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Bi-Univalent Functions Based on Binomial Series-Type Convolution Operator Related with Telephone Numbers

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Abstract: This paper introduces two novel subclasses of the function class Σ for bi-univalent functions, leveraging generalized telephone numbers and Binomial series through convolution. The exploration is conducted within the domain of the open unit disk. We delve into the analysis of initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, deriving insights and findings for functions belonging to these new subclasses. Additionally, Fekete-Szegö inequalities are established for these functions. Furthermore, the study unveils a range of new subclasses of Σ, some of which are special cases, yet have not been previously explored in conjunction with telephone numbers. These subclasses emerge as a result of hybrid-type convolution operators. Concluding from our results, we present several corollaries, which stand as fresh contributions in the domain of involution numbers involving hybrid-type convolution operators.

Keywords: univalent functions; analytic functions; bi-univalent functions; binomial series; convolution operator; involution numbers; coefficient bounds

MSC: 30C45; 30C50; 30C55

1. Introduction

In this article, we will study Bi-Univalent Functions Based on Binomial Series-Type Convolution Operator Related with Telephone Numbers. For this purpose, we will first give the basic definitions and theorems we need. Let $\mathcal{A}$ represent the class of functions that can be written as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

these functions are analytic in the unit disk which defined below and here a_n represents the coefficients,

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $\mathcal{S}$ be the class made up of all functions that are univalent on the open unit disk and taken from class $\mathcal{A}$. The most well-known and important subclasses of this class are the starlike and convex classes. Two conversant subclasses of $\mathcal{A}$ are correspondingly the class of starlike functions and convex functions of order α ($0 \leq \alpha < 1$). These classes are familiarised by Robertson [1] and are defined with their analytical description as

$$\mathcal{S}^*(\alpha) := \left\{ h \in \mathcal{A} : \Re\left(\frac{zh'(z)}{h(z)} \right) > \alpha, \, z \in \mathbb{U} \right\}$$

and

$$\mathcal{C}(\alpha) := \left\{ h \in \mathcal{A} : \Re\left(1 + \frac{zh''(z)}{h'(z)} \right) > \alpha, \, z \in \mathbb{U} \right\}.$$

It is well known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}$ and $\mathcal{C}(\alpha) \subset \mathcal{S}$. In the interpretation of Alexander's relation, $h \in \mathcal{C}(\alpha)$ if and only if $zh'(z)$ for $z \in \mathbb{U}$, belongs to $\mathcal{S}^*(\alpha)$ for each $0 \leq \alpha < 1$.

For $\alpha = 0$ the class $\mathcal{S}^* := \mathcal{S}^*(0)$ condenses to the well-known class of normalized starlike univalent functions and $\mathcal{C} := \mathcal{C}(0)$ reduces to the normalized convex univalent functions.

The classes formed by the starlike and convex functions and the subclasses of these classes have been studied a lot in the past and still maintain their popularity today.

With the f function of type (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard Product of these functions is denoted by $f * h$ and defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{2}$$

Let the functions f and g be analytic, the subordination of the f function to the g function is denoted by $f(z) \prec g(z)$. The important thing here is to prove the existence of an analytic function ϖ that satisfies the conditions $\varpi(0) = 0$ and $|\varpi(z)| < 1$ when $f(z) = g(\varpi(z))$ is defined on the open unit disk. Lately Ma and Minda [2] amalgamated various subclasses of starlike and convex functions for which either of the quantity $\frac{z\,f'(z)}{f(z)}$ or $\frac{(z\,f'(z))'}{f'(z)}$ is subordinate to a more general superordinate function $Y(z) = 1 + M_1 z + M_2 z^2 + M_3 z^3 + \cdots, M_1 > 0$. For $f \in \mathcal{A}$, the class of Ma-Minda starlike functions is given by $\frac{z\,f'(z)}{f(z)} \prec Y(z)$ and Ma-Minda convex functions is by $\frac{(z\,f'(z))'}{f'(z)} \prec Y(z)$. They concentrated on some results, such as covering theorems, growth theorems, and distortion bounds. Several subfamilies of the collection $\mathcal{S}$ have been looked at as specific options for the class $\mathcal{S}^*(Y(z))$ throughout the past few years. In the study that has lately been examined, the families mentioned below are particularly noteworthy.

(i). $\mathcal{S}_{\mathcal{L}}^* \equiv \mathcal{S}^*(\sqrt{1+z})$ [3], $\mathcal{S}_{\exp}^* \equiv \mathcal{S}^*(\exp(z))$ [4], $\mathcal{S}_{\tanh}^* \equiv \mathcal{S}^*(1 + \tanh(z))$ [5],

(ii). $\mathcal{S}_{\cos}^* \equiv \mathcal{S}^*(\cos(z))$ [6], $\mathcal{S}_{pet}^* \equiv \mathcal{S}^*\left(1 + \sinh^{-1} z\right)$ [7], $\mathcal{S}_{\cosh}^* \equiv \mathcal{S}^*(\cosh(z))$ [8],

(iii). $\mathcal{S}_{\sin}^* \equiv \mathcal{S}^*(1 + \sin(z))$ [9], $\mathcal{S}_{car}^* \equiv \mathcal{S}^*\left(1 + z + \frac{1}{2}z^2\right)$ [10],

(iv). $\mathcal{S}_{(n-1)\mathcal{L}}^* \equiv \mathcal{S}^*(\Psi_{n-1}(z))$ [11] with $\Psi_{n-1}(z) = 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n$ for $n \geq 2$.

Main idea of this article, we made an attempt to define two new subclasses of the function class of bi-univalent functions defined in the open unit disk, involving Binomial series by convolution and find the initial Taylor coefficient estimate $|a_2|$ and $|a_3|$, relating with generalized telephone numbers. Therefore, before moving on to our general section on Coefficient Bounds, we need to give some general definitions, theorems and examples for detailed examination.

