \in \mathbb{R},$$

where $\overline{h(x)}$ denotes the conjugate of $h(x)$. Obviously, the function f is continuous on $\mathbb{R}$ and $\|f(x)\| \le 1$ for all $x \in \mathbb{R}$. Let $\widetilde{y}$ be the solution to $D(y) = f$ given by

$$\widetilde{y}(x) = \frac{-1}{V}\int_0^\infty ((r-q)e^{-pu} + (p-r)e^{-qu} + (q-p)e^{-ru})f(x+u)du, \quad x \in \mathbb{R}. \tag{22}$$

Since f is bounded and $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$, and $\operatorname{Re} r > 0$, it follows that $\widetilde{y}(x)$ is bounded on $\mathbb{R}$. Furthermore, $\|D(\widetilde{y})\|_\infty \le 1$ and the Ulam stability of D for $\varepsilon = 1$ with the constant K leads to the existence of $y_0 \in \ker D$, which is given by Equation (3) such that

$$\|\widetilde{y} - y_0\|_\infty \le K. \tag{23}$$

If $(C_1, C_2, C_3) \neq (0,0,0)$, we get, in view of the boundedness of $\widetilde{y}$

$$\lim_{x\to\infty} \|\widetilde{y}(x) - y_0(x)\| = +\infty, \tag{24}$$

which contradicts relation (23). Therefore, $C_1 = C_2 = C_3 = 0$, and relation (23) becomes

$$\|\widetilde{y}(x)\| \le K, \quad \text{for all } x \in \mathbb{R}. \tag{25}$$

Taking $x = 0$ in Equation (25) we get
$\|\widetilde{y}(0)\| \le K$; i.e.,

$$\frac{1}{|V|}\left|\int_0^\infty (r-q)e^{-pu} + (p-r)e^{-qu} + (q-p)e^{-ru})f(u)du\right| \le K,$$

or, equivalently,

$$\frac{1}{|V|}\left|\int_0^\infty h(u)f(u)du\right| = \frac{1}{|V|}\int_0^\infty \frac{|h(u)|^2}{|h(u)| + \theta e^{-u}}du \le K, \quad \forall\theta > 0. \tag{26}$$

We show next that

$$\lim_{\theta\to 0}\int_0^\infty \frac{|h(u)|^2}{|h(u)|+\theta e^{-u}}du=\int_0^\infty |h(u)|du.$$

Indeed,

$$\begin{aligned}\left|\int_0^\infty \frac{|h(u)|^2}{|h(u)|+\theta e^{-u}}du-\int_0^\infty |h(u)|du\right| &\leq \int_0^\infty\left|\frac{|h(u)|^2}{|h(u)|+\theta e^{-u}}-|h(u)|\right|du\\ &= \theta\int_0^\infty \frac{|h(u)|e^{-u}}{|h(u)|+\theta e^{-u}}du\\ &\leq \theta\int_0^\infty e^{-u}du=\theta,\quad \theta>0,\end{aligned}$$

Consequently, letting $\theta\to 0$ in Equation (26), we get $K_D\leq K$, which contradicts the supposition $K<K_D$.

(ii) The case where $\operatorname{Re} p<0$, $\operatorname{Re} q<0$, and $\operatorname{Re} r<0$ follows analogously for

$$f(x)=\frac{\overline{h(-x)}}{|h(-x)|+\theta e^{x}}s,$$

for $s\in X$, $\|s\|=1$, $x\in\mathbb{R}$, and $\theta>0$, where h is defined by

$$h(x)=(r-q)e^{px}+(p-r)e^{qx}+(q-p)e^{rx},\quad x\in\mathbb{R}.$$

(iii) Consider $\operatorname{Re} p>0$, $\operatorname{Re} q>0$, and $\operatorname{Re} r<0$. Let

$$h_1(x)=(r-q)e^{-px}+(p-r)e^{-qx},\quad h_2(x)=(q-p)e^{rx},\quad x\in\mathbb{R}.$$

