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Boundedness and Essential Norm of an Operator between Weighted-Type Spaces of Holomorphic Functions on a Unit Ball

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Abstract: The boundedness of a sum-type operator between weighted-type spaces is characterized and its essential norm is estimated.

Keywords: bounded operator; essential norm; weighted-type space; unit ball

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1. Introduction

By $\mathbb{N}_k$, where $k \in \mathbb{Z}$, we denote the set $\{n \in \mathbb{Z} : n \geq k\}$. Let $B(a,r) = \{z \in \mathbb{C}^n : |z-a| < r\}$, where $a \in \mathbb{C}^n$, $r \geq 0$, $|z| = \sqrt{\langle z, z\rangle}$ and $\langle z, w\rangle = \sum_{j=1}^n z_j\overline{w_j}$, $z, w \in \mathbb{C}^n$. Further, let $\mathbb{B} = B(0,1)$, $\mathbb{S} = \partial\mathbb{B}$, $dV(z)$ be the n-dimensional Lebesgue measure on $\mathbb{B}$, $H(\mathbb{B})$ be the space of holomorphic functions on $\mathbb{B}$ and $S(\mathbb{B})$ be the family of holomorphic self-maps of $\mathbb{B}$. For some basics on the functions in $H(\mathbb{B})$, consult, e.g., [1]. For some other presentations of the theory, see also [2,3]. If $f \in C(\mathbb{B})$ and $f(z) \geq 0$, $z \in \mathbb{B}$, then we call it a weight function and write $f \in W(\mathbb{B})$. $\mu \in W(\mathbb{B})$ is radial if $\mu(z) = \mu(|z|)$, $z \in \mathbb{B}$. If $\mu \in W(\mathbb{B})$ is radial and non-increasing in $|z|$, and $\lim_{|z|\to 1}\mu(z) = 0$, then it is typical. If X is a normed space, then $B_X = \{x : \|x\|_X \leq 1\}$.

Let X and Y be two normed spaces. A linear operator $T : X \to Y$ is bounded if there is $C \geq 0$ such that $\|Tf\|_Y \leq C\|f\|_X$, $f \in X$, and we write $T \in L(X,Y)$. The operator is compact if it maps bounded sets into relatively compact ones ([4–7]), and we write $T \in \mathcal{K}(X,Y)$. The essential norm of $T \in L(X,Y)$ is

$$\|T\|_{e,X\to Y} = \inf\{\|T+K\|_{X\to Y} : K \in \mathcal{K}(X,Y)\}.$$

If $\mu \in W(\mathbb{B})$, then the space of $f \in H(\mathbb{B})$, such that

$$\|f\|_{H^\infty_\mu} = \sup_{z\in\mathbb{B}} \mu(z)|f(z)| < +\infty,$$

is the weighted-type space $H^\infty_\mu(\mathbb{B}) = H^\infty_\mu$. The little weighted-type space $H^\infty_{\mu,0}(\mathbb{B}) = H^\infty_{\mu,0}$ contains $f \in H(\mathbb{B})$ such that $\lim_{|z|\to 1}\mu(z)|f(z)| = 0$. For some information on these function spaces see, e.g., [8–14]. For several technical and theoretical reasons, these spaces are suitable choices for studying concrete linear operators from or to them.

Each $\varphi \in S(\mathbb{B})$ induces the composition operator $C_\varphi f(z) = f(\varphi(z))$. Each $u \in H(\mathbb{B})$ induces the multiplication operator $M_u f(z) = u(z)f(z)$. The radial derivative of $f \in H(\mathbb{B})$ is $\Re f(z) = \sum_{j=1}^n z_j D_j f(z)$, where $D_j f(z) = \frac{\partial f}{\partial z_j}(z)$, $j = \overline{1,n}$. If $n = 1$, then $D_1 f := Df = f'$, where by $D^k f$ we denote the differentiation operator of the kth order $f^{(k)}$ (for $k = 0$, the identity operator is obtained).

There has been some interest in these operators, integral-type operators (for some of them see, e.g., [14–16]), and their products. Besides the products of C_φ and M_u, there have been some investigations into the products of D and C_φ. One of the first papers on these products was [17], where $D \circ C_\varphi$ between Bergman and Hardy spaces was studied. Ohno in [18] studied the products between Hardy spaces. S. Li and S. Stević then studied the operators between various spaces (see, e.g., [19], where we studied the products from H^∞ and the Bloch space to nth weighted-type spaces, and the related references therein). For some later investigations of the operators see, e.g., [20–22]. The operator $D \circ M_u$ on Bloch-type spaces was studied in [23].

Motivated by the above-mentioned product-type operators, researchers started investigating some more complex operators. The operator $D^m_{\varphi,u} := M_u C_\varphi D^m$ is a natural generalization of the product $C_\varphi D$ and has been investigated in depth. One of the first studies of the operator was conducted in [24]. Zhu studied the operator from Bergman-type spaces to some weighted-type spaces. The research was continued in [25], where the operator from Bloch-type spaces to weighted Bergman spaces was studied, and in [26], where the operator on weighted Bergman spaces was studied. In several papers, we have studied the operator between various spaces of holomorphic functions (see [27], where we studied the operator from the mixed-norm space to the nth weighted-type space, and the related references therein). For some later studies of the operator, see, e.g., [28–33]. The operator $\mathfrak{R}^m_{u,\varphi} = M_u C_\varphi \mathfrak{R}^m$, which is an n-dimensional variant of $D^m_{\varphi,u}$, was introduced in [34] (see also [35]).

The sum $M_u C_\varphi + M_u C_\varphi D$ was studied first in [36], whereas the sum $M_u C_\varphi D^n + M_u C_\varphi D^{n+1}$ for an arbitrary $n \in \mathbb{N}_0$ was studied in [37]. For some other studies of these and related operators, see, e.g., [38–43].

Motivated, among others, by the investigations in [34–37,43], S. Stević introduced several sums of operators, including the following:

$$\mathfrak{S}^m_{\vec{u},\varphi} = \sum_{j=1}^{m} M_{u_j} C_\varphi \mathfrak{R}^j = \sum_{j=1}^{m} \mathfrak{R}^j_{u_j,\varphi}, \tag{1}$$

where $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{1,m}$, and $\varphi \in S(\mathbb{B})$, and investigated them, e.g., in [44,45].

For some other concrete operators, see, for example, [46–53]. Some of them are product-type operators containing an integral-type operator. In [50], the products of integral-type operators and C_φ from a mixed norm space to Bloch-type spaces were studied. Another product-type operator, which includes an integral-type operator, acting from $Q_k(p,q)$ to α-Bloch spaces, was studied in [48].

Here, we continue our research in [27,34,36,37,44,45] by studying the boundedness and compactness and estimating the essential norm of the operators $\mathfrak{S}^m_{\vec{u},\varphi}$ acting between weighted-type spaces of holomorphic functions.

By C we denote some positive constants. If we write $a \lesssim b$ (respectively, $a \gtrsim b$), then there is $C > 0$ such that $a \le Cb$ (respectively, $a \ge Cb$). If $a \lesssim b$ and $b \lesssim a$, then we write $a \asymp b$.

2. Auxiliary Results

Lemma 1. *Let $m \in \mathbb{N}$, $\mu \in W(\mathbb{B})$ and*

$$\frac{\mu(z)}{\mu(w)} \le C_r < +\infty, \tag{2}$$

for $z, w \in \mathbb{B}$, such that $|z - w| < r(1 - |z|)$ for some $r \in (0,1)$. Then,

$$|\mathfrak{R}^m f(z)| \lesssim \frac{|z|}{(1-|z|)^m \mu(z)} \|f\|_{H^\infty_\mu}, \tag{3}$$

for $f \in H^\infty_\mu(\mathbb{B})$ *and* $z \in \mathbb{B}$.

