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Generalized Reynolds Operators on Lie-Yamaguti Algebras

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Abstract: In this paper, the notion of generalized Reynolds operators on Lie-Yamaguti algebras is introduced, and the cohomology of a generalized Reynolds operator is established. The formal deformations of a generalized Reynolds operator are studied using the first cohomology group. Then, we show that a Nijenhuis operator on a Lie-Yamaguti algebra gives rise to a representation of the deformed Lie-Yamaguti algebra and a 2-cocycle. Consequently, the identity map will be a generalized Reynolds operator on the deformed Lie-Yamaguti algebra. We also introduce the notion of a Reynolds operator on a Lie-Yamaguti algebra, which can serve as a special case of generalized Reynolds operators on Lie-Yamaguti algebras.

Keywords: generalized Reynolds operator; Lie-Yamaguti algebra; Nijenhuis operator; Reynolds operator

MSC: 17B38; 17B60; 17B56; 17D99

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1. Introduction

The notion of a Rota-Baxter operator on an associative algebra was introduced by Baxter [1] in his study of fluctuation theory in probability. Then Kupershmidt [2] introduced the notion of a relative Rota-Baxter operator (also called $\mathcal{O}$- operator) on a Lie algebra. Reynolds operators were introduced by Reynolds [3] in his study of fluctuation theory in fluid dynamics. In [4], Kampé de Fériet coined the concept of the Reynolds operator and regarded the operator as a mathematical subject in general. Generalized Reynolds operators (also called twisted Rota-Baxter operators) introduced by Uchino [5] in the context of associative algebras are algebraic analogue of twisted Poisson structure. The cohomology and deformations of twisted Rota-Baxter operators on associative algebras was studied by Das [6]. Twisted Rota-Baxter operators have been introduced and widely studied for other algebraic structures such as Lie algebras [7], Leibniz algebras [8] and 3-Lie algebras [9,10].

As a generalization of a Lie algebra and a Lie-triple system, the notion of a Lie-Yamaguti algebra was introduced by Kinyon and Weinstein [11] in their study of Courant algebroids. This structure can be traced back to Nomizu's work on the invariant affine connections on homogeneous spaces [12] and Yamaguti's work on Lie triple systems [13] and general Lie triple algebras [14]. Recently, there has been significant research focused on various aspects of Lie-Yamaguti algebras in both mathematics and physics. These include deformations [15,16], quasi-derivations [17], Nijenhuis operators [18], modules over quadratic spaces and representations [19] of Lie-Yamaguti algebras, equivariant Lie-Yamaguti algebras [20], relative Rota-Baxter operators [21,22], relative differential operators [23] and weighted Rota-Baxter operators [24] on Lie-Yamaguti algebras.

Motivated by the mentioned work on the generalized Reynolds operators and considering the importance of Lie-Yamaguti algebra, cohomology and deformation, this paper aims to study the cohomology theory and deformations of generalized Reynolds operators on Lie-Yamaguti algebras.

This paper is organized as follows. In Section 2, we briefly recall basics about representations and cohomology of Lie-Yamaguti algebras. Section 3 introduces the notion of generalized Reynolds operators on Lie-Yamaguti algebras. Moreover, we construct new generalized Reynolds operators out of an old one by suitable modifications. Section 4 introduces the cohomology of a generalized Reynolds operator on a Lie-Yamaguti algebra. In Section 5, we use the cohomological approach to study formal deformations of generalized Reynolds operators. In Section 6, we study two special classes of generalized Reynolds operators on Lie-Yamaguti algebras which are provided by Nijenhuis operators and Reynolds operators on Lie-Yamaguti algebras.

2. Preliminaries

Throughout this paper, we work on an algebraically closed field $\mathbb{K}$ of characteristics different from 2 and 3. We recall some basic definitions of Lie-Yamaguti algebra from [11,14].

Definition 1 ([11]). *A Lie-Yamaguti algebra is a 3-tuple $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ in which L is a vector space together with a binary operation $[\cdot,\cdot]$ and a ternary operation $\{\cdot,\cdot,\cdot\}$ on L such that*

$$
\begin{aligned}
&(LY01)\ \ [x,y] = -[y,x],\\
&(LY02)\ \ \{x,y,z\} = -\{y,x,z\},\\
&(LY03)\ \ ([[x,y],z] + \{x,y,z\}) + c.p. = 0,\\
&(LY04)\ \ \{[x,y],z,a\} + \{[z,x],y,a\} + \{[y,z],x,a\} = 0,\\
&(LY05)\ \ \{a,b,[x,y]\} = [\{a,b,x\},y] + [x,\{a,b,y\}],\\
&(LY06)\ \ \{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} + \{x,\{a,b,y\},z\} + \{x,y,\{a,b,z\}\},
\end{aligned}
$$

for all $x,y,z,a,b \in L$ and where c.p. denotes the sum over cyclic permutation of x,y,z, that is $([[x,y],z] + \{x,y,z\}) + c.p. = ([[x,y],z] + \{x,y,z\}) + ([[z,x],y] + \{z,x,y\}) + ([[y,z],x] + \{y,z,x\})$.

A homomorphism between two Lie-Yamaguti algebras $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ and $(L', [\cdot,\cdot]', \{\cdot,\cdot,\cdot\}')$ is a linear map $\varphi : L \to L'$ satisfying

$$\varphi([x,y]) = [\varphi(x), \varphi(y)]', \quad \varphi(\{x,y,z\}) = \{\varphi(x), \varphi(y), \varphi(z)\}', \ \ \forall\, x,y,z \in L.$$

Yamaguti introduced the concept of representation of Lie-Yamaguti algebra in [14].

Definition 2 ([14]). *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ be a Lie-Yamaguti algebra and V be a vector space. A representation of $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ on V consists of a linear map $\rho : L \to \mathrm{End}(V)$ and two bilinear maps $D, \theta : L \times L \to \mathrm{End}(V)$ such that*

$$
\begin{aligned}
&(R01)\ D(x,y) - \theta(y,x) + \theta(x,y) + \rho([x,y]) - \rho(x)\rho(y) + \rho(y)\rho(x) = 0,\\
&(R02)\ D([x,y],z) + D([y,z],x) + D([z,x],y) = 0,\\
&(R03)\ \theta([x,y],a) = \theta(x,a)\rho(y) - \theta(y,a)\rho(x),\\
&(R04)\ D(a,b)\rho(x) = \rho(x)D(a,b) + \rho(\{a,b,x\}),\\
&(R05)\ \theta(x,[a,b]) = \rho(a)\theta(x,b) - \rho(b)\theta(x,a),\\
&(R06)\ D(a,b)\theta(x,y) = \theta(x,y)D(a,b) + \theta(\{a,b,x\},y) + \theta(x,\{a,b,y\}),\\
&(R07)\ \theta(a,\{x,y,z\}) = \theta(y,z)\theta(a,x) - \theta(x,z)\theta(a,y) + D(x,y)\theta(a,z),
\end{aligned}
$$

for all $x,y,z,a,b \in L$. In this case, we also call V a L-module.

It can be concluded from (R06) that

$$(R06)'\ D(a,b)D(x,y) = D(x,y)D(a,b) + D(\{a,b,x\},y) + D(x,\{a,b,y\}).$$

Example 1. *Let* $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ *be a Lie-Yamaguti algebra. We define linear maps* $\mathrm{ad} : L \to \mathrm{End}(L), \mathcal{L}, \mathcal{R} : \otimes^2 L \to \mathrm{End}(L)$ *by*

$$\mathrm{ad}(x)(z) := [x,z], \mathcal{L}(x,y)(z) := \{x,y,z\}, \mathcal{R}(x,y)(z) := \{z,x,y\},$$

for all $x, y, z \in L$*. Then* $(L; \mathrm{ad}, \mathcal{L}, \mathcal{R})$ *forms a representation of* L *on itself, called the adjoint representation.*

Representations of a Lie-Yamaguti algebra can be characterized by the semidirect product Lie-Yamaguti algebras.