1.1. Integral Operator

Fractional calculus was first studied in the late 17th century. Fractional calculus has a wide range of applications, for example, fluid flow models, electrochemical analysis, groundwater flow problems, structural damping models, acoustic wave equations for complex media, quantum theory, economy, finance, biology, human sciences, etc. Since its application area is very wide, it is a multidisciplinary subject and will increase its popularity and importance even more today and in the near future. References [12–16] can be consulted for some studies. Fractional derivative operator is a field that grows day by day and new studies are made. Many operators have been defined recently, which is clear proof of how important the subject is. Some of these operators are defined via a fractional integral. Thanks to these operators, we can process and analyze data in many different disciplines. Some common fractional derivatives operators are: Riemann–Liouville, Hadamard, Caputo

and Erdélyi–Kober fractional operators, which have been proposed and implemented. We recall the operator $\mathcal{L}_\kappa^\sigma : \mathbb{U} \to \mathbb{U}$, studied by Babalola [17], is defined by

$$\mathcal{L}_\kappa^\sigma f(z) := \left(\rho_\sigma * \rho_{\sigma,\kappa}^{-1} * f \right)(z), \tag{3}$$

where

$$\rho_{\sigma,\kappa}(z) = \frac{z}{(1-z)^{\sigma-\kappa+1}}, \quad \sigma - \kappa + 1 > 0, \quad \text{and} \quad \rho_\sigma = \rho_{\sigma,0}.$$

and $\rho_{\sigma,\kappa}^{-1}$ is given by

$$\left(\rho_{\sigma,\kappa} * \rho_{\sigma,\kappa}^{-1} \right)(z) = \frac{z}{1-z} \qquad (\sigma, \kappa \in \mathbb{N} := \{1, 2, 3, \cdots\}).$$

If the function f is defined in type (1) and belongs to class $\mathcal{A}$, the Equation (3) can be written as follows

$$\mathcal{L}_\kappa^\sigma f(z) = z + \sum_{n=2}^\infty \left(\frac{\Gamma(\sigma+n)}{\Gamma(\sigma+1)} \cdot \frac{(\sigma-\kappa)!}{(\sigma+n-\kappa-1)!} \right) a_n\, z^n \qquad (z \in \mathbb{U}).$$

Using the binomial series, we have:

$$(1-\delta)^j = \sum_{\ell=0}^j \binom{j}{\ell}(-\delta)^\ell \qquad \text{where} \quad j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}.$$

For a function f belonging to the class $\mathcal{A}$, Srivastava and Sheza M. El-Deeb [18] introduced the linear derivative operator as follows:

$$\mathcal{D}_{n,\delta,\kappa}^{\sigma,0} f(z) = f(z),$$

$$\begin{aligned}
\mathcal{D}_{n,\delta,\kappa}^{\sigma,1} f(z) &= \mathcal{D}_{n,\delta,\kappa}^\sigma f(z) \\
&= (1-\delta)^n \mathcal{L}_\kappa^\sigma f(z) + \left[1 - (1-\delta)^n \right] z (\mathcal{L}_\kappa^\sigma f)'(z) \\
&= z + \sum_{j=2}^\infty \left[1 + (j-1)c^n(\delta) \right] \left(\frac{\Gamma(\sigma+j)}{\Gamma(\sigma+1)} \cdot \frac{(\sigma-\kappa)!}{(\sigma+j-\kappa-1)!} \right) a_j\, z^j,
\end{aligned}$$

and, in general,

$$\begin{aligned}
\mathcal{D}_{j,\delta,\kappa}^{\sigma,m} f(z) &= \mathcal{D}_{j,\delta,\kappa}^\sigma \left(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1} f(z) \right) \\
&= (1-\delta)^j \mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1} f(z) + \left[1 - (1-\delta)^j \right] z \left(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1} f(z) \right)' \\
&= z + \sum_{n=2}^\infty \left[1 + (n-1)c^j(\delta) \right]^m \left(\frac{\Gamma(\sigma+n)}{\Gamma(\sigma+1)} \cdot \frac{(\sigma-\kappa)!}{(\sigma+n-\kappa-1)!} \right) a_n\, z^n \\
&= z + \sum_{n=2}^\infty \mathcal{V}_n\, a_n\, z^n \qquad (\delta > 0;\ j, \sigma, \kappa \in \mathbb{N};\ m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{4}
\end{aligned}$$

where

$$\mathcal{V}_n = \left[1 + (n-1)c^j(\delta) \right]^m \left(\frac{\Gamma(\sigma+n)}{\Gamma(\sigma+1)} \cdot \frac{(\sigma-\kappa)!}{(\sigma+n-\kappa-1)!} \right) \tag{5}$$

and

$$c^j(\delta) = -\sum_{\ell=1}^j \binom{j}{\ell}(-\delta)^\ell \qquad (j \in \mathbb{N}).$$

It follows from (4) that

$$c^j(\delta)\, z \left(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m} f(z)\right)' = \mathcal{D}_{j,\delta,\kappa}^{\sigma,m+1} f(z) - \left[1 - c^j(\delta)\right] \mathcal{D}_{j,\delta,\kappa}^{\sigma,m} f(z). \tag{6}$$

1.2. Generalized Telephone Numbers (GTN)

The usual involution numbers, also used in definition telephone numbers, are assumed by the recurrence relation

$$\mathcal{Q}(n) = \mathcal{Q}(n-1) + (n-1)\mathcal{Q}(n-2) \quad for \quad n \geq 2$$

here the following initial condition is provided

$$\mathcal{Q}(0) = \mathcal{Q}(1) = 1$$

first published in 1800 by Heinrich August Rothe by which they may easily be calculated [19]. One way to explain this recurrence is to partition the $\mathcal{Q}(n)$ connection patterns of the n subscribers to a telephone system into the patterns in which the first person is not calling anyone else and the patterns in which the first person is making a call. There are $\mathcal{Q}(n-1)$ connection patterns in which the first person is disconnected, explaining the first term of the recurrence. If the first person is connected to someone, there are $n-1$ choices for that person, and $\mathcal{Q}(n-2)$ patterns of connection for the remaining $n-2$ people, explaining the second term of the recurrence [20]. $\mathcal{Q}(n)$ is the number of involutions (self-inverse permutations) in the symmetric group (see, for example, [19,20]). Relation between involution numbers and symmetric groups were first studied in the 1800s. Since involutions correspond to standard Young tableaux, it is clear that the nth involution number is also the number of Young tableaux on the set $1, 2, \ldots, n$ (for more information, see [21]). According to John Riordan, the above recurrence relation, in fact, produces the number of connection patterns in a telephone system with n subscribers (see [22]). In 2017, Wlochand Wolowiec-Musial [23] introduced generalized telephone numbers $\mathcal{Q}(\wp, n)$ defined for integers $n \geq 0$ and $\wp \geq 1$ by the following recursion:

$$\mathcal{Q}(\wp, n) = \wp \mathcal{Q}(\wp, n-1) + (n-1)\mathcal{Q}(\wp, n-2)$$

here the following initial conditions are provided

$$\mathcal{Q}(\wp, 0) = 1, \mathcal{Q}(\wp, 1) = \wp,$$

and studied some features. In 2019, Bednarz et al. [24] introduced a new generalization of telephone numbers by

$$\mathcal{Q}_\wp(n) = \mathcal{Q}_\wp(n-1) + \wp(n-1)\mathcal{Q}_\wp(n-2); n \geq 2, \wp \geq 1$$

here the following initial conditions are provided

$$\mathcal{Q}_\wp(0) = \mathcal{Q}_\wp(1) = 1.$$

They examined and researched the main features of this class that they introduced. Moreover, they investigated the connections of these numbers with the congruences and gave some proofs. Lately, they derived the exponential generating function and they gave the definiton of the summation formula for $\mathcal{Q}_\wp(n)$

$$e^{x + \wp \frac{x^2}{2}} = \sum_{n=0}^{\infty} \mathcal{Q}_\wp(n) \frac{x^n}{n!} \quad (\wp \geq 1).$$

It is clear that $\mathcal{Q}(n)$ will be obtained when $\wp = 1$. In addition, the following equations are obtained for different values of n:

1. $\mathcal{Q}_\wp(0) = \mathcal{Q}_\wp = 1$
2. $\mathcal{Q}_\wp(2) = 1 + \wp$
3. $\mathcal{Q}_\wp(3) = 1 + 3\wp$
4. $\mathcal{Q}_\wp(4) = 1 + 6\wp + 3\wp^2$
5. $\mathcal{Q}_\wp(5) = 1 + 10\wp + 15\wp^2$
6. $\mathcal{Q}_\wp(6) = 1 + 15\wp + 45\wp^2 + 15\wp^3.$

and due to Deniz [25], now we consider the following analytic function

$$\Xi(z) \qquad := e^{(z+\wp\frac{z^2}{2})} =$$
$$1 + z + \frac{1+\wp}{2}z^2 \quad + \frac{1+3\wp}{6}z^3 + \frac{3\wp^2 + 6\wp + 1}{24}z^4 + \frac{1 + 10\wp + 15\wp^2}{120}z^5 + \cdots. \qquad (7)$$

for $z \in \mathbb{U}$. Here, the Ξ function defined in $\mathbb{U}$ is chosen as an analytic function with a positive real part and Ξ satisfies the conditions $\Xi(0) = 1$, $\Xi'(0) > 0$, and Ξ maps open unit disk onto a region starlike with respect to 1 and symmetric with respect to the real axis. In recent years, researchers who have focused their studies on Generalized Telephone Numbers have defined a new class and presented appropriate solutions by addressing problems such as coefficient relations, Fekete-Szegö inequalities of this class. Based on these studies, similar results were obtained for f^{-1}. In addition, with the help of convolution products for analytic functions normalized in $\mathbb{U}$, different applications and special cases of Fekete-Szegö inequality are examined and some important problems and applications are examined in [26]. In the light of this information, similar discussions can be made for bi-univalent functions.

Now we recall and define a new subclass of bi-univalent functions in the following section.

1.3. Bi-Univalent Functions Σ

Let f belongs of class $\mathcal{S}$. In this case, we know that the function f has an inverse f^{-1}, and this inverse function is defined as follows:

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \qquad (8)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). Note that the functions

$$f_1(z) = \frac{z}{1-z}, \qquad f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \qquad f_3(z) = -\log(1-z)$$

with their corresponding inverses

$$f_1^{-1}(w) = \frac{w}{1+w}, \qquad f_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}, \qquad f_3^{-1}(w) = \frac{e^w - 1}{e^w}$$

are elements of Σ. In the past years, Srivastava et al.'s reference article [27] has been a pioneer for many researchers and the importance of the subject has been better understood after this article. Afterwards, different studies on this subject were carried out by many researchers. Recently there has been triggering interest to study bi-univalent function class

Σ and obtained non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1,2\}; \ \ \mathbb{N} := \{1,2,3,\cdots\}$$

is still an open problem (for more detail see [28–33]). By using the hybrid-type convolution operator $\mathcal{D}_{n,\delta,\lambda}^{\sigma,m}$ and motivated by certain recent study on bi-univalent functions which still remain popular today [34–39]. We define a subclass in association with generalized telephone numbers (GTN) [25,26].

Definition 1. *The f function belonging to the class* Σ *in type (1) is said to belong to the class* $\mathcal{B}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\lambda,z)$ *if f satisfies the following inequalities :*

$$\left(\frac{z^{1-\lambda}(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z))'}{[\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z)]^{1-\lambda}} \right) \prec \Xi(z) \tag{9}$$

and

$$\left(\frac{w^{1-\lambda}(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w))'}{[\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w)]^{1-\lambda}} \right) \prec \Xi(w), \tag{10}$$

here $0 \le \lambda \le 1$; $z, w \in \mathbb{U}$ *and it is assumed that the g function is as in (8).*

The new subclasses of the Σ class created by the special selection of the parameters in this definition can be defined as in the following two examples.

