axioms

MDPI

Article

Some Axioms and Identities of L-Moments from Logistic Distribution with Generalizations

Khalaf S. Sultan [1,*] and Nashmiah R. AL-Shamari [2]

[1] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt
[2] Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; naalshammari@ksu.edu.sa
* Correspondence: ksultan@azhar.edu.eg

Abstract: In this paper, we derive the L-moments for some distributions, such as logistic, generalized logistic, doubly truncated logistic, and doubly truncated generalized logistic distributions. We also establish some new axioms and identities, including recurrence relations satisfied by the L-moment from the underlying derivations. In addition, we establish some new general recurrence relations satisfied by the L-moment from any distribution.

Keywords: order statistics; L-moments; logistic distribution; generalized logistic distribution

MSC: 62G20; 62G32; 62E17

Citation: Sultan, K.S.; AL-Shamari, N.R. Some Axioms and Identities of L-Moments from Logistic Distribution with Generalizations. *Axioms* **2023**, *12*, 928. https://doi.org/10.3390/axioms12100928

Academic Editors: Nuno Bastos, Touria Karite and Amir Khan

Received: 26 August 2023
Revised: 22 September 2023
Accepted: 26 September 2023
Published: 28 September 2023

1. Introduction

Order statistics play an important role in the statistical inference of parametric and nonparametric statistics, estimation theory, and hypothesis testing. Order statistics have also found important applications, including life testing, reliability theory, characterization, statistical quality control, detection of outliers, analysis of censored data, goodness-of-fit tests, single image processing, and many other fields. Order statistics received attention from numerous researchers, among them Arnold et al. [1] and David and Nagaraja [2]. For a detailed discussion on the moments of order statistics, one can refer to [3].

Like other statistical moments, L-moments characterize the geometry of distributions, summarization, and description of theoretical probability distributions (observed data samples), estimation of parameters and quantiles of probability distributions, and hypotheses testing for probability distributions. L-moments are directly analogous to that and have similar interpretations as the moments. This makes L-moments conceptually accessible to many potential users.

Hosking [4] has defined the L-moments as based on linear combinations of differences in the expectations of order statistics, which are based on powers (exponents) of differences. They can be defined for any random variable whose mean exists. Hosking [5] concludes that "L-moments can provide good summary measures of distributional shape and may be preferable to moments for this purpose". Sillitto [6] has introduced population L-moments as alternatives to the classical population central moments determined by the population distribution. Greenwood et al. [7] have introduced probability weighted moments, which are an alternative statistical "moment" that, like the moments, characterize the geometry of distributions and are useful for parameter estimation. Karian and Dudewicz [8] have studied the method of L-moments in some of their examples, where the overall performance appears comparable to the overall performance of the percentile method, where the method of percentiles and the method of L-moments are related in the sense that they both are based on order statistics.

Sahu et al. [9] have described regionalization procedures for hydrological and climatological assessment of ungauged watersheds, where different techniques together with

L-moments are being utilized by many researchers and hydrologists for almost every extreme event, viz., extreme rainfall, low flow, flood, and drought. Domański et al. [10] have presented an application of L-moment statistics and the respective L-moment ratio diagrams to assess control performance, in particular, in terms of control system sustainability. In addition, the evolution in their performance over time is depicted visually. L-moment diagrams are common in extreme event analysis and are considered a very powerful tool in this field at the regional level. Anderson [11] has shown that the results of L-moments and L-moment ratios were less sensitive than traditional moments for the Barabási–Albert, Erdös–Rényi, and Watts–Strogratz network models when his research centered on finding the statistical moments, network measures, and statistical tests that are most sensitive to various node degradations for these three different network models. Fallahgoul et al. [12] have developed and applied a novel semiparametric estimation method based on L-moments. Unlike conventional moments, L-moments are linear in the data and therefore robust to outliers. Additionally, an extensive empirical analysis of portfolio choice under nonexpected utility demonstrated the effectiveness of the L-moment approach.

In this paper, we display the L-moments and the sample L-moments, some of their general properties, and how to use the sample L-moments to develop the method of L-moments for estimating the parameters that are described in Section 2. In Section 3, we establish general recurrence relations between L-moments for any distribution. Next, we derive the exact explicit expressions for L-moments of underlying distributions, namely, logistic distribution, generalized logistic distribution, doubly truncated logistic distribution, and doubly truncated generalized logistic distribution in Section 4. Then, in Section 5, we establish some recurrence relations by L-moments from specific distributions. Finally, we provide our conclusions in Section 6.

2. L-Moments

In this section, we present the definitions of the probability weight moments, L-moments, and L-moment ratios. Next, we establish some properties of L-moments and L-moment ratios.

2.1. Population of L-Moments

The probability weighted moments of a random variable X with a pdf $f(x)$, cdf $F(x)$, and quantile x_u are defined by the expectations as

$$M_{p,r,s} = E\left[X^p (F(X))^r (1-F(X))^s\right] = \int_0^1 x_u^p u^r (1-u)^s du,$$

where $p,\ r,$ and s are integers. The most common probability weighted moment is

$$\beta_r = M_{1,r,0} = E\left[X(F(X))^r\right] = \int_0^1 x_u\, u^r du = \tfrac{1}{r+1} E[X_{r+1:r+1}] \quad \text{for } r = 0,1,2,\ldots, \tag{1}$$

where

$$\begin{aligned} E[X_{r:n}] &= \mu_{r:n} = \textstyle\int_{-\infty}^{\infty} x f_{r:n}(x) dx \\ &= \textstyle\int_{-\infty}^{\infty} x C_{r:n} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x) dx, -\infty < x < \infty,\ C_{r:n} = \frac{n!}{(r-1)!(n-r)!}, \end{aligned} \tag{2}$$

gives the single moments for order statistics of $X_{r:n}$, $1 \le r \le n$, $n = 1,2,3,\ldots$ (see [1]).

Landwehr et al. [13–15] have considered the L-moments as beginning with the statistical needs for researchers of surface-water hydrology with an interest in floods and extreme rainfall hydrology. Historically, L-moments were developed from probability weighted moments. The core theory of L-moments for univariate applications was unified in the late 1980s to early 1990s. Hosking [16] has confirmed that probability weighted moments (or L-moments) are sometimes more popular than maximum likelihood because of their good performance for small samples. Additionally, L-moments can serve as a good choice for the starting values in the iterative numerical procedure required to obtain maximum likelihood estimates.

Hosking [4] has unified discussion and estimation of distributions using L-moments and used particular ratios of them as measures of skewness and kurtosis. They can be defined for any random variable whose mean exists. Hosking has also defined the theoretical L-moments from r^{th}-shifted Legendre polynomials:

$$\lambda_r = \int_0^1 x_u \, P^*_{r-1}(u) \, du \text{ for } r \geq 1, \tag{3}$$

where

$$P^*_{r-1}(u) = \sum_{k=0}^{r-1} p^*_{r-1,k} u^k, \tag{4}$$

$$p^*_{r-1,k} = (-1)^{r-1-k} \binom{r-1}{k} \binom{r-1+k}{k}. \tag{5}$$

is the shifted Legendre polynomial (see [17]) and x_u is a quantile function. The first few L-moments are

$$\lambda_1 = E[X] = \int_0^1 x_u \, du,$$
$$\lambda_2 = \int_0^1 x_u \times (2u-1) \, du,$$
$$\lambda_3 = \int_0^1 x_u \times (6u^2 - 6u + 1) \, du,$$
$$\lambda_4 = \int_0^1 x_u \times (20u^3 - 30u^2 + 12u - 1) \, du.$$

The L-moment ratios of X are the quantities

$$\tau_r = \lambda_r / \lambda_2 \text{ for } r = 3, 4, 5, \ldots,$$

satisfying $|\tau_r| < 1$. Note that $\tau_3 = \lambda_3/\lambda_2$ is called L-skewness and $\tau_4 = \lambda_4/\lambda_2$ is called L-kurtosis. The L-moments λ_1 and λ_2 and the L-moment ratios τ_3 and τ_4 are the most useful quantities for summarizing probability distributions. The most important property is that if X and Y are random variables with L-moments λ_r and λ^*_r, respectively, and suppose that $Y = aX + b$, then,

$$\lambda^*_1 = a\lambda_1 + b,$$
$$\lambda^*_r = (sign\ a)^r |a| \lambda_r,\ r \geq 2,$$
$$\tau^*_r = (sign\ a)^r \tau_r, r \geq 3.$$

Hosking [5] concludes that "L-moments can provide good summary measures of distributional shape and may be preferable to moments for this purpose". Royston [18] and Vogel and Fennessey [19] have discussed the advantages of L-skewness and L-kurtosis over their classical counterparts.

The system of linear equations relating L-moments λ_r to probability weighted moments β_r can be obtained (see [20]) for $r \geq 0$ as follows:

$$\lambda_{r+1} = \sum_{m=0}^{r} p^*_{r,m} \, \beta_m. \tag{6}$$

The first four L-moments in terms of probability weighted moments are

$$\lambda_1 = \beta_0,$$
$$\lambda_2 = 2\beta_1 - \beta_0,$$
$$\lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0,$$
$$\lambda_4 = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0.$$

Note that $\lambda_1 = E[X]$ is the L-location or the mean of the distribution, while λ_2 is a measure of the scale or dispersion of the random variable X, so λ_2 is the L-scale.