Proof. For any fixed $r \in (0,1)$, the Cauchy–Schwarz and Cauchy inequalities imply

$$|\Re f(z)| \lesssim \frac{|z|}{1-|z|} \sup_{w\in B(z,r(1-|z|))} |f(w)| \tag{4}$$

for $z \in \mathbb{B}$ and $f \in H(\mathbb{B})$. From (2), we have

$$|f(w)| \le \frac{C_r}{\mu(z)}\mu(w)|f(w)| \le \frac{C_r}{\mu(z)}\|f\|_{H^\infty_\mu}$$

for each $w \in B(z, r(1-|z|))$. By the above two inequalities, we have

$$|\Re f(z)| \lesssim \frac{|z|}{(1-|z|)\mu(z)}\|f\|_{H^\infty_\mu},$$

that is, (3) holds when $m = 1$.

Next, assume that for $k \in \mathbb{N}_2$,

$$\left|\Re^{k-1} f(z)\right| \lesssim \frac{|z|}{(1-|z|)^{k-1}\mu(z)}\|f\|_{H^\infty_\mu}, \tag{5}$$

for every $f \in H^\infty_\mu(\mathbb{B})$ and $z \in \mathbb{B}$.

If we replace f by $\Re^{k-1} f$ in (4), then we obtain

$$|\Re^k f(z)| \lesssim \frac{|z|}{1-|z|} \sup_{w\in B(z,r(1-|z|))} |\Re^{k-1} f(w)|.$$

Since it holds that

$$\frac{1}{\mu(w)} \le \frac{C_r}{\mu(z)} \quad \text{and} \quad 1-|w| > (1-r)(1-|z|)$$

for each $w \in B(z, r(1-|z|))$, (5) implies

$$\begin{aligned}\sup_{w\in B(z,r(1-|z|))} |\Re^{k-1} f(w)| &\lesssim \sup_{w\in B(z,r(1-|z|))} \frac{|w|}{(1-|w|)^{k-1}\mu(w)}\|f\|_{H^\infty_\mu} \\ &\lesssim \frac{C_r}{(1-r)^{k-1}} \frac{1}{(1-|z|)^{k-1}\mu(z)}\|f\|_{H^\infty_\mu}.\end{aligned}$$

Thus,

$$\left|\Re^k f(z)\right| \lesssim \frac{|z|}{(1-|z|)^k\mu(z)}\|f\|_{H^\infty_\mu},$$

from which (3) holds for each $m \in \mathbb{N}$. □

Lemma 2. *Let* $w \in \mathbb{B}$ *and* μ *be a typical weight such that*

$$\frac{\mu(r)}{(1-r)^\alpha} \le C\frac{\mu(\rho)}{(1-\rho)^\alpha}, \tag{6}$$

for $\delta \le r \le \rho < 1$ *and some* $\delta \in (0,1)$, $\alpha > 0$ *and* $C > 0$. *Then, for* $w \in \mathbb{B}$ *and* $k \in \mathbb{N}_0$, *the function*

$$f^\alpha_{w,k}(z) = \frac{(1-|w|^2)^{\alpha+k}}{(1-\langle z,w\rangle)^{\alpha+k}\mu(w)}, \tag{7}$$

belongs to $H^\infty_{\mu,0}(\mathbb{B})$.

Moreover, we have

$$\sup_{w\in\mathbb{B}} \|f^\alpha_{w,k}\|_{H^\infty_\mu} \lesssim 1. \tag{8}$$

Proof. We have

$$\mu(z)|f^\alpha_{w,k}(z)| = \frac{(1-|w|^2)^{\alpha+k}\mu(z)}{|1-\langle z,w\rangle|^{\alpha+k}\mu(w)} \le \frac{(1-|w|^2)^{\alpha+k}\mu(z)}{(1-|z||w|)^{\alpha+k}\mu(w)} \le 2^{\alpha+k}\frac{\mu(z)}{\mu(w)} \tag{9}$$

$$\le 2^{\alpha+k}\frac{\mu(0)}{\mu(w)} \tag{10}$$

for $z \in \mathbb{B}$, which implies $f^\alpha_{w,k} \in H^\infty_\mu(\mathbb{B})$ for each $w \in \mathbb{B}$. From (9) and since $\lim_{|z|\to 1}\mu(z) = 0$, we obtain $f^\alpha_{w,k} \in H^\infty_{\mu,0}(\mathbb{B})$.

From (10) and since μ is radial and non-increasing, it follows that

$$\|f^\alpha_{w,k}\|_{H^\infty_\mu} \le 2^{\alpha+k}\frac{\mu(0)}{\mu(\delta)} \tag{11}$$

for $|w| \le \delta$.

Now, assume that $\delta \le |w| < 1$. Since μ is radial and non-increasing and (6) holds, we get

$$\begin{aligned}\mu(z)|f^\alpha_{w,k}(z)| \le& \frac{(1-|w|^2)^{\alpha+k}\mu(z)}{(1-|z||w|)^{\alpha+k}\mu(w)}\\ =&\frac{\mu(|z|)}{\mu(|w||z|)}\cdot\frac{(1-|w|^2)^k}{(1-|z||w|)^k}\cdot\frac{(1-|w|^2)^\alpha\mu(|w||z|)}{(1-|z||w|)^\alpha\mu(|w|)} \le 2^{k+\alpha}C,\end{aligned} \tag{12}$$

when $|w||z| \ge \delta$.

If $\delta \le |w| < 1$ and $|w||z| < \delta$, then we have

$$\begin{aligned}\mu(z)|f^\alpha_{w,k}(z)| \le& \frac{\mu(|z|)}{\mu(|w||z|)}\cdot\frac{(1-|w|^2)^k}{(1-|z||w|)^k}\cdot\frac{(1-|w|^2)^\alpha\mu(|w||z|)}{(1-|z||w|)^\alpha\mu(|w|)}\\ \le& \frac{2^{k+\alpha}\mu(0)}{(1-\delta)^\alpha}\frac{(1-|w|)^\alpha}{\mu(|w|)} \le \frac{C2^{k+\alpha}\mu(0)}{\mu(\delta)},\end{aligned} \tag{13}$$

From (12) and (13), we have

$$\|f^\alpha_{w,k}\|_{H^\infty_\mu} \le \frac{C2^{k+\alpha}\mu(0)}{\mu(\delta)} \tag{14}$$

for $\delta \le |w| < 1$.

From (11) and (14), relation (8) follows. □

Remark 1. *If*

$$\lim_{r\to 1-0}\frac{\mu(r)}{(1-r)^{\alpha+k}} = +\infty \tag{15}$$

then $f^\alpha_{w,k} \to 0$ *as* $|w| \to 1$ *uniformly on compacts of* $\mathbb{B}$.

For our next lemma, see [34,35].

Lemma 3. *Let $s \geq 0$, $w \in \mathbb{B}$, and $g_{w,s}(z) = (1 - \langle z, w \rangle)^{-s}$. Then,*

$$\Re^k g_{w,s}(z) = s \frac{P_k(\langle z, w \rangle)}{(1 - \langle z, w \rangle)^{s+k}}, \tag{16}$$

where $P_k(w) = s^{k-1}w^k + p_{k-1}^{(k)}(s)w^{k-1} + \cdots + p_2^{(k)}(s)w^2 + w$, and where $p_j^{(k)}(s)$, $j = \overline{2, k-1}$, are non-negative polynomials for $s > 0$.

$$\Re^k g_{w,s}(z) = \sum_{t=1}^{k} a_t^{(k)} \left(\prod_{j=0}^{t-1} (s + j) \right) \frac{\langle z, w \rangle^t}{(1 - \langle z, w \rangle)^{s+t}}, \tag{17}$$

where $(a_t^{(k)})$, $t = \overline{1, k}$, $k \in \mathbb{N}$, are defined as

$$a_1^{(k)} = a_k^{(k)} = 1, \ k \in \mathbb{N}; \tag{18}$$

and for $2 \leq t \leq k-1$, $k \geq 3$,

$$a_t^{(k)} = t a_t^{(k-1)} + a_{t-1}^{(k-1)}. \tag{19}$$

Lemma 4. *Assume $\mu \in W(\mathbb{B})$ satisfies condition (6), where $\alpha > 0$, $m \in \mathbb{N}$, $w \in \mathbb{B}$, $f_{w,t}^{\alpha}$ is defined in (7), and $(a_t^{(k)})_{t=\overline{1,k}}$, $k = \overline{1, m}$, are defined in (18) and (19). Then, for each $l \in \{1, \ldots, m\}$, there is*

$$h_w^{(l)}(z) = \sum_{k=0}^{m} c_k^{(l)} f_{w,k}^{\alpha}(z) \tag{20}$$

where $c_k^{(l)}$, $k = \overline{0, m}$, are numbers such that

$$\Re^j h_w^{(l)}(w) = 0, \ 0 \leq j < l, \tag{21}$$

$$\Re^j h_w^{(l)}(w) = a_l^{(j)} \frac{|w|^{2l}}{(1 - |w|^2)^l \mu(w)}, \ l \leq j \leq m, \tag{22}$$

hold. Moreover, we have $\sup_{w \in \mathbb{B}} \|h_w^{(l)}\|_{H_\mu^\infty} < +\infty$.