Example 1. *For* $\lambda = 0$, *the f function belonging to the class* Σ *in type (1) is said to belong to the class* $\mathcal{S}_{j,\delta,\Sigma}^{\sigma,m;\wp}$ *if f satisfies the following inequalities:*

$$\left(\frac{z(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z))'}{\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z)} \right) \prec \Xi(z) \tag{11}$$

and

$$\left(\frac{w(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w))'}{\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w)} \right) \prec \Xi(w) \tag{12}$$

in here $z, w \in \mathbb{U}$ *and it is assumed that the g function is as in (8).*

Example 2. *For* $\lambda = 1$, *the f function belonging to the class* Σ *in type (1) is said to belong to the class* $\mathcal{R}_{j,\delta,\Sigma}^{\sigma,m;\wp}$ *if f satisfies the following inequalities:*

$$\left(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z))' \right) \prec \Xi(z) \tag{13}$$

and

$$\left((\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w))' \right) \prec \Xi(w) \tag{14}$$

in here $z, w \in \mathbb{U}$ *and it is assumed that the g function is as in (8).*

In [40], Obradovic et al. gave some criteria for univalence expressing by $\Re(f'(z)) > 0$, for the linear combinations

$$\tau\left(1 + \frac{zf''(z)}{f'(z)} \right) + (1-\tau)\frac{1}{f'(z)} > 0, \qquad (\tau \ge 1, z \in \mathbb{U}).$$

According to the above definitions, Lashin [41] defined the new subclasses of bi-univalent function.

Definition 2. *A function f belonging to the class Σ in type* (1) *is considered to be in the class $\mathcal{M}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\tau,\Xi)$ if f satisfies the following inequalities:*

$$\tau\left(1 + \frac{z(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z))''}{(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z))'}\right) + (1-\tau)\frac{1}{(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z))'} \prec \Xi(z) \tag{15}$$

and

$$\tau\left(1 + \frac{w(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w))''}{(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w))'}\right) + (1-\tau)\frac{1}{(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w))'} \prec \Xi(w) \tag{16}$$

where $z,w \in \mathbb{U}$, $\tau \geq 1$, and it is assumed that the function g is as defined in (8).

Example 3. *A function f belonging to the class Σ in type* (1) *is considered to be in the class $\mathcal{M}_{j,\delta,\Sigma}^{\sigma,m;\wp}(1,\Xi) \equiv \mathcal{K}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\Xi)$ if f satisfies the following inequalities:*

$$\left(1 + \frac{z(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z))''}{(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z))'}\right) \prec \Xi(z) \text{ and } \left(1 + \frac{w(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w))''}{(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w))'}\right) \prec \Xi(w)$$

where $z,w \in \mathbb{U}$, and it is assumed that the function g is as defined in (8).

2. Coefficient Bounds

To establish our main results, we require the following lemma.

Lemma 1 (see [42]). *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions h, analytic in $\mathbb{U}$, for which*

$$\Re\{h(z)\} > 0 \qquad (z \in \mathbb{U}),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

We begin by estimating the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{B}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\lambda,\Xi)$. Let $P(z)$ be defined by

$$P(z) : = \frac{1 + \varpi(z)}{1 - \varpi(z)} = 1 + c_1 z + c_2 z^2 + \cdots.$$

It is evident that

$$\varpi(z) = \frac{P(z) - 1}{P(z) + 1}$$

$$= \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right]. \tag{17}$$

Since $\varpi(z)$ is a Schwarz function, it follows that $\Re(p_1(z)) > 0$ and $p_1(0) = 1$. Therefore,

$$\Psi(\varpi(z)) = e^{\left(\frac{P(z)-1}{P(z)+1} + \kappa \frac{\left[\frac{P(z)-1}{P(z)+1}\right]^2}{2}\right)}$$

$$= 1 + \frac{c_1}{2}z + \left(\frac{c_2}{2} + \frac{(\wp-1)c_1^2}{8}\right)z^2 + \left(\frac{c_3}{2} + (\wp-1)\frac{c_1 c_2}{4} + \frac{(1-3\wp)}{48}c_1^3\right)z^3 + \ldots \tag{18}$$

Define the functions $p(z)$ and $q(z)$ as follows:

$$p(z) := \frac{1 + u(z)}{1 - u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(z) := \frac{1 + v(z)}{1 - v(z)} = 1 + q_1 z + q_2 z^2 + \cdots$$

or, equivalently,

$$u(z) := \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2} \right) z^2 + \cdots \right]$$

and

$$v(z) := \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2} \right) z^2 + \cdots \right].$$

Subsequently, $p(z)$ and $q(z)$ are analytic in $\mathbb{U}$ with $p(0) = 1 = q(0)$. Furthermore, since both u and v map from $\mathbb{U}$ to $\mathbb{U}$, the functions $p(z)$ and $q(z)$ exhibit a positive real part in $\mathbb{U}$, and they satisfy the inequalities:

$$|p_i| \leq 2 \quad \text{and} \quad |q_i| \leq 2. \tag{19}$$

For the scope of our study, we introduce the notation:

$$\mathcal{V}_2 = \mathcal{V}_{\vartheta,m}^{a,c}(2) = \left[1 + c^j(\delta) \right]^m \left(\frac{\Gamma(\sigma + 2)}{\Gamma(\sigma + 1)} \cdot \frac{(\sigma - \lambda)!}{(\sigma + 1 - \lambda)!} \right), \tag{20}$$

$$\mathcal{V}_3 = \left[1 + 2c^j(\delta) \right]^m \left(\frac{\Gamma(\sigma + 3)}{\Gamma(\sigma + 1)} \cdot \frac{(\sigma - \lambda)!}{(\sigma + 2 - \lambda)!} \right). \tag{21}$$

In the subsequent theorem, we embark on the initial exploration of the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to this novel subclass $\mathcal{B}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\lambda, \Xi)$.

Theorem 1. *Let assume that the f function is as in* (1) *and in the class* $\mathcal{B}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\lambda, \Xi)$. *Then*

$$|a_2| \leq \min \left\{ \begin{array}{l} \frac{1}{(\lambda+1)\mathcal{V}_2}, \\ \sqrt{\frac{2}{|\{(\lambda-1)(\lambda+2)-(\wp-1)(\lambda+1)^2\}\mathcal{V}_2^2 + 2(\lambda+2)\mathcal{V}_3|}} \end{array} \right. \tag{22}$$

and

$$|a_3| \leq \min \left\{ \begin{array}{l} \frac{1}{(\lambda+1)^2\mathcal{V}_2^2} + \frac{1}{(2+\lambda)\mathcal{V}_3}, \\ \frac{2}{\{(\lambda-1)(\lambda+2)-(\wp-1)(\lambda+1)^2\}\mathcal{V}_2^2 + 2(\lambda+2)\mathcal{V}_3} + \frac{1}{(2+\lambda)\mathcal{V}_3}. \end{array} \right. \tag{23}$$