Proposition 1. *Let* $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ *be a Lie-Yamaguti algebra and* V *be a vector space. Let* $\rho : L \to \mathrm{End}(V)$ *and* $D, \theta : L \times L \to \mathrm{End}(V)$ *be linear maps. Then* (ρ, D, θ) *is a representation of* $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ *on* V *if and only if* $L \oplus V$ *is a Lie-Yamaguti algebra under the following maps:*

$$\begin{aligned}
&[x+u, y+v]_\ltimes := [x,y] + \rho(x)(v) - \rho(y)(u),\\
&\{x+u, y+v, z+w\}_\ltimes := \{x,y,z\} + D(x,y)(w) - \theta(x,z)(v) + \theta(y,z)(u),
\end{aligned}$$

for all $x, y, z \in L$ *and* $u, v, w \in V$*. In the case, the Lie-Yamaguti algebra* $L \oplus V$ *is called a semidirect product of* L *and* V*, denoted by* $L \ltimes V = (L \oplus V, [\cdot,\cdot]_\ltimes, \{\cdot,\cdot,\cdot\}_\ltimes)$*.*

Let us recall the cohomology theory on Lie-Yamaguti algebras in [14]. Let $(V; \rho, D, \theta)$ be a representation of a Lie-Yamaguti algebra $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$, and we denote the set of $n+1$-cochains by $C^{n+1}_{\mathrm{LY}}(L, V)$, where

$$C^{n+1}_{\mathrm{LY}}(L,V) = \begin{cases} \mathrm{Hom}(\underbrace{\wedge^2 L \otimes \cdots \otimes \wedge^2 L}_{n}, V) \times \mathrm{Hom}(\underbrace{\wedge^2 L \otimes \cdots \otimes \wedge^2 L}_{n} \otimes L, V) & n \geq 1,\\ \mathrm{Hom}(L, V) & n = 0. \end{cases}$$

In the sequel, we recall the coboundary map of $n+1$-cochains on Lie-Yamaguti algebra L with the coefficients in the representation $(V; \rho, D, \theta)$:

If $n \geq 1$, for any $(f,g) \in C^{n+1}_{\mathrm{LY}}(L,V)$, $K_i = x_i \wedge y_i \in \wedge^2 L, (i = 1, 2, \cdots, n+1), z \in L$, the coboundary map $\delta = (\delta_I, \delta_{II}) : C^{n+1}_{\mathrm{LY}}(L,V) \to C^{n+2}_{\mathrm{LY}}(L,V), (f,g) \mapsto (\delta_I(f,g), \delta_{II}(f,g))$ is given as follows:

$$\begin{aligned}
&\delta_I(f,g)(K_1, \cdots, K_{n+1})\\
=&(-1)^n(\rho(x_{n+1})g(K_1, \cdots, K_n, y_{n+1}) - \rho(y_{n+1})g(K_1, \cdots, K_n, x_{n+1})\\
&- g(K_1, \cdots, K_n, [x_{n+1}, y_{n+1}])) + \sum_{k=1}^{n} (-1)^{k+1} D(K_k) f(K_1, \cdots, \widehat{K_k} \cdots, K_{n+1})\\
&+ \sum_{1 \leq k < l \leq n+1} (-1)^k f(K_1, \cdots, \widehat{K_k} \cdots, \{x_k, y_k, x_l\} \wedge y_l + x_l \wedge \{x_k, y_k, y_l\}, \cdots, K_{n+1}),
\end{aligned}$$

$$\begin{aligned}
&\delta_{II}(f,g)(K_1, \cdots, K_{n+1}, z)\\
=&(-1)^n(\theta(y_{n+1}, z)g(K_1, \cdots, K_n, x_{n+1}) - \theta(x_{n+1}, z)g(K_1, \cdots, K_n, y_{n+1}))\\
&+ \sum_{k=1}^{n+1} (-1)^{k+1} D(K_k) g(K_1, \cdots, \widehat{K_k} \cdots, K_{n+1}, z)\\
&+ \sum_{1 \leq k < l \leq n+1} (-1)^k g(K_1, \cdots, \widehat{K_k} \cdots, \{x_k, y_k, x_l\} \wedge y_l + x_l \wedge \{x_k, y_k, y_l\}, \cdots, K_{n+1}, z)\\
&+ \sum_{k=1}^{n+1} (-1)^k g(K_1, \cdots, \widehat{K_k} \cdots, K_{n+1}, \{x_k, y_k, z\}).
\end{aligned}$$

where $\widehat{}$ denotes omission.

For the case that $n=0$, for any $f\in C^1_{\mathrm{LY}}(L,V)$, the coboundary map $\delta=(\delta_I,\delta_{II})$: $C^1_{\mathrm{LY}}(L,V)\to C^2_{\mathrm{LY}}(L,V), f\to(\delta_I(f),\delta_{II}(f))$ is given by:

$$\begin{aligned}\delta_I(f)(x,y)=&\rho(x)f(y)-\rho(y)f(x)-f([x,y]),\\ \delta_{II}(f)(x,y,z)=&D(x,y)f(z)+\theta(y,z)f(x)-\theta(x,z)f(y)-f(\{x,y,z\}).\end{aligned}$$

The corresponding cohomology groups are denoted by $\mathcal{H}^*_{\mathrm{LY}}(L,V)$.

3. Generalized Reynolds Operators on Lie-Yamaguti Algebras

In this section, we introduce Generalized Reynolds operators on Lie-Yamaguti algebras and provide some new constructions.

Let $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$ be a Lie-Yamaguti algebra and $(V;\rho,D,\theta)$ be a representation of it. Then $H=(H_1,H_2)\in C^2_{\mathrm{LY}}(L,V)$ is a 2-cocycle, if $\delta(H_1,H_2)=0$, i.e., (H_1,H_2) satisfies

$$\begin{aligned}&\{x,y,H_1(x_1,y_1)\}-H_1(\{x,y,x_1\},y_1)-H_1(x_1,\{x,y,y_1\})+H_2(x,y,[x_1,y_1])\\&-[x_1,H_2(x,y,y_1)]-[H_2(x,y,x_1),y_1]=0, \qquad (1)\\&\{x,y,H_2(x_1,y_1,z))\}-\{H_2(x,y,x_1),y_1,z\}-\{x_1,H_2(x,y,y_1),z\}-\{x_1,y_1,H_2(x,y,z)\}\\&+H_2(x,y,\{x_1,y_1,z\})-H_2(\{x,y,x_1\},y_1,z)-H_2(x_1,\{x,y,y_1\},z)\\&-H_2(x_1,y_1,\{x,y,z\})=0, \qquad (2)\end{aligned}$$

for all $x,y,z,x_1,y_1\in L$.

Definition 3. *A linear map $T{:}V\to L$ is said to a generalized Reynolds operators if T satisfies*

$$\begin{aligned}[][Tu,Tv]=&T(\rho(Tu)v-\rho(Tv)u+H_1(Tu,Tv)), \qquad (3)\\ \{Tu,Tv,Tw\}=&T(D(Tu,Tv)w+\theta(Tv,Tw)u-\theta(Tu,Tw)v+H_2(Tu,Tv,Tw)), \qquad (4)\end{aligned}$$

for $u,v,w\in V$.

Remark 1. *(i) When a Lie-Yamaguti algebra reduces to a Lie triple system, that is $[\cdot,\cdot]=0$, we get the notion of a generalized Reynolds operator on a Lie triple system immediately.*
(ii) When a Lie-Yamaguti algebra reduces to a Lie algebra, that is $\{\cdot,\cdot,\cdot\}=0$, we get the notion of a generalized Reynolds operator on a Lie algebra. See [7] for more details about generalized Reynolds operators on Lie algebras.

Example 2. *Any relative Rota-Baxter operator (in particular, Rota-Baxter operator of weight 0) on a Lie-Yamaguti algebra is a generalized Reynolds operator with $(H_1,H_2)=0$. See [21,22,24] for more details about relative Rota-Baxter operators and weighted Rota-Baxter operators on Lie-Yamaguti algebras.*

Example 3. *Let $(V;\rho,D,\theta)$ be a representation of a Lie-Yamaguti algebra $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$. Suppose that $h\in C^1_{\mathrm{LY}}(L,V)$ is an invertible 1-cochain. Take $H_1=-\delta_I(h)$ and $H_2=-\delta_{II}(h)$. Then*

$$\begin{aligned}&H_1(Tu,Tv)=-\delta_I(h)(Tu,Tv)=-\rho(Tu)v+\rho(Tv)u+h([Tu,Tv]),\\&H_2(Tu,Tv,Tw)=-\delta_{II}(h)(Tu,Tv,Tw)\\&=-D(Tu,Tv)h(Tw)-\theta(Tv,Tw)h(Tu)+\theta(Tu,Tw)h(Tv)+h(\{Tu,Tv,Tw\}),\end{aligned}$$

for $u,v,w\in V$. This shows that $T=h^{-1}$ is a generalized Reynolds operator.

Let $T{:}V\to L$ be a generalized Reynolds operator. Suppose $(V';\rho',D',\theta')$ is a representation of another Lie-Yamaguti algebra $(L',[\cdot,\cdot]',\{\cdot,\cdot,\cdot\}')$, and $(H'_1,H'_2)\in C^2_{\mathrm{LY}}(L',V')$ is a 2-cocycle. Let $T':V'\to L'$ be a generalized Reynolds operator.

Definition 4. *A morphism of generalized Reynolds operators from T to T' consists of a pair (φ, ϕ) of a Lie-Yamaguti algebra morphism $\varphi : L \to L'$ and a linear map $\phi : V \to V'$ satisfying*

$$\begin{aligned}
&\varphi \circ T = T' \circ \phi,\\
&\phi(\rho(x)u) = \rho'(\varphi(x))\phi(u),\\
&\phi(D(x,y)u) = D'(\varphi(x), \varphi(y))\phi(u), \phi(\theta(x,y)u) = \theta'(\varphi(x), \varphi(y))\phi(u),\\
&\phi \circ H_1 = H_1' \circ (\varphi \otimes \varphi), \phi \circ H_2 = H_2' \circ (\varphi \otimes \varphi \otimes \varphi),
\end{aligned}$$

for$x, y \in L, u \in V$.