Proof. Let $d_k = \alpha + k$, $k \in \mathbb{N}_0$. Replace the constants $c_k^{(l)}$ in (20) by c_k. Then, from (17), we get

$$\begin{aligned}
h_w^{(l)}(w) &= \frac{c_0 + c_1 + \cdots + c_m}{\mu(w)}, \\
\Re h_w^{(l)}(w) &= \frac{(d_0 c_0 + d_1 c_1 + \cdots + d_m c_m)|w|^2}{(1 - |w|^2)\mu(w)}, \\
&\vdots \\
\Re^m h_w^{(l)}(w) &= a_1^{(m)} \frac{(d_0 c_0 + d_1 c_1 + \cdots + d_m c_m)|w|^2}{(1 - |w|^2)\mu(w)} + \cdots \\
&\quad + a_l^{(m)} \frac{(d_0 \cdots d_{l-1} c_0 + d_1 \cdots d_l c_1 + \cdots + d_m \cdots d_{m+l-1} c_m)|w|^{2l}}{(1 - |w|^2)^l \mu(w)} + \cdots \\
&\quad + a_m^{(m)} \frac{(d_0 \cdots d_{m-1} c_0 + d_1 \cdots d_m c_1 + \cdots + d_m \cdots d_{2m-1} c_m)|w|^{2m}}{(1 - |w|^2)^m \mu(w)}.
\end{aligned} \tag{23}$$

Lemma 2.5 in [19] shows that the determinant of the system,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ d_0 & d_1 & \cdots & d_m \\ \vdots & \vdots & & \vdots \\ \prod_{k=0}^{l} d_k & \prod_{k=0}^{l} d_{k+1} & \cdots & \prod_{k=0}^{l} d_{m+k} \\ \vdots & \vdots & & \vdots \\ \prod_{k=0}^{m-1} d_k & \prod_{k=0}^{m-1} d_{k+1} & \cdots & \prod_{k=0}^{m-1} d_{m+k} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \\ \vdots \\ \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{24}$$

is not equal to zero. This implies that there is a unique solution $c_k = c_k^{(l)}$, $k = \overline{0, m}$, to (24). For these c_k values, (20) satisfies (21) and (22). Finally, Lemma 2 implies $\sup_{w\in\mathbb{B}} \|h_w^{(l)}\|_{H_\mu^\infty} < +\infty$. □

The following lemma is well known as a characterization of the compactness of a closed set in the little weighted-type space. Its proof is a slight modification of the proof of Lemma 1 in [54]. Thus, we omit the proof.

Lemma 5. *A closed subset K of $H^\infty_{\nu,0}(\mathbb{B})$ is compact if and only if it is bounded and*

$$\lim_{|z|\to 1} \sup_{f\in K} \nu(z)|f(z)| = 0.$$

Lemma 6. *Let Y be a Banach space of holomorphic functions on $\mathbb{B}$ and μ be a typical weight function on $\mathbb{B}$. Then, $T : H^\infty_{\mu,0}(\mathbb{B}) \to Y$ is compact if and only if it is weakly compact.*

Proof. Let

$$N_1 = \left\{ g \in L^1(\mathbb{B}) : \int_{\mathbb{B}} \mu(z) f(z) g(z) dV(z) = 0 \text{ for every } f \in H^\infty_\mu(\mathbb{B}) \right\}.$$

Since $(H^\infty_{\mu,0}(\mathbb{B}))^* = L^1(\mathbb{B})/N_1$ [8,13], the compactness of $T : H^\infty_{\mu,0}(\mathbb{B}) \to Y$ is equivalent to the compactness of $T^* : Y^* \to L^1(\mathbb{B})/N_1$. The space $L^1(\mathbb{B})/N_1$ has the Schur property, so $T^* : Y^* \to L^1(\mathbb{B})/N_1$ is weakly compact, which is equivalent to $T : H^\infty_{\mu,0}(\mathbb{B}) \to Y$ being weakly compact. □

3. Boundedness

First, we consider the operator $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ for $\{\mu, \nu\} \subset W(\mathbb{B})$. To analyze $\mathfrak{S}^m_{\vec{u},\varphi}$, the growth condition for $|\Re^m f|$ in Lemma 1 and the functions $f^\alpha_{w,k}$ and $h^{(l)}_w$ defined in Lemmas 2 and 4, respectively, play an important role in our argument. The class of all typical weights satisfying conditions (2) and (6) is denoted by $W_1(\mathbb{B})$.

Theorem 1. *Let $k \in \mathbb{N}$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$, $\mu \in W_1(\mathbb{B})$ and $\nu \in W(\mathbb{B})$. Then, $\Re^k_{u,\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is bounded if and only if*

$$J_k := \sup_{z\in\mathbb{B}} \frac{\nu(z)|u(z)||\varphi(z)|}{(1 - |\varphi(z)|)^k \mu(\varphi(z))} < \infty. \tag{25}$$

Furthermore, if it is bounded, then we have

$$\|\Re^k_{u,\varphi}\|_{H^\infty_\mu \to H^\infty_\nu} \asymp J_k.$$

Proof. By Lemma 1, we have

$$\nu(z)|\Re^k_{u,\varphi}f(z)| \lesssim \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k\mu(\varphi(z))}\|f\|_{H^\infty_\mu}$$

for $z \in \mathbb{B}$ and $f \in H^\infty_\mu(\mathbb{B})$. By this inequality, we see that condition (25) implies $\Re^k_{u,\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is bounded and $\|\Re^k_{u,\varphi}\|_{H^\infty_\mu \to H^\infty_\nu} \lesssim J_k$.

Now, we assume that $\Re^k_{u,\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is bounded. For a fixed $w \in \mathbb{B}$, we put $G_w(z) = f^\alpha_{\varphi(w),1}(z)$. Lemma 2 shows that $G_w \in H^\infty_\mu(\mathbb{B})$ and $\sup_{w\in\mathbb{B}}\|G_w\|_{H^\infty_\mu} \lesssim 1$. By Lemma 3, we have

$$\Re^k G_w(z) = \frac{(1-|\varphi(w)|)^{\alpha+1}}{\mu(\varphi(w))}\frac{(\alpha+1)P_k(\langle z,\varphi(w)\rangle)}{(1-\langle z,\varphi(w)\rangle)^{k+\alpha+1}}$$

for some polynomial P_k whose coefficients are all non-negative. Since $|\varphi(w)| \lesssim |\varphi(w)|^2 \lesssim P_k(|\varphi(w)|^2)$ if $|\varphi(w)| > 1/2$, we have

$$\begin{aligned}\nu(w)|\Re^k_{u,\varphi}G_w(w)| &= (\alpha+1)\frac{\nu(w)|u(w)|P_k(|\varphi(w)|^2)}{(1-|\varphi(w)|^2)^k\mu(\varphi(w))}\\ &\gtrsim \frac{\nu(w)|u(w)||\varphi(w)|}{(1-|\varphi(w)|)^k\mu(\varphi(w))},\end{aligned}$$