2.2. Sample L-Moments and Method of L-Moments

For any distribution with finite means, Hosking [4] defined the sample L-moments $\hat{\lambda}_r$ as follows:

$$\hat{\lambda}_r = \frac{1}{r\binom{n}{r}} \sum_{i=1}^{n} \left(\sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \binom{i-1}{r-1-j} \binom{n-i}{j} \right) x_{i:n},$$

where $x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{n:n}$ are the sample order statistics. We see that the statistic $\hat{\lambda}_1$ is the sample mean, the sample L-scale $\hat{\lambda}_2$ is half Gini's mean difference (see [21]), $\hat{\lambda}_3$ is used by Sillitto [6] as a measure of symmetry and by Locke and Spurrier [22] to test for symmetry, and $\hat{\lambda}_4$ is used by Hosking [4] as a measure of kurtosis. The r^{th} sample L-moment ratios are the following quantities (see [23]):

$$\hat{\tau}_r = \hat{\lambda}_r / \hat{\lambda}_2 \, , r = 3, 4, 5, \ldots .$$

Note that $\hat{\tau}_3 = \hat{\lambda}_3/\hat{\lambda}_2$ is a measure of skewness, and $\hat{\tau}_4 = \hat{\lambda}_4/\hat{\lambda}_2$ is a measure of kurtosis. These are, respectively, the sample L-skewness and sample L-kurtosis. The quantities $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\tau}_3$, and $\hat{\tau}_4$ are useful summary statistics for a data sample. They can be used to identify the distribution from which a sample was drawn and applied to estimate parameters when fitting a distribution to a sample by equating the sample and population L-moments (see [24]).

From a random sample of size n, obtained from a probability distribution, the method of L-moments (LMOMs) is to equate the L-moments of the distribution to the sample L-moments such that $\lambda_r = \hat{\lambda}_r$ for the p number of unknown parameters is chosen for a distribution (see [25]).

3. General Relationships Based on L-Moments

The moments of order statistics have acquired considerable interest in recent years and, in fact, have been tabulated quite extensively for many distributions. Many authors have investigated and derived several recurrence relations because one could list the following four main reasons why these recurrence relations for the moments of order statistics are important:

1. They reduce the number of direct computations greatly;
2. They usefully express the higher order moments of order statistics in terms of the lower order moments and hence make the evaluation of higher order moments easy;
3. They are very useful in checking the computation of the moments of order statistics;
4. Results can be used for characterizing the distributions.

Now, for the same main reasons in the moments of order statistics, Hosking [26] has studied the recurrence relations between trimmed L-moments with different degrees of trimming, and he found the relation between trimmed L-moments and L-moments.

In order to establish new general recurrence relations between the L-moments, we need to review the most important lemmas that are necessary later in the theorem:

Lemma 1. *If*

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k},$$

where

$$\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2} & , n \text{ even}, \\ \frac{n-1}{2} & , n \text{ odd}. \end{cases}$$

is the Legendre polynomial (see [27]) of degree $n = 0, 1, 2, \ldots$ for $x \in [-1, 1]$ and $P_n^*(x)$ is the shifted Legendre polynomial of degree $n = 0, 1, 2, \ldots.$ on the interval $[0, 1]$ in Equation (4), we then have

$$\frac{d}{dx} P_n^*(x) = 2P_n'(2x-1) \text{ where } P_n'(x) = \frac{d}{dx} P_n(x). \tag{7}$$

The shifted Legendre polynomial satisfies the following recurrence relations, $n = 0, 1, \ldots..,$

$$P_{n+1}^*(x) = P_n^*(x) - \frac{2}{n+1}(1-x)\sum_{i=0}^{n} (2i+1)P_i^*(x), \tag{8}$$

and

$$P_{n+1}^*(x) = 2\sum_{i=0}^{n} (2i+1)\, q_{i+1}^*(x) - P_n^*(x). \tag{9}$$

where

$$q_{i+1}^*(x) = \int_0^x P_i^*(t)dt = \sum_{k=0}^{i} \frac{1}{k+1} p_{i,k}^* x^{k+1} \text{ for } i > 0,$$

is the integrated shifted Legendre polynomial.

Proof. To prove (7), by compensating x for $(2x-1)$ in the differentiation of the Legendre polynomial

$$P_n'(x) = \frac{d}{dx} P_n(x) = \sum_{r=0}^{[(n-1)/2]} (2n-4r-1)P_{n-2r-1}(x),$$

(see [28]) and use $P_n^*(x) = P_n(2x-1)$ (see [23]), we obtain

$$P_n'(2x-1) = \frac{d}{dx} P_n(2x-1) = \sum_{i=0}^{[(n-1)/2]} (2n-4i-1)P_{n-2i-1}^*(x). \tag{10}$$

By the comparison between the differentiation of shifted Legendre polynomials,

$$\frac{d}{dx} P_n^*(x) = 2\sum_{i=0}^{[(n-1)/2]} (2n-4i-1)P_{n-2i-1}^*(x),$$

(see [29–31]) and $P_n'(2x-1)$ in (10), we can express the relationship (7).

To prove (8), we have the recursive formula for Legendre polynomials (see [28]) for $n = 0, 1, 2, \ldots.,$

$$P_{n+1}(x) = P_n(x) - \frac{1}{n+1}(1-x)\sum_{i=0}^{n} (2i+1)P_i(x), \tag{11}$$

and then compensate x for $(2x-1)$ in (11) and use $P_n^*(x) = P_n(2x-1)$ (see [23]).

Now, for Equation (9), by bringing a recursive formula for Legendre polynomials (see [28]) for $n = 0, 1, 2, \ldots.$, this relates the polynomials and their derivatives to each other as follows:

$$P_{n+1}'(x) = \sum_{i=0}^{n} (2i+1)P_i(x) - P_n'(x), \tag{12}$$

where we compensate x to $(2x-1)$ in (12), use $P_n^*(x) = P_n(2x-1)$ (see [23]) and (7); we have,

$$\frac{d}{dx} P_{n+1}^*(x) = 2\sum_{i=0}^{n} (2i+1)P_i^*(x) - \frac{d}{dx} P_n^*(x), \tag{13}$$

and afterward integrating both sides with respect to t from $t = 0$ to $t = x$ in (13).

Hence,

$$P_{n+1}^*(x) - P_{n+1}^*(0) = 2\sum_{i=0}^{n} (2i+1)\int_0^x P_i^*(t)dt - (P_n^*(x) - P_n^*(0)), \tag{14}$$

and using that $P_n^*(0) = (-1)^n \forall n = 0, 1, 2 \ldots.$ (see [23]). □

Theorem 1. *Let X be a continuous random variable with cdf $u = F(x)$ and quantile function x_u; $0 \le u \le 1$. Then, L-moments λ_r satisfy the following recurrence relations:*

$$\lambda_{r+2} = \frac{2r+1}{r+1}(2A_{r+1} - \lambda_{r+1}) - \frac{r}{r+1}\lambda_r,\ r = 0, 1, \ldots.., \tag{15}$$

$$\lambda_{r+2} = \lambda_{r+1} - \frac{2}{r+1}\sum\nolimits_{i=0}^{r} (2i+1)(\lambda_{i+1} - A_{i+1}),\ r = 0, 1, \ldots.., \tag{16}$$

$$\lambda_{r+2} = 2(2r+1)B_{r+1} + \lambda_r,\ r = 1, 2, \ldots.., \tag{17}$$

$$\lambda_{r+2} = 2\sum\nolimits_{i=0}^{r} (2i+1)\ B_{i+1} - \lambda_{r+1},\ r = 0, 1, \ldots.., \tag{18}$$

where $A_{r+1} = \sum_{k=0}^{r} p^*_{r,k}\beta_{k+1}$, $B_{r+1} = \sum_{k=0}^{r} \frac{1}{k+1} p^*_{r,k}\beta_{k+1}$, and $p^*_{r,k}$ are in (5) and β_{k+1} is in (1).

Proof. For (15), we have a recurrence relation between shifted Legendre polynomials for $n = 0, 1, 2\ldots$, (see [29–31]):

$$P^*_{r+1}(u) = \frac{2r+1}{r+1}(2u-1)P^*_r(u) - \frac{r}{r+1}P^*_{r-1}(u),\ r = 0, 1, \ldots..,$$

By multiplying both sides by x_u and integrating over u, we obtain

$$\lambda_{r+2} = \frac{2r+1}{r+1}\left[2\int_0^1 uP^*_r(u)x_u du - \lambda_{r+1}\right] - \frac{r}{r+1}\lambda_r. \tag{19}$$

Then,

$$\int_0^1 uP^*_r(u)x_u du = \int_0^1 u\left(\sum_{k=0}^{r} p^*_{r,k}u^k\right)x_u du = \sum_{k=0}^{r} p^*_{r,k}\int_0^1 u^{k+1}x_u du = \sum_{k=0}^{r} p^*_{r,k}\beta_{k+1} = A_{r+1}. \tag{20}$$

using (20) in (19), the proof is complete. For (16), the same technique as the method of proof for (15) is used, but begins by using (8).

Now, also for (17) and (18), they have the same technique as the method of proof, begun by using the recurrence relation between shifted Legendre polynomials for $n = 0, 1, 2\ldots$, (see [29–31]):

$$2(2n+1)\ q^*_{n+1}(x) = P^*_{n+1}(x) - P^*_{n-1}(x),$$

and (9), respectively, and multiplying both sides by x_u and integrating over u. □

All Equations (15)–(18) in Theorem 1 are equal to $\lambda_2, \lambda_3, \ldots.$, those given equations relating λ_r to β_r obtained by Zafirakou-Koulouris et al. [20] in (6).

4. L-Moments from the Logistic Distributions

In this section, we present some statistical distributions, like logistic, generalized logistic, doubly truncated logistic and doubly truncated generalized logistic with their first four implicit L-moments. Then, we derive the LMOMs for the unknown parameters from these distributions.

4.1. L-Moments of the Logistic Distribution

The pdf of a logistic distribution with the location parameter ζ (the mode, median, and mean) and scale parameter α is reported by Balakrishnan [32] and Walck [33]:

$$f(x) = \frac{1}{\alpha}\frac{e^{-\left(\frac{x-\zeta}{\alpha}\right)}}{\left[1 + e^{-\left(\frac{x-\zeta}{\alpha}\right)}\right]^2}, -\infty < x < \infty, -\infty < \zeta < \infty,\ \alpha > 0,$$

and the cdf is

$$F(x) = \frac{1}{1 + e^{-\left(\frac{x-\zeta}{\alpha}\right)}}, -\infty < x < \infty, -\infty < \zeta < \infty,\ \alpha > 0.$$

For $0 < u < 1$, the quantile is

$$x_u = \zeta + \alpha \ln\left(\frac{u}{1-u}\right), -\infty < \zeta < \infty,\ \alpha > 0.$$

The mean of the logistic distribution is $E[X] = \zeta$. The random variable of standard logistic Z can be obtained by putting $\zeta = 0$ and $\alpha = 1$.