Given a 2-cocycle (H_1, H_2) in the cochain complex of L with coefficients in V, one can construct the twisted semidirect product algebra. More precisely, the direct sum $L \oplus V$ carries a Lie-Yamaguti algebra structure with the bracket given by

$$\begin{aligned}
[x+u, y+v]_H :&= [x,y] + \rho(x)(v) - \rho(y)(u) + H_1(x,y),\\
\{x+u, y+v, z+w\}_H :&= \{x,y,z\} + D(x,y)(w) - \theta(x,z)(v) + \theta(y,z)(u) + H_2(x,y,z),
\end{aligned}$$

for $x, y, z \in L, u, v, w \in V$.

We denote this twisted semidirect product Lie-Yamaguti algebra by $L \ltimes_H V$. Using this twisted semidirect product, one can characterize generalized Reynolds operators by their graph.

Proposition 2. *A linear map $T : V \to L$ is a generalized Reynolds operator if and only if its graph $Gr(T) = \{Tu + u \mid u \in V\}$ is a subalgebra of the twisted semidirect product $L \ltimes_H V$.*

Proof. Let $T{:}V \to L$ be a linear operator. Then, for all $u, v, w \in V$, we have

$$\begin{aligned}
&[Tu+u, Tv+v]_H = [Tu, Tv] + \rho(Tu)v - \rho(Tv)u + H_1(Tu, Tv),\\
&\{Tu+u, Tv+v, Tw+w)\}_H\\
&= \{Tu, Tv, Tw\} + D(Tu, Tv)w - \theta(Tu, Tw)v + \theta(Tv, Tw)u + H_2(Tu, Tv, Tw),
\end{aligned}$$

which implies that the graph $Gr(T)$ is a subalgebra of $L \ltimes_H V$ if and only if T satisfies Equations (3) and (4), which means that T is a generalized Reynolds operator. □

Since V and $Gr(T)$ are isomorphic as vector spaces, we get the following result immediately.

Proposition 3. *Let $T{:}V \to L$ be a generalized Reynolds operator on Lie-Yamaguti algebra $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ with respect to the representation $(V; \rho, D, \theta)$. Then $(V, [\cdot,\cdot]_T, \{\cdot,\cdot,\cdot\}_T)$ is a Lie-Yamaguti algebra, where*

$$[u,v]_T = \rho(Tu)v - \rho(Tv)u + H_1(Tu, Tv), \tag{5}$$

$$\{u,v,w\}_T = D(Tu, Tv)w - \theta(Tu, Tw)v + \theta(Tv, Tw)u + H_2(Tu, Tv, Tw), \tag{6}$$

for all $u, v, w \in V$. Moreover, T is a homomorphism from $(V, [\cdot,\cdot]_T, \{\cdot,\cdot,\cdot\}_T)$ to $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$.

At the end of this section, we construct new generalized Reynolds operators out of an old one by suitable modifications. We start with the following.

Proposition 4. *Let $(V; \rho, D, \theta)$ be a representation of a Lie-Yamaguti algebra $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$. For any 2-cocycle $(H_1, H_2) \in C^2_{\mathrm{LY}}(L, V)$ and 1-cochain $h \in C^1_{\mathrm{LY}}(L, V)$, the Lie-Yamaguti algebra $L \ltimes_H V$ and $L \ltimes_{H+\delta h} V$ are isomorphic.*

Proof. We define an isomorphism $\Phi_h : L \ltimes_H V \to L \ltimes_{H+\delta h} V$ of the underlying vector spaces by $\Phi_h(x+u) := x + u - h(x)$, for $x + u \in L \ltimes_H V$. Moreover, we have

$$\begin{aligned}
&\Phi_h[x+u, y+v]_H \\
&= [x,y] + \rho(x)v - \rho(y)u + H_1(x,y) - h([x,y]) \\
&= [x,y] + \rho(x)v - \rho(y)u + H_1(x,y) - \rho(x)h(y) + \rho(y)h(x) + \delta_I(h)(x,y) \\
&= [x+u-h(x), y+v-h(y)]_{H+\delta_I h} \\
&= [\Phi_h(x+u), \Phi_h(y+v)]_{H+\delta_I h}, \\
&\Phi_h\{x+u, y+v, z+w\}_H \\
&= \{x,y,z\} + D(x,y)(w) - \theta(x,z)(v) + \theta(y,z)(u) + H_2(x,y,z) - h(\{x,y,z\}) \\
&= \{x,y,z\} + D(x,y)(w) - \theta(x,z)(v) + \theta(y,z)(u) + H_2(x,y,z) \\
&\qquad - D(x,y)h(z) - \theta(y,z)h(x) + \theta(x,z)h(y) + \delta_{II}(h)(x,y,z) \\
&= \{x+u-h(x), y+v-h(y), z+w-h(z)\}_{H+\delta_{II}h} \\
&= \{\Phi_h(x+u), \Phi_h(y+v), \Phi_h(z+w)\}_{H+\delta_{II}h}.
\end{aligned}$$

This shows that Φ_h is in fact an isomorphism of Lie-Yamaguti algebras. □

Proposition 5. *Let $T:V \to L$ be a generalized Reynolds operator, for any 1-cochain $h \in C^1_{\mathrm{LY}}(L,V)$, if the linear map $(Id_V - h \circ T) : V \to V$ is invertible, then the map $T \circ (Id_V - h \circ T)^{-1} : V \to L$ is a generalized Reynolds operator.*

Proof. Consider the subalgebra $Gr(T) \subset L \ltimes_H V$ of the twisted semidirect product. Thus by Proposition 4, we get that $\Phi_h(Gr(T)) = \{Tu + u - h(Tu) \mid u \in V\} \subset L \ltimes_{H+\delta h} V$ is a subalgebra. Since the map $(Id_V - h \circ T) : V \to V$ is invertible, we have $\Phi_h(Gr(T))$ is the graph of the linear map $T \circ (Id_V - h \circ T)^{-1}$. Hence by Proposition 2, the map $T \circ (Id_V - h \circ T)^{-1}$ is a generalized Reynolds operator. □

Let $T:V \to L$ be a generalized Reynolds operator. Suppose $B \in C^1_{\mathrm{LY}}(L,V)$ is a 1-cocycle. Then B is said to be T-admissible if the linear map $(Id_V + B \circ T) : V \to V$ is invertible. With this notation, we have the following.

Proposition 6. *Let $B \in C^1_{\mathrm{LY}}(L,V)$ be a T-admissible 1-cocycle. Then the map $T \circ (Id_V + B \circ T)^{-1} : V \to L$ is a generalized Reynolds operator.*

Proof. Consider the deformed subspace

$$\tau_B(Gr(T)) = \{Tu + u + B \circ Tu \mid u \in V\} \subset L \ltimes_H V.$$

Since B is a 1-cocycle, $\tau_B(Gr(T)) \subset L \ltimes_H V$ turns out to be a subalgebra. Further, the map $(Id_V + B \circ T)$ is invertible implies that $\tau_B(Gr(T))$ is the graph of the map $T \circ (Id_V + B \circ T)^{-1}$. Then it follows from Proposition 2 that $T \circ (Id_V + B \circ T)^{-1} : V \to L$ is a generalized Reynolds operator. □

The generalized Reynolds operator in the above proposition is called the gauge transformation of T associated with B. We denote this generalized Reynolds operator simply by T_B.

Proposition 7. *Let $T:V \to L$ be a generalized Reynolds operator and $B \in C^1_{\mathrm{LY}}(L,V)$ be a T-admissible 1-cocycle. Then the Lie-Yamaguti algebra structures on V induced from the generalized Reynolds operators T and T_B are isomorphic.*

Proof. Consider the linear isomorphism $(Id_V + B \circ T) : V \to V$. Moreover, for any $u, v, w \in V$, we have

$$\begin{aligned}
&[(Id_V + B\circ T)u, (Id_V + B\circ T)v]_{T_B} \\
&= \rho(Tu)(Id_V + B\circ T)v - \rho(Tv)(Id_V + B\circ T)u + H_1(Tu, Tv) \\
&= \rho(Tu)v - \rho(Tv)u + \rho(Tu)(B\circ T)v - \rho(Tv)(B\circ T)u + H_1(Tu, Tv) \\
&= \rho(Tu)v - \rho(Tv)u + B[Tu, Tv] + H_1(Tu, Tv) \\
&= [u, v]_T + B\circ T([u, v]_T) \\
&= (Id_V + B\circ T)([u, v]_T), \\
&\{(Id_V + B\circ T)u, (Id_V + B\circ T)v, (Id_V + B\circ T)w\}_{T_B} \\
&= D(Tu, Tv)(Id_V + B\circ T)w - \theta(Tu, Tw)(Id_V + B\circ T)v + \theta(Tv, Tw)(Id_V + B\circ T)u \\
&\quad + H_2(Tu, Tv, Tw) \\
&= D(Tu, Tv)w - \theta(Tu, Tw)v + \theta(Tv, Tw)u + D(Tu, Tv)(B\circ T)w - \theta(Tu, Tw)(B\circ T)v \\
&\quad + \theta(Tv, Tw)(B\circ T)u + H_2(Tu, Tv, Tw) \\
&= \{u, v, w\}_T + B\circ T(\{u, v, w\}_T) \\
&= (Id_V + B\circ T)(\{u, v, w\}_T).
\end{aligned}$$

Thus $Id_V + B \circ T$ is an isomorphism of Lie-Yamaguti algebras from $(V, [\cdot,\cdot]_T, \{\cdot,\cdot,\cdot\}_T)$ to $(V, [\cdot,\cdot]_{T_B}, \{\cdot,\cdot,\cdot\}_{T_B})$. □

4. Cohomology of Generalized Reynolds Operators

In this section, we define cohomology of a generalized Reynolds operator T as the cohomology of the Lie-Yamaguti algebra $(V, [\cdot,\cdot]_T, \{\cdot,\cdot,\cdot\}_T)$ constructed in Proposition 3 with coefficients in a suitable representation on L. In the next section, we will use this cohomology to study deformations of T.

