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A New Pseudo-Type κ-Fold Symmetric Bi-Univalent Function Class

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Abstract: We introduce and study a new pseudo-type κ-fold symmetric bi-univalent function class that meets certain subordination conditions in this article. For functions in the newly formed class, the initial coefficient bounds are obtained. For members in this class, the Fekete–Szegö issue is also estimated. In addition, we uncover pertinent links to previous results and give a few observations.

Keywords: regular; subordination; Fekete–Szegö inequality; bi-univalent

MSC: 30C45; 30C50

1. Preliminaries

Let $\{\varsigma \in \mathbb{C} : |\varsigma| < 1\} = \mathfrak{D}$, where $\mathbb{C}$ is the set of all complex numbers. Let $\mathcal{A}$ denote the class of all regular functions of the type

$$s(\varsigma) = \varsigma + \sum_{j=2}^{\infty} d_j \varsigma^j \tag{1}$$

with $s(0) = s'(0) - 1 = 0$, $\varsigma \in \mathfrak{D}$ and $\mathcal{S}$ denote the subfamily of functions $\in \mathcal{A}$ which are univalent in $\mathfrak{D}$. For $\tau \geq 1$, the class of τ-pseudo-convex functions is defined as

$$\mathcal{K}^\tau = \left\{ s \in \mathcal{A} : \Re\left(\frac{\{(\varsigma s'(\varsigma))'\}^\tau}{s(\varsigma)} \right) > 0, \quad \varsigma \in \mathfrak{D} \right\},$$

the class of τ-pseudo-starlike functions is given by

$$\mathcal{S}^\tau = \left\{ s \in \mathcal{A} : \Re\left(\frac{\varsigma\{s'(\varsigma)\}^\tau}{s(\varsigma)} \right) > 0, \quad \varsigma \in \mathfrak{D} \right\}$$

and the class of τ-pseudo-bounded turning is introduced as

$$\mathcal{R}^\tau = \{ s \in \mathcal{A} : \Re(s'(\varsigma))^\tau > 0, \quad \}, \ (\varsigma \in \mathfrak{D}),$$

The class $\mathcal{K}^\tau$ was explored by Guney and Murugusundaramoorthy [1] and the class $\mathcal{S}^\tau$ was examined in [2]. We note that $\mathcal{S}^1 = \mathcal{S}$. Al-Amiri and Reade [3] presented the class $\mathfrak{M}(\nu)$ $(\nu < 1)$ of functions $s \in \mathcal{A}$ with $s'(\varsigma) \neq 0$ *in* $\mathfrak{D}$ which satisfy

$$\Re\left(\nu \frac{(\varsigma s'(\varsigma))'}{s'(\varsigma)} + (1-\nu)s'(\varsigma) \right) > 0, \quad (\varsigma \in \mathfrak{D}).$$

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In [4], Sukhjit Singh and Sushma Gupta gave certain criteria for univalence by proving $\Re(s'(\varsigma)) > 0$, whenever

$$\Re\left(\nu\frac{(\varsigma s'(\varsigma))'}{s'(\varsigma)} + (1-\nu)s'(\varsigma)\right) > \xi, \quad (0 \le \nu < 1, 0 \le \xi < 1, \varsigma \in \mathfrak{D}).$$

The Koebe theorem (see [5]) ensures that $s(\mathfrak{D})$, $s \in \mathcal{S}$, contains a disc of radius $1/4$. Thus, any function s admits an inverse $g = s^{-1}$ defined by $g(s(\varsigma)) = \varsigma$, and $s(g(\varkappa)) = \varkappa$, $|\varkappa| < r_0(s)$, $r_0(s) \ge 1/4, \varsigma \in \mathfrak{D}, \varkappa \in \mathfrak{D}$, where

$$g(\varkappa) = \varkappa - d_2\varkappa^2 + (2d_2^2 - d_3)\varkappa^3 - (5d_2^3 - 5d_2d_3 + d_4)\varkappa^4 + \cdots \tag{2}$$

If $s \in \mathcal{S}$ and $s^{-1} \in \mathcal{S}$, then a member s of $\mathcal{A}$ given by (1) is called bi-univalent in $\mathfrak{D}$ and the collection of such functions in $\mathfrak{D}$ is symbolized by σ. For a brief study, and to know some interesting properties of the family σ, see [6]. Some subfamilies of the family σ that are comparable to the well-known subfamilies of the family S have been introduced by Tan [7], Brannan and Taha [8], and Srivastava et al. [9]. In fact, as sequels to the above subfamilies of σ, a number of different subfamilies of σ have since then been explored by many authors (see, for example, [10–14]). Most of these works are devoted to the study of the Fekete–Szegö issue of functions in various subfamilies of σ.

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{R} = (-\infty, +\infty)$.