The r^{th} probability weighted moment in (1) can be obtained by (see [34])

$$\beta_r = \frac{\zeta}{r+1} + \frac{\alpha}{r+1}[\psi(r+1) + \gamma] = \frac{1}{r+1}[\zeta + \alpha[\psi(r+1) + \gamma]], r = 0, 1, 2, \ldots,$$

where $\gamma = -\psi(1) = 0.577216$ is Euler's constant and $\psi(\cdot)$ is the digamma function, which is defined as

$$\psi(r) = \frac{\Gamma'(r)}{\Gamma(r)} = \frac{d}{dr} \ln \Gamma(r), \quad r \neq 0, -1, -2, \ldots,$$

and $\Gamma(.)$ is a gamma function. Thus, the first four β_r can be computed as follows:

$$\beta_0 = \zeta,\ \beta_1 = \frac{\zeta + \alpha}{2},\ \beta_2 = \frac{\zeta}{3} + \frac{\alpha}{2} \text{ and } \beta_3 = \frac{\zeta}{4} + \frac{11\alpha}{24},$$

where $\psi(1) = -\gamma,\ \psi(2) = 1 - \gamma\ ,\ \psi(3) = \frac{3}{2} - \gamma$ and $\psi(4) = \frac{11}{6} - \gamma$. Then, the first four L-moments in (6) are given as (see [34])

$$\begin{gathered} \lambda_1 = \beta_0 = \zeta,\ \lambda_2 = 2\beta_1 - \beta_0 = \alpha,\ \lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0 = 0,\ \tau_3 = \frac{\lambda_3}{\lambda_2} = 0, \\ \lambda_4 = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0 = \frac{\alpha}{6} \text{ and } \tau_4 = \frac{\lambda_4}{\lambda_2} = \frac{1}{6}. \end{gathered} \quad (21)$$

The L-moment estimators for location parameter ζ and scale parameter α can be obtained from the first and second L-moments (λ_1, λ_2) in (21) as

$$\hat{\zeta} = \hat{\lambda}_1 \text{ and } \hat{\alpha} = \hat{\lambda}_2. \quad (22)$$

4.2. L-Moments of the Generalized Logistic Distribution

The generalized logistic distribution has three parameters and is thus fit to the mean, scale, and shape of a data set. The pdf and cdf of the generalized logistic distribution are given, respectively, for $-\infty < \zeta < \infty$ and $\alpha > 0$, as reported by Burr [35] and Asquith [25]:

$$f(x) = \frac{1}{\alpha} \frac{\left[1-\delta\left(\frac{x-\zeta}{\alpha}\right)\right]^{\frac{1}{\delta}-1}}{\left[1+\left[1-\delta\left(\frac{x-\zeta}{\alpha}\right)\right]^{1/\delta}\right]^2} \quad \begin{array}{l} , -\infty < x \leq \zeta + \frac{\alpha}{\delta} \text{ if } 0 < \delta < 1, \\ , \zeta + \frac{\alpha}{\delta} \leq x < \infty \text{ if } -1 < \delta < 0, \end{array}$$

and

$$F(x) = \frac{1}{1+\left[1-\delta\left(\frac{x-\zeta}{\alpha}\right)\right]^{1/\delta}} \quad \begin{array}{l} , -\infty < x \leq \zeta + \frac{\alpha}{\delta} \text{ if } 0 < \delta < 1, \\ , \zeta + \frac{\alpha}{\delta} \leq x < \infty \text{ if } -1 < \delta < 0. \end{array}$$

For $0 < u < 1$, the quantile is

$$x_u = \zeta + \frac{\alpha}{\delta}\left[1 - \left(\frac{1-u}{u}\right)^{\delta}\right], -\infty < \zeta < \infty,\ \alpha > 0,\ \delta \neq 0.$$

The random variable of the standard generalized logistic Z can be obtained by putting $\zeta = 0$ and $\alpha = 1$. The first four moments, $k = 1,2,3,4$ of the standard generalized logistic random variable are as follows (see [3]):

$$E\left[Z^k\right] = \frac{1}{\delta^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\beta(1-j\delta, 1+j\delta), \quad |\delta| < \frac{1}{k}.$$

where $\beta(1-j\delta,\ j\delta+1)$ is the beta function and can be defined by the integral

$$\beta(a,\ b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt,\ a,b > 0.$$

Now, we derive the first moment for the order statistics of the standard generalized logistic random variable.

Lemma 2. *The moments of order statistics in (2) of the standard generalized logistic random variable $Z_{j:n}$ are*

$$\mu_{j:n} = \frac{1}{\delta}\left(1 - \frac{\Gamma(j-\delta)\Gamma(n-j+1+\delta)}{\Gamma(j)\Gamma(n-j+1)}\right), \quad -1 < \delta < 1. \tag{23}$$

Proof. The j^{th} moment of order statistics is

$$\begin{aligned}\mu_{j:n} &= E[Z_{j:n}] = \int zf_{j:n}(z)dz = \tfrac{n!}{(j-1)!(n-j)!}\int z[F(z)]^{j-1}f(z)[1-F(z)]^{n-j}dz \\ &= \tfrac{n!}{(j-1)!(n-j)!}\int_0^1 z_u u^{j-1}(1-u)^{n-j}du \\ &= \tfrac{n!}{(j-1)!(n-j)!}\tfrac{1}{\delta}\int_0^1\left(u^{j-1}(1-u)^{n-j} - u^{j-1-\delta}(1-u)^{n-j+\delta}\right)du \\ &= \tfrac{n!}{(j-1)!(n-j)!}\tfrac{1}{\delta}(\beta(j,n-j+1) - \beta(j-\delta,n-j+\delta+1)),\end{aligned}$$

after some simplification, we obtain the required result. □

Note that:

- By letting $n = j = 1$ in Lemma 2, we deduce the first moment established for a standard generalized logistic distribution.
- By letting the shape parameter $\delta \to 0$ in Lemma 2, we deduce the moment of order statistics of the standard logistic distribution (see [36]):

$$\mu_{r:n} = E[Z_{r:n}] = \psi(j) - \psi(n-j+1),\ j = 1,2,\ldots,n. \tag{24}$$

Now, the r^{th}, $r = 0,1,2,\ldots$, probability weighted moment in (1) for generalized logistic distribution can be stated as follows:

$$\begin{aligned}\beta_r &= (\zeta + \alpha\mu_{r+1:r+1})/(1+r) = \tfrac{1}{r+1}\left(\zeta + \tfrac{\alpha}{\delta}\right) - \tfrac{\alpha}{\delta}\beta(r+1-\delta,\delta+1) \\ &= \tfrac{1}{r+1}\left(\zeta + \tfrac{\alpha}{\delta}\right) - \tfrac{\alpha}{\delta}\beta(1-\delta,\delta+1)\tfrac{(1-\delta)^{(r)}}{\Gamma(r+2)}, \quad -1 < \delta < 1,\end{aligned}$$

where

$$(1-\delta)^{(r)} = \frac{\Gamma(1-\delta+r)}{\Gamma(1-\delta)} = \prod_{i=1}^{r}(i-\delta),$$

are rising factorials.

Therefore, the L-moments in (6) are (see [25])

$$\begin{gathered}\lambda_1 = \left(\zeta + \tfrac{\alpha}{\delta}\right) - \tfrac{\alpha}{\delta}\beta(1-\delta,\delta+1),\ \lambda_2 = \alpha\beta(1-\delta,\delta+1),\ \lambda_3 = -\alpha\delta\beta(1-\delta,\delta+1),\ \tau_3 = -\delta, \\ \lambda_4 = \tfrac{1+5\delta^2}{6}\alpha\beta(1-\delta,\delta+1) \text{ and } \tau_4 = \tfrac{1+5\delta^2}{6}.\end{gathered} \tag{25}$$

The L-moments estimators for location parameter ζ, scale parameter α, and shape parameter δ can be obtained from the first and second L-moments (λ_1, λ_2) and L-skewness τ_3 ($\tau_3 = \lambda_3/\lambda_2$ is a function of δ only) in (25), which are measures of location, scale, and skewness, respectively, as

$$\hat{\zeta} = \hat{\lambda}_1 - \frac{\hat{\alpha}}{\hat{\delta}}(1 - \beta(1-\hat{\delta}, \hat{\delta}+1)), \;\; \hat{\alpha} = \frac{\hat{\lambda}_2}{\beta(1-\hat{\delta}, \hat{\delta}+1)} \text{ and } \hat{\delta} = -\hat{\tau}_3. \tag{26}$$

4.3. L-Moments of the Doubly Truncated Logistic Distribution

The standard doubly truncated logistic distribution has been extended by Balakrishnan and Rao [3] with pdf:

$$f(z) = \frac{1}{P-Q} e^{-z}/(1+e^{-z})^2, \quad Q_1 \le z \le P_1,$$

and with cdf (see [32]):

$$F(z) = \frac{1}{P-Q}\left[\frac{1}{1+e^{-z}} - Q\right], \quad Q_1 \le z \le P_1,$$

where Q and $1-P$ $(0 < Q < P < 1)$ are given by

$$P = F(P_1) \text{ and } Q = F(Q_1),$$

where $F(\cdot)$ is given in the standard logistic distribution. Then,

$$Q_1 = \log\left(\frac{Q}{1-Q}\right) \text{ and } P_1 = \log\left(\frac{P}{1-P}\right).$$

The quantile is

$$z_u = \log\left[\frac{u(P-Q)+Q}{1-[u(P-Q)+Q]}\right], \;\; 0 < u < 1.$$

The first moment of Z is given by

$$E[Z] = \frac{PP_1 - QQ_1 + \log\left[\frac{1-P}{1-Q}\right]}{P-Q}.$$

Note that by letting $Q \to 0$ and $P \to 1$, we deduce the first moment for the logistic distribution, which is equal to zero.