Proposition 8. *Let $T\colon V \to L$ be a generalized Reynolds operator. Define linear maps $\rho_T : V \to \mathrm{End}(L)$ and $\theta_T, D_T\colon \otimes^2 V \to \mathrm{End}(L)$ by*

$$\begin{aligned}
\rho_T(u)x &:= [Tu, x] + T(\rho(x)u + H_1(x, Tu)), \\
\theta_T(u, v)x &:= \{x, Tu, Tv\} - T(D(x, Tu)v - \theta(x, Tv)u + H_2(Tu, Tv, x)), \\
D_T(u, v)x &:= \{Tu, Tv, x\} - T(\theta(Tv, x)u - \theta(Tu, x)v + H_2(Tu, Tv, x)),
\end{aligned}$$

for all $u, v \in V, x \in L$. Then $(L; \rho_T, \theta_T, D_T)$ is a representation of the Lie-Yamaguti algebra $(V, [\cdot,\cdot]_T, \{\cdot,\cdot,\cdot\}_T)$.

Proof. By a direct calculation using (LY01)–(LY06), (R01)–(R07) and (1)–(6), for all $u, v, w, u_1, u_2 \in V, x \in L$, we have

$$\begin{aligned}
&D_T(u, v)x - \theta_T(v, u)x + \theta_T(u, v)x + \rho_T([u, v]_T)x - \rho_T(u)\rho_T(v)x + \rho_T(v)\rho_T(u)x \\
&= \{Tu, Tv, x\} - T(\theta(Tv, x)u - \theta(Tu, x)v + H_2(Tu, Tv, x)) - \{x, Tv, Tu\} + T(D(x, Tv)u \\
&\quad - \theta(x, Tu)v + H_2(Tv, Tu, x)) + \{x, Tu, Tv\} - T(D(x, Tu)v - \theta(x, Tv)u + H_2(Tu, Tv, x)) \\
&\quad + [T[u, v]_T, x] + T(\rho(x)[u, v]_T + H_1(x, T[u, v]_T)) - [Tu, [Tv, x]] - T(\rho([Tv, x])u + H_1([Tv, x], Tu)) \\
&\quad - [Tu, T(\rho(x)v)] - T(\rho(T(\rho(x)v))u + H_1(T(\rho(x)v), Tu)) - [Tu, TH_1(x, Tv))] \\
&\quad - T(\rho(TH_1(x, Tv)))u + H_1(TH_1(x, Tv)), Tu)) + [Tv, [Tu, x]] + T(\rho([Tu, x])v + H_1([Tu, x], Tv)) \\
&\quad + [Tv, T(\rho(x)u)] + T(\rho(T(\rho(x)u))v + H_1(T(\rho(x)u), Tv)) + [Tv, TH_1(x, Tu))] \\
&\quad + T(\rho(TH_1(x, Tu)))v + H_1(TH_1(x, Tu)), Tv)) \\
&= 0,
\end{aligned}$$

$$
\begin{aligned}
&D_T([u,v]_T,w)x + D_T([v,w]_T,u)x + D_T([w,u]_T,v)x\\
&= \{T[u,v]_T,Tw,x\} - T(\theta(Tw,x)[u,v]_T - \theta(T[u,v]_T,x)w + H_2(T[u,v]_T,Tw,x))\\
&\quad + \{T[v,w]_T,Tu,x\} - T(\theta(Tu,x)[v,w]_T - \theta(T[v,w]_T,x)u + H_2(T[v,w]_T,Tu,x))\\
&\quad + \{T[w,u]_T,Tv,x\} - T(\theta(Tv,x)[w,u]_T - \theta(T[w,u]_T,x)v + H_2(T[w,u]_T,Tv,x))\\
&= 0,\\
&\theta_T([u,v]_T,w)x - \theta_T(u,w)\rho_T(v)x + \theta_T(v,w)\rho_T(u)x\\
&= \{x,T[u,v]_T,Tw\} - T(D(x,T[u,v]_T)w - \theta(x,Tw)[u,v]_T + H_2(T[u,v]_T,Tw,x))\\
&\quad - \{[Tv,x],Tu,Tw\} + T(D([Tv,x],Tu)w - \theta([Tv,x],Tw)u + H_2(Tu,Tw,[Tv,x]))\\
&\quad - \{T(\rho(x)v),Tu,Tw\} + T(D(T(\rho(x)v),Tu)w - \theta(T(\rho(x)v),Tw)u + H_2(Tu,Tw,T(\rho(x)v)))\\
&\quad - \{TH_1(x,Tv),Tu,Tw\} + T(D(TH_1(x,Tv),Tu)w - \theta(TH_1(x,Tv),Tw)u + H_2(Tu,Tw,TH_1(x,Tv)))\\
&\quad + \{[Tu,x],Tv,Tw\} - T(D([Tu,x],Tv)w - \theta([Tu,x],Tw)v + H_2(Tv,Tw,[Tu,x]))\\
&\quad + \{T(\rho(x)u),Tv,Tw\} - T(D(T(\rho(x)u),Tv)w - \theta(T(\rho(x)u),Tw)v + H_2(Tv,Tw,T(\rho(x)u)))\\
&\quad + \{TH_1(x,Tu),Tv,Tw\} - T(D(TH_1(x,Tu),Tv)w - \theta(TH_1(x,Tu),Tw)v + H_2(Tv,Tw,TH_1(x,Tu)))\\
&= 0,\\
&D_T(u,v)\rho_T(w)x - \rho_T(w)D_T(u,v)x - \rho_T(\{u,v,w\}_T)x\\
&= \{Tu,Tv,[Tw,x]\} - T(\theta(Tv,[Tw,x])u - \theta(Tu,[Tw,x])v + H_2(Tu,Tv,[Tw,x]))\\
&\quad + \{Tu,Tv,T(\rho(x)w)\} - T(\theta(Tv,T(\rho(x)w))u - \theta(Tu,T(\rho(x)w))v + H_2(Tu,Tv,T(\rho(x)w)))\\
&\quad + \{Tu,Tv,TH_1(x,Tw)\} - T(\theta(Tv,TH_1(x,Tw))u - \theta(Tu,TH_1(x,Tw))v + H_2(Tu,Tv,TH_1(x,Tw)))\\
&\quad - [Tw,\{Tu,Tv,x\}] - T(\rho(\{Tu,Tv,x\})w + H_1(\{Tu,Tv,x\},Tw)) + [Tw,T(\theta(Tv,x)u)]\\
&\quad + T(\rho(T(\theta(Tv,x)u))w + H_1(T(\theta(Tv,x)u),Tw)) - [Tw,T\theta(Tu,x)v]\\
&\quad - T(\rho(T\theta(Tu,x)v)w + H_1(T\theta(Tu,x)v,Tw)) + [Tw,TH_2(Tu,Tv,x)]\\
&\quad + T(\rho(TH_2(Tu,Tv,x))w + H_1(TH_2(Tu,Tv,x),Tw)) - [T\{u,v,w\}_T,x]\\
&\quad - T(\rho(x)\{u,v,w\}_T + H_1(x,T\{u,v,w\}_T))\\
&= 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\theta_T(w,[u,v]_T)x - \rho_T(u)\theta_T(w,v)x + \rho_T(v)\theta_T(w,u)x = 0,\\
&D_T(u,v)\theta_T(u_1,u_2)x - \theta_T(u_1,u_2)D_T(u,v)x - \theta_T(\{u,v,u_1\}_T,u_2)x - \theta_T(u_1,\{u,v,u_2\}_T)x = 0,\\
&\theta_T(u_1,\{u,v,w\}_T)x - \theta_T(v,w)\theta_T(u_1,u)x + \theta_T(u,w)\theta_T(u_1,v)x - D_T(u,v)\theta_T(u_1,w)x = 0.
\end{aligned}
$$

Therefore, we deduce that $(L;\rho_T,\theta_T,D_T)$ is a representation of the Lie-Yamaguti algebra $(V,[\cdot,\cdot]_T,\{\cdot,\cdot,\cdot\}_T)$. □