and thus we obtain

$$\frac{\nu(w)|u(w)||\varphi(w)|}{(1-|\varphi(w)|)^k\mu(\varphi(w))} \lesssim \|\Re^k_{u,\varphi}\|_{H^\infty_\mu\to H^\infty_\nu} \tag{26}$$

for any $w \in \mathbb{B}$ with $|\varphi(w)| > 1/2$. If $|\varphi(w)| \le 1/2$, then $f_j(z) = z_j \in H^\infty_\mu(\mathbb{B})$ $(z \in \mathbb{B}, j = \overline{1,n})$ shows

$$\nu(w)|u(w)||\varphi_j(w)| \le \|\Re^k_{u,\varphi}f_j\|_{H^\infty_\nu} \le \|\Re^k_{u,\varphi}\|_{H^\infty_\mu\to H^\infty_\nu}\|\mu\|_\infty,$$

from which, together with $|\varphi(w)| \asymp \sum_{j=1}^n |\varphi_j(w)|$, we have

$$\frac{\nu(w)|u(w)||\varphi(w)|}{(1-|\varphi(w)|)^k\mu(\varphi(w))} \lesssim \nu(w)|u(w)||\varphi(w)| \lesssim \|\Re^k_{u,\varphi}\|_{H^\infty_\mu\to H^\infty_\nu} \tag{27}$$

for any $w \in \mathbb{B}$ with $|\varphi(w)| \le 1/2$. Combining (26) and (27), we get

$$J_k \lesssim \|\Re^k_{u,\varphi}\|_{H^\infty_\mu\to H^\infty_\nu} < \infty$$

for each $k \in \mathbb{N}$. Thus, we accomplish the proof. □

Corollary 1. *Under the assumptions of Theorem 1, the followings statements are equivalent:*

(a) $\Re^k_{u,\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ *is bounded;*
(b) $\Re^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ *is bounded;*
(c) *The condition* (25) *holds.*

Proof. In fact, since $G_w(z) = f^\alpha_{\varphi(w),1}(z)$ in the proof of Theorem 1 is in $H^\infty_{\mu,0}(\mathbb{B})$, the argument of Theorem 1 still holds in the case of $\Re^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$. That is, we also see that $\Re^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is bounded if and only if u and φ satisfy (25). Hence, Theorem 1 implies the desired claim. □

Theorem 2. *Let $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$ $(j = \overline{1,m})$, $\varphi \in S(\mathbb{B})$, $\mu \in W_1(\mathbb{B})$, and $\nu \in W(\mathbb{B})$. Then, all operators $\mathfrak{R}^j_{u_j,\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ $(j = \overline{1,m})$ are bounded if and only if $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is bounded and*

$$\sup_{z\in\mathbb{B}} \nu(z)|u_j(z)||\varphi(z)| < \infty, \tag{28}$$

for $j = \overline{1,m}$.

Proof. If $\mathfrak{R}^j_{u_j,\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ $(j = \overline{1,m})$ are bounded, then $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is also bounded. As in the proof of Theorem 1, condition (28) can be verified by the functions $f_l(z) = z_l \in H^\infty_\mu(\mathbb{B})$ $(l = \overline{1,n})$.

To prove the other direction, we assume that $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is bounded and (28) is true for $j = \overline{1,m}$. By Theorem 1, it is enough to prove

$$J_j = \sup_{z\in\mathbb{B}} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^j \mu(\varphi(z))} < \infty \tag{29}$$

for $j = \overline{1,m}$.

If $|\varphi(w)| > 0$, by Lemma 4, then there is $h^{(m)}_{\varphi(w)} \in H^\infty_\mu(\mathbb{B})$ such that

$$\mathfrak{R}^j h^{(m)}_{\varphi(w)}(\varphi(w)) = 0$$

for $0 \le j \le m-1$,

$$\mathfrak{R}^m h^{(m)}_{\varphi(w)}(\varphi(w)) = \frac{|\varphi(w)|^{2m}}{(1-|\varphi(w)|^2)^m \mu(\varphi(w))}$$

and $\sup_{w\in\mathbb{B}} \|h^{(m)}_{\varphi(w)}\|_{H^\infty_\mu} < \infty$. By considering the boundedness of $\mathfrak{S}^m_{\vec{u},\varphi}$, we have

$$\begin{aligned}\|\mathfrak{S}^m_{\vec{u},\varphi}\|_{H^\infty_\mu \to H^\infty_\nu} &\gtrsim \|\mathfrak{S}^m_{\vec{u},\varphi} h^{(m)}_{\varphi(w)}\|_{H^\infty_\nu} \\ &\ge \nu(w)\left|\sum_{j=1}^m u_j(w)\mathfrak{R}^j h^{(m)}_{\varphi(w)}(\varphi(w))\right| \\ &= \frac{\nu(w)|u_m(w)||\varphi(w)|^{2m}}{(1-|\varphi(w)|^2)^m \mu(\varphi(w))}.\end{aligned}$$

Hence, it follows that

$$\sup_{|\varphi(w)|>1/2} \frac{\nu(w)|u_m(w)||\varphi(w)|}{(1-|\varphi(w)|)^m \mu(\varphi(w))} \lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{H^\infty_\mu \to H^\infty_\nu} < \infty.$$

By (28), we have

$$\sup_{|\varphi(w)|\le 1/2} \frac{\nu(w)|u_m(w)||\varphi(w)|}{(1-|\varphi(w)|)^m \mu(\varphi(w))} \lesssim \sup_{|\varphi(w)|\le 1/2} \nu(w)|u_m(w)||\varphi(w)| < \infty,$$

and so $J_m < \infty$.

Next, we assume that (29) holds for $j = \overline{s+1,m}$, for $s \in \{1,2,\dots,m-1\}$. For $h^{(s)}_{\varphi(w)}$ as in Lemma 4, we see that $\sup_{w\in\mathbb{B}} \|h^{(s)}_{\varphi(w)}\|_{H^\infty_\mu} < \infty$ and

$$\begin{aligned}\nu(w)\left|\sum_{j=s}^m a^{(j)}_s u_j(w)\frac{|\varphi(w)|^{2s}}{(1-|\varphi(w)|^2)^s \mu(\varphi(w))}\right| &\le \sup_{z\in\mathbb{B}} \nu(z)\left|\sum_{j=1}^m u_j(z)\mathfrak{R}^j h^{(s)}_{\varphi(w)}(\varphi(z))\right| \\ &\lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{H^\infty_\mu \to H^\infty_\nu}.\end{aligned} \tag{30}$$

From (30), it follows that

$$\frac{\nu(w)|u_s(w)||\varphi(w)|^{2s}}{(1-|\varphi(w)|^2)^s\mu(\varphi(w))} \lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{H^\infty_\mu\to H^\infty_\nu} + \sum_{j=s+1}^{m}\frac{\nu(w)|u_j(w)||\varphi(w)|^{2s}}{(1-|\varphi(w)|^2)^s\mu(\varphi(w))},$$

so that we get

$$\sup_{|\varphi(w)|>1/2}\frac{\nu(w)|u_s(w)||\varphi(w)|}{(1-|\varphi(w)|^2)^s\mu(\varphi(w))} \lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{H^\infty_\mu\to H^\infty_\nu} + \sum_{j=s+1}^{m} J_j.$$

On the other hand, by (28), we have

$$\sup_{|\varphi(z)|\le 1/2}\frac{\nu(w)|u_s(w)||\varphi(w)|}{(1-|\varphi(w)|^2)^s\mu(\varphi(w))} \lesssim \sup_{|\varphi(w)|\le 1/2}\nu(w)|u_s(w)||\varphi(w)| < \infty.$$

Hence, (29) holds for $j=s$ and thus for $j=\overline{1,m}$. □

For the same reasons as in Corollary 1, we get the following corollary.