Next, we find the first four L-moments for the doubly truncated logistic distribution. In the following lemma, we derive the moment of order statistics of the random variable from a doubly truncated logistic distribution.

Lemma 3. *The moment of order statistics from the doubly truncated logistic distribution is given by, for $j = 1, 2, \ldots, n$,*

$$\begin{aligned} \mu_{j:n} &= \tfrac{n!}{(j-1)!(n-j)!} \sum_{i=0}^{n-j} \binom{n-j}{i} \tfrac{(-1)^i(-Q)^{i+j-1}}{(P-Q)^{i+j}} \left[PP_1 - QQ_1 + \log\left[\tfrac{1-P}{1-Q}\right]\right] \\ &+ \tfrac{n!}{(j-1)!(n-j)!} \sum_{i=0}^{n-j} \sum_{l=1}^{i+j-1} \binom{n-j}{i} \binom{i+j-1}{l} \tfrac{(-1)^i(-Q)^{i+j-1-l}}{(P-Q)^{i+j}(l+1)} \\ &\times \left[P^{l+1}P_1 - Q^{l+1}Q_1 + \log\left[\tfrac{1-P}{1-Q}\right] + \sum_{s=0}^{l-1} \tfrac{1}{s+1}\left(P^{s+1} - Q^{s+1}\right)\right]. \end{aligned} \tag{27}$$

Proof. The j^{th} moment of order statistics is

$$\begin{aligned}
\mu_{j:n} &= E\left[Z_{j:n}\right] = \tfrac{n!}{(j-1)!(n-j)!}\int_{Q_1}^{P_1} z[F(z)]^{j-1}f(z)[1-F(z)]^{n-j}dz \\
&= \tfrac{n!}{(j-1)!(n-j)!}\int_{Q_1}^{P_1} z\left[\tfrac{1}{P-Q}\left[\tfrac{1}{1+e^{-z}}-Q\right]\right]^{j-1}\left[\tfrac{1}{P-Q}\tfrac{e^{-z}}{(1+e^{-z})^2}\right] \\
&\quad\times\left[1-\left[\tfrac{1}{P-Q}\left[\tfrac{1}{1+e^{-z}}-Q\right]\right]\right]^{n-j}dz \\
&= \tfrac{n!}{(j-1)!(n-j)!}\sum_{i=0}^{n-j}\sum_{l=0}^{i+j-1}\binom{n-j}{i}\binom{i+j-1}{l}\tfrac{(-1)^i(-Q)^{i+j-1-l}}{(P-Q)^{i+j}}I_1,
\end{aligned} \tag{28}$$

where

$$I_1=\int_Q^P \log\left(\frac{t}{1-t}\right)t^l dt=\frac{1}{l+1}\left[P^{l+1}\log\left(\frac{P}{1-P}\right)-Q^{l+1}\log\left(\frac{Q}{1-Q}\right)-\int_Q^P\frac{t^l}{1-t}dt\right], \tag{29}$$

substituting (29) into (28), we obtain

$$\begin{aligned}
\mu_{j:n} &= \tfrac{n!}{(j-1)!(n-j)!}\sum_{i=0}^{n-j}\binom{n-j}{i}\tfrac{(-1)^i(-Q)^{i+j-1}}{(P-Q)^{i+j}}\left[P\log\left(\tfrac{P}{1-P}\right)-Q\log\left(\tfrac{Q}{1-Q}\right)-I_2\right] \\
&+\tfrac{n!}{(j-1)!(n-j)!}\sum_{i=0}^{n-j}\sum_{l=1}^{i+j-1}\binom{n-j}{i}\binom{i+j-1}{l}\tfrac{(-1)^i(-Q)^{i+j-1-l}}{(P-Q)^{i+j}(l+1)} \\
&\times\left[P^{l+1}\log\left(\tfrac{P}{1-P}\right)-Q^{l+1}\log\left(\tfrac{Q}{1-Q}\right)-I_3\right],
\end{aligned} \tag{30}$$

where

$$I_2=\int_Q^P\frac{1}{1-t}dt=-\log(1-t)|_{t=Q}^{t=P}=-\log(1-P)+\log(1-Q), \tag{31}$$

and

$$I_3=\int_Q^P\frac{t^l}{1-t}dt=-\sum_{s=0}^{l-1}\int_Q^P t^s+\int_Q^P\frac{1}{1-t}dt=-\sum_{s=0}^{l-1}\frac{1}{s+1}\left(P^{s+1}-Q^{s+1}\right)-\log(1-P)+\log(1-Q). \tag{32}$$

Finally, by substituting (31) and (32) in (30) and doing some simplification, we obtain the required result. □

Note that:

- By letting $n=j=1$ in Lemma 3, we deduce the first moment established for the doubly truncated logistic distribution.
- Furthermore, letting $Q\to 0$ and $P\to 1$ in Lemma 3 and using Proposition 1 as follows, we deduce the single moments order statistics for the logistic distribution established in (24).

Proposition 1. *Let $j=1,2,\ldots n$ and $n-j$ a non-negative integer. Then,*

$$\sum_{i=0}^{n-j}\binom{n-j}{i}(-1)^i\frac{1}{i+j}=\frac{(j-1)!(n-j)!}{n!},$$

$$\frac{n!}{(j-1)!(n-j)!}\sum_{i=0}^{n-j}\binom{n-j}{i}(-1)^i\frac{1}{i+j}\psi(i+j)=\psi(j)-\psi(n-j+1)-\gamma$$

where γ is Euler's constant.

Proof. For the first equation, we proceed by induction on n. As $n = 1$, it is $1 = 1$, and the proposition immediately follows. Assume now the proposition for n and observe that, since $\binom{n+1-j}{i} = \binom{n-j}{i} + \binom{n-j}{i-1}$, then for $n+1$ it holds:

$$\sum_{i=0}^{n-j+1} \binom{n-j+1}{i} (-1)^i \frac{1}{i+j} = \sum_{i=0}^{n-j} \binom{n-j}{i} (-1)^i \frac{1}{i+j} - \sum_{i=0}^{n-j} \binom{n-j}{i} (-1)^i \frac{1}{i+1+j}.$$

The hypothesis of induction yields

$$\sum_{i=0}^{n-j} \binom{n-j}{i} (-1)^i \frac{1}{i+j} = \frac{(j-1)!(n-j)!}{n!},$$

and

$$\sum_{i=0}^{n-j} \binom{n-j}{i} (-1)^i \frac{1}{i+1+j} = \frac{j!(n-j)!}{(n+1)!} = j\frac{(j-1)!(n-j)!}{(n+1)!}.$$

Therefore, the proposition is proved.

Now for the second equation, we proceed by induction on n. As $n = 1$, it is $\psi(1) = -\gamma$, and the proposition immediately follows. Assume now the proposition for n and observe that, since $\binom{n+1-j}{i} = \binom{n-j}{i} + \binom{n-j}{i-1}$, then for $n+1$ it holds:

$$\begin{aligned} \tfrac{(n+1)!}{(j-1)!(n-j+1)!} & \sum_{i=0}^{n-j+1} \binom{n-j+1}{i} (-1)^i \tfrac{1}{i+j} \psi(i+j) \\ & = \tfrac{(n+1)!}{(j-1)!(n-j+1)!} \sum_{i=0}^{n-j} \binom{n-j}{i} (-1)^i \tfrac{1}{i+j} \psi(i+j) \\ & - \tfrac{(n+1)!}{(j-1)!(n-j+1)!} \sum_{i=0}^{n-j} \binom{n-j}{i} (-1)^i \tfrac{1}{i+1+j} \psi(i+1+j). \end{aligned}$$

The hypothesis of induction yields

$$\frac{(n+1)!}{(j-1)!(n-j+1)!} \sum_{i=0}^{n-j} \binom{n-j}{i} (-1)^i \frac{1}{i+j} \psi(i+j) = \frac{n+1}{n-j+1} (\psi(j) - \psi(n-j+1) - \gamma),$$

and

$$\begin{aligned} & \tfrac{(n+1)!}{(j-1)!(n-j+1)!} \sum_{i=0}^{n-j} \binom{n-j}{i} (-1)^i \tfrac{1}{i+1+j} \psi(i+1+j) \\ & = \tfrac{1}{n-j+1} + \tfrac{j}{n-j+1} (\psi(j) - \psi(n-j+1) - \gamma), \left(\text{by using } \psi(1+j) = \psi(j) + \tfrac{1}{j}\right). \end{aligned}$$

Therefore, we perform some simplification by using $1/(n-j+1) = \psi(n-j+2) - \psi(n-j+1)$, and obtain the required result. □

Lemma 4. *The L-moments for the doubly truncated logistic distribution are given by*

$$\begin{aligned} \lambda_1 & = \tfrac{PP_1 - QQ_1 + \log\left[\frac{1-P}{1-Q}\right]}{P-Q}, \ \lambda_2 = \tfrac{P - Q - PP_1Q + PQQ_1 - (-1+P+Q)\log\left[\frac{1-P}{1-Q}\right]}{(P-Q)^2}, \\ \lambda_3 & = \tfrac{1}{(P-Q)^3} \big(2(-1+Q)Q + P\big(2 + Q^2(P_1 - Q_1)\big) + P^2(-2 + P_1Q - QQ_1) \\ & \quad + \big(2 + (-3+P)P - 3Q + 4PQ + Q^2\big) \log\left[\tfrac{1-P}{1-Q}\right]\big), \\ \lambda_4 & = \tfrac{1}{6(P-Q)^4} \big(Q(-30 + (45 - 16Q)Q) + P^3(16 - 6P_1Q + 6QQ_1) \\ & \quad + 6P\big(5 + Q^2(-7 - P_1Q + QQ_1)\big) + 3P^2(-15 + 2Q(7 - 3P_1Q + 3QQ_1)) \\ & \quad - 6(-1+P+Q)\big(5 + P^2 + (-5+Q)Q + P(-5+8Q)\big) \log\left[\tfrac{1-P}{1-Q}\right]\big). \end{aligned} \tag{33}$$