Let $\delta^T = (\delta^T_I,\delta^T_{II}) : C^{n+1}_{\mathrm{LY}}(V,L) \to C^{n+2}_{\mathrm{LY}}(V,L)$ be the corresponding coboundary operator of the Lie-Yamaguti algebra $(V,[\cdot,\cdot]_T,\{\cdot,\cdot,\cdot\}_T)$ with coefficients in the representation $(L;\rho_T,\theta_T,D_T)$. More precisely, $\delta^T(f,g) = (\delta^T_I(f,g),\delta^T_{II}(f,g))$ is given by

$$
\begin{aligned}
&\delta^T_I(f,g)(U_1,\cdots,U_{n+1})\\
=&(-1)^n([Tu_{n+1},g(U_1,\cdots,U_n,v_{n+1})] + T\rho(u_{n+1})g(U_1,\cdots,U_n,v_{n+1})\\
&+ TH_1(g(U_1,\cdots,U_n,v_{n+1}),Tu_{n+1}) - [Tv_{n+1},g(U_1,\cdots,U_n,u_{n+1})]\\
&- T\rho(v_{n+1})g(U_1,\cdots,U_n,u_{n+1}) - TH_1(g(U_1,\cdots,U_n,u_{n+1}),Tv_{n+1})\\
&- g(U_1,\cdots,U_n,\rho(Tu_{n+1})v_{n+1} - \rho(Tv_{n+1})u_{n+1} + H_1(Tu_{n+1},Tv_{n+1})))\\
&+ \sum_{k=1}^{n}(-1)^{k+1}(\{u_k,v_k,f(U_1,\cdots,\widehat{U_k}\cdots,U_{n+1})\} - T\theta(Tv_k,f(U_1,\cdots,\widehat{U_k}\cdots,U_{n+1}))u_k\\
&+ T\theta(Tu_k,f(U_1,\cdots,\widehat{U_k}\cdots,U_{n+1}))v_k - TH_2(Tu_k,v_k,f(U_1,\cdots,\widehat{U_k}\cdots,U_{n+1})))
\end{aligned}
$$

$$+\sum_{1\le k<l\le n+1}(-1)^k f(U_1,\cdots,\widehat{U_k}\cdots,(D(Tu_k,Tv_k)u_l-\theta(Tu_k,Tu_l)v_k+\theta(Tv_k,Tu_l)u_k+H_2(Tu_k,Tv_k,Tu_l))\wedge v_l$$
$$+u_l\wedge(D(Tu_k,Tv_k)v_l-\theta(Tu_k,Tv_l)v_k+\theta(Tv_k,Tv_l)u_k+H_2(Tu_k,Tv_k,Tv_l)),\cdots,U_{n+1}),$$

$$\begin{aligned}
&\delta^T_{II}(f,g)(U_1,\cdots,U_{n+1},w)\\
=&(-1)^n(\{g(U_1,\cdots,U_n,u_{n+1}),Tv_{n+1},Tw\}-T(D(g(U_1,\cdots,U_n,u_{n+1}),Tv_{n+1})w\\
&-\theta(g(U_1,\cdots,U_n,u_{n+1}),Tw)v_{n+1}+H_2(Tv_{n+1},Tw,g(U_1,\cdots,U_n,u_{n+1})))\\
&-\{g(U_1,\cdots,U_n,v_{n+1}),Tu_{n+1},Tw\}+T(D(g(U_1,\cdots,U_n,v_{n+1}),Tu_{n+1})w\\
&-\theta(g(U_1,\cdots,U_n,v_{n+1}),Tw)u_{n+1}+H_2(Tu_{n+1},Tw,g(U_1,\cdots,U_n,v_{n+1}))))\\
&+\sum_{k=1}^{n+1}(-1)^{k+1}(\{Tu_k,Tv_k,g(U_1,\cdots,\widehat{U_k}\cdots,U_{n+1},w)\}-T(\theta(Tv_k,g(U_1,\cdots,\widehat{U_k}\cdots,U_{n+1},w))u_k\\
&-\theta(Tu_k,g(U_1,\cdots,\widehat{U_k}\cdots,U_{n+1},w))v_k+H_2(Tu_k,Tv_k,g(U_1,\cdots,\widehat{U_k}\cdots,U_{n+1},w))))\\
&+\sum_{1\le k<l\le n+1}(-1)^k g(U_1,\cdots,\widehat{U_k}\cdots,(D(Tu_k,Tv_k)u_l-\theta(Tu_k,Tu_l)v_k+\theta(Tv_k,Tu_l)u_k+H_2(Tu_k,Tv_k,Tu_l))\wedge v_l\\
&+u_l\wedge(D(Tu_k,Tv_k)v_l-\theta(Tu_k,Tv_l)v_k+\theta(Tv_k,Tv_l)u_k+H_2(Tu_k,Tv_k,Tv_l)),\cdots,U_{n+1},w)\\
&+\sum_{k=1}^{n+1}(-1)^k g(U_1,\cdots,\widehat{U_k}\cdots,U_{n+1},D(Tu_k,Tv_k)w-\theta(Tu_k,Tw)v_k+\theta(Tv_k,Tw)u_k+H_2(Tu_k,Tv_k,Tw)).
\end{aligned}$$

where $U_i=u_i\wedge v_i\in\wedge^2V,(i=1,2,\cdots,n+1),w\in V$.

For the case that $n=0$, for any $f\in C^1_{\mathrm{LY}}(V,L)$, the coboundary map $\delta^T=(\delta^T_I,\delta^T_{II})$: $C^1_{\mathrm{LY}}(V,L)\to C^2_{\mathrm{LY}}(V,L),f\mapsto(\delta^T_I(f),\delta^T_{II}(f))$ is given by:

$$\begin{aligned}
\delta^T_I(f)(u,v)=&[Tu,f(v)]+T(\rho(f(v))u+H_1(f(v),Tu))-[Tv,f(u)]\\
&-T(\rho(f(u))v+H_1(f(u),Tv))-f(\rho(Tu)v-\rho(Tv)u+H_1(Tu,Tv)),\\
\delta^T_{II}(f)(u,v,w)=&\{Tu,Tv,f(w)\}-T(\theta(Tv,f(w))u-\theta(Tu,f(w))v+H_2(Tu,Tv,f(w)))\\
&+\{f(u),Tv,Tw\}-T(D(f(u),Tv)w-\theta(f(u),Tw)v+H_2(Tv,Tw,f(u)))\\
&-\{f(v),Tu,Tw\}+T(D(f(v),Tu)w-\theta(f(v),Tw)u+H_2(Tu,Tw,f(v)))\\
&-f(D(Tu,Tv)w-\theta(Tu,Tw)v+\theta(Tv,Tw)u+H_2(Tu,Tv,Tw)).
\end{aligned}$$

Proposition 9. *Let T be a generalized Reynolds operator on a Lie-Yamaguti algebra $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$ with respect to the representation $(V;\rho,D,\theta)$. For any $K=a\wedge b\in L\wedge L$, we define $\wp(K):V\to L$ by*

$$\wp(K)v:=T(D(K)v+H_2(K,Tv))-\{K,Tv\},\forall v\in V.$$

Then $\wp(K)$ is a 1-cocycle on the Lie-Yamaguti algebra $(V,[\cdot,\cdot]_T,\{\cdot,\cdot,\cdot\}_T)$ with coefficients in the representation $(L;\rho_T,\theta_T,D_T)$.

Proof. For any $u,v,w\in V$, we have

$$\begin{aligned}
&\delta^T_I(\wp(K))(u,v)\\
=&[Tu,\wp(K)(v)]+T(\rho(\wp(K)(v))u+H_1(\wp(K)(v),Tu))-[Tv,\wp(K)(u)]\\
&-T(\rho(\wp(K)(u))v+H_1(\wp(K)(u),Tv))-\wp(K)(\rho(Tu)v-\rho(Tv)u+H_1(Tu,Tv))\\
=&[Tu,T(D(K)v+H_2(K,Tv))-\{K,Tv\}]+T(\rho(T(D(K)v+H_2(K,Tv))-\{K,Tv\})u\\
&+H_1(T(D(K)v+H_2(K,Tv))-\{K,Tv\},Tu))-[Tv,T(D(K)u+H_2(K,Tu))-\{K,Tu\}]\\
&-T(\rho(T(D(K)u+H_2(K,Tu))-\{K,Tu\})v+H_1(T(D(K)u+H_2(K,Tu))-\{K,Tu\},Tv))\\
&-T(D(K)(\rho(Tu)v-\rho(Tv)u+H_1(Tu,Tv))+H_2(K,T(\rho(Tu)v-\rho(Tv)u+H_1(Tu,Tv))))\\
&-\{K,T(\rho(Tu)v-\rho(Tv)u+H_1(Tu,Tv))\}\\
=&0,
\end{aligned}$$