Take an arbitrary $\theta>0$, $s\in X$, $\|s\|=1$ and define the function $f:X\to\mathbb{R}$ by

$$f(x)=\begin{cases}\frac{\overline{h_1(x)}}{|h_1(x)|+\theta e^{-x}}s, & \text{if } x\in[0,+\infty)\\ \frac{-\overline{h_2(-x)}}{|h_2(-x)|+\theta e^{x}}s, & \text{if } x\in(-\infty,0)\end{cases}\tag{27}$$

It can be seen that f is continuous and $\|f\|_\infty\leq 1$.
Let $\widetilde{y}$ be the solution to $D(y)=f$ given by

$$\widetilde{y}(x)=\frac{-1}{V}\int_0^\infty((r-q)e^{-pu}+(p-r)e^{-qu})f(x+u)du+\frac{1}{V}\int_0^\infty(q-p)e^{ru}f(x-u)du,\tag{28}$$

Since f is bounded, taking account of the signs of the roots p,q, and r, it follows that $\widetilde{y}(x)$ is bounded.
On the other hand, all the elements of $\ker D$ are unbounded, except $y=0$. We conclude that the relation

$$\|\widetilde{y}-y\|_\infty\leq K$$

takes place only for $y=0$; therefore,

$$\|\widetilde{y}(x)\|\leq K,\quad \forall x\in\mathbb{R}.$$

For $x=0$, it follows that $\|\widetilde{y(0)}\|\leq K$, which is equivalent to

$$\frac{1}{|V|}\int_0^\infty\left(\frac{|h_1(u)|^2}{|h_1(u)|+\theta e^{-u}}+\frac{|h_2(u)|^2}{|h_2(u)|+\theta e^{-u}}\right)du\leq K.\tag{29}$$

We prove that

$$\lim_{\theta\to 0}\int_0^\infty \left(\frac{|h_1(u)|^2}{|h_1(u)|+\theta e^{-u}}+\frac{|h_2(u)|^2}{|h_2(u)|+\theta e^{-u}}\right)du=\int_0^\infty (|h_1(u)|+|h_2(u)|)du.$$

Indeed,

$$\begin{aligned}
&\left|\int_0^\infty \left(\frac{|h_1(u)|^2}{|h_1(u)|+\theta e^{-u}}+\frac{|h_2(u)|^2}{|h_2(u)|+\theta e^{-u}}\right)du-\int_0^\infty (|h_1(u)|+|h_2(u)|)du\right| \\
&\qquad\le \int_0^\infty \left|\frac{|h_1(u)|^2}{|h_1(u)|+\theta e^{-u}}-|h_1(u)|+\frac{|h_2(u)|^2}{|h_2(u)|+\theta e^{-u}}-|h_2(u)|\right|du \\
&\qquad= \theta\int_0^\infty \left(\frac{|h_1(u)|}{|h_1(u)|+\theta e^{-u}}+\frac{|h_2(u)|}{|h_2(u)|+\theta e^{-u}}\right)e^{-u}du \\
&\qquad\le 2\theta\int_0^\infty e^{-u}du=2\theta.
\end{aligned}$$

Now, letting $\theta\to 0$ in Equation (29), it follows that $K_D<K$, which is a contradiction.

(iv) The case where $\operatorname{Re} p>0$, $\operatorname{Re} q<0$, and $\operatorname{Re} r<0$ follows analogously for

$$f(x)=\begin{cases}\frac{\overline{h_1(x)}}{|h_1(x)|+\theta e^{-x}}s, & \text{if } x\in[0,+\infty)\\ \frac{-\overline{h_2(-x)}}{|h_2(-x)|+\theta e^{x}}s, & \text{if } x\in(-\infty,0)\end{cases} \tag{30}$$

for $s\in X$, $\|s\|=1$, $x\in\mathbb{R}$, and $\theta>0$, where h_1 and h_2 are defined by

$$h_1(x)=(r-q)e^{-px},\quad h_2(x)=(p-r)e^{qu}+(q-p)e^{rx},\quad x\in\mathbb{R},$$

respectively.