Proof. It follows from (9) and (10) that

$$\left(\frac{z^{1-\lambda}(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z))'}{[\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z)]^{1-\lambda}} \right) = \Xi(u(z)) \tag{24}$$

and

$$\left(\frac{w^{1-\lambda}(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w))'}{[\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w)]^{1-\lambda}} \right) = \Xi(v(w)), \tag{25}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the following forms:

$$\Xi(u(z)) = 1 + \frac{1}{2}p_1 z + \left(\frac{p_2}{2} + \frac{(\wp - 1)p_1^2}{8} \right) z^2 + \cdots \tag{26}$$

and

$$\Xi(v(w)) = 1 + \frac{1}{2}q_1 w + \left(\frac{q_2}{2} + \frac{(\wp - 1)q_1^2}{8} \right) w^2 + \cdots, \tag{27}$$

respectively. Now, by equating the coefficients in (24) and (25), we have

$$(1 + \lambda)\mathcal{V}_2 a_2 = \frac{1}{2}p_1, \tag{28}$$

$$\left[\frac{(\lambda - 1)(\lambda + 2)}{2}\mathcal{V}_2^2 a_2^2 + (\lambda + 2)\mathcal{V}_3 a_3 \right] = \frac{p_2}{2} + \frac{(\wp - 1)p_1^2}{8}, \tag{29}$$

$$-(\lambda + 1)\mathcal{V}_2 a_2 = \frac{1}{2}q_1 \tag{30}$$

and

$$\left[\left(2(\lambda + 2)\mathcal{V}_3 + \frac{(\lambda - 1)(\lambda + 2)}{2}\mathcal{V}_2^2 \right) a_2^2 - (\lambda + 2)\mathcal{V}_3 a_3 \right] = \frac{q_2}{2} + \frac{(\wp - 1)q_1^2}{8}. \tag{31}$$

From (28) and (30), we can determine that

$$a_2 = \frac{p_1}{2(1 + \lambda)\mathcal{V}_2} = \frac{-q_1}{2(1 + \lambda)\mathcal{V}_2}, \tag{32}$$

which implies

$$p_1 = -q_1 \tag{33}$$

and

$$8(\lambda + 1)^2 \mathcal{V}_2^2 a_2^2 = p_1^2 + q_1^2. \tag{34}$$

Thus we have

$$a_2^2 = \frac{p_1^2 + q_1^2}{8(\lambda + 1)^2 \mathcal{V}_2^2} \tag{35}$$

and

$$(\lambda + 1)^2 \mathcal{V}_2^2 a_2^2 = \frac{p_1^2 + q_1^2}{8}. \tag{36}$$

By adding (29) and (31), and utilizing (32) as well as (33), we get

$$\left[(\lambda - 1)(\lambda + 2)\mathcal{V}_2^2 + 2(\lambda + 2)\mathcal{V}_3 \right] a_2^2 = \frac{p_2 + q_2}{2} + \frac{(\wp - 1)}{8}(p_1^2 + q_1^2). \tag{37}$$

Thus, by using (36)

$$a_2^2 = \frac{p_2 + q_2}{2\left[\{(\lambda - 1)(\lambda + 2) - (\wp - 1)(\lambda + 1)^2\}\mathcal{V}_2^2 + 2(\lambda + 2)\mathcal{V}_3 \right]} \tag{38}$$

Applying Lemma 1 to the coefficients p_2 and q_2, yields the immediate result

$$|a_2|^2 \leq \frac{2}{|\{(\lambda - 1)(\lambda + 2) - (\wp - 1)(\lambda + 1)^2\}\mathcal{V}_2^2 + 2(\lambda + 2)\mathcal{V}_3|}. \tag{39}$$

Hence,

$$|a_2| \leq \sqrt{\frac{2}{|\{(\lambda - 1)(\lambda + 2) - (\wp - 1)(\lambda + 1)^2\}\mathcal{V}_2^2 + 2(\lambda + 2)\mathcal{V}_3|}}.$$

This yields the bound on $|a_2|$ as stated in (22). To establish the bound on $|a_3|$, we subtract (31) from (29), resulting in

$$\left[2(\lambda + 2)\mathcal{V}_3 a_3 - 2(\lambda + 2)\mathcal{V}_3 a_2^2\right] = \frac{1}{2}(p_2 - q_2) + \frac{\wp - 1}{8}(p_1^2 - q_1^2). \tag{40}$$

Using (32), (33) and (40) we can deduce that

$$
\begin{aligned}
a_3 &= a_2^2 + \frac{p_2 - q_2}{4(2 + \lambda)\mathcal{V}_3} \\
&= \frac{p_1^2 + q_1^2}{8(\lambda + 1)^2 \mathcal{V}_2^2} + \frac{p_2 - q_2}{4(2 + \lambda)\mathcal{V}_3}.
\end{aligned}
\tag{41}
$$

Applying Lemma 1 once more to for the coefficients p_2, q_2, we immediately obtain

$$|a_3| \leq \frac{1}{(\lambda + 1)^2 \mathcal{V}_2^2} + \frac{1}{(2 + \lambda)\mathcal{V}_3}$$

also,

$$|a_3| \leq \frac{2}{|\{(\lambda - 1)(\lambda + 2) - (\wp - 1)(\lambda + 1)^2\}\mathcal{V}_2^2 + 2(\lambda + 2)\mathcal{V}_3|} + \frac{1}{(2 + \lambda)\mathcal{V}_3}.$$

This completes the proof of Theorem 1. $\square$

As a consequence of our results, by appropriately setting the parameter, we present the following corollaries, which are novel and have not been studied for the case of involution numbers involving hybrid-type convolution operators.

When we fix $\lambda = 0$ in Theorem 1, the following corollary emerges.

Corollary 1. *Let assume that the f function is as in* (1) *and in the class* $\mathcal{S}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\Xi)$. *Then*

$$|a_2| \leq \min\left\{ \begin{array}{l} \frac{1}{\mathcal{V}_2}, \\ \sqrt{\frac{2}{|4\mathcal{V}_3 - (\wp + 1)\mathcal{V}_2^2|}} \end{array} \right. \tag{42}$$

and

$$|a_3| \leq \min\left\{ \begin{array}{l} \frac{1}{\mathcal{V}_2^2} + \frac{1}{2\mathcal{V}_3}, \\ \frac{2}{|-(1+\wp)\mathcal{V}_2^2 + 4\mathcal{V}_3|} + \frac{1}{2\mathcal{V}_3}. \end{array} \right. \tag{43}$$

Fixing $\lambda = 1$ in Theorem 1, we have the following corollary.