If, for $\kappa \in \mathbb{N}$, $s(e^{\frac{2\pi i}{\kappa}}\varsigma) = e^{\frac{2\pi i}{\kappa}}s(\varsigma)$, $\varsigma \in \mathfrak{D}$, then a regular function s is called a κ-fold symmetric (κ-FS). The function s, defined by $s(\varsigma) = (f(\varsigma^\kappa))^{1/\kappa}$, $\kappa \in \mathbb{N}$, $f \in \mathcal{S}$, is univalent and maps $\mathfrak{D}$ into a κ-fold symmetry region. We indicate by $\mathcal{S}_\kappa$ the class of κ-fold symmetric univalent (κ-FSU) functions in $\mathfrak{D}$. A function $s \in \mathcal{S}_\kappa$ has the following form:

$$s(\varsigma) = \varsigma + \sum_{j=1}^{\infty} d_{\kappa j+1} \cdot \varsigma^{\kappa j+1} \qquad (\kappa \in \mathbb{N}; \varsigma \in \mathfrak{D}). \tag{3}$$

Clearly $\mathcal{S}_1 = \mathcal{S}$.

Similar to the idea of $\mathcal{S}_\kappa$, Srivastava et al. [15] investigated the class σ_κ of κ-fold symmetric bi-univalent (κ-FSBU) functions. A few intriguing findings were made, including the series

$$\begin{aligned} s^{-1}(\varkappa) = \varkappa - d_{\kappa+1}\varkappa^{\kappa+1} + [(1+\kappa)d_{\kappa+1}^2 - d_{2\kappa+1}]\varkappa^{2\kappa+1} \\ - \left[\frac{1}{2}(1+\kappa)(2+3\kappa)d_{\kappa+1}^3 - (2+3\kappa)d_{\kappa+1}d_{2\kappa+1} + d_{3\kappa+1}\right]\varkappa^{3\kappa+1} + \cdots . \end{aligned} \tag{4}$$

when $s \in \sigma_\kappa$.

Note that the functions

$$s_1(\varsigma) = \left(\frac{1}{2}log\left(\frac{1+\varsigma^\kappa}{1-\varsigma^\kappa}\right)\right)^{1/\kappa}, s_2(\varsigma) = \left(\frac{\varsigma^\kappa}{1-\varsigma^\kappa}\right)^{1/\kappa}, \quad s_3(\varsigma) = (-log(1-\varsigma^\kappa))^{1/\kappa}, \ \cdots .$$

with the corresponding inverses

$$g_1(\varkappa) = \left(\frac{e^{2\varkappa^\kappa} - 1}{e^{2\varkappa^\kappa} - 1}\right)^{1/\kappa}, \ g_2(\varkappa) = \left(\frac{\varkappa^\kappa}{1+\varkappa^\kappa}\right)^{1/\kappa}, \ g_3(\varkappa) = \left(\frac{e^{\varkappa^\kappa} - 1}{e^{\varkappa^\kappa}}\right)^{1/m}, \ \cdots .$$

are elements of σ_κ. We obtain (2) from (4) on taking $\kappa = 1$.

The focus on the initial coefficients of functions in some subfamilies of σ_κ is an interesting topic and this opened an area for many developments. New subfamilies of σ_κ were introduced and examined in depth by many researchers (see, for example, [16–19]). We mention here some recent works on this topic. Initial coefficient bounds for new subfamilies of σ_κ were determined in [20]. The Fekete–Szegö (FS) issue $|d_{2m+1} - \delta d_{m+1}^2|$, $\delta \in \mathbb{R}$ (see [21]) for certain special families of σ_κ was examined by Swamy et al. [22,23]; and another special family of σ_κ satisfying certain subordination conditions was examined by Aldawish et al. [24]; initial coefficients estimates for elements belonging to certain new families of σ_κ were obtained by Breaz and Cotîrlă in [25] (see [26–28]), indicating the developments in this domain.

For functions s_1 and s_2 regular in $\mathfrak{D}$, s_1 is said to subordinate s_2, if there is a Schwarz function ψ in $\mathfrak{D}$, such that $\psi(0) = 0$, $|\psi(z)| < 1$ and $s_1(z) = s_2(\psi(z)), z \in \mathfrak{D}$. This subordination is indicated as $s_1 \prec s_2$. If $s_2 \in \mathcal{S}$, then $s_1(z) \prec s_2(z)$ is equivalent to $s_1(0) = s_2(0)$ and $s_1(\mathfrak{D}) \subset s_2(\mathfrak{D})$.