Corollary 2. *Under the assumptions of Theorem 2, the followings statements are equivalent:*

(a) *All the operators $\mathfrak{R}^j_{u_j,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ $(j=\overline{1,m})$ are bounded;*

(b) *$\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is bounded and (28) holds for $j=\overline{1,m}$;*

(c) *$\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_{\mu}(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is bounded and (28) holds for $j=\overline{1,m}$.*

Theorem 3. *Let $k\in\mathbb{N}$, $u\in H(\mathbb{B})$, $\varphi\in S(\mathbb{B})$, $\mu\in W_1(\mathbb{B})$, and $\nu\in W(\mathbb{B})$. Then, the boundedness of $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ is equivalent to the boundedness of $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ and*

$$\lim_{|z|\to 1}\nu(z)|u(z)||\varphi(z)| = 0. \tag{31}$$

Proof. First, suppose that $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is bounded and (31) holds. Since

$$\begin{aligned}\nu(z)|\mathfrak{R}^k_{u,\varphi}p(z)| &= \nu(z)|u(z)\mathfrak{R}^k p(\varphi(z))| \\ &\le \nu(z)|u(z)||\varphi(z)||\nabla[\mathfrak{R}^{k-1}p](\varphi(z))| \\ &\le \nu(z)|u(z)||\varphi(z)|\sup_{w\in\mathbb{B}}|\nabla[\mathfrak{R}^{k-1}p](w)|\end{aligned}$$

for any polynomial p, (31) implies $\mathfrak{R}^k_{u,\varphi}p \in H^\infty_{\nu,0}(\mathbb{B})$. Since the set of all polynomials is dense in $H^\infty_{\mu,0}(\mathbb{B})$, for any $f\in H^\infty_{\mu,0}(\mathbb{B})$ there is a sequence of polynomials $(p_j)_{j\in\mathbb{N}}$ such that $\|f-p_j\|_{H^\infty_\mu} \to 0$ as $j\to\infty$. Using the boundedness of $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$, we have

$$\|\mathfrak{R}^k_{u,\varphi}f - \mathfrak{R}^k_{u,\varphi}p_j\|_{H^\infty_\nu} \le \|\mathfrak{R}^k_{u,\varphi}\|_{H^\infty_{\mu,0}\to H^\infty_\nu}\|f-p_j\|_{H^\infty_\mu} \to 0$$

as $j\to\infty$. Since $\mathfrak{R}^k_{u,\varphi}p_j \in H^\infty_{\nu,0}(\mathbb{B})$ and $H^\infty_{\nu,0}(\mathbb{B})$ is closed in $H^\infty_\nu(\mathbb{B})$, $\mathfrak{R}^k_{u,\varphi}f \in H^\infty_{\nu,0}(\mathbb{B})$, then $\mathfrak{R}^k_{u,\varphi}(H^\infty_{\mu,0}(\mathbb{B})) \subset H^\infty_{\nu,0}(\mathbb{B})$. Hence, $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ is bounded.

Now, assume that $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ is bounded. Since $H^\infty_{\nu,0}(\mathbb{B}) \subset H^\infty_\nu(\mathbb{B})$ and the norms on the spaces $H^\infty_{\nu,0}(\mathbb{B})$ and $H^\infty_\nu(\mathbb{B})$ are the same, it immediately follows that the boundedness of $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ implies the boundedness of $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$.

In order to derive the condition (31), we consider the functions $f_j(z) = z_j$ for $j = \overline{1,n}$. Since μ is typical, we see $f_j \in H^\infty_{\mu,0}(\mathbb{B})$. The boundedness of $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ implies that $\mathfrak{R}^k_{u,\varphi} f_j = u \cdot \varphi_j \in H^\infty_{\nu,0}(\mathbb{B})$, that is,

$$\lim_{|z|\to 1} \nu(z)|u(z)||\varphi_j(z)| = 0,$$

from which (31) easily follows. □

Theorem 4. *Let $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$ $(j = \overline{1,m})$, $\varphi \in S(\mathbb{B})$, $\mu \in W_1(\mathbb{B})$, and $\nu \in W(\mathbb{B})$. Then, $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ is bounded and*

$$\lim_{|z|\to 1} \nu(z)|u_j(z)||\varphi(z)| = 0, \quad j = \overline{1,m} \tag{32}$$

if and only if $\mathfrak{R}^j_{u_j,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ $(j = \overline{1,m})$ are bounded.

Proof. Suppose that $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ is bounded and (32) holds. Theorem 3 shows that it is enough to prove that $\mathfrak{R}^j_{u_j,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu}(\mathbb{B})$ are bounded for $j = \overline{1,m}$. For this purpose, it is sufficient to show the boundedness of $\mathfrak{R}^j_{u_j,\varphi} : H^\infty_{\mu}(\mathbb{B}) \to H^\infty_{\nu}(\mathbb{B})$, so we may prove that

$$\sup_{z\in\mathbb{B}} \frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j \mu(\varphi(z))} < \infty \tag{33}$$

for $j = \overline{1,m}$. Now, looking back at the proof of Theorem 2, by Lemma 4, there exists a function $h^{(m)}_{\varphi(w)} \in H^\infty_\mu(\mathbb{B})$ satisfying

$$\mathfrak{R}^j h^{(m)}_{\varphi(w)}(\varphi(w)) = 0$$

for each $0 \le j \le m-1$,

$$\mathfrak{R}^m h^{(m)}_{\varphi(w)}(\varphi(w)) = \frac{|\varphi(w)|^{2m}}{(1-|\varphi(w)|^2)^m \mu(\varphi(w))}$$

and $\sup_{w\in\mathbb{B}} \|h^{(m)}_{\varphi(w)}\|_{H^\infty_\mu} < \infty$. According to Lemma 2, we see $h^{(m)}_{\varphi(w)} \in H^\infty_{\mu,0}(\mathbb{B})$. Hence, as in the proof of Theorem 2, we obtain

$$\sup_{|\varphi(w)|>1/2} \frac{\nu(w)|u_m(w)||\varphi(w)|}{(1-|\varphi(w)|)^m \mu(\varphi(w))} \lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{H^\infty_{\mu,0}\to H^\infty_{\nu,0}} < \infty.$$

On the other hand, the assumption (32) indicates

$$\sup_{z\in\mathbb{B}} \nu(z)|u_m(z)||\varphi(z)| < \infty,$$

and so we obtain

$$\sup_{|\varphi(w)|\le 1/2} \frac{\nu(w)|u_m(w)||\varphi(w)|}{(1-|\varphi(w)|)^m \mu(\varphi(w))} \lesssim \sup_{|\varphi(w)|\le 1/2} \nu(w)|u_m(w)||\varphi(z)| < \infty.$$

Thus, (33) holds for $j = m$. We can also prove that (33) holds for all $j = \overline{1,m}$ by exactly the same argument as in the proof of Theorem 2. Hence, Theorem 1 implies $\mathfrak{R}^j_{u_j,\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ are bounded, and so $\mathfrak{R}^j_{u_j,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ are bounded. The other direction is trivial from Theorem 3. □

4. Essential Norm and Compactness

Here, we investigate the essential norm and the compactness of $\mathfrak{R}^k_{u,\varphi}$ and $\mathfrak{S}^m_{\vec{u},\varphi}$. To characterize the compactness of T, it is well known that it is sufficient to evaluate $\|T\|_e$. To estimate the essential norm of $\mathfrak{R}^k_{u,\varphi}$ or $\mathfrak{S}^m_{\vec{u},\varphi}$, we need the properties of the test functions $f^\alpha_{w,k}$ and $h^{(l)}_w$ in Lemmas 2 and 4, respectively, plus the fact that $f^\alpha_{w,k}$ and $h^{(l)}_w$ converge weakly to 0 as $|w| \to 1-0$. Since this weak convergence is verified by the condition (15) on $\mu \in W(\mathbb{B})$, we continue to assume that $\mu \in W_1(\mathbb{B})$ and add further condition (15). The class of such weights we denote by $W_{\alpha,k}(\mathbb{B})$.

Theorem 5. *Let $k \in \mathbb{N}$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$, $\mu \in W_{\alpha,k}(\mathbb{B})$ and $\nu \in W(\mathbb{B})$. Suppose that $\mathfrak{R}^k_{u,\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is bounded. Then,*

$$\|\mathfrak{R}^k_{u,\varphi}\|_e \asymp \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))}. \tag{34}$$

Proof. If $\|\varphi\|_\infty < 1$, then $\mathfrak{R}^k_{u,\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is compact, implying $\|\mathfrak{R}^k_{u,\varphi}\|_e = 0$, whereas the limit in (34) is taken over an empty set, so the theorem vacuously holds.