□

Theorem 5. *Suppose that p is a* **double root** *and r a simple root of the characteristic equation with $\operatorname{Re} p\neq 0$ and $\operatorname{Re} r\neq 0$. Then, the best Ulam constant of D is given by*

$$K_D=\begin{cases}\frac{1}{|r-p|^2}\int_0^\infty \left|((r-p)u-1)e^{-pu}+e^{ru}\right|du, & \text{if } \operatorname{Re} p>0,\operatorname{Re} r>0;\\ \frac{1}{|r-p|^2}\int_0^\infty |((r-p)u-1)e^{-pu}|+|e^{ru}|du, & \text{if } \operatorname{Re} p>0,\operatorname{Re} r<0;\\ \frac{1}{|r-p|^2}\int_0^\infty |((p-r)u-1)e^{pu}+e^{ru}|du, & \text{if } \operatorname{Re} p<0,\operatorname{Re} r<0.\end{cases} \tag{31}$$

Proof. First, let $\operatorname{Re} p>0$ and $\operatorname{Re} r<0$.

$$K_D=\frac{1}{(r-p)^2}\int_0^\infty |((r-p)u-1)e^{-pu}|+|e^{ru}|du.$$

Let $\theta>0$ and $s\in X$, $\|s\|=1$. Consider $h_1(u)=e^{-pu}((r-p)u-1)$, $h_2(u)=e^{ru}$, and $u\in\mathbb{R}$ and define

$$f(u)=\begin{cases}\frac{\overline{h_1(u)}}{|h_1(u)|+\theta e^{-u}}s, & \text{if } u\ge 0\\ \frac{\overline{h_2(-u)}}{|h_2(-u)|+\theta e^{u}}s, & \text{if } u<0.\end{cases} \tag{32}$$

The function f is continuous on $\mathbb{R}$ and $\|f(u)\|\le 1$ for all $u\in\mathbb{R}$. Suppose that D admits an Ulam constant $K<K_D$. Let $\widetilde{y}$ given by

$$\widetilde{y}(x)=\frac{1}{(r-p)^2}\int_0^\infty \left(((r-p)u-1)f(x+u)e^{-pu}+f(x-u)e^{ru}\right)du \tag{33}$$

be the solution to $D(y) = f$, where f is given by Equation (32). The relation $\|D(\widetilde{y})\|_\infty \leq 1$ leads to the existence of a unique $y \in \ker D$ such that

$$\|\widetilde{y} - y\|_\infty \leq K, \tag{34}$$

in view of the Ulam stability of D with the constant K. The function $\widetilde{y}$ is bounded since f is bounded and $\operatorname{Re} p > 0$ and $\operatorname{Re} q < 0$. On the other hand, all the elements of $\ker D$ are unbounded except $y = 0$. We conclude that Equation (34) applies only for $y = 0$; therefore,

$$\|\widetilde{y}(x)\| \leq K, \quad \forall x \in \mathbb{R}.$$

For $x = 0$, it follows that $\|\widetilde{y(0)}\| \leq K$, which is equivalent to

$$\frac{1}{|r-p|^2} \int_0^\infty \left(\frac{|h_1(u)|^2}{|h_1(u)| + \theta e^{-u}} + \frac{|h_2(u)|^2}{|h_2(u)| + \theta e^{-u}} \right) du \leq K. \tag{35}$$

We prove that

$$\lim_{\theta \to 0} \int_0^\infty \left(\frac{|h_1(u)|^2}{|h_1(u)| + \theta e^{-u}} + \frac{|h_2(u)|^2}{|h_2(u)| + \theta e^{-u}} \right) du = \int_0^\infty (|h_1(u)| + |h_2(u)|) du.$$

Indeed,

$$\begin{aligned}
&\left| \int_0^\infty \left(\frac{|h_1(u)|^2}{|h_1(u)| + \theta e^{-u}} + \frac{|h_2(u)|^2}{|h_2(u)| + \theta e^{-u}} \right) du - \int_0^\infty (|h_1(u)| + |h_2(u)|) du \right| \\
&\qquad \leq \int_0^\infty \left| \frac{|h_1(u)|^2}{|h_1(u)| + \theta e^{-u}} - |h_1(u)| + \frac{|h_2(u)|^2}{|h_2(u)| + \theta e^{-u}} - |h_2(u)| \right| du \\
&\qquad = \theta \int_0^\infty \left(\frac{|h_1(u)|}{|h_1(u)| + \theta e^{-u}} + \frac{|h_2(u)|}{|h_2(u)| + \theta e^{-u}} \right) e^{-u} du \\
&\qquad \leq 2\theta \int_0^\infty e^{-u} du = 2\theta.
\end{aligned}$$

Now, letting $\theta \to 0$ in Equation (35), it follows that $K_D < K$, which is a contradiction.