Inspired by the efforts of Al-Amiri [3] and the authors of [19], we introduce a new class $\mathfrak{P}^\tau_{\sigma_\kappa}(\eta, \nu, \varphi)$, $\eta \in \mathbb{C}^* = \mathbb{C} - \{0\}, 0 \le \nu \le 1$, and $\varphi(\varsigma)$ is a regular function, such that $\Re(\varphi(\varsigma)) > 0$, $\varphi'(0) > 0$, $\varphi(0) = 1$, $\varphi(\mathfrak{D})$ is symmetric with respect to the real axis. In Section 2, we estimate the upper bounds of $|d_{\kappa+1}|$, $|d_{2\kappa+1}|$ and $|d_{2\kappa+1} - \delta d^2_{\kappa+1}|$ ($\delta \in \mathbb{R}$), for functions that belong to the class $\mathfrak{P}^\tau_{\sigma_\kappa}(\eta, \nu, \varphi)$. We consider two special cases $\mathfrak{Q}^\varrho_{\sigma_\kappa}(\eta, \nu, \tau) = \mathfrak{P}^\tau_{\sigma_\kappa}(\eta, \nu, \left(\frac{1+\varsigma}{1-\varsigma}\right)^\varrho), 0 < \varrho \le 1$ and $\mathfrak{X}^\xi_{\sigma_\kappa}(\eta, \nu, \tau) = \mathfrak{P}^\tau_{\sigma_\kappa}(\eta, \nu, \frac{1+(1-2\xi)\varsigma}{1-\varsigma})$, $0 \le \xi < 1$, in Section 3 and Section 4, respectively. We also identify connections to existing results and present a few new observations.

2. The Class $\mathfrak{P}^\tau_{\sigma_\kappa}(\eta, \nu, \varphi)$

Throughout this paper, $s^{-1}(\varkappa) = g(\varkappa)$ is as in (4), $\eta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, \varsigma \in \mathfrak{D}, \varkappa \in \mathfrak{D}$ and $\varphi(\varsigma)$ will be a regular function such that $\Re(\varphi(\varsigma)) > 0$, $\varphi'(0) > 0$, $\varphi(0) = 1$, and $\varphi(\mathfrak{D})$ is symmetric with respect to the real axis. An expansion of $\varphi(\varsigma)$ has the form:

$$\varphi(\varsigma) = 1 + B_1\varsigma + B_2\varsigma^2 + B_3\varsigma^3 + \cdots (B_1 > 0). \tag{5}$$

Let **P** be the class of regular functions of the type $p(\varsigma) = 1 + p_1\varsigma + p_2\varsigma^2 + p_3\varsigma^3 + \cdots$, $\Re(p(\varsigma)) > 0$. A κ-FS function $p_\kappa \in \mathbf{P}$ is of the form $p_\kappa(\varsigma) = 1 + p_\kappa\varsigma^\kappa + p_{2\kappa}\varsigma^{2\kappa} + p_{3\kappa}\varsigma^{3\kappa} + \cdots$ (see [29]).

Let $\mathfrak{h}(\varsigma)$ and $\mathfrak{p}(\varkappa)$ be regular in $\mathfrak{D}$ with $\max\{|\mathfrak{h}(\varsigma)|, |\mathfrak{p}(\varkappa)|\} < 1$ and $\mathfrak{h}(0) = 0 = \mathfrak{p}(0)$. We suppose that $\mathfrak{h}(\varsigma) = h_\kappa\varsigma^\kappa + h_{2\kappa}\varsigma^{2\kappa} + h_{3\kappa}\varsigma^{3\kappa} + \cdots$ and $\mathfrak{p}(\varkappa) = p_\kappa\varkappa^\kappa + p_{2\kappa}\varkappa^{2\kappa} + p_{3\kappa}\varkappa^{3\kappa} + \cdots$. Also, we assume that

$$|h_\kappa| < 1; |h_{2\kappa}| \le 1 - |h_\kappa|^2; |p_\kappa| < 1; |p_{2\kappa}| \le 1 - |p_\kappa|^2. \tag{6}$$

After simple computations, using (5), we have

$$\varphi(\mathfrak{h}(\varsigma)) = 1 + B_1 h_\kappa\varsigma^\kappa + (B_1 h_{2\kappa} + B_2 h^2_\kappa)\varsigma^{2\kappa} + \ldots \tag{7}$$

and

$$\varphi(\mathfrak{p}(\varkappa)) = 1 + B_1 p_\kappa\varkappa^\kappa + (B_1 p_{2\kappa} + B_2 p^2_\kappa)\varkappa^{2\kappa} + \ldots. \tag{8}$$

Definition 1. *A function $s \in \sigma_\kappa$ of the form (3) is said to be in the class $\mathfrak{P}^\tau_{\sigma_\kappa}(\eta, \nu, \varphi)$ if*

$$\frac{1}{\eta}\left(\nu\frac{\{(\varsigma s'(\varsigma))'\}^\tau}{s'(\varsigma)} + (1-\nu)(s'(\varsigma))^\tau - 1\right) + 1 \prec \varphi(\varsigma)$$

and

$$\frac{1}{\eta}\left(\nu\frac{\{(\varkappa g'(\varkappa))'\}^\tau}{g'(\varkappa)} + (1-\nu)(g'(\varkappa))^\tau - 1\right) + 1 \prec \varphi(\varkappa),$$

where $g = s^{-1}$, $\tau \ge 1$, $\eta \in \mathbb{C}^*$, and $0 \le \nu < 1$.