Proof. The r^{th}, $r = 0, 1, 2, \ldots$, probability weighted moments are obtained easily by the Lemma 3 as

$$\begin{aligned}\beta_r &= \int_{Q_1}^{P_1} z[F(z)]^r f(z)dz = \int_0^1 z_u u^r du = \tfrac{1}{1+r}\mu_{r+1:r+1}, \\ &= \tfrac{(-Q)^r}{(P-Q)^{r+1}}\left[PP_1 - QQ_1 + \log\left[\tfrac{1-P}{1-Q}\right]\right] \\ &+ \tfrac{1}{(P-Q)^{r+1}}\sum_{l=1}^{r}\binom{r}{l}\tfrac{(-Q)^{r-l}}{(l+1)}\left[P^{l+1}P_1 - Q^{l+1}Q_1 + \log\left[\tfrac{1-P}{1-Q}\right] + \sum_{s=0}^{l-1}\tfrac{1}{s+1}\left(P^{s+1} - Q^{s+1}\right)\right],\end{aligned}$$

and by using (6), the proof is completed. □

The L-moment estimators for location parameter ζ and scale parameter α of the random variable of doubly truncated logistic $X = \alpha Z + \zeta$ can be obtained from the first and second L-moments (λ_1, λ_2) in (33) and using the linear transformation as

$$\hat{\zeta} = \hat{\lambda}_1^* - \hat{\alpha}\lambda_1 \text{ and } \hat{\alpha} = \frac{\hat{\lambda}_2^*}{\lambda_2}. \tag{34}$$

where $\hat{\lambda}_1^*$ and $\hat{\lambda}_2^*$ are the sample L-moments of X.

4.4. L-Moments of the Doubly Truncated Generalized Logistic Distribution

The doubly truncated standard generalized logistic pdf

$$f(z) = \tfrac{1}{P-Q}\tfrac{(1-\delta z)^{\frac{1}{\delta}-1}}{\left[1+(1-\delta z)^{1/\delta}\right]^2}, \quad Q_1 < z < P_1 < \tfrac{1}{\delta} \text{ if } 0 < \delta < 1,$$
$$, \tfrac{1}{\delta} < Q_1 < z < P_1 \text{ if } -1 < \delta < 0,$$

with cdf

$$F(z) = \tfrac{1}{P-Q}\left[\tfrac{1}{1+(1-\delta z)^{1/\delta}} - Q\right], \quad Q_1 < z < P_1 < \tfrac{1}{\delta} \text{ if } 0 < \delta < 1,$$
$$, \tfrac{1}{\delta} < Q_1 < z < P_1 \text{ if } -1 < \delta < 0,$$

where Q and $1 - P$ $(0 < Q < P < 1)$ are given by

$$P = F(P_1) \text{ and } Q = F(Q_1),$$

where $F(\cdot)$ is given in the standard generalized logistic distribution. Then,

$$Q_1 = \frac{1}{\delta}\left[1 - \left(\frac{1-Q}{Q}\right)^{\delta}\right] \text{ and } P_1 = \frac{1}{\delta}\left[1 - \left(\frac{1-P}{P}\right)^{\delta}\right].$$

The quantile is

$$z_u = \frac{1}{\delta}\left[1 - \left[\frac{1 - [u(P-Q)+Q]}{u(P-Q)+Q}\right]^{\delta}\right], \quad 0 < u < 1.$$

The k^{th}, $k = 1, 2, 3, 4$, moment of Z is

$$E[Z^k] = \frac{\sum_{j=0}^{k}(-1)^j\binom{k}{j}[\beta(P; 1 - j\delta, j\delta + 1) - \beta(Q; 1 - j\delta, j\delta + 1)]}{\delta^k(P-Q)}, \quad |\delta| < \frac{1}{k}.$$

where $\beta(\cdot\,;\,1-j\delta,\ j\delta+1)$ is the lower incomplete beta function and can be defined by the variable limit integrals

$$\beta(x;\,a,\,b)=\int_0^x t^{a-1}(1-t)^{b-1}dt,\ 0\le x\le 1,\ a,b>0.$$

Note that by letting $Q\to 0$ and $P\to 1$, we deduce the moment for the generalized logistic distribution. Furthermore, by letting the shape parameter $\delta\to 0$, we deduce the mean of the standard doubly truncated logistic distribution.

Now, we are about to find the first four L-moments for the doubly truncated generalized logistic distribution. In the following lemma, we derive the first moment for the order statistic of the random variable from a doubly truncated generalized logistic distribution.

Lemma 5. *The moments of order statistics from the doubly truncated generalized logistic distribution are given by, for* $j=1,2,\ldots,n$,

$$\begin{aligned}\mu_{j:n}&=\tfrac{1}{\delta}\Bigg[1-\tfrac{n!}{(j-1)!(n-j)!}\sum_{i=0}^{n-j}\sum_{l=0}^{i+j-1}\binom{n-j}{i}\binom{i+j-1}{l}\tfrac{(-1)^i(-Q)^{i+j-1-l}}{(P-Q)^{i+j}}\\&\times[\beta(P;1-\delta+l,1+\delta)-\beta(Q;1-\delta+l,1+\delta)]\Bigg],\ \ |\delta|<1.\end{aligned}\tag{35}$$

Proof. The j^{th} moment of order statistics

$$\begin{aligned}\mu_{j:n}&=E\big[Z_{j:n}\big]=\tfrac{n!}{(j-1)!(n-j)!}\int_{Q_1}^{P_1}z[F(z)]^{j-1}f(z)[1-F(z)]^{n-j}dz\\&=\tfrac{n!}{(j-1)!(n-j)!}\int_0^1 z_u u^{j-1}(1-u)^{n-j}du=\tfrac{n!}{(j-1)!(n-j)!}\tfrac{1}{\delta}[I_1-I_2],\end{aligned}\tag{36}$$

where

$$I_1=\int_0^1 u^{j-1}(1-u)^{n-j}du=\beta(j,n-j+1),\tag{37}$$

and

$$\begin{aligned}I_2&=\int_0^1 u^{j-1}(1-u)^{n-j}\left[\tfrac{1-[u(P-Q)+Q]}{[u(P-Q)+Q]}\right]^{\delta}du\\&=\sum_{i=0}^{n-j}\sum_{l=0}^{i+j-1}\binom{n-j}{i}\binom{i+j-1}{l}\tfrac{(-1)^i(-Q)^{i+j-1-l}}{(P-Q)^{i+j}}\\&\times[\beta(P;1-\delta+l,1+\delta)-\beta(Q;1-\delta+l,1+\delta)],\quad|\delta|<1.\end{aligned}\tag{38}$$

Substituting (37) and (38) in (36), we obtain (35) and thus complete the proof. □

Note that:

- By letting $n=j=1$ in Lemma 5, we deduce the first moment established for the doubly truncated generalized logistic distribution.
- Furthermore, by letting $Q\to 0$ and $P\to 1$ in Lemma 5 and using Proposition 2, we have the single moments order statistics established in (23) from the generalized logistic distribution.
- By letting the shape parameter $\delta\to 0$ in Lemma 5, we deduce the first moment for the order statistic of the random variable from the doubly truncated logistic distribution in Lemma 3.

Proposition 2. *Let $j = 1, 2, \ldots n$ and $n - j$ a non-negative integer. Then,*

$$\frac{n!}{(j-1)!(n-j)!}\sum_{i=0}^{n-j}\binom{n-j}{i}(-1)^i\beta(i+j-\delta,1+\delta) = \frac{\Gamma(j-\delta)\Gamma(n-j+1+\delta)}{\Gamma(j)\Gamma(n-j+1)},$$

where $|\delta| < 1$.

Proof. We proceed by induction on n. As $n = 1$, it is $\beta(1-\delta,1+\delta) = \Gamma(1-\delta)\Gamma(1+\delta)$, and the proposition immediately follows. Assume now the proposition for n and observe that, since $\binom{n+1-j}{i} = \binom{n-j}{i} + \binom{n-j}{i-1}$, then for $n+1$ it holds:

$$\begin{aligned}\frac{(n+1)!}{(j-1)!(n-j+1)!} & \sum_{i=0}^{n-j+1}\binom{n-j+1}{i}(-1)^i\beta(i+j-\delta,1+\delta) \\ &= \frac{(n+1)!}{(j-1)!(n-j+1)!}\sum_{i=0}^{n-j}\binom{n-j}{i}(-1)^i\beta(i+j-\delta,1+\delta) \\ &- \frac{(n+1)!}{(j-1)!(n-j+1)!}\sum_{i=0}^{n-j}\binom{n-j}{i}(-1)^i\beta(i+1+j-\delta,1+\delta).\end{aligned}$$