Corollary 2. *Let assume that the f function is as in* (1) *and in the class* $\mathcal{R}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\Xi)$. *Then*

$$|a_2| \leq \min\left\{ \begin{array}{l} \frac{1}{2\mathcal{V}_2}, \\ \sqrt{\frac{1}{3\mathcal{V}_3 - 2(\wp - 1)\mathcal{V}_2^2}} \end{array} \right. \tag{44}$$

and

$$|a_3| \leq \min\left\{ \begin{array}{l} \frac{1}{4\mathcal{V}_2^2} + \frac{1}{3\mathcal{V}_3} \\ \frac{2}{|-4(\wp - 1)\mathcal{V}_2^2 + 6\mathcal{V}_3|} + \frac{1}{3\mathcal{V}_3}. \end{array} \right. \tag{45}$$

In the subsequent theorem, we are embarking on the initial exploration of the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions within this novel subclass $\mathcal{M}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\tau, \Xi)$.

Theorem 2. *Let assume that the f function is as in* (1) *and* $f \in \mathcal{M}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\tau,\Xi)$, $\tau \geq 1$. *Then*

$$|a_2| \leq \min \begin{cases} \frac{1}{2(2\tau-1)\mathcal{V}_2}, \\ \frac{1}{\sqrt{|(1+\tau)-2(2\tau-1)^2(\wp-1)|\mathcal{V}_2^2}} \end{cases} \tag{46}$$

and

$$|a_3| \leq \min \begin{cases} \frac{2}{3(3\tau-1)\mathcal{V}_3} + \frac{1}{4(2\tau-1)^2\mathcal{V}_2^2}, \\ \frac{2}{3(3\tau-1)\mathcal{V}_3} + \frac{1}{|(1+\tau)-2(2\tau-1)^2(\wp-1)|\mathcal{V}_2^2}. \end{cases} \tag{47}$$

Proof. It follows from (15) and (16) that

$$\tau\left(1 + \frac{(z\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z))''}{(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z))'}\right) + (1-\tau)\frac{1}{(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}f(z))'} = \Xi(u(z)) \tag{48}$$

and

$$\tau\left(1 + \frac{w(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w))''}{(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w))'}\right) + (1-\tau)\frac{1}{(\mathcal{D}_{j,\delta,\kappa}^{\sigma,m-1}g(w))'} = \Xi(v(w)). \tag{49}$$

From (48) and (49), we have

$$1 + 2(2\tau-1)\mathcal{V}_2 a_2 z + \left[3(3\tau-1)\mathcal{V}_3 a_3 + 4(1-2\tau)\mathcal{V}_2^2 a_2^2\right]z^2 + \cdots$$
$$= 1 + \frac{1}{2}p_1 z + \left(\frac{p_2}{2} + \frac{(\wp-1)p_1^2}{8}\right)z^2 + \cdots,$$

and

$$1 - 2(2\tau-1)\mathcal{V}_2 a_2 w + \left(2(5\tau-1)\mathcal{V}_2^2 a_2^2 - 3(3\tau-1)\mathcal{V}_3 a_3\right)w^2 - \cdots$$
$$= 1 + \frac{1}{2}q_1 w + \left(\frac{q_2}{2} + \frac{(\wp-1)q_1^2}{8}\right)w^2 + \cdots.$$

By equating the coefficients, we obtain

$$2(2\tau-1)\mathcal{V}_2 a_2 = \frac{1}{2}p_1, \tag{50}$$

$$3(3\tau-1)\mathcal{V}_3 a_3 + 4(1-2\tau)\mathcal{V}_2^2 a_2^2 = \frac{p_2}{2} + \frac{(\wp-1)p_1^2}{8}, \tag{51}$$

$$-2(2\tau-1)\mathcal{V}_2 a_2 = \frac{1}{2}q_1, \tag{52}$$

and

$$2(5\tau-1)\mathcal{V}_2^2 a_2^2 - 3(3\tau-1)\mathcal{V}_3 a_3 = \frac{q_2}{2} + \frac{(\wp-1)q_1^2}{8}. \tag{53}$$

Using (50) and (52), we obtain

$$p_1 = -q_1 \tag{54}$$

From (50) by using (19),

$$|a_2| \leq \frac{1}{2(2\tau-1)\mathcal{V}_2}. \tag{55}$$

Also

$$32(2\tau - 1)^2 \mathcal{V}_2^2 a_2^2 = p_1^2 + q_1^2$$

$$a_2^2 = \frac{p_1^2 + q_1^2}{32(2\tau - 1)^2 \mathcal{V}_2^2}. \tag{56}$$

Thus by (19), we get

$$|a_2| \leq \frac{1}{4(2\tau - 1)\mathcal{V}_2} = \frac{1}{4(2\tau - 1)\mathcal{V}_2}. \tag{57}$$

Now from (51), (53) and using (56), we obtain

$$\left(2(1 + \tau) - 4(2\tau - 1)^2(\wp - 1)\right)\mathcal{V}_2^2 a_2^2 = \frac{p_2 + q_2}{2}. \tag{58}$$

Thus, by (58) we obtain

$$a_2^2 = \frac{p_2 + q_2}{4[(1 + \tau) - 2(2\tau - 1)^2(\wp - 1)]\mathcal{V}_2^2}$$

$$|a_2| \leq \frac{1}{\sqrt{|(1 + \tau) - 2(2\tau - 1)^2(\wp - 1)|\mathcal{V}_2^2}}.$$

By using (51) and (53), and then substituting (54), we get

$$a_3 = \frac{p_2 - q_2}{6(3\tau - 1)\mathcal{V}_3} + a_2^2. \tag{59}$$

Taking the modulus of both sides, we obtain

$$|a_3| \leq \frac{2}{3(3\tau - 1)\mathcal{V}_3} + |a_2^2|. \tag{60}$$

Using (55) and (57), we get

$$|a_3| \leq \frac{2}{3(3\tau - 1)\mathcal{V}_3} + \frac{1}{4(2\tau - 1)^2 \mathcal{V}_2^2}.$$