Now, assume $\|\varphi\|_\infty = 1$. Let $r \in (0,1)$. Put $C_r f(z) = f(rz)$. Since C_r is compact on $H^\infty_\mu(\mathbb{B})$, the operator $\mathfrak{R}^k_{u,\varphi} C_r : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is also compact, from which it follows that

$$\|\mathfrak{R}^k_{u,\varphi}\|_e \le \sup_{\|f\|_{H^\infty_\mu} \le 1} \|\mathfrak{R}^k_{u,\varphi} f - \mathfrak{R}^k_{u,\varphi} C_r f\|_{H^\infty_\nu}. \tag{35}$$

Now, we fix $f \in H^\infty_\mu$, which satisfies $\|f\|_{H^\infty_\mu} \le 1$ and $\rho \in (0,1)$ arbitrarily. Using the mean value theorem, the fact that $\mathfrak{R}(f(rz)) = (\mathfrak{R} f)(rz)$, and the Cauchy inequality, we have

$$\begin{aligned}
&\sup_{|\varphi(z)|\le\rho} \nu(z)|\mathfrak{R}^k_{u,\varphi} f(z) - \mathfrak{R}^k_{u,\varphi} C_r f(z)| \\
\le &\sup_{|\varphi(z)|\le\rho} \nu(z)|u(z)||\mathfrak{R}^k f(\varphi(z)) - \mathfrak{R}^k f(r\varphi(z))| \\
\le &\sup_{|\varphi(z)|\le\rho} \nu(z)|u(z)|(1-r)|\varphi(z)| \sup_{|w|\le\rho} |\nabla[\mathfrak{R}^k f](w)| \\
\lesssim &\sup_{|\varphi(z)|\le\rho} \nu(z)|u(z)|\frac{(1-r)|\varphi(z)|}{1-\rho} \sup_{|w|\le\frac{1+\rho}{2}} |\mathfrak{R}^k f(w)|.
\end{aligned}$$

Combining Lemma 1 with this, we obtain

$$\begin{aligned}
&\sup_{|\varphi(z)|\le\rho} \nu(z)|\mathfrak{R}^k_{u,\varphi} f(z) - \mathfrak{R}^k_{u,\varphi} C_r f(z)| \\
\lesssim &\frac{(1+\rho)(1-r)}{(1-\rho)^{k+1}\mu(\frac{1+\rho}{2})} \|f\|_{H^\infty_\mu} \sup_{|\varphi(z)|\le\rho} \nu(z)|u(z)||\varphi(z)|.
\end{aligned} \tag{36}$$

Since $f_j(z) = z_j \in H^\infty_\mu(\mathbb{B})$ $(j = \overline{1,n})$ and the boundedness of $\mathfrak{R}^k_{u,\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ shows $u \cdot \varphi_j \in H^\infty_\nu(\mathbb{B})$ for $j = \overline{1,n}$, we get

$$\sup_{|\varphi(z)|\le\rho} \nu(z)|u(z)||\varphi(z)| < \infty.$$

By letting $r \to 1$ in (36), we have

$$\sup_{\|f\|_{H^\infty_\mu}\le 1} \sup_{|\varphi(z)|\le\rho} \nu(z)|\mathfrak{R}^k_{u,\varphi} f(z) - \mathfrak{R}^k_{u,\varphi} C_r f(z)| \to 0. \tag{37}$$

Further, Lemma 1 yields

$$\sup_{\|f\|_{H_\mu^\infty}\le 1}\ \sup_{|\varphi(z)|>\rho} \nu(z)|\Re^k_{u,\varphi}f(z)-\Re^k_{u,\varphi}C_rf(z)|$$
$$\lesssim \sup_{|\varphi(z)|>\rho}\frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k\mu(\varphi(z))}. \tag{38}$$

From (35), (37) and (38), it follows that

$$\|\Re^k_{u,\varphi}\|_e \lesssim \sup_{|\varphi(z)|>\rho}\frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k\mu(\varphi(z))}.$$

By letting $\rho\to 1$, we obtain the upper estimate

$$\|\Re^k_{u,\varphi}\|_e \lesssim \limsup_{|\varphi(z)|\to 1}\frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k\mu(\varphi(z))}.$$

To prove the lower estimate for $\|\Re^k_{u,\varphi}\|_e$, we take a sequence $(z_j)_{j\in\mathbb{N}}\subset\mathbb{B}$ such that $|\varphi(z_j)|\to 1$ as $j\to\infty$. Put $G_j=f^\alpha_{\varphi(z_j),1}$, where $f^\alpha_{w,1}$ are as in Lemma 2. Then, $\sup_{j\ge 1}\|G_j\|_{H_\mu^\infty}<\infty$. As we pointed out in Remark 1, the assumption (15) on μ implies that $G_j\to 0$ uniformly on compact subsets of $\mathbb{B}$ as $j\to\infty$.

A duality argument employed in $H_\mu^\infty(\mathbb{B})$ [8,13] implies that $G_j\to 0$ weakly in $H_\mu^\infty(\mathbb{B})$ as $j\to\infty$, and so $\|KG_j\|_{H_\nu^\infty}\to 0$ as $j\to\infty$ for any compact operator $K: H_\mu^\infty(\mathbb{B})\to H_\nu^\infty(\mathbb{B})$. Hence, Lemma 3 gives

$$\begin{aligned}\|\Re^k_{u,\varphi}\|_e &\gtrsim \limsup_{j\to\infty}(\|\Re^k_{u,\varphi}G_j\|_{H_\nu^\infty}-\|KG_j\|_{H_\nu^\infty})\\ &\ge \limsup_{j\to\infty}\nu(z_j)|u(z_j)|\frac{(\alpha+1)|P_k(|\varphi(z_j)|^2)|}{(1-|\varphi(z_j)|^2)^k\mu(\varphi(z_j))}\\ &\gtrsim \limsup_{j\to\infty}\frac{\nu(z_j)|u(z_j)||\varphi(z_j)|}{(1-|\varphi(z_j)|)^k\mu(\varphi(z_j))}.\end{aligned}$$

That is, the lower estimate

$$\|\Re^k_{u,\varphi}\|_e \gtrsim \limsup_{|\varphi(z)|\to 1}\frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k\mu(\varphi(z))}$$

holds. The proof is accomplished. □

Corollary 3. *Under the assumptions of Theorem 5, the followings statements are equivalent:*

(a) $\Re^k_{u,\varphi}: H_\mu^\infty(\mathbb{B})\to H_\nu^\infty(\mathbb{B})$ *is compact;*

(b) $\Re^k_{u,\varphi}: H_{\mu,0}^\infty(\mathbb{B})\to H_\nu^\infty(\mathbb{B})$ *is compact;*

(c) *u and φ satisfy the following condition*

$$\lim_{|\varphi(z)|\to 1}\frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k\mu(\varphi(z))}=0. \tag{39}$$

Proof. By Theorem 5, it is enough to prove the equivalence (b) $\Leftrightarrow$ (c). To do this, we estimate the essential norm of the bounded operator $\Re^k_{u,\varphi}: H_{\mu,0}^\infty(\mathbb{B})\to H_\nu^\infty(\mathbb{B})$. The upper estimate for this operator is obtained by the arguments in the proof of Theorem 5. On the other hand, we use the weak convergence of the sequence $(G_j)_{j\in\mathbb{N}}$ to 0 in $H_\mu^\infty(\mathbb{B})$ for the lower estimate. In fact, an application of the Hahn–Banach extension theorem implies that $G_j \to 0$ weakly in $H^\infty_{\mu,0}(\mathbb{B})$ as $j \to \infty$. Thus, we also see that the essential norm of $\Re^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ can be evaluated from below by

$$\limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))}.$$

This indicates that (b) ⇔ (c) is true. □

Theorem 6. *Let $m \in \mathbb{N}$, $\{u_1, u_2, \dots, u_m\} \subset H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$, $\mu \in W_{\alpha,0}(\mathbb{B})$ and $\nu \in W(\mathbb{B})$. Suppose $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$ is bounded and (28) holds for $j = \overline{1,m}$. Then,*

$$\|\mathfrak{S}^m_{\vec{u},\varphi}\|_e \asymp \max_{j=\overline{1,m}} \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j \mu(\varphi(z))}.$$