The hypothesis of induction yields

$$\frac{(n+1)!}{(j-1)!(n-j+1)!}\sum_{i=0}^{n-j}\binom{n-j}{i}(-1)^i\beta(i+j-\delta,1+\delta) = \frac{n+1}{n-j+1}\frac{\Gamma(j-\delta)\Gamma(n-j+1+\delta)}{\Gamma(j)\Gamma(n-j+1)},$$

and

$$\begin{aligned}\frac{(n+1)!}{(j-1)!(n-j+1)!} & \sum_{i=0}^{n-j}\binom{n-j}{i}(-1)^i\beta(i+1+j-\delta,1+\delta) \\ &= \frac{j}{n-j+1}\frac{\Gamma(1+j-\delta)\Gamma(n-j+1+\delta)}{\Gamma(1+j)\Gamma(n-j+1)}, \\ & \text{(by using } \Gamma(1+j-\delta) = (j-\delta)\Gamma(j-\delta) \text{ and } \Gamma(1+j) = j\Gamma(j)\text{)} \\ &= \frac{j-\delta}{n-j+1}\frac{\Gamma(j-\delta)\Gamma(n-j+1+\delta)}{\Gamma(j)\Gamma(n-j+1)},\end{aligned}$$

therefore, we perform some simplification by using

$$(n-j+1+\delta)\Gamma(n-j+1+\delta)/(n-j+1)\Gamma(n-j+1) = \Gamma(n-j+2+\delta)/\Gamma(n-j+2),$$

and we obtain the required result. □

Lemma 6. *The first four L-moments for doubly truncated generalized logistic distribution are*

$$\begin{aligned}\lambda_1 &= \tfrac{1}{(P-Q)\delta}[(P-Q) - (\beta(P;1-\delta,1+\delta) - \beta(Q;1-\delta,1+\delta))], \\ \lambda_2 &= \tfrac{1}{(P-Q)^2\delta}[(P+Q)(\beta(P;1-\delta,1+\delta) - \beta(Q;1-\delta,1+\delta)) \\ & -2(\beta(P;2-\delta,1+\delta) - \beta(Q;2-\delta,1+\delta))], \\ \lambda_3 &= \tfrac{1}{(P-Q)^3\delta}\big[-\big(P^2+4PQ+Q^2\big)(\beta(P;1-\delta,1+\delta) - \beta(Q;1-\delta,1+\delta)) \\ & +6(P+Q)(\beta(P;2-\delta,1+\delta) - \beta(Q;2-\delta,1+\delta)) \\ & -6(\beta(P;3-\delta,1+\delta) - \beta(Q;3-\delta,1+\delta))\big], \\ \lambda_4 &= \tfrac{1}{(P-Q)^4\delta}\big[\big(P^3+9P^2Q+9PQ^2+Q^3\big)(\beta(P;1-\delta,1+\delta) - \beta(Q;1-\delta,1+\delta)) \\ & -12\big(P^2+3PQ+Q^2\big)(\beta(P;2-\delta,1+\delta) - \beta(Q;2-\delta,1+\delta)) \\ & +30(P+Q)(\beta(P;3-\delta,1+\delta) - \beta(Q;3-\delta,1+\delta)) \\ & -20(\beta(P;4-\delta,1+\delta) - \beta(Q;4-\delta,1+\delta))\big].\end{aligned} \tag{39}$$

and using the above L-moments, we can obtain τ_3 and τ_4.

Proof. By applying Lemma 5, β_r becomes:

$$\begin{aligned}\beta_r &= \int_{Q_1}^{P_1} z[F(z)]^r f(z)dz = \int_0^1 z_u u^r du = \tfrac{1}{1+r}\mu_{r+1:r+1} \\ &= \tfrac{1}{\delta}\left(\tfrac{1}{r+1} - \frac{\sum_{l=0}^{r}\binom{r}{l}(-Q)^{r-l}(\beta(P;1-\delta+l,1+\delta)-\beta(Q;1-\delta+l,1+\delta))}{(P-Q)^{r+1}}\right), \ |\delta|<1.\end{aligned}$$

Since β_r is given as

$$\begin{aligned}\beta_0 &= \tfrac{1}{\delta}\Big[1 - \tfrac{1}{P-Q}(\beta(P;1-\delta,1+\delta) - \beta(Q;1-\delta,1+\delta))\Big], \\ \beta_1 &= \tfrac{1}{\delta}\Big[\tfrac{1}{2} - \tfrac{1}{(P-Q)^2}(-Q[\beta(P;1-\delta,1+\delta) - \beta(Q;1-\delta,1+\delta)] \\ &\quad +[\beta(P;2-\delta,1+\delta) - \beta(Q;2-\delta,1+\delta)])], \\ \beta_2 &= \tfrac{1}{\delta}\Big[\tfrac{1}{3} - \tfrac{1}{(P-Q)^3}\left(Q^2[\beta(P;1-\delta,1+\delta) - \beta(Q;1-\delta,1+\delta)]\right. \\ &\quad -2Q[\beta(P;2-\delta,1+\delta) - \beta(Q;2-\delta,1+\delta)] \\ &\quad +[\beta(P;3-\delta,1+\delta) - \beta(Q;3-\delta,1+\delta)])], \\ \beta_3 &= \tfrac{1}{\delta}\Big[\tfrac{1}{4} - \tfrac{1}{(P-Q)^4}\left(-Q^3[\beta(P;1-\delta,1+\delta) - \beta(Q;1-\delta,1+\delta)]\right. \\ &\quad +3Q^2[\beta(P;2-\delta,1+\delta) - \beta(Q;2-\delta,1+\delta)] \\ &\quad -3Q[\beta(P;3-\delta,1+\delta) - \beta(Q;3-\delta,1+\delta)] \\ &\quad +[\beta(P;4-\delta,1+\delta) - \beta(Q;4-\delta,1+\delta)])].\end{aligned}$$

and by using (6), the proof is completed. □

If we denote λ_r in (39) by $\lambda_r(\delta)$, then the L-moments estimators for location parameter ζ, scale parameter α, and shape parameter δ of the random variable of doubly truncated generalized logistic $X = \alpha Z + \zeta$ can be obtained from the first and second L-moments $(\lambda_1(\delta), \lambda_2(\delta))$ and L-skewness $\tau_3(\delta)(\tau_3(\delta) = \lambda_3(\delta)/\lambda_2(\delta))$ in (39) and using the linear transformation, which are measures of location, scale, and skewness, respectively, as solved numerically in the three systems of the nonlinear equations:

$$\hat{\zeta} = \hat{\lambda}_1^* - \hat{\alpha}\lambda_1(\hat{\delta}), \hat{\alpha} = \frac{\hat{\lambda}_2^*}{\lambda_2(\hat{\delta})}, \text{ and } \hat{\tau}_3^* = \tau_3(\hat{\delta}). \quad (40)$$

where $\hat{\lambda}_1^*$ and $\hat{\lambda}_2^*$ are the sample L-moments of X and $\hat{\tau}_3^*$ is the sample L-moment ratios.

5. Particular Relationships Based on L-Moments

In this section, we establish some particular recurrence relations between the L-moments satisfying for logistic, generalized logistic, doubly truncated logistic, and doubly truncated generalized logistic distributions that enables computation and allows for evaluation of all the L-moments $\lambda_r(r \geq 2)$, starting from λ_1 in a simple recurrent manner, where the calculation of L-moments in the traditional way of greater degrees depends on special functions that need more mathematical calculations and special programs.

The following lemma is important throughout the results in this section.

Lemma 7. *For $r = 0, 1, 2, 3, \ldots$, the relation between the L-moments in (3) and moments of order statistics in (2) are*

$$\mu_{r+1:r+1} = (r+1)\sum_{i=0}^{r} c_{r,i}\lambda_{i+1}, \quad (41)$$

and

$$\mu_{1:r+1} = (r+1)\sum_{i=0}^{r} (-1)^i c_{r,i}\lambda_{i+1}, \quad (42)$$

where the coefficients $c_{r,i}$ are given as

$$c_{r,i} = (2i+1)\int_0^1 u^r P_i^*(u)\,du = (2i+1)\sum_{k=0}^{i} p_{i,k}^* \frac{1}{r+k+1}, i = 0,1,2,\ldots, \quad (43)$$

and $p_{r,k}^*$ is given in (5).

Proof. The function u^r, which is sequence integrable on $[0,1]$, may be expressed in terms of $P_i^*(u)$ as (see [37])

$$u^r = \sum_{i=0}^{r} c_{r,i} P_i^*(u), 0 \le u \le 1.$$

Multiplying both sides by x_u and integrating over u, we obtain

$$\int_0^1 x_u u^r du = \sum_{i=0}^{r} c_{r,i} \int_o^1 x_u P_i^*(u) du,$$

then (41) is proved.

The function $(1-u)^r$, which is sequence integrable on $[0,1]$, may be expressed in terms of $P_i^*(1-u)$ as (see [37])

$$(1-u)^r = \sum_{i=0}^{r} c_{r,i} P_i^*(1-u),\ \ 0 \le 1-u \le 1,$$

by using the property of a shifted Legendre polynomial function from Hetyei [38]:

$$(-1)^i P_i^*(-u) = P_i^*(u+1),$$

then,

$$P_i^*(1-u) = P_i^*(-u+1) = (-1)^i P_i^*(u).$$

So, we have

$$(1-u)^r = \sum_{i=0}^{r} (-1)^i c_{r,i} P_i^*(u).$$

Again, multiplying both sides by x_u and integrating over u, we obtain

$$\int_0^1 x_u (1-u)^r du = \sum_{i=0}^{r} (-1)^i c_{r,i} \int_o^1 x_u P_i^*(u) du,$$

then (42) is proved. □

5.1. Relations for Logistic Distribution

In this subsection, we establish recurrence relations satisfied by L-moments from a logistic distribution.

Lemma 8. *For $r = 1,2,\ldots.$, then the L-moments from standard logistic distribution satisfy*

$$\lambda_{r+1} = \frac{1}{(r+1)(-1)^r c_{r,r}} \left[\sum_{i=0}^{r-1} (-1)^i (-(r+1)c_{r,i} + r c_{r-1,i}) \lambda_{i+1} - \frac{1}{r} \right]. \quad (44)$$

where λ_1 and $c_{.,.}$ are given in (21) and (43), respectively.

Proof. The recurrence relation of order statistics from standard logistic distribution follows (see [3]):

$$\mu_{1:r+1} = \mu_{1:r} - \frac{1}{r}, r \ge 1,$$

Substituting from (42), we have

$$(r+1)\sum_{i=0}^{r}(-1)^i c_{r,i}\lambda_{i+1} = r\sum_{i=0}^{r-1}(-1)^i c_{r-1,i}\lambda_{i+1} - \frac{1}{r}.$$

Therefore,

$$\begin{aligned}(r+1)(-1)^r c_{r,r}\lambda_{r+1} &= -(r+1)\textstyle\sum_{i=0}^{r-1}(-1)^i c_{r,i}\lambda_{i+1} + r\sum_{i=0}^{r-1}(-1)^i c_{r-1,i}\lambda_{i+1} - \frac{1}{r} \\ &= \textstyle\sum_{i=0}^{r-1}(-1)^i(-(r+1)c_{r,i} + rc_{r-1,i})\lambda_{i+1} - \frac{1}{r},\end{aligned}$$

by simplifying the resulting expression, we obtain the relation. □

5.2. Relations for Generalized Logistic Distribution

In this subsection, we establish recurrence relations satisfied by L-moments from a generalized logistic distribution.