The proof of the other cases is obtained similarly by converting the signs of $\operatorname{Re} p$ and $\operatorname{Re} r$, respectively. □

Theorem 6. *Suppose that p is a* **triple root** *of the characteristic equation with $\operatorname{Re} p \neq 0$. Then, the best Ulam constant of D is given by*

$$K_D = \frac{1}{|\operatorname{Re} p|^3}.$$

Proof. Consider first the case where $\operatorname{Re} p > 0$. Suppose that D is stable with an Ulam constant $K < K_D$. Take $s \in X$, $\|s\| = 1$ and let $f : \mathbb{R} \to X$ be given by

$$f(x) = e^{i\Im p x} s, \quad \forall x \in \mathbb{R}, \tag{36}$$

where $\Im p$ denotes the imaginary part of the root p.

The function of the right-hand side of Equation (19) is a solution to the equation $Dy = f$ and, in the following, we denote it by $\widetilde{y}$. Consequently,

$$\widetilde{y}(x) = -\frac{1}{2} \int_0^\infty u^2 f(x+u) e^{-pu} du, \quad x \in \mathbb{R}. \tag{37}$$

Replacing f given by Equation (36) in Equation (37), it follows that

$$\widetilde{y}(x) = -\frac{1}{2}e^{i\Im px}s\int_0^\infty u^2 e^{-\operatorname{Re} pu}du, \quad x \in \mathbb{R}.$$

The substitution $\operatorname{Re} pu = v$ leads to

$$\widetilde{y}(x) = -\frac{1}{2(\operatorname{Re} p)^3}e^{i\Im px}s\int_0^\infty v^2 e^{-v}dv = -\frac{1}{(\operatorname{Re} p)^3}e^{i\Im px}s, \quad x \in \mathbb{R}.$$

Since

$$\|D(\widetilde{y})\|_\infty = \|f\|_\infty = 1, \tag{38}$$

it follows in view of the Ulam stability of D, for $\varepsilon = 1$, that there exists a solution y_0 to $D(y) = 0$,

$$y_0 = (C_1 + C_2 x + C_3 x^2)e^{px}, \quad x \in \mathbb{R}$$

such that

$$\|\widetilde{y} - y_0\|_\infty \leq K \tag{39}$$

If $(C_1, C_2, C_3) \neq (0, 0, 0)$, we have

$$\lim_{x\to\infty} \|\widetilde{y}(x) - y_0(x)\| = +\infty,$$

which contradicts Equation (39), since $\widetilde{y}$ is bounded. If $(C_1, C_2, C_3) = (0, 0, 0)$, then Equation (39) becomes

$$\|\widetilde{y}\|_\infty \leq K \Longleftrightarrow \frac{1}{|\operatorname{Re} p|^3} \leq K \Longleftrightarrow K \leq K_D,$$

which contradicts the supposition $K < K_D$. The case where $\operatorname{Re} p < 0$ follows analogously. The theorem is proved. □

3. Conclusions

In this paper, we obtain a result regarding the Ulam stability for the linear differential operator of the third order with constant coefficients and we demonstrate its best Ulam constant. In this way, we give sharper estimates between an approximate solution and an exact solution of the associated differential equation. These results are connected to the notion of the perturbation of a continuous dynamical system governed by such differential equations. The novelty of the paper consists in the fact that we give a complete answer to the problem of the best Ulam constant for this operator, treating both the cases of simple and multiple roots.

Various aspects of third-order differential equations have been studied in recent years due to their applications in physics, engineering, biology, and social sciences. In particular, oscillation criteria for linear and nonhomogeneous third-order differential equations have been obtained with respect to oscillations of the corresponding homogeneous equations (see [21,22]). From this point of view, the problem of Ulam stability for these equations is important since it establishes a relation between the solutions to the nonhomogeneous and homogeneous equations.

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