$$
\begin{aligned}
&\delta_{II}^{T}(\wp(K))(u,v,w)\\
=&\{Tu,Tv,\wp(K)(w)\}-T(\theta(Tv,\wp(K)(w))u-\theta(Tu,\wp(K)(w))v+H_2(Tu,Tv,\wp(K)(w)))\\
&+\{\wp(K)(u),Tv,Tw\}-T(D(\wp(K)(u),Tv)w-\theta(\wp(K)(u),Tw)v+H_2(Tv,Tw,\wp(K)(u)))\\
&-\{\wp(K)(v),Tu,Tw\}+T(D(\wp(K)(v),Tu)w-\theta(\wp(K)(v),Tw)u+H_2(Tu,Tw,\wp(K)(v)))\\
&-\wp(K)(D(Tu,Tv)w-\theta(Tu,Tw)v+\theta(Tv,Tw)u+H_2(Tu,Tv,Tw))\\
=&\{Tu,Tv,T(D(K)w+H_2(K,Tw))-\{K,Tw\}\}-T(\theta(Tv,T(D(K)w+H_2(K,Tw))\\
&-\{K,Tw\})u-\theta(Tu,T(D(K)w+H_2(K,Tw))-\{K,Tw\})v+H_2(Tu,Tv,T(D(K)w+H_2(K,Tw))\\
&-\{K,Tw\}))+\{T(D(K)u+H_2(K,Tu))-\{K,Tu\},Tv,Tw\}-T(D(T(D(K)u+H_2(K,Tu))-\{K,Tu\},Tv)w\\
&-\theta(T(D(K)u+H_2(K,Tu))-\{K,Tu\},Tw)v+H_2(Tv,Tw,T(D(K)u+H_2(K,Tu))-\{K,Tu\}))\\
&-\{T(D(K)v+H_2(K,Tv))-\{K,Tv\},Tu,Tw\}+T(D(T(D(K)v+H_2(K,Tv))-\{K,Tv\},Tu)w\\
&-\theta(T(D(K)v+H_2(K,Tv))-\{K,Tv\},Tw)u+H_2(Tu,Tw,T(D(K)v+H_2(K,Tv))-\{K,Tv\}))\\
&-T(D(K)(D(Tu,Tv)w-\theta(Tu,Tw)v+\theta(Tv,Tw)u+H_2(Tu,Tv,Tw))\\
&+H_2(K,T(D(Tu,Tv)w-\theta(Tu,Tw)v+\theta(Tv,Tw)u+H_2(Tu,Tv,Tw))))\\
&-\{K,T(D(Tu,Tv)w-\theta(Tu,Tw)v+\theta(Tv,Tw)u+H_2(Tu,Tv,Tw))\}\\
=&0.
\end{aligned}
$$

This finishes the proof. □

Now, we give a cohomology of generalized Reynolds operators on Lie-Yamaguti algebras.

Definition 5. *Let T be a generalized Reynolds operator on a Lie-Yamaguti algebra $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$ with respect to the representation $(V;\rho,D,\theta)$. Define the set of p-cochains by*

$$\mathcal{C}_T^p(V,L)=\begin{cases} C_{\mathrm{LY}}^p(V,L) & p\geq 1,\\ L\wedge L & p=0.\end{cases}$$

Define $\partial^T:\mathcal{C}_T^p(V,L)\to\mathcal{C}_T^{p+1}(V,L)$ by

$$\partial^T=\begin{cases} \delta^T & p\geq 1,\\ \wp & p=0.\end{cases}$$

Then $(\oplus_{p=0}^{\infty}\mathcal{C}_T^p(V,L),\partial^T)$ is a cochain complex. Denote the set of p-cocycles by $\mathcal{Z}_T^p(V,L)$ and the set of p-coboundaries by $\mathcal{B}_T^p(V,L)$. Denote by

$$\mathcal{H}_T^p(V,L):=\frac{\mathcal{Z}_T^p(V,L)}{\mathcal{B}_T^p(V,L)}$$

the p-th cohomology group which will be taken to be the p-th cohomology group for the generalized Reynolds operator T.

5. Formal Deformations of Generalized Reynolds Operator

Let $\mathbb{K}[[t]]$ be a ring of power series of one variable t, and let $L[[t]]$ be the set of formal power series over L. If $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$ is a Lie-Yamaguti algebra, then there is a Lie-Yamaguti algebra structure over the ring $\mathbb{K}[[t]]$ on $L[[t]]$ given by

$$[\sum_{i=0}^{\infty}x_it^i,\sum_{i=0}^{\infty}y_jt^j]=\sum_{s=0}^{\infty}\sum_{i+j=s}[x_i,y_j]t^s,\{\sum_{i=0}^{\infty}x_it^i,\sum_{i=0}^{\infty}y_jt^j,\sum_{k=0}^{\infty}z_kt^k\}=\sum_{s=0}^{\infty}\sum_{i+j+k=s}\{x_i,y_j,z_k\}t^s. \quad (7)$$

For any representation $(V;\rho,D,\theta)$ of a Lie-Yamaguti algebra $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$, there is a nature representation of the Lie-Yamaguti algebra $L[[t]]$ on the $\mathbb{K}[[t]]$-module $V[[t]]$, which is given by

$$\rho(\sum_{i=0}^{\infty}x_it^i)(\sum_{i=0}^{\infty}v_jt^j)=\sum_{s=0}^{\infty}\sum_{i+j=s}\rho(x_i)v_jt^s, \tag{8}$$

$$D(\sum_{i=0}^{\infty}x_it^i,\sum_{i=0}^{\infty}y_jt^j)(\sum_{k=0}^{\infty}v_kt^k)=\sum_{s=0}^{\infty}\sum_{i+j+k=s}D(x_i,y_j)v_kt^s, \tag{9}$$

$$\theta(\sum_{i=0}^{\infty}x_it^i,\sum_{i=0}^{\infty}y_jt^j)(\sum_{k=0}^{\infty}v_kt^k)=\sum_{s=0}^{\infty}\sum_{i+j+k=s}\theta(x_i,y_j)v_kt^s. \tag{10}$$

Similarly, the 2-cocycle (H_1,H_2) can be extended to a 2-cocycle (denoted by the same notation (H_1,H_2)) on the Lie-Yamaguti algebra $L[[t]]$ with coefficients in $V[[t]]$. Consider a power series

$$T_t=\sum_{i=0}^{\infty}T_it^i,\ T_i\in \mathrm{Hom}(V,L), \tag{11}$$

that is, $T_t\in \mathrm{Hom}(V,L)[[t]]=\mathrm{Hom}(V,L[[t]])$. Extend it to be a $\mathbb{K}[[t]]$-module map from $V[[t]]$ to $L[[t]]$ which is still denoted by T_t.

Definition 6. *If $T_t=\sum_{i=0}^{\infty}T_it^i$ with $T_0=T$ satisfies*

$$[T_tu,T_tv]=T_t(\rho(T_tu)v-\rho(T_tv)u+H_1(T_tu,T_tv)), \tag{12}$$

$$\{T_tu,T_tv,T_tw\}=T_t(D(T_tu,T_tv)w+\theta(T_tv,T_tw)u-\theta(T_tu,T_tw)v+H_2(T_tu,T_tv,T_tw)), \tag{13}$$

for all $u,v,w\in V$, we say that T_t is a formal deformation of the generalized Reynolds operator T.

By applying Equations (7)–(11) to expand Equations (12) and (13) and collecting coefficients of t^n, we see that Equations (12) and (13) are equivalent to the system of equations

$$\sum_{i+j=n}[T_iu,T_jv]=\sum_{i+j=n}T_i(\rho(T_ju)v-\rho(T_jv)u)+\sum_{i+j+k=n}T_i(H_1(T_ju,T_kv)), \tag{14}$$

$$\begin{aligned}\sum_{i+j+k=n}\{T_iu,T_jv,T_kw\}=&\sum_{i+j+k=n}T_i(D(T_ju,T_kv)w+\theta(T_jv,T_kw)u-\theta(T_ju,T_kw)v)\\&+\sum_{i+j+k+l=n}T_i(H_1(T_ju,T_kv,T_lw)).\end{aligned} \tag{15}$$

Note that (14) and (15) hold for $n=0$ as $T_0=T$ is a generalized Reynolds operator.

Proposition 10. *Let $T_t=\sum_{i=0}^{\infty}T_it^i$ is a formal deformation of a generalized Reynolds operator T on a Lie-Yamaguti algebra $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$ with respect to the representation $(V;\rho,D,\theta)$. Then T_1 is a 1-cocycle of the generalized Reynolds operator T, called the infinitesimal of the deformation T_t.*

Proof. When $n=1$, Equations (14) and (15) are equivalent to

$$\begin{aligned}[T_1u,Tv]+[Tu,T_1v]=&\,T_1(\rho(Tu)v-\rho(Tv)u)+T(\rho(T_1u)v-\rho(T_1v)u)+T_1(H_1(Tu,Tv))\\&+T(H_1(T_1u,Tv))+T(H_1(Tu,T_1v)),\end{aligned}$$

$$\begin{aligned}&\{T_1u,Tv,Tw\}+\{Tu,T_1v,Tw\}+\{Tu,Tv,T_1w\}\\&=T_1(D(Tu,Tv)w+\theta(Tv,Tw)u-\theta(Tu,Tw)v)+T(D(T_1u,Tv)w+\theta(T_1v,Tw)u-\theta(T_1u,Tw)v)\\&+T(D(Tu,T_1v)w+\theta(Tv,T_1w)u-\theta(Tu,T_1w)v)+T_1(H_1(Tu,Tv,Tw))\\&+T(H_1(T_1u,Tv,Tw))+T(H_1(Tu,T_1v,Tw))+T(H_1(Tu,Tv,T_1w)).\end{aligned}$$

This implies that $\partial_I^T(T_1)(u,v) = 0$ and $\partial_{II}^T(T_1)(u,v,w) = 0$. Hence the linear term T_1 is a 1-cocycle in the cohomology of the generalized Reynolds operator T. □

Definition 7. *Let T be a generalized Reynolds operator on a Lie-Yamaguti algebra $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ with respect to the representation $(V;\rho,D,\theta)$. Two formal deformations $T'_t = \sum_{i=0}^{\infty} T'_i t^i$ and $T_t = \sum_{i=0}^{\infty} T_i t^i$ are said to be equivalent if there exist an element $K = a \wedge b \in L \wedge L$ such that two linear maps*

$$\phi_t = Id_L + t\{K, -\} + \sum_{i=2}^{\infty} \phi_i t^i, \quad \phi_i \in \mathrm{End}(L), \tag{16}$$

$$\varphi_t = Id_V + t(D(K) - + H_2(K, T-)) + \sum_{i=2}^{\infty} \varphi_i t^i, \quad \varphi_i \in \mathrm{End}(V), \tag{17}$$

meet the following equations:

$$\begin{aligned}
&\phi_t[x,y] = [\phi_t(x), \phi_t(y)], \quad \phi_t\{x,y,z\} = \{\phi_t(x), \phi_t(y), \phi_t(z)\},\\
&\varphi_t(D(x,y)v) = D(\phi_t(x), \phi_t(y))\varphi_t(v), \quad \varphi_t(\theta(x,y)v) = \theta(\phi_t(x), \phi_t(y))\varphi_t(v),\\
&\varphi_t(\rho(x)v) = \rho(\phi_t(x))\varphi_t(v),\\
&T_t(\varphi_t(v)) = \phi_t(T'_t(v)),
\end{aligned} \tag{18}$$

for all $x,y,z \in L, v \in V$.