Now by using (58) in (60),

$$|a_3| \leq \frac{2}{3(3\tau - 1)\mathcal{V}_3} + |a_2^2|$$

$$= \frac{2}{3(3\tau - 1)\mathcal{V}_3} + \frac{1}{|(1 + \tau) - 2(2\tau - 1)^2(\wp - 1)|\mathcal{V}_2^2}.$$

□

Corollary 3. *Let assume that the f function is as in (1) and* $f \in \mathcal{K}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\Xi)$. *Then*

$$|a_2| \leq \min \begin{cases} \frac{1}{2\mathcal{V}_2}, \\ \frac{1}{\sqrt{|2 - 2(\wp - 1)|\mathcal{V}_2^2}} \end{cases} \tag{61}$$

and

$$|a_3| \leq \min \begin{cases} \frac{2}{6\mathcal{V}_3} + \frac{1}{4\mathcal{V}_2^2}, \\ \frac{2}{6\mathcal{V}_3} + \frac{1}{|2 - 2(\wp - 1)|\mathcal{V}_2^2}. \end{cases} \tag{62}$$

3. Fekete-Szegö Inequalities

For $f \in \mathcal{A}$, Fekete and Szegö [43] introduced the generalized functional $|a_3 - \aleph a_2^2|$, where $\aleph$ is some real number. In [44] Zaprawa provided the Fekete and Szegö results

for $f \in \Sigma$. We prove Fekete-Szegö inequalities for functions f in the new subclasses $\mathcal{B}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\lambda, \Xi)$ and $\mathcal{M}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\tau, \Xi)$ using the following lemmas proven by Zaprawa [44].

Lemma 2 ([44]). *Let $k \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$ then*

$$|(k+1)z_1 + (k-1)z_2| \leq \begin{cases} 2|k|R, & |k| \geq 1 \\ \\ 2R & |k| \leq 1. \end{cases} \tag{63}$$

Lemma 3 ([44]). *Let $k, l \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$ then*

$$|(k+l)z_1 + (k-l)z_2| \leq \begin{cases} 2|k|R, & |k| \geq |l| \\ \\ 2|l|R & |k| \leq |l|. \end{cases} \tag{64}$$

Now, we obtain Fekete-Szegö inequalities for $f \in \mathcal{B}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\lambda, \Xi)$:

Theorem 3. *For $\aleph \in \mathbb{R}$, let assume that the f function is as in (1) and $f \in \mathcal{B}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\lambda, \Xi)$, then*

$$\left| a_3 - \aleph a_2^2 \right| \leq \begin{cases} \frac{1}{(2+\lambda)\mathcal{V}_3} & ; & 0 \leq |h(\aleph)| \leq \frac{1}{4(2+\lambda)\mathcal{V}_3} \\ 4|h(\aleph)| & ; & |h(\aleph)| \geq \frac{1}{4(2+\lambda)\mathcal{V}_3} \end{cases}$$

where

$$h(\aleph) = \frac{1-\aleph}{2\left[\{(\lambda-1)(\lambda+2) - (\wp-1)(\lambda+1)^2\}\mathcal{V}_2^2 + 2(\lambda+2)\mathcal{V}_3\right]}. \tag{65}$$

Proof. From (41), we have

$$a_3 - \aleph a_2^2 = \frac{p_2 - q_2}{4(2+\lambda)\mathcal{V}_3} + (1-\nu)a_2^2. \tag{66}$$

By substituting (38) in (66), we have

$$\begin{aligned} a_3 - \aleph a_2^2 &= \frac{p_2 - q_2}{4(2+\lambda)\mathcal{V}_3} \\ &+ \frac{(1-\aleph)(p_2 + q_2)}{2\left[\{(\lambda-1)(\lambda+2) - (\wp-1)(\lambda+1)^2\}\mathcal{V}_2^2 + 2(\lambda+2)\mathcal{V}_3\right]} \\ &= \left(h(\aleph) + \frac{1}{4(2+\lambda)\mathcal{V}_3}\right)p_2 + \left(h(\nu) - \frac{1}{4(2+\lambda)\mathcal{V}_3}\right)q_2, \end{aligned} \tag{67}$$

where

$$h(\aleph) = \frac{1-\aleph}{2\left[\{(\lambda-1)(\lambda+2) - (\wp-1)(\lambda+1)^2\}\mathcal{V}_2^2 + 2(\lambda+2)\mathcal{V}_3\right]}.$$

Thus by taking modulus of (67), we conclude that

$$\left| a_3 - \aleph a_2^2 \right| \leq \begin{cases} \frac{1}{(2+\lambda)\mathcal{V}_3} & ; & 0 \leq |h(\aleph)| \leq \frac{1}{4(2+\lambda)\mathcal{V}_3} \\ 4|h(\aleph)| & ; & |h(\aleph)| \geq \frac{1}{4(2+\lambda)\mathcal{V}_3} \end{cases} \tag{68}$$

where $h(\aleph)$ is given by (65). $\square$

By taking $\aleph = 1$ in above Theorem one can easily state the following:

Remark 1. *Let the function f be assumed by* (1) *and* $f \in \mathcal{B}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\lambda, \Xi)$. *Then*

$$\left| a_3 - a_2^2 \right| \leq \frac{1}{(2+\lambda)\mathcal{V}_3}.$$

By taking $\lambda = 0$ and $\lambda = 1$, we can state the following:

Corollary 4. *For* $\aleph \in \mathbb{R}$, *let assume that the f function is as in* (1) *and* $f \in \mathcal{S}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\Xi)$, *then*

$$\left| a_3 - \aleph a_2^2 \right| \leq \begin{cases} \frac{1}{2\mathcal{V}_3} & ; \quad 0 \leq |h(\aleph)| \leq \frac{1}{8\mathcal{V}_3} \\ 4|h(\aleph)| & ; \quad |h(\aleph)| \geq \frac{1}{8\mathcal{V}_3} \end{cases}$$

where $h(\aleph) = \frac{1-\aleph}{2[4\mathcal{V}_3 - (\wp+1)\mathcal{V}_2^2]}.$

Corollary 5. *For* $\aleph \in \mathbb{R}$, *let assume that the f function is as in* (1) *and* $f \in \mathcal{R}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\Xi)$, *then*

$$\left| a_3 - \aleph a_2^2 \right| \leq \begin{cases} \frac{1}{3\mathcal{V}_3} & ; \quad 0 \leq |h(\aleph)| \leq \frac{1}{12\mathcal{V}_3} \\ 4|h(\aleph)| & ; \quad |h(\aleph)| \geq \frac{1}{12\mathcal{V}_3} \end{cases}$$

where $h(\aleph) = \frac{1-\aleph}{2[6\mathcal{V}_3 - 4(\wp-1)\mathcal{V}_2^2]}.$

Now, we prove Fekete-Szegö inequalities for $f \in \mathcal{M}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\tau, \Xi)$.