Proof. The case $\|\varphi\|_\infty < 1$ is treated as in Theorem 5. Now, assume $\|\varphi\|_\infty = 1$. For a fixed $r \in (0,1)$, the operator C_r is compact on $H^\infty_\mu(\mathbb{B})$. Fix $f \in H^\infty_\mu(\mathbb{B})$ with $\|f\|_{H^\infty_\mu} \le 1$ and $\rho \in (0,1)$ arbitrarily. Since

$$|\mathfrak{S}^m_{\vec{u},\varphi} f(z) - \mathfrak{S}^m_{\vec{u},\varphi} C_r f(z)| = \left| \sum_{j=1}^m u_j(z)(\Re^j f(\varphi(z)) - \Re^j f(r\varphi(z))) \right|$$

for each $z \in \mathbb{B}$, from Lemma 1, we have

$$\begin{aligned} \sup_{|\varphi(z)|>\rho} \nu(z)|\mathfrak{S}^m_{\vec{u},\varphi} f(z) - \mathfrak{S}^m_{\vec{u},\varphi} C_r f(z)| &\lesssim \sum_{j=1}^m \sup_{|\varphi(z)|>\rho} \frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j \mu(\varphi(z))} \|f\|_{H^\infty_\mu} \\ &\lesssim \max_{j=\overline{1,m}} \sup_{|\varphi(z)|>\rho} \frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j \mu(\varphi(z))}. \end{aligned}$$

By noting (28), the same argument which derives (36) and (37) implies

$$\sup_{\|f\|_{H^\infty_\mu}\le 1} \sup_{|\varphi(z)|\le\rho} \nu(z)|\mathfrak{S}^m_{\vec{u},\varphi} f(z) - \mathfrak{S}^m_{\vec{u},\varphi} C_r f(z)| \to 0$$

as $r \to 1$. Hence, these inequalities give the upper estimate

$$\|\mathfrak{S}^m_{\vec{u},\varphi}\|_e \lesssim \max_{j=\overline{1,m}} \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j \mu(\varphi(z))}.$$

Let $(z_k)_{k\in\mathbb{N}} \subset \mathbb{B}$ be such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ and put $h^{(s)}_k = h^{(s)}_{\varphi(z_k)}$ for each $s = \overline{1,m}$, where $h^{(s)}_w$ are as in Lemma 4. Then, we see that $\sup_{k\ge 1} \|h^{(s)}_k\|_{H^\infty_\mu} < \infty$ and $\mu \in W_{\alpha,0}(\mathbb{B})$ imply that $h^{(s)}_k \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to \infty$ for $s = \overline{1,m}$.

Since a duality argument employed in $H^\infty_\mu(\mathbb{B})$ implies $h^{(s)}_k \to 0$ weakly in $H^\infty_\mu(\mathbb{B})$, we see that $\|Kh^{(s)}_k\|_{H^\infty_\mu} \to 0$ as $k \to \infty$ for compact $K : H^\infty_\mu(\mathbb{B}) \to H^\infty_\nu(\mathbb{B})$. From (21) and (22), we have that

$$\Re^j h^{(s)}_k(\varphi(z_k)) = 0, \quad 1 \le j < s, \tag{40}$$

and

$$\Re^j h^{(s)}_k(\varphi(z_k)) = a^{(j)}_s \frac{|\varphi(z_k)|^{2s}}{(1-|\varphi(z_k)|^2)^s \mu(\varphi(z_k))}, \quad s \le j \le m, \tag{41}$$

hold for $s = \overline{1,m}$. Hence, it follows from (40) and (41) that

$$\begin{aligned}\|\mathfrak{S}^m_{\vec{u},\varphi}\|_e &\gtrsim \limsup_{k\to\infty}(\|\mathfrak{S}^m_{\vec{u},\varphi}h_k^{(m)}\|_{H_\nu^\infty} - \|Kh_k^{(m)}\|_{H_\nu^\infty}) \\ &\geq \limsup_{k\to\infty}\nu(z_k)\left|\sum_{j=1}^m u_j(z_k)\mathfrak{R}^j h_k^{(m)}(\varphi(z_k))\right| \\ &= \limsup_{k\to\infty}\frac{\nu(z_k)|u_m(z_k)||\varphi(z_k)|^{2m}}{(1-|\varphi(z_k)|^2)^m\mu(\varphi(z_k))} \\ &\gtrsim \limsup_{k\to\infty}\frac{\nu(z_k)|u_m(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|)^m\mu(\varphi(z_k))},\end{aligned}$$

and so

$$\|\mathfrak{S}^m_{\vec{u},\varphi}\|_e \gtrsim \limsup_{|\varphi(z)|\to 1}\frac{\nu(z)|u_m(z)||\varphi(z)|}{(1-|\varphi(z)|)^m\mu(\varphi(z))}. \tag{42}$$

Now, we assume that for $s \in \{1,\dots,m-1\}$,

$$\|\mathfrak{S}^m_{\vec{u},\varphi}\|_e \gtrsim \limsup_{|\varphi(z)|\to 1}\frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j\mu(\varphi(z))} \tag{43}$$

holds for $j = \overline{s+1,m}$. Equations (40) and (41) imply

$$\limsup_{k\to\infty}\nu(z_k)\left|\sum_{j=s}^m a_s^{(j)}u_j(z_k)\frac{|\varphi(z_k)|^{2s}}{(1-|\varphi(z_k)|^2)^s\mu(\varphi(z_k))}\right| \lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_e,$$

from which we easily get

$$\begin{aligned}&\limsup_{k\to\infty}\frac{\nu(z_k)|u_s(z_k)||\varphi(z_k)|^{2s}}{(1-|\varphi(z_k)|^2)^s\mu(\varphi(z_k))} \\ &\lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_e + \liminf_{k\to\infty}\sum_{j=s+1}^m a_s^{(j)}\frac{\nu(z_k)|u_j(z_k)||\varphi(z_k)|^{2s}}{(1-|\varphi(z_k)|^2)^s\mu(\varphi(z_k))} \\ &\lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_e.\end{aligned}$$

This indicates that (43) holds for $j = s$, and therefore holds for any $j \in \{1,\dots,m\}$. Hence, we obtain the lower estimate

$$\|\mathfrak{S}^m_{\vec{u},\varphi}\|_e \gtrsim \max_{j=\overline{1,m}}\limsup_{|\varphi(z)|\to 1}\frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j\mu(\varphi(z))}.$$

We complete the proof. □

The following result is proved exactly by the previous arguments.

Corollary 4. *Under the assumptions of Theorem 6, the followings statements are equivalent:*

(a) $\mathfrak{S}^m_{\vec{u},\varphi} : H_\mu^\infty(\mathbb{B}) \to H_\nu^\infty(\mathbb{B})$ *is compact;*

(b) $\mathfrak{S}^m_{\vec{u},\varphi} : H_{\mu,0}^\infty(\mathbb{B}) \to H_\nu^\infty(\mathbb{B})$ *is compact;*

(c) u_j *and* φ *satisfy the following condition*

$$\lim_{|\varphi(z)|\to 1}\frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j\mu(\varphi(z))} = 0 \tag{44}$$

for $j = \overline{1,m}$.