Theorem 1. *Let T be a generalized Reynolds operator on a Lie-Yamaguti algebra $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ with respect to the representation $(V;\rho,D,\theta)$. Two formal deformations $T'_t = \sum_{i=0}^{\infty} T'_i t^i$ and $T_t = \sum_{i=0}^{\infty} T_i t^i$ of T are equivalent, then T'_1 and T_1 define the same cohomology class in $\mathcal{H}_T^1(V, L)$.*

Proof. Let ϕ_t and φ_t are two linear maps defined in Equations (16) and (17) such that two deformations $T'_t = \sum_{i=0}^{\infty} T'_i t^i$ and $T_t = \sum_{i=0}^{\infty} T_i t^i$ are equivalent. By Equation (18), we have $T'_1(v) - T_1(v) = T(D(K)v + H_2(K, Tv)) - \{K, Tv\}$. From Proposition 9, we can get $T'_1(v) - T_1(v) = \partial^T(K)v \in \mathcal{B}_T^1(V, L)$, which implies that T'_1 and T_1 are in the same cohomology class. □

6. Nijenhuis Operators and Reynolds Operators on Lie-Yamaguti Algebras

First, we show that a Nijenhuis operator on a Lie-Yamaguti algebra gives rise to a generalized Reynolds operator on a Lie-Yamaguti algebra.

Recall from [18] that a Nijenhuis operator on a Lie-Yamaguti algebra $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ is a linear map $N : L \to L$ satisfies

$$\begin{aligned}
[Nx, Ny] =& N([Nx, y] + [x, Ny] - N[x, y]),\\
\{Nx, Ny, Nz\} =& N(\{Nx, Ny, z\} + \{Nx, y, Nz\} + \{x, Ny, Nz\}) -\\
& N^2(\{Nx, y, z\} + \{x, Ny, z\} + \{x, y, Nz\}) + N^3\{x, y, z\},
\end{aligned}$$

for all $x, y, z \in L$. In this case the vector space L carries a new Lie-Yamaguti bracket $([\cdot,\cdot]_N, \{\cdot,\cdot,\cdot\}_N)$, which is defined by

$$[x, y]_N = [Nx, y] + [x, Ny] - N[x, y], \tag{19}$$

$$\begin{aligned}
\{x, y, z\}_N =& \{Nx, Ny, z\} + \{Nx, y, Nz\} + \{x, Ny, Nz\} -\\
& N(\{Nx, y, z\} + \{x, Ny, z\} + \{x, y, Nz\}) + N^2\{x, y, z\}.
\end{aligned} \tag{20}$$

The Lie-Yamaguti algebra $(L, [\cdot,\cdot]_N, \{\cdot,\cdot,\cdot\}_N)$ will be called the deformed Lie-Yamaguti algebra, and denoted by L_N. It is obvious that N is a homomorphism from the deformed Lie-Yamaguti algebra $(L, [\cdot,\cdot]_N, \{\cdot,\cdot,\cdot\}_N)$ to $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$.

Lemma 1. *Let N be a Nijenhuis operator on a Lie-Yamaguti algebras $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$. Define $\rho_N : L_N \to \mathrm{End}(L)$ and $\theta_N, D_N : \wedge^2 L_N \to \mathrm{End}(L)$ by*

$$\rho_N(x)a := [Nx,a], \;\; D_N(x,y)a := \{Nx,Ny,a\}, \;\; \theta_N(x,y)a := \{a,Nx,Ny\}, \tag{21}$$

for all $x,y \in L_N, a \in L$. Then $(L;\rho_N,D_N,\theta_N)$ is a representation of the deformed Lie-Yamaguti algebra L_N.

Proof. By using (LY01)–(LY06), (19)–(21), for any $x,y,z,x_1,x_2 \in L_N, a \in L$, we have

$$\begin{aligned}
&D_N(x,y)a - \theta_N(y,x)a + \theta_N(x,y)a + \rho_N([x,y]_N)a - \rho_N(x)\rho_N(y)a + \rho_N(y)\rho_N(x)a \\
&= \{Nx,Ny,a\} - \{a,Ny,Nx\} + \{a,Nx,Ny\} + [[Nx,Ny],a] - [Nx,[Ny,a]] + [Ny,[Nx,a]] \\
&= 0, \\
&D_N([x,y]_N,z)a + D_N([y,z]_N,x)a + D_N([z,x]_N,y)a \\
&= \{[Nx,Ny],Nz,a\} + \{[Ny,Nz],Nx,a\} + \{[Nz,Nx],Ny,a\} \\
&= 0, \\
&\theta_N([x,y]_N,z)a - \theta_N(x,z)\rho_N(y)a + \theta_N(y,z)\rho_N(x)a \\
&= \{a,[Nx,Ny],Nz\} - \{[Ny,a],Nx,Nz\} + \{[Nx,a],Ny,Nz\} \\
&= 0, \\
&D_N(x,y)\rho_N(z)a - \rho_N(z)D_N(x,y)a - \rho_N(\{x,y,z\}_N)a \\
&= \{Nx,Ny,[Nz,a]\} - [Nz,\{Nx,Ny,a\}] - [\{Nx,Ny,Nz\},a] \\
&= 0, \\
&\theta_N(z,[x,y]_N)a - \rho_N(x)\theta_N(z,y)a + \rho_N(y)\theta_N(z,x)a \\
&= \{a,Nz,[Nx,Ny]\} - [Nx,\{a,Nz,Ny\}] + [Ny,\{a,Nz,Nx\}] \\
&= 0, \\
&D_N(x,y)\theta_N(x_1,x_2)a - \theta_N(x_1,x_2)D_N(x,y)a - \theta_N(\{x,y,x_1\}_N,x_2)a - \theta_N(x_1,\{x,y,x_2\}_N)a \\
&= \{Nx,Ny,\{a,Nx_1,Nx_2\}\} - \{\{Nx,Ny,a\},Nx_1,Nx_2\} - \{a,\{Nx,Ny,Nx_1\},Nx_2\} \\
&\quad - \{a,Nx_1,\{Nx,Ny,Nx_2\}\} \\
&= 0, \\
&\theta_N(x_1,\{x,y,z\}_N)a - \theta_N(y,z)\theta_N(x_1,x)a + \theta_N(x,z)\theta_N(x_1,y)a - D_N(x,y)\theta_N(x_1,z)a \\
&= \{a,Nx_1,\{Nx,Ny,Nz\}\} - \{\{a,Nx_1,Nx\},Ny,Nz\} + \{\{a,Nx_1,Ny\},Nx,Nz\} \\
&\quad - \{Nx,Ny,\{a,Nx_1,Nz\}\} \\
&= 0.
\end{aligned}$$

Therefore, we deduce that $(L;\rho_N,D_N,\theta_N)$ is a representation of the deformed Lie-Yamaguti algebra L_N. □

Theorem 2. *Let N be a Nijenhuis operator on a Lie-Yamaguti algebra $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$. Define the map $H_1^N : \wedge^2 L_N \to L$ and $H_2^N : \wedge^3 L_N \to L$ by*

$$H_1^N(x,y) = -N[x,y], \tag{22}$$

$$H_2^N(x,y,z) = -N(\{Nx,y,z\} + \{x,Ny,z\} + \{x,y,Nz\} - N\{x,y,z\}), \tag{23}$$

for all $x,y \in L_N, z \in L$. Then (H_1^N, H_2^N) is a 2-cocycle of L_N with coefficients in $(L;\rho_N,D_N,\theta_N)$. Moreover the identity map $Id : L \to L_N$ is a generalized Reynolds operator on L_N with respect to the representation $(L;\rho_N,D_N,\theta_N)$.