Lemma 9. *For $r = 1, 2, \ldots,$ then the L-moments from standard generalized logistic distribution satisfy*

$$\lambda_{r+1} = \frac{1}{(r+1)(-1)^r c_{r,r}}\left[\sum_{i=0}^{r-1}(-1)^i(-(r+1)c_{r,i} + (r+\delta)c_{r-1,i})\lambda_{i+1} - \frac{1}{r}\right]. \tag{45}$$

where λ_1 and $c_{.,.}$ are given in (25) and (43), respectively.

Proof. The recurrence relation for the single moments of order statistics follows (see [3]):

$$\mu_{1:r+1} = \left(1+\frac{\delta}{r}\right)\mu_{1:n} - \frac{1}{r}\ ,\ r \geq 1,$$

Substituting from (42), we have

$$(r+1)\sum_{i=0}^{r}(-1)^i c_{r,i}\lambda_{i+1} = \left(1+\frac{\delta}{r}\right)r\sum_{i=0}^{r-1}(-1)^i c_{r-1,i}\lambda_{i+1} - \frac{1}{r}.$$

Therefore,

$$\begin{aligned}(r+1)(-1)^r c_{r,r}\lambda_{r+1} &= -(r+1)\textstyle\sum_{i=0}^{r-1}(-1)^i c_{r,i}\lambda_{i+1} + \left(1+\frac{\delta}{r}\right)r\sum_{i=0}^{r-1}(-1)^i c_{r-1,i}\lambda_{i+1} - \frac{1}{r} \\ &= \textstyle\sum_{i=0}^{r-1}(-1)^i(-(r+1)c_{r,i} + (r+\delta)c_{r-1,i})\lambda_{i+1} - \frac{1}{r},\end{aligned}$$

by simplifying the resulting expression, we obtain the relation. □

Letting the shape parameter $\delta \to 0$ in Lemma 9, we deduce the recurrence relation for L-moments from the standard logistic distribution in Lemma 8.

5.3. Relations for Doubly Truncated Logistic Distribution

Recurrence relations for doubly truncated logistic distribution are given by Lemma 10 in this subsection.

Lemma 10.

$$\lambda_2 = (1-B)\lambda_1 - AP_1 - D_1 \tag{46}$$

and for $r \geq 2$,

$$\begin{aligned}\lambda_{r+1} &= \tfrac{1}{(r+1)(-1)^r c_{r,r}}\Big[\textstyle\sum_{i=0}^{r-2}(-1)^i[-(r+1)c_{r,i} + rBc_{r-1,i} + (r-1)Ac_{r-2,i}]\lambda_{i+1} \\ &\quad +(-1)^{r-1}[-(r+1)c_{r,r-1} + rBc_{r-1,r-1}]\lambda_r + D_r\Big],\end{aligned} \tag{47}$$

where λ_1 and $c_{.,.}$ are given in (33) and (43), respectively, and

$$A = \frac{P_2}{P-Q},\ B = \frac{(2P-1)}{P-Q},\ \text{and}\ D_m = -\frac{1}{P-Q}\left(Q_1 Q_2 + \frac{1}{m}\right)\ for\ m \geq 1. \tag{48}$$

Proof. First, before beginning the proof, denote that

$$P_2 = P(1-P)/(P-Q)\ \text{and}\ Q_2 = Q(1-Q)/(P-Q),$$

and we simplify the following recurrence relations (see [3]):

$$\mu_{1:2} = Q_1 + \frac{1}{P-Q}[P_2(P_1 - Q_1) + (2P-1)(\mu_{1:1} - Q_1) - 1],$$

for $n \geq 2$,

$$\mu_{1:n+1} = Q_1 + \frac{1}{P-Q}\left[P_2(\mu_{1:n-1} - Q_1) + (2P-1)(\mu_{1:n} - Q_1) - \frac{1}{n}\right].$$

Note that by letting $Q \to 0$ and $P \to 1$, we have the recurrence relation for the single moments of the standard logistic distribution, so that we can rewrite them as

$$\mu_{1:2} = AP_1 + B\mu_{1:1} + D_1, \tag{49}$$

and for $n \geq 2$:

$$\mu_{1:n+1} = A\mu_{1:n-1} + B\mu_{1:n} + D_n, \tag{50}$$

where A, B, and D_m are given in (48).

Now, to prove (46), we have (49), which gives

$$\mu_{1:1} = \lambda_1, \tag{51}$$

and $\mu_{1:2}$ can be found as follows by using (42):

$$\mu_{1:2} = 2\sum_{i=0}^{1}(-1)^i c_{1,i}\lambda_{i+1} = \lambda_1 - \lambda_2, \tag{52}$$

So, by substituting (51) and (52) into (49), it reduces to

$$\lambda_1 - \lambda_2 = AP_1 + B\lambda_1 + D_1.$$

By ordering this equation, we obtain the relation in (46).

Now, the second equation in the lemma can be proved by using (50), where we can find $\mu_{1:r-1}$, $\mu_{1:r}$ and $\mu_{1:r+1}$ by using (42), as follows:

$$\mu_{1:r-1} = (r-1)\sum_{i=0}^{r-2}(-1)^i c_{r-2,i}\lambda_{i+1}, \tag{53}$$

$$\mu_{1:r} = r\sum_{i=0}^{r-1}(-1)^i c_{r-1,i}\lambda_{i+1} = r(-1)^{r-1}c_{r-1,r-1}\lambda_r + r\sum_{i=0}^{r-2}(-1)^i c_{r-1,i}\lambda_{i+1}, \tag{54}$$

$$\begin{aligned}\mu_{1:r+1} &= (r+1)\sum_{i=0}^{r}(-1)^i c_{r,i}\lambda_{i+1} \\ &= (r+1)(-1)^r c_{r,r}\lambda_{r+1} + (r+1)(-1)^{r-1}c_{r,r-1}\lambda_r + (r+1)\sum_{i=0}^{r-2}(-1)^i c_{r,i}\lambda_{i+1}.\end{aligned} \tag{55}$$

Upon substituting (53), (54), and (55) in (50) and simplifying the resulting expression, we obtain the relation given in (47). □

Note that by letting $Q \to 0$ and $P \to 1$ in Lemma 10, we obtain the simple recurrence relations between L-moments of logistic distribution in Lemma 8.

5.4. Relations for Doubly Truncated Generalized Logistic Distribution

In this subsection, we establish the recurrence relation for single moment order statistics from the standard doubly truncated generalized logistic distribution in Lemma 11. Then, recurrence relations for the doubly truncated generalized logistic distribution between the L-moments are given by Lemma 12.

Lemma 11. *For* $n \geq 2$,

$$\mu_{1:n+1} = A\mu_{1:n-1} + B_n\mu_{1:n} + D_n, \tag{56}$$

and

$$\mu_{1:2} = AP_1 + B_1\mu_{1:1} + D_1, \tag{57}$$

where

$$A = \frac{P_2}{P-Q},\ B_m = \frac{1}{P-Q}\left[(2P-1) + \frac{\delta}{m}\right],\ \text{and}\ D_m = -\frac{1}{P-Q}\left(Q_1Q_2 + \frac{1}{m}\right)\ \text{for}\ m \geq 1. \tag{58}$$

Proof. *For* $n \geq 1$, denoting that

$$P_2 = P(1-P)/(P-Q)\ \text{and}\ Q_2 = Q(1-Q)/(P-Q),$$

let us consider the characterizing differential equation for the doubly truncated generalized logistic population as follows:

$$\begin{aligned}(1-\delta z)f(z) &= (1-2Q)F(z) - (P-Q)[F(z)]^2 + Q_2 \\ &= (1-P-Q)F(z) + (P-Q)F(z)[1-F(z)] + Q_2,\end{aligned}$$

and

$$f_{1:n}(z) = nf(z)[1-F(z)]^{n-1}, Q_1 < z < P_1,$$

then,

$$\begin{aligned}&1-\delta\mu_{1:n} \\ &= n\Bigg[(1-P-Q)\int_{Q_1}^{P_1} F(z)[1-F(z)]^{n-1}dz + (P-Q)\int_{Q_1}^{P_1} F(z)[1-F(z)]^{n}dz \\ &\quad + Q_2\int_{Q_1}^{P_1}[1-F(z)]^{n-1}dz\Bigg],\end{aligned} \tag{59}$$

By integrating by parts, treating 1 for integration, and the rest of the integrands for differentiation, we obtain

$$1-\delta\mu_{1:n} = n[(1-P-Q)(\mu_{1:n-1} - \mu_{1:n}) + (P-Q)(\mu_{1:n} - \mu_{1:n+1}) + Q_2(\mu_{1:n-1} - Q_1)], \tag{60}$$

The relation in (56) follows simply by rewriting (60).
Relation (57) is obtained by setting $n = 1$ in (59) and simplifying. □

Note that:

- By letting the shape parameter $\delta \to 0$ in Lemma 11, we deduce the recurrence relations established in (49) and (50) for the single moments of order statistics from the doubly truncated logistic distribution.
- By letting $Q \to 0$ and $P \to 1$, we deduce the recurrence relations for the generalized logistic distribution, established in the proof of Lemma 9.

Lemma 12.

$$\lambda_2 = (1-B_1)\lambda_1 - AP_1 - D_1, \tag{61}$$

and for $r \geq 2$,

$$\lambda_{r+1} = \frac{1}{(r+1)(-1)^r c_{r,r}} \Big[\sum_{i=0}^{r-2} (-1)^i [-(r+1)c_{r,i} + rB_r c_{r-1,i} + (r-1)Ac_{r-2,i}] \lambda_{i+1} + (-1)^{r-1} [-(r+1)c_{r,r-1} + rB_r c_{r-1,r-1}] \lambda_r + D_r \Big], \quad (62)$$

where λ_1 *and* $c_{\cdot,\cdot}$ *are given in (39) and (43), respectively, and* A, B_r, and D_r *are given in (58).*

Proof. This lemma has the same proof method that we used in Lemma 10, but by taking (56) and (57) to prove (61) and (62), respectively. □

Note that:

- By letting $Q \to 0$ and $P \to 1$ in Lemma 12, we have the recurrence relations between L-moments established in Lemma 9 from generalized logistic distribution.
- By letting the shape parameter $\delta \to 0$ in Lemma 12, we obtain the recurrence relations between L-moments of the doubly truncated logistic distribution in Lemma 10.