Remark 1. *(i) The subclass $\mathfrak{P}^\tau_{\sigma_\kappa}(\eta, 0, \varphi) \equiv \mathscr{H}^\tau_{\sigma_\kappa}(\eta, \varphi)$, and was explored in [24].*

(ii) $\mathfrak{P}^1_{\sigma_\kappa}(\eta, \nu, \varphi) \equiv \mathscr{I}_{\sigma_\kappa}(\eta, \nu, \varphi)$ is the subclass of functions $s \in \sigma_\kappa$ satisfying

$$\frac{1}{\eta}\left(\nu\frac{(\varsigma s'(\varsigma))'}{s'(\varsigma)} + (1-\nu)s'(\varsigma) - 1\right) + 1 \prec \varphi(\varsigma)$$

and its inverse $g = s^{-1}$ satisfies

$$\frac{1}{\eta}\left(\nu\frac{(\varkappa g'(\varkappa))'}{g'(\varkappa)} + (1-\nu)g'(\varkappa) - 1\right) + 1 \prec \varphi(\varkappa),$$

where $\eta \in \mathbb{C}^*$ and $0 \leq \nu < 1$.

Theorem 1. *If the function s given by (3) belongs to the family $\mathfrak{P}^{\tau}_{\sigma_\kappa}(\eta,\nu,\varphi)$ and $\delta \in \mathbb{R}$, then*

$$|d_{\kappa+1}| \leq \frac{|\eta|B_1\sqrt{2B_1}}{\sqrt{|\{M(1+\kappa)+[N\tau(\tau-1)+(1-(1+\kappa)\tau)2\nu](1+\kappa)^2\}\eta B_1^2 - 2L^2B_2| + 2L^2B_1}}, \tag{9}$$

$$|d_{2\kappa+1}| \leq \begin{cases} \frac{B_1|\eta|}{M} & ;\ 0 < B_1 < \frac{2L^2}{|\eta|M(1+\kappa)} \\ \frac{B_1|\eta|}{M} + \left(\frac{1+\kappa}{2} - \frac{L^2}{|\eta|B_1M}\right)\frac{2\eta^2B_1^3}{|\{M(1+\kappa)+[N\tau(\tau-1)+(1-(1+\kappa)\tau)2\nu](1+\kappa)^2\}\eta B_1^2 - 2L^2B_2| + 2L^2B_1} & ;\ B_1 \geq \frac{2L^2}{|\eta|M(1+\kappa)}, \end{cases} \tag{10}$$

and

$$|d_{2\kappa+1} - \delta d_{\kappa+1}^2| \leq \begin{cases} \frac{B_1|\eta|}{M} & ;\ |1+\kappa-2\delta| < J \\ \frac{|\eta|^2B_1^3|\kappa-2\delta+1|}{|\{M(1+\kappa)+[N\tau(\tau-1)+(1-(1+\kappa)\tau)2\nu](1+\kappa)^2\}\eta B_1^2 - 2L^2B_2|} & ;\ |1+\kappa-2\delta| \geq J, \end{cases} \tag{11}$$

where

$$J = \left|\frac{\{M(1+\kappa)+[N\tau(\tau-1)+2\nu(1-(1+\kappa)\tau)](1+\kappa)^2\}\eta B_1^2 - 2L^2B_2}{\eta M B_1^2}\right|. \tag{12}$$

$$L = (1+\kappa)(\tau(1+\nu\kappa)-\nu), \tag{13}$$

$$M = (\tau(1+2\nu\kappa)-\nu)(1+2\kappa) \tag{14}$$

and

$$N = 1+\nu\kappa(2+\kappa). \tag{15}$$

Proof. Let the function s of the form (3) belong to the family $\mathfrak{P}^{\tau}_{\sigma_\kappa}(\eta,\nu,\varphi)$. Then, we have regular functions $\mathfrak{h},\mathfrak{p} : \mathfrak{D} \longrightarrow \mathfrak{D}$, $\mathfrak{h}(0) = \mathfrak{p}(0) = 0$ satisfying

$$\frac{1}{\eta}\left(\nu\frac{\{(\varsigma s'(\varsigma))'\}^{\tau}}{s'(\varsigma)} + (1-\nu)(s'(\varsigma))^{\tau} - 1\right) + 1 = \varphi(\mathfrak{h}(\varsigma)), \tag{16}$$

and

$$\frac{1}{\eta}\left(\nu\frac{\{(\varkappa g'(\varkappa))'\}^{\tau}}{g'(\varkappa)} + (1-\nu)(g'(\varkappa))^{\tau} - 1\right) + 1 = \varphi(\mathfrak{p}(\varkappa)). \tag{17}$$