Proof. For all $x_1,y_1,x_2,y_2,z \in L_N$, by using (19)–(23), we have

$$
\begin{aligned}
&\delta_I(H_1^N, H_2^N)(x_1, y_1, x_2, y_2) \\
=& -\rho_N(x_2)H_2^N(x_1, y_1, y_2) + \rho_N(y_2)H_2^N(x_1, y_1, x_2) + H_2^N(x_1, y_1, [x_2, y_2]_N) \\
&+ D_N(x_1, y_1)H_1^N(x_2, y_2) - H_1^N(\{x_1, y_1, x_2\}_N, y_2) - H_1^N(x_2, \{x_1, y_1, y_2\}_N) \\
=&[Nx_2, N(\{Nx_1, y_1, y_2\} + \{x_1, Ny_1, y_2\} + \{x_1, y_1, Ny_2\} - N\{x_1, y_1, y_2\})] \\
&- [Ny_2, N(\{Nx_1, y_1, x_2\} + \{x_1, Ny_1, x_2\} + \{x_1, y_1, Nx_2\} - N\{x_1, y_1, x_2\})] \\
&- N(\{Nx_1, y_1, [x_2, y_2]_N\} + \{x_1, Ny_1, [x_2, y_2]_N\} + \{x_1, y_1, [Nx_2, Ny_2]\} - N\{x_1, y_1, [x_2, y_2]_N\}) \\
&- \{Nx_1, Ny_1, N[x_2, y_2]\} + N[\{x_1, y_1, x_2\}_N, y_2] + N(x_2, \{x_1, y_1, y_2\}_N) \\
=&0, \\
&\delta_{II}(H_1^N, H_2^N)(x_1, y_1, x_2, y_2, z) \\
=& -\theta_N(y_2, z)H_2^N(x_1, y_1, x_2) + \theta_N(x_2, z)H_2^N(x_1, y_1, y_2) + D_N(x_1, y_1)H_2^N(x_2, y_2, z) \\
&- D_N(x_2, y_2)H_2^N(x_1, y_1, z) - H_2^N(\{x_1, y_1, x_2\}_N, y_2, z) - H_2^N(x_2, \{x_1, y_1, y_2\}_N, z) \\
&- H_2^N(x_2, y_2, \{x_1, y_1, z\}_N) + H_2^N(x_1, y_1, \{x_2, y_2, z\}_N) \\
=&\{N(\{Nx_1, y_1, x_2\} + \{x_1, Ny_1, x_2\} + \{x_1, y_1, Nx_2\} - N\{x_1, y_1, x_2\}), Ny_2, Nz\} \\
&- \{N(\{Nx_1, y_1, y_2\} + \{x_1, Ny_1, y_2\} + \{x_1, y_1, Ny_2\} - N\{x_1, y_1, y_2\}), Nx_2, Nz\} \\
&- \{Nx_1, Ny_1, N(\{Nx_2, y_2, z\} + \{x_2, Ny_2, z\} + \{x_2, y_2, Nz\} - N\{x_2, y_2, z\})\} \\
&+ \{Nx_2, Ny_2, N(\{Nx_1, y_1, z\} + \{x_1, Ny_1, z\} + \{x_1, y_1, Nz\} - N\{x_1, y_1, z\})\} \\
&+ N(\{\{Nx_1, Ny_1, Nx_2\}, y_2, z\} + \{\{x_1, y_1, x_2\}_N, Ny_2, z\} + \{\{x_1, y_1, x_2\}_N, y_2, Nz\}) \\
&+ N(\{Nx_2, \{x_1, y_1, y_2\}_N, z\} + \{x_2, \{Nx_1, Ny_1, Ny_2\}, z\} + \{x_2, \{x_1, y_1, y_2\}_N, Nz\}) \\
&+ N(\{Nx_2, y_2, \{x_1, y_1, z\}_N\} + \{x_2, Ny_2, \{x_1, y_1, z\}_N\} + \{x_2, y_2, \{Nx_1, Ny_1, Nz\}\}) \\
&- N^2\{\{x_1, y_1, x_2\}_N, y_2, z\} - N^2\{x_2, \{x_1, y_1, y_2\}_N, z\} - N^2\{x_2, y_2, \{x_1, y_1, z\}_N\} \\
&- N(\{Nx_1, y_1, \{x_2, y_2, z\}_N\} + \{x_1, Ny_1, \{x_2, y_2, z\}_N\} + \{x_1, y_1, \{Nx_2, Ny_2, Nz\}\}) \\
&+ N^2\{x_1, y_1, \{x_2, y_2, z\}_N\} \\
=&0.
\end{aligned}
$$

Thus, we deduce that (H_1^N, H_2^N) is a 2-cocycle of L_N with coefficients in $(L; \rho_N, D_N, \theta_N)$. Moreover, by (21)–(23), it is easy to prove that (3) and (4) are equivalent to (19) and (20), which implies that the identity map $Id : L \to L_N$ is a generalized Reynolds operator on L_N with respect to the representation $(L; \rho_N, D_N, \theta_N)$. □

Next, we introduce the notion of a Reynolds operator on a Lie-Yamaguti algebra, which turns out to be a special generalized Reynolds operator.

Definition 8. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ be a Lie-Yamaguti algebra. A linear map $R : L \to L$ is called a Reynolds operator if*

$$[Rx, Ry] = R([Rx, y] + [x, Ry] - [Rx, Ry]), \tag{24}$$
$$\{Rx, Ry, Rz\} = R(\{Rx, Ry, z\} + \{x, Ry, Rz\} + \{Rx, y, Rz\} - \{Rx, Ry, Rz\}), \tag{25}$$

for all $x, y, z \in L$. Moreover, a Lie-Yamaguti algebra L with a Reynolds operator R is called a Reynolds Lie-Yamaguti algebra. We denote it by $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, R)$.

The following results give the relation between Reynolds operators and generalized Reynolds operators on Lie-Yamaguti algebras.

Proposition 11. *Let R be a Reynolds operator on a Lie-Yamaguti algebra $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$. Then R is a generalized Reynolds operator on L with respect to the adjoint representation $(L; \mathrm{ad}, \mathcal{L}, \mathcal{R})$, where $(H_1, H_2) \in C^2_{\mathrm{LY}}(L, L)$ is defined by*

$$H_1(x,y) = -[x,y], H_2(x,y,z) = -\{x,y,z\}, \forall x,y,z \in L.$$

Proof. Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ be a Lie-Yamaguti algebra. By (1) and (2), the Lie-Yamaguti bracket $([\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ is a 2- cocycle with coefficients in the adjoint representation $(L; \mathrm{ad}, \mathcal{L}, \mathcal{R})$, which implies that R is a generalized Reynolds operator on L with respect to the adjoint representation. □

Proposition 12. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, R)$ be a Reynolds Lie-Yamaguti algebra. Define multiplications $[\cdot,\cdot]_R$ and $\{\cdot,\cdot,\cdot\}_R$ on L by*

$$[x,y]_R = [Rx,y] + [x,Ry] - [Rx,Ry], \tag{26}$$
$$\{x,y,z\}_R = \{Rx,Ry,z\} + \{x,Ry,Rz\} + \{Rx,y,Rz\} - \{Rx,Ry,Rz\}, \tag{27}$$

for all $x,y,z \in L$. Then $(L, [\cdot,\cdot]_R, \{\cdot,\cdot,\cdot\}_R, R)$ is a Reynolds Lie-Yamaguti algebra. Moreover, R is a homomorphism from the Lie-Yamaguti algebra $(L, [\cdot,\cdot]_R, \{\cdot,\cdot,\cdot\}_R)$ to $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$.

Proof. By Propositions 3 and 11, $(L, [\cdot,\cdot]_R, \{\cdot,\cdot,\cdot\}_R)$ is a Lie-Yamaguti algebra and R is a homomorphism from the Lie-Yamaguti algebra $(L, [\cdot,\cdot]_R, \{\cdot,\cdot,\cdot\}_R)$ to $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$. For $x,y,z \in L$, by (24)–(27), we have

$$\begin{aligned}
[Rx,Ry]_R &= [R^2x,Ry] + [Rx,R^2y] - [R^2x,R^2y] \\
&= R([Rx,y]_R + [x,Ry]_R - [Rx,Ry]_R), \\
\{Rx,Ry,Rz\}_R &= \{R^2x,R^2y,Rz\} + \{Rx,R^2y,R^2z\} + \{R^2x,Ry,R^2z\} - \{R^2x,R^2y,R^2z\} \\
&= R(\{Rx,Ry,z\}_R + \{x,Ry,Rz\}_R + \{Rx,y,Rz\}_R - \{Rx,Ry,Rz\}_R),
\end{aligned}$$

which implies that R is a Reynolds operator on the Lie-Yamaguti algebra $(L, [\cdot,\cdot]_R, \{\cdot,\cdot,\cdot\}_R)$. □

7. Conclusions

In the current study, the cohomology theory of generalized Reynolds operators on Lie-Yamaguti algebras is proposed to control the formal deformations of generalized Reynolds operators on Lie-Yamaguti algebras. More precisely, the notion of generalized Reynolds operators on Lie-Yamaguti algebras is introduced, and some new constructions are given. Then, the cohomology theory of generalized Reynolds operators on Lie-Yamaguti algebras is established. As an application, infinitesimals of formal deformations are classified by the first cohomology group. Finally, we show that a Nijenhuis operator on a Lie-Yamaguti algebra gives rise to a generalized Reynolds operator on a Lie-Yamaguti algebra and introduce the notion of a Reynolds operator on a Lie-Yamaguti algebra, which turns out to be a special generalized Reynolds operator. In particular, we obtain generalized Reynolds operators and Reynolds operators on a Lie triple system when a Lie-Yamaguti algebra reduces to a Lie triple system.

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