The results in Lemmas 8–12 can be applied in different fields that have actual data sets from the logistics and generalized logistics distributions. These include network analysis (see [11]), statistical inference, (see [39,40]), and rainfall modeling (see [41]).

6. Conclusions

In this paper, the L-moments are derived for some distributions, such as logistic, generalized logistic, doubly truncated logistic, and doubly truncated generalized logistic. Methods of estimation by L-moment are used to obtain the unknown parameters for logistic, generalized logistic, doubly truncated logistic, and doubly truncated generalized logistic distributions. Finally, some new recurrence relations based on L-moment are established and used for calculating the higher moments, where sometimes calculating the moments of order statistics for certain distributions may not be explicit, so recurrence relations are used to calculate higher order moments using lower order moments to reduce the risk of approximation in numerical calculations, which is very helpful. In the future, theoretical results can be utilized in several directions, such as the process of estimating unknown values using the modified moments method, and to some applications for linear moments, especially in electrical engineering, architecture, natural sciences and network analysis.

Author Contributions: Conceptualization, K.S.S. and N.R.A.-S.; methodology, K.S.S. and N.R.A.-S.; software, K.S.S. and N.R.A.-S.; validation, K.S.S. and N.R.A.-S.; formal analysis, K.S.S. and N.R.A.-S.; investigation, K.S.S. and N.R.A.-S.; resources, K.S.S. and N.R.A.-S.; writing—original draft preparation, N.R.A.-S.; writing—review and editing, K.S.S. and N.R.A.-S.; visualization, K.S.S. and N.R.A.-S.; supervision, K.S.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the authors.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors would like to thank the referees for their helpful comments, which improved the presentation of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Arnold, B.C.; Balakrishnan, N.; Nagaraja, H.N. *A First Course in Order Statistics*; Wiley: New York, NY, USA, 1992.
2. David, H.; Nagaraja, H.N. *Order Statistics*, 3rd ed.; Wiley: New York, NY, USA, 2003.
3. Balakrishnan, N.; Rao, C.R. (Eds.) *Handbook of Statistics: Order Statistics: Theory and Methods*, 1st ed.; Elsevier Science (North-Holland): Amsterdam, The Netherlands, 1998; Volume 16.
4. Hosking, J.R.M. L-Moments: Analysis and estimation of distributions using linear combinations of order statistics. *J. R. Stat. Soc. Ser. B Methodol.* **1990**, *52*, 105–124. [CrossRef]

5. Hosking, J.R.M. Moments or L moments? An example comparing two measures of distributional shape. *Am. Stat.* **1992**, *46*, 186–189. [CrossRef]
6. Sillitto, G.P. Derivation of approximants to the inverse distribution function of a continuous univariate population from the order statistics of a sample. *Biometrika* **1969**, *56*, 641–650. [CrossRef]
7. Greenwood, J.A.; Landwehr, J.M.; Matalas, N.C.; Wallis, J.R. Probability weighted moments: Definition and relation to parameters of several distributions expressible in inverse form. *Water Resour. Res.* **1979**, *15*, 1049–1054. [CrossRef]
8. Karian, Z.A.; Dudewicz, E.J. Comparison of GLD fitting methods: Superiority of percentile fits to moments in L^2 norm. *J. Iran. Stat. Soc.* **2003**, *2*, 171–187.
9. Sahu, R.T.; Verma, M.K.; Ahmad, I. Regional Frequency Analysis Using L-Moment Methodology-A Review. In *Recent Trends in Civil Engineering (Lecture Notes in Civil Engineering)*; Pathak, K.K., Bandara, J.M.S.J., Agrawal, R., Eds.; Springer: Singapore, 2021; Volume 77, pp. 811–832.
10. Domański, P.D.; Jankowski, R.; Dziuba, K.; Góra, R. Assessing Control Sustainability Using L-Moment Ratio Diagrams. *Electronics* **2023**, *12*, 2377. [CrossRef]
11. Anderson, T.S. Statistical L-moment and L-moment Ratio Estimation and their Applicability in Network Analysis. Ph.D. Thesis, Air Force Institute of Technology, Air University, OH, USA, 2019.
12. Fallahgoul, H.; Mancini, L.; Stoyanov, S.V. *An L-Moment Approach for Portfolio Choice under Non-Expected Utility*; Working Paper 18–65; Swiss Finance Institute Research Paper: Geneva, Switzerland, 2023.
13. Landwehr, J.M.; Matalas, N.C.; Wallis, J.R. Estimation of parameters and quantiles of Wakeby distributions. *Water Resour. Res.* **1979**, *15*, 1362–1379.
14. Landwehr, J.M.; Matalas, N.C.; Wallis, J.R. Probability weighted moments compared with some traditional techniques in estimating Gumbel parameters and quantiles. *Water Resour. Res.* **1979**, *15*, 1055–1064. [CrossRef]
15. Landwehr, J.M.; Matalas, N.C.; Wallis, J.R. Quantile estimation with more or less floodlike distributions. *Water Resour. Res.* **1980**, *16*, 547–555. [CrossRef]
16. Hosking, J.R.M. Maximum-likelihood estimation of the parameters of the generalized extreme-value distribution. *Appl. Stat.* **1985**, *34*, 301–310. [CrossRef]
17. Hosking, J.R.M. *Some Theoretical Results Concerning L-Moments*; Research Report RC14492; T. J. Watson Research Center (IBM Research Division): Yorktown Heights, NY, USA, 1989.
18. Royston, P. Which measures of skewness and kurtosis are best? *Stat. Med.* **1992**, *11*, 333–343. [CrossRef]
19. Vogel, R.M.; Fennessey, N.M. L moment diagrams should replace product moment diagrams. *Water Resour. Res.* **1993**, *29*, 1745–1752. [CrossRef]
20. Zafirakou-Koulouris, A.; Vogel, R.M.; Craig, S.M.; Habermeier, J. L-moment diagrams for censored observations. *Water Resour. Res.* **1998**, *34*, 1241–1249. [CrossRef]
21. Elamir, E.A.; Seheult, A.H. Control charts based on linear combinations of order statistics. *J. Appl. Stat.* **2001**, *28*, 457–468. [CrossRef]
22. Locke, C.; Spurrier, J. The use of U-statistics for testing normality against non-symmetric altematives. *Biometrika* **1976**, *63*, 143–147. [CrossRef]
23. Hosking, J.R.M. *Fortran Routines for Use with the Method of L-Moments*, 3rd ed.; Research Report RC20525; T. J. Watson Research Center (IBM Research Division): Yorktown Heights, NY, USA, 1996.
24. Hosking, J.R.M. The four-parameter kappa distribution. *IBM J. Res. Dev.* **1994**, *38*, 251–258. [CrossRef]
25. Asquith, W.H. Univariate Distributional Analysis with L-Moment Statistics Using R. Ph.D. Thesis, Texas Tech University, Lubbock, TX, USA, 2011.
26. Hosking, J.R.M. Some theory and practical uses of trimmed L-moments. *J. Stat. Plan. Inference* **2007**, *137*, 3024–3039. [CrossRef]
27. Koepp, W. *Hypergeometric Summation: An Algorithmic Approach to Summation and Special Functions Identities*; Vieweg: Braunschweig, Germany, 1998.
28. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions*; Dover: New York, NY, USA, 1972.
29. Sadov, S. Coupling of the Legendre Polynomials with Kernels |x-y|α and ln|x-y|. Available online: http://arxiv.org/abs/math/0310063v1 (accessed on 3 January 2023).
30. Obsieger, B. *Numerical Methods III—Approximation of Functions*; University-Books.eu; University of Rijeka: Rijeka, Croatia, 2011.
31. Cher, C.-T. Identification of linear Distributed systems by using Legendre polynomials. *J. Lee-Ming Inst. Technol.* **1985**, *3*, 285–295.
32. Balakrishnan, N. *Handbook of the Logistic Distribution*; Marcel Dekker: New York, NY, USA, 1992.
33. Walck, C. *Handbook on Statistical Distributions for Experimentalists*; Report number SUF-PFY/96-01; University of Stockholm: Stockholm, Sweden, 2007.
34. Hamdan, M.S. The Properties of L-moments Compared to Conventional Moments. Master's Thesis, The Islamic University of Gaza, Gaza, Palestine, 2009.
35. Burr, I.W. Cumulative frequency functions. *Ann. Math. Stat.* **1942**, *13*, 215–232. [CrossRef]
36. Gupta, S.S.; Balakrishnan, N. *Logistic Order Statistics and Their Properties*; Defense Technical Information Center: Fort Belvoir, VA, USA, 1990.
37. Sweilam, N.H.; Khader, M.M.; Mahdy, A.M.S. Computational methods for fractional differential equations generated by optimization problem. *J. Fract. Calc. Appl.* **2012**, *3*, 1–12.

38. Hetyei, G. Shifted Jacobi Polynomials and Delannoy Number. Available online: http://arxiv.org/abs/0909.5512?context=math.CO (accessed on 17 April 2023).
39. Usman, S.; Ishfaq, A.; Ibrahim, M.A.; Nursel, K.; Muhammad, H. Variance estimation based on L-moments and auxiliary information. *Math. Popul. Stud.* **2022**, *29*, 31–46.
40. Usman, S.; Ishfaq, A.; Ibrahim, M.A.; Nadia, H.; Muhammad, H. A novel family of variance estimators based on L-moments and calibration approach under stratified random sampling. *Commun. Stat.-Simul. Comput.* **2023**, *52*, 3782–3795.
41. Nain, M.; Hooda, B.K. Regional Frequency Analysis of Maximum Monthly Rainfall in Haryana State of India Using L-Moments. *J. Reliab. Stat. Stud.* **2021**, *14*, 33–56. [CrossRef]