Using (3) in (16) and (17) we obtain:

$$\frac{1}{\eta}\left(\nu\frac{[\{(\varsigma s'(\varsigma))'\}^{\tau}}{s'(\varsigma)} + (1-\nu)(s'(\varsigma))^{\tau} - 1\right) + 1 =$$

$$\frac{1}{\eta}\{Ld_{\kappa+1}\varsigma^{\kappa}+[\,Md_{2\kappa+1}+ \\ (1+\kappa)^2\left(\frac{N\tau(\tau-1)}{2}+\nu(1-(1+\kappa)\tau)\right)d_{\kappa+1}^2\Big]\varsigma^{2\kappa}+\cdots\Big\}+1 \tag{18}$$

and

$$\frac{1}{\eta}\left(\nu\frac{\{(\varkappa g'(\varkappa))'\}^{\tau}}{g'(\varkappa)}+(1-\nu)(g'(\varkappa))^{\tau}-1\right)+1=$$

$$\frac{1}{\eta}\Big\{-Ld_{\kappa+1}\varkappa^{\kappa}+\Big[M((1+\kappa)d_{\kappa+1}^2-d_{2\kappa+1})+ \\ (1+\kappa)^2\left(\frac{N\tau(\tau-1)}{2}+\nu(1-(1+\kappa)\tau)\right)d_{\kappa+1}^2\Big]\varkappa^{2\kappa}+\cdots\Big\}+1, \tag{19}$$

where L, M, and N are as in (13), (14), and (15), respectively.

Comparing (7) and (18), we obtain

$$Ld_{\kappa+1}=\eta B_1 h_{\kappa} \tag{20}$$

and

$$Md_{2\kappa+1}+\left(\frac{N\tau(\tau-1)}{2}+\nu(1-(1+\kappa)\tau)\right)(1+\kappa)^2 d_{\kappa+1}^2=\eta[B_1 h_{2\kappa}+B_2 h_{\kappa}^2]\,. \tag{21}$$

Comparing (8) and (19), we obtain

$$-Ld_{\kappa+1}=\eta B_1 p_{\kappa} \tag{22}$$

and

$$M((\kappa+1)d_{\kappa+1}^2-d_{2\kappa+1})+\left(\tfrac{N\tau(\tau-1)}{2}+\nu(1-(1+\kappa)\tau)\right)(1+\kappa)^2 d_{\kappa+1}^2 \\ =\eta[B_1 p_{2\kappa}+B_2 p_{\kappa}^2]\,, \tag{23}$$

From (20) and (22), we obtain

$$h_{\kappa}=-p_{\kappa} \tag{24}$$

and

$$2L^2 d_{\kappa+1}^2=\eta^2 B_1^2(h_{\kappa}^2+p_{\kappa}^2). \tag{25}$$

We add (21) and (23) and then use (25) to obtain

$$[\{M(1+\kappa)+[N\tau(\tau-1)+(1-(1+\kappa)\tau)2\nu](1+\kappa)^2\}\eta B_1^2-2L^2B_2]d_{\kappa+1}^2 \\ =\eta^2 B_1^3(h_{2\kappa}+p_{2\kappa}) \tag{26}$$

By using (6) and (20) in (26) for the coefficients $h_{2\kappa}$ and $p_{2\kappa}$, we obtain

$$[|\{M(1+\kappa)+[N\tau(\tau-1)+(1-(1+\kappa)\tau)2\nu](1+\kappa)^2\}\eta B_1^2-2L^2B_2|+2L^2B_1]|d_{\kappa+1}|^2 \\ \le 2\eta^2 B_1^3, \tag{27}$$

which implies (9).

We subtract (23) from (21) to find the bound on $|d_{2\kappa+1}|$:

$$d_{2\kappa+1}=\frac{\eta B_1(h_{2\kappa}-p_{2\kappa})}{2M}+\left(\frac{1+\kappa}{2}\right)d_{\kappa+1}^2. \tag{28}$$

In view of (20), (24), (28) and applying (6), we obtain

$$|d_{2\kappa+1}| \leq \frac{|\eta|B_1}{M} + \left(\frac{1+\kappa}{2} - \frac{L^2}{|\eta|B_1 M}\right) \times \tag{29}$$

$$\times \frac{2\eta^2 B_1^3}{|\{M(1+\kappa) + [N\tau(\tau-1) + (1-(1+\kappa)\tau)2\nu](1+\kappa)^2\}\eta B_1^2 - 2L^2 B_2| + 2L^2 B_1},$$

which obtains (10), the desired assessment.

From (26) and (28), for $\delta \in \mathbb{R}$, we obtain

$$d_{2\kappa+1} - \delta d_{\kappa+1}^2 = \frac{\eta B_1}{2}\left[\left(\mathfrak{Y}(\delta) + \frac{1}{M}\right)h_{2\kappa} + \left(\mathfrak{Y}(\delta) - \frac{1}{M}\right)p_{2\kappa}\right],$$

where

$$\mathfrak{Y}(\delta) = \frac{\eta B_1^2(\kappa - 2\delta + 1)}{\{M(1+\kappa) + [N\tau(\tau-1) + (1-(1+\kappa)\tau)2\nu](\kappa+1)^2\}\eta B_1^2 - 2L^2 B_2}.$$

In view of (6), we conclude that

$$|d_{2\kappa+1} - \delta d_{\kappa+1}^2| \leq \begin{cases} \frac{|\eta|B_1}{M} & ; 0 \leq |\mathfrak{Y}(\delta)| < \frac{1}{M} \\ |\eta|B_1|\mathfrak{Y}(\delta)| & ; |\mathfrak{Y}(\delta)| \geq \frac{1}{M}, \end{cases}$$

form which we obtain (11) with J as in (12). So the proof is completed. □

Remark 2. *We obtain Corollary 1 of [24] if $\nu = 0$ in Theorem 1.*

Choosing $\tau = 1$ in $\mathfrak{P}_{\sigma_\kappa}^{\tau}(\eta, \nu, \varphi)$, we have the corollary given below:

Corollary 1. *Let $\delta \in \mathbb{R}$ and let the function s given by (3) be in the family $\mathscr{I}_{\sigma_\kappa}(\eta, \nu, \varphi)$. Then,*

$$|d_{\kappa+1}| \leq \frac{|\eta|B_1\sqrt{2B_1}}{\sqrt{|\{(1+\kappa)M_1 - 2\nu\kappa(1+\kappa)^2\}\eta B_1^2 - 2L_1^2 B_2| + 2L_1^2 B_1}},$$

$$|d_{2\kappa+1}| \leq \begin{cases} \frac{|\eta|B_1}{M_1} & ; 0 < B_1 < \frac{2L_1^2}{(1+\kappa)M_1|\eta|} \\ \frac{|\eta|B_1}{M_1} + \left(\frac{1+\kappa}{2} - \frac{L_1^2}{B_1 M_1|\eta|}\right)\frac{2\eta^2 B_1^3}{|\{(1+\kappa)M_1 - 2\nu\kappa(1+\kappa)^2\}\eta B_1^2 - 2L_1^2 B_2| + 2L_1^2 B_1} & ; B_1 \geq \frac{2L_1^2}{(1+\kappa)M_1|\eta|} \end{cases}$$

and

$$|d_{2\kappa+1} - \delta d_{\kappa+1}^2| \leq \begin{cases} \frac{|\eta|B_1}{M_1} & ; |\kappa - 2\delta + 1| < J_1 \\ \frac{B_1^3|\kappa - 2\delta + 1||\eta|^2}{|\{(\kappa+1)M_1 - 2\nu\kappa(\kappa+1)^2\}\eta B_1^2 - 2L_1^2 B_2|} & ; |\kappa - 2\delta + 1| \geq J_1, \end{cases}$$

where

$$J_1 = \left|\frac{\{(1+\kappa)M_1 - 2\nu\kappa(1+\kappa)^2\}\eta B_1^2 - 2L_1^2 B_2}{M_1 B_1^2 \eta}\right|,$$

$$L_1 = (1+\kappa)((\kappa-1)\nu + 1) \tag{30}$$

and

$$M_1 = (1+2\kappa)((2\kappa-1)\nu + 1), \tag{31}$$

Remark 3. *If $\nu = 0$ and $\eta = 1$ in Corollary 1 are allowed, then the first and second theorems of Tang et al. [19] are obtained.*

Choosing $\kappa = 1$ in Theorem 1, we have

Corollary 2. *If $s \in \mathfrak{P}^{\tau}_{\sigma_1}(\eta,\nu,\varphi)$ is given by (1) and $\delta \in \mathbb{R}$, then*

$$|d_2| \le \frac{|\eta| B_1\sqrt{B_1}}{\sqrt{|\{M_2+2(N_2\tau(\tau-1)+2\nu(1-2\tau))\}\eta B_1^2 - L_2^2 B_2| + L_2^2 B_1}},$$
$$|d_3| \le \begin{cases} \frac{|\eta| B_1}{M_2} & ; 0 < B_1 < \frac{L_2^2}{|\eta| M_2} \\ \frac{|\eta| B_1}{M_2} + \left(1 - \frac{L_2^2}{|\eta| B_1 M_2}\right)\frac{\eta^2 B_1^3}{|\{M_2+2(N_2\tau(\tau-1)+2\nu(1-2\tau))\}\eta B_1^2 - L_2^2 B_2| + L_2^2 B_1}; & B_1 \ge \frac{L_2^2}{|\eta| M_2}, \end{cases}$$

and

$$|d_3 - \delta d_2^2| \le \begin{cases} \frac{|\eta| B_1}{M_2} & ; |1-\delta| < J_2 \\ \frac{|\eta|^2 B_1^3 |1-\delta|}{|\{M_2+2(N_2\tau(\tau-1)+2\nu(1-2\tau))\}\eta B_1^2 - L_2^2 B_2|}; & |1-\delta| \ge J_2, \end{cases}$$

where

$$J_2 = \left| \frac{\{M_2+2(N_2\tau(\tau-1)+2\nu(1-2\tau))\}\eta B_1^2 - L_2^2 B_2}{\eta M_2 B_1^2} \right|,$$

$$L_2 = 2((1+\nu)\tau - \nu), \tag{32}$$

$$M_2 = 3((1+2\nu)\tau - \nu) \tag{33}$$

and

$$N_2 = 3\nu + 1. \tag{34}$$

Setting $\eta = \tau = 1$ in Corollary 2, we obtain the following.