Theorem 7. *Let $k \in \mathbb{N}$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$, $\mu \in W_{\alpha,k}(\mathbb{B})$ and $\nu \in W(\mathbb{B})$. Suppose that $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ is bounded. Then,*

$$\|\mathfrak{R}^k_{u,\varphi}\|_e \asymp \limsup_{|z|\to 1} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))}.$$

Proof. Assume $\|\varphi\|_\infty < 1$. Since $\mathfrak{R}^k_{u,\varphi}$ is compact from $H^\infty_{\mu,0}(\mathbb{B})$ into $H^\infty_{\nu,0}(\mathbb{B})$, $\|\mathfrak{R}^k_{u,\varphi}\|_e = 0$ holds. On the other hand, we obtain

$$\frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))} \le \frac{\nu(z)|u(z)||\varphi(z)|}{(1-\|\varphi\|_\infty)^k \mu(\|\varphi\|_\infty)} \tag{45}$$

for each $z \in \mathbb{B}$. We consider the function $f_j(z) = z_j$ for $j = \overline{1,n}$. Since μ is typical, we see $f_j \in H^\infty_{\mu,0}(\mathbb{B})$. The boundedness of $\mathfrak{R}^k_{u,\varphi}$ implies that $\mathfrak{R}^k_{u,\varphi} f_j = u \cdot \varphi_j \in H^\infty_{\nu,0}(\mathbb{B})$, that is,

$$\lim_{|z|\to 1} \nu(z)|u(z)||\varphi(z)| \asymp \lim_{|z|\to 1} \sum_{j=1}^n \nu(z)|u(z)||\varphi_j(z)| = 0. \tag{46}$$

Thus, (45) and (46) give that

$$\limsup_{|z|\to 1} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))} = 0 = \|\mathfrak{R}^k_{u,\varphi}\|_e.$$

Now, we assume that $\|\varphi\|_\infty = 1$. In view of Theorem 5, it is sufficient to prove

$$\limsup_{|z|\to 1} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))} = \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))}. \tag{47}$$

Take a sequence $(z_l)_{l\in\mathbb{N}} \subset \mathbb{B}$ such that

$$\limsup_{|z|\to 1} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))} = \lim_{l\to\infty} \frac{\nu(z_l)|u(z_l)||\varphi(z_l)|}{(1-|\varphi(z_l)|)^k \mu(\varphi(z_l))}. \tag{48}$$

If $\sup_{l\in\mathbb{N}} |\varphi(z_l)| < 1$, then (46) shows that the second limit in (48) is zero. Since the following inequality obviously holds

$$\limsup_{|z|\to 1} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))} \ge \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))},$$

we see that (47) holds as the upper limit of both sides is zero.

If $\sup_{l\in\mathbb{N}} |\varphi(z_l)| = 1$, then we can choose a subsequence $(\varphi(z_{l_i}))_{i\in\mathbb{N}}$ such that $|\varphi(z_{l_i})| \to 1$ as $i \to \infty$. Thus,

$$\begin{aligned}\limsup_{|z|\to 1} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))} &= \lim_{i\to\infty} \frac{\nu(z_{l_i})|u(z_{l_i})||\varphi(z_{l_i})|}{(1-|\varphi(z_{l_i})|)^k \mu(\varphi(z_{l_i}))} \\ &\le \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))},\end{aligned}$$

which proves that (47) really holds. □

Theorem 8. *Let $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$ $(j = \overline{1,m})$, $\varphi \in S(\mathbb{B})$, $\mu \in W_{\alpha,0}(\mathbb{B})$ and $\nu \in W(\mathbb{B})$. If $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ is bounded and (32) holds, then*

$$\|\mathfrak{S}^m_{\vec{u},\varphi}\|_e \asymp \max_{j=\overline{1,m}} \limsup_{|z|\to 1} \frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j \mu(\varphi(z))}.$$

Proof. Since $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ is bounded and (32) holds, it follows from Theorems 3 and 4 that all operators $\mathfrak{R}^j_{u_j,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu}(\mathbb{B})$ $(j = \overline{1,m})$ are bounded. Hence, by Theorem 2, we see that $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu}(\mathbb{B})$ is bounded and (28) holds for $j = \overline{1,m}$. Theorem 6 gives

$$\|\mathfrak{S}^m_{\vec{u},\varphi}\|_e \asymp \max_{j=\overline{1,m}} \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j \mu(\varphi(z))}.$$

By exactly the same argument as in (47), we obtain

$$\limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j \mu(\varphi(z))} = \limsup_{|z|\to 1} \frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j \mu(\varphi(z))}$$

for $j = \overline{1,m}$, from which it follows that

$$\|\mathfrak{S}^m_{\vec{u},\varphi}\|_e \asymp \max_{j=\overline{1,m}} \limsup_{|z|\to 1} \frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j \mu(\varphi(z))}.$$

The proof is accomplished. □

Theorem 9. *Let $k \in \mathbb{N}$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$, $\mu \in W_{\alpha,k}(\mathbb{B})$ and $\nu \in W(\mathbb{B})$. The following statements are equivalent:*

(a) $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ *is compact;*

(b) $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ *is compact;*

(c) $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ *is bounded;*

(d) $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ *is weakly compact;*

(e) *The following condition holds:*

$$\lim_{|z|\to 1} \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))} = 0.$$

Proof. By Lemma 6, we get the equivalence (a) ⇔ (d). The equivalence (a) ⇔ (e) follows from Theorem 7 immediately. (b) ⇒ (c) is obvious. If we prove implications (e) ⇒ (b) and (c) ⇒ (d), we accomplish the proof. By Lemma 1, we have

$$\nu(z)|\mathfrak{R}^k_{u,\varphi} f(z)| \lesssim \frac{\nu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|)^k \mu(\varphi(z))} \|f\|_{H^\infty_\mu}$$

for each $f \in H^\infty_\mu(\mathbb{B})$, from which we see $\mathfrak{R}^k_{u,\varphi}(H^\infty_\mu(\mathbb{B})) \subset H^\infty_{\nu,0}(\mathbb{B})$ and

$$\lim_{|z|\to 1} \sup_{\|f\|_{H^\infty_\mu} \le 1} \nu(z)|\mathfrak{R}^k_{u,\varphi} f(z)| = 0.$$

By Lemma 5, we see that (e) ⇒ (b) holds. Now, we assume (c) is true. Then, $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ is bounded. A duality argument and weak-star density of $H^\infty_{\mu,0}(\mathbb{B})$ in $H^\infty_\mu(\mathbb{B})$ shows

$$(\mathfrak{R}^k_{u,\varphi})^{**} = \mathfrak{R}^k_{u,\varphi} \quad \text{on } H^\infty_\mu(\mathbb{B}) = (H^\infty_{\mu,0}(\mathbb{B}))^{**}.$$

Therefore,

$$(\mathfrak{R}^k_{u,\varphi})^{**}((H^\infty_{\mu,0}(\mathbb{B}))^{**}) = \mathfrak{R}^k_{u,\varphi}(H^\infty_\mu(\mathbb{B})) \subset H^\infty_{\nu,0}(\mathbb{B}).$$

This and Gantmacher's theorem [55] imply the weak compactness of $\mathfrak{R}^k_{u,\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ follows. Namely, we have proven the implication (c) ⇒ (d). □

In exactly the same way as in Theorem 9, we also obtain the following result.

Theorem 10. *Let $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$ $(j = \overline{1,m})$, $\varphi \in S(\mathbb{B})$, $\mu \in W_{\alpha,0}(\mathbb{B})$ and $\nu \in W(\mathbb{B})$. Suppose that (32) holds for $j = \overline{1,m}$. Then, the following statements are equivalent:*

(a) $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ *is compact;*

(b) $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_{\mu}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ *is compact;*

(c) $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_{\mu}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ *is bounded;*

(d) $\mathfrak{S}^m_{\vec{u},\varphi} : H^\infty_{\mu,0}(\mathbb{B}) \to H^\infty_{\nu,0}(\mathbb{B})$ *is weakly compact;*

(e) *The following conditions hold:*

$$\lim_{|z|\to 1} \frac{\nu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|)^j \mu(\varphi(z))} = 0, \qquad j = \overline{1,m}.$$

5. Conclusions

We studied the boundedness of a recently introduced operator between weighted-type spaces of holomorphic functions and estimated its essential norm. To do this, we gave some methods, ideas and tricks which may be useful in investigations of related concrete linear operators, which will be the focus of our further investigations.

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