Theorem 4. *For* $\nu \in \mathbb{R}$, *let assume that the f function is as in* (1) *and* $f \in \mathcal{M}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\tau, \Xi)$, *then*

$$\left| a_3 - \nu a_2^2 \right| \leq \begin{cases} \frac{2}{3(3\tau-1)\mathcal{V}_3} & ; 0 \leq |h(\nu)| \leq \frac{1}{6(3\tau-1)\mathcal{V}_3} \\ 4|h(\nu)| & ; |h(\nu)| \geq \frac{1}{6(3\tau-1)\mathcal{V}_3} \end{cases}$$

where

$$h(\nu) = \frac{1-\nu}{4[(1+\tau) - 2(2\tau-1)^2(\wp-1)]\mathcal{V}_2^2}.$$

Proof. From (59), we have

$$a_3 - \nu a_2^2 = \frac{p_2 - q_2}{6(3\tau-1)\mathcal{V}_3} + +(1-\nu)a_2^2. \tag{69}$$

By substituting (58) in (69), we have

$$
\begin{aligned}
a_3 - \nu a_2^2 &= \frac{p_2 - q_2}{6(3\tau-1)\mathcal{V}_3} + a_2^2 + \frac{(p_2 + q_2)(1-\nu)}{4[(1+\tau) - 2(2\tau-1)^2(\wp-1)]\mathcal{V}_2^2} \\
&= \left(h(\nu) + \frac{1}{6(3\tau-1)\mathcal{V}_3} \right) p_2 + \left(h(\nu) - \frac{1}{6(3\tau-1)\mathcal{V}_3} \right) q_2,
\end{aligned}
\tag{70}
$$

where

$$h(\nu) = \frac{1-\nu}{4[(1+\tau) - 2(2\tau-1)^2(\wp-1)]\mathcal{V}_2^2}. \tag{71}$$

Thus by taking modulus of (70), we get

$$\left| a_3 - \nu a_2^2 \right| \leq \begin{cases} \dfrac{2}{3(3\tau - 1)\mathcal{V}_3} & ; 0 \leq |h(\nu)| \leq \dfrac{1}{6(3\tau - 1)\mathcal{V}_3} \\ 4|h(\nu)| & ; |h(\nu)| \geq \dfrac{1}{6(3\tau - 1)\mathcal{V}_3} \end{cases} \tag{72}$$

where $h(\nu)$ is given by (71). $\square$

By taking $\nu = 1$ in above theorem, we can easily state the following:

Remark 2. *Let assume that the f function is as in* (1) *and* $f \in \mathcal{M}_{\Sigma}^{a,b,c}(\tau, \Xi)$. *Then*

$$\left| a_3 - a_2^2 \right| \leq \frac{2}{3(3\tau - 1)\mathcal{V}_3}.$$

Corollary 6. *For* $\nu \in \mathbb{R}$, *let assume that the f function is as in* (1) *and* $f \in \mathcal{K}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\Xi)$, *then*

$$\left| a_3 - \nu a_2^2 \right| \leq \begin{cases} \dfrac{2}{6\mathcal{V}_3} & ; 0 \leq |h(\nu)| \leq \dfrac{1}{12\mathcal{V}_3} \\ 4|h(\nu)| & ; |h(\nu)| \geq \dfrac{1}{12\mathcal{V}_3} \end{cases}$$

where $h(\nu) = \frac{1-\nu}{4[2-2(\wp-1)]\mathcal{V}_2^2}$.

4. Discussion

The research presented in this paper follows the same path as the previous studies that introduced new classes of bi-univalent functions, building upon the pioneering article by Srivastava et al. [27], which involves generalized telephone numbers. We then extended this approach to define a new function class and derived results concerning the initial Taylor coefficients for this class.

Furthermore, by specific parameter choices, our newly defined subclasses $\mathcal{B}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\lambda, \Xi)$ and $\mathcal{M}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\tau, \Xi)$ give rise to various other subclasses of analytic functions, such as $\mathcal{S}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\Xi)$, $\mathcal{R}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\Xi)$, and $\mathcal{K}_{j,\delta,\Sigma}^{\sigma,m;\wp}(\Xi)$. These subclasses have not been previously explored in connection with telephone numbers. Furthermore, by tailoring the parameters, we've attempted to discretize the new results, presenting novel discussions in this direction.

The main contributions of our work lie in providing new and improved results for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, which further enhances the understanding of the discussed classes.

5. Conclusions

Our motivation in this study is to unlock a plethora of interesting and valuable applications of a diverse array of telephone numbers within the realm of Geometric Function Theory. We firmly believe that this research will serve as a catalyst, inspiring numerous researchers to expand upon this concept by delving into meromorphic bi-univalent functions. Additionally, new classes could be formulated based on specific hybrid-type convolution operators, incorporating Poisson, Borel, and Pascal distribution series. Another avenue to explore is subordination with Gegenbauer and Legendre polynomials, as seen in recent studies [35–39,45] within the context of the Σ class.

By defining subclasses akin to starlike functions concerning the symmetric points of Σ in relation to telephone numbers, we could potentially unify and extend various classes of analytic bi-univalent functions. This approach could pave the way for comprehensive discussions on new extensions and detailed examinations of enhanced improvements to initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Moreover, our future plans include delving into second Hankel determinant and Toeplitz determinant inequality results, as previously explored in [45,46].

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