Corollary 3. *If $s \in \mathfrak{P}^{1}_{\sigma_1}(1,\nu,\varphi)$ is given by (1) and $\delta \in \mathbb{R}$, then*

$$|d_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{|(3-\nu)B_1^2 - 4B_2| + 4B_1}},$$
$$|d_3| \le \begin{cases} \frac{B_1}{3(\nu+1)} & ; 0 < B_1 < \frac{4}{3(\nu+1)} \\ \frac{B_1}{3(\nu+1)} + \left(1 - \frac{4}{3(\nu+1)B_1}\right)\frac{B_1^3}{|(3-\nu)B_1^2 - 4B_2| + 4B_1}; & B_1 \ge \frac{4}{3(\nu+1)}, \end{cases}$$

and

$$|d_3 - \delta d_2^2| \le \begin{cases} \frac{B_1}{3(\nu+1)} & ; |1-\delta| < \left|\frac{(3-\nu)B_1^2 - 4B_2}{3(\nu+1)B_1^2}\right| \\ \frac{B_1^3|1-\delta|}{|(3-\nu)B_1^2 - 4B_2|}; & |1-\delta| \ge \left|\frac{(3-\nu)B_1^2 - 4B_2}{3(\nu+1)B_1^2}\right|. \end{cases}$$

Remark 4. *When $\nu = 0$ is selected in Corollary 3, we obtain Corollaries 1 and 4 of Tang et al. [19] (also see [30]).*

3. The Class $\mathfrak{Q}^{\tau}_{\sigma_\kappa}(\eta,\nu,\varrho)$

Let $\varphi(\varsigma) = 1 + 2\varrho\varsigma + 2\varrho^2\varsigma^2 + \cdots = \left(\frac{1+\varsigma}{1-\varsigma}\right)^{\varrho}$ in Definition 1. Then, we have the subclass of all $s \in \sigma_\kappa$ satisfying

$$\left| arg\left[\frac{1}{\eta}\left(\nu\frac{\{(\varsigma s'(\varsigma))'\}^{\tau}}{s'(\varsigma)} + (1-\nu)(s'(\varsigma))^{\tau} - 1\right) + 1\right]\right| < \frac{\varrho\pi}{2}$$

and

$$\left| arg\left[\frac{1}{\eta}\left(\nu\frac{\{(\varkappa g'(\varkappa))'\}^{\tau}}{g'(\varkappa)} + (1-\nu)(g'(\varkappa))^{\tau} - 1\right) + 1\right]\right| < \frac{\varrho\pi}{2},$$

where $g = s^{-1}$, $0 < \varrho \le 1$, $\eta \in \mathbb{C}^*, \tau \ge 1$, and $0 \le \nu < 1$. We denote this class by $\mathfrak{Q}^{\tau}_{\sigma_\kappa}(\eta,\nu,\varrho) = \mathfrak{P}^{\tau}_{\sigma_\kappa}(\eta,\nu,\left(\frac{1+\varsigma}{1-\varsigma}\right)^{\varrho})$.

Remark 5. *(i) The family* $\mathfrak{Q}^{\tau}_{\sigma_\kappa}(\eta,0,\varrho) \equiv \mathscr{B}^{\tau}_{\sigma_\kappa}(\eta,\varrho)$, *and was explored in [24], where* $\eta \in \mathbb{C}^*$, $\tau \geq 1$ *and* $0 < \varrho \leq 1$.

ii) $\mathfrak{Q}^{1}_{\sigma_\kappa}(\eta,\nu,\varrho) \equiv \mathscr{D}_{\sigma_\kappa}(\eta,\nu,\varrho)$ *is the subfamily of all* $s \in \sigma_\kappa$ *satisfying*

$$\left|arg\left[\frac{1}{\eta}\left(\nu\frac{(\varsigma s'(\varsigma))'}{s'(\varsigma)} + (1-\nu)s'(\varsigma) - 1\right) + 1\right]\right| < \frac{\varrho\pi}{2}$$

and its inverse $g = s^{-1}$ *satisfies*

$$\left|arg\left[\frac{1}{\eta}\left(\nu\frac{(\varkappa g'(\varkappa))'}{g'(\varkappa)} + (1-\nu)g'(\varkappa) - 1\right) + 1\right]\right| < \frac{\varrho\pi}{2},$$

where $0 < \varrho \leq 1$, $\eta \in \mathbb{C}^*$ *and* $0 \leq \nu < 1$.

Taking $\varphi(\varsigma) = \left(\frac{1+\varsigma}{1-\varsigma}\right)^{\varrho}$ in Theorem 1, we obtain

Corollary 4. *If the function s given by (3) belongs to the family* $\in \mathfrak{Q}^{\tau}_{\sigma_\kappa}(\eta,\nu,\varrho)$ *and* $\delta \in \mathbb{R}$, *then*

$$|d_{\kappa+1}| \leq \frac{2\varrho|\eta|}{\sqrt{\varrho|\{M(1+\kappa)+[N\tau(\tau-1)+(1-(1+\kappa)\tau)2\nu](1+\kappa)^2\}\eta - L^2| + L^2}},$$

$$|d_{2\kappa+1}| \leq \begin{cases} \frac{2\varrho|\eta|}{M} & ; 0 < \varrho < \frac{L^2}{M(1+\kappa)|\eta|} \\ \frac{2\varrho|\eta|}{M} + \left(1+\kappa - \frac{L^2}{\varrho M|\eta|}\right)\frac{2\varrho^2\eta^2}{\varrho|\{(1+\kappa)M+[N\tau(\tau-1)+(1-(1+\kappa)\tau)2\nu](1+\kappa)^2\}\eta - L^2| + L^2} & ; \varrho \geq \frac{L^2}{M(1+\kappa)|\eta|}, \end{cases}$$

and

$$|d_{2\kappa+1} - \delta d^2_{\kappa+1}| \leq \begin{cases} \frac{2\varrho|\eta|}{M} & ; |\kappa - 2\delta + 1| < J_3 \\ \frac{2\varrho|\kappa-2\delta+1||\eta|^2}{|\{(1+\kappa)M+[\tau(\tau-1)N+(1-(\kappa+1)\tau)2\nu](\kappa+1)^2\}\eta - L^2|} & ; |\kappa - 2\delta + 1| \geq J_3, \end{cases}$$

where

$$J_3 = \left|\frac{\{M(1+\kappa) + [N\tau(\tau-1) + (1-(1+\kappa)\tau)2\nu](1+\kappa)^2\}\eta - L^2}{\eta M}\right|,$$

L, M, and N are as in (13), (14), and (15), respectively.

Remark 6. *For* $\nu = 0$ *in Corollary 4, we obtain Corollary 4 in [24].*

Choosing $\tau = 1$ in $\mathfrak{Q}^{\tau}_{\sigma_\kappa}(\eta,\nu,\varphi)$, we obtain the corollary given below:

Corollary 5. *If the function s given by (3) belongs to the family* $\mathscr{D}_{\sigma_\kappa}(\eta,\nu,\varrho)$ *and* $\delta \in \mathbb{R}$, *then*

$$|d_{\kappa+1}| \leq \frac{2\varrho|\eta|}{\sqrt{\varrho|\{M_1(1+\kappa) - 2\nu\kappa(1+\kappa)^2\}\eta - L_1^2| + L_1^2}},$$

$$|d_{2\kappa+1}| \leq \begin{cases} \frac{2\varrho|\eta|}{M_1} & ; 0 < \varrho < \frac{L_1^2}{M_1(1+\kappa)|\eta|} \\ \frac{2\varrho|\eta|}{M_1} + \left(1+\kappa - \frac{L_1^2}{\varrho M_1|\eta|}\right)\frac{2\varrho^2\eta^2}{\varrho|\{M_1(1+\kappa) - 2\nu\kappa(1+\kappa)^2\}\eta - L_1^2| + L^2} & ; \varrho \geq \frac{L_1^2}{(1+\kappa)M_1|\eta|} \end{cases}$$

and

$$|d_{2\kappa+1} - \delta d^2_{\kappa+1}| \leq \begin{cases} \frac{2\varrho|\eta|}{M_1} & ; |\kappa - 2\delta + 1| < J_4 \\ \frac{2\varrho|\kappa-2\delta+1||\eta|^2}{|\{M_1(\kappa+1) - 2\nu\kappa(\kappa+1)^2\}\eta - L_1^2|} & ; |\kappa - 2\delta + 1| \geq J_4, \end{cases}$$

where

$$J_4 = \left|\frac{\{M_1(1+\kappa) - 2\nu\kappa(1+\kappa)^2\}\eta - L_1^2}{\eta M_1}\right|,$$

L_1 and M_1 are as in (30) and (31), respectively.

Corollary 4 yields the following if $\kappa = 1$:

Corollary 6. *If $s \in \mathfrak{Q}^{\tau}_{\sigma_1}(\eta, \nu, \varrho)$ is given by (1) and $\delta \in \mathbb{R}$, then*

$$|d_2| \leq \frac{2|\eta|\varrho}{\sqrt{\varrho|\{2M_2+4(N_2\tau(\tau-1)+2\nu(1-2\tau))\}\eta - L_2^2| + L_2^2}},$$
$$|d_3| \leq \begin{cases} \frac{2|\eta|\varrho}{M_2} & ; 0 < \varrho < \frac{L_2^2}{2|\eta|M_2} \\ \frac{2|\eta|\varrho}{M_2} + \left(2 - \frac{L_2^2}{|\eta|\varrho M_2}\right)\frac{2\eta^2\varrho^2}{\varrho|\{2M_2+4(N_2\tau(\tau-1)+2\nu(1-2\tau))\}\eta - L_2^2| + L_2^2} & ; \varrho \geq \frac{L_2^2}{2|\eta|M_2}, \end{cases}$$

and

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{2|\eta|\varrho}{M_2} & ; |1-\delta| < J_5 \\ \frac{2|\eta|^2\varrho|1-\delta|}{|\{2M_2+4(N_2\tau(\tau-1)+2\nu(1-2\tau))\}\eta - L_2^2|} & ; |1-\delta| \geq J_5, \end{cases}$$

where

$$J_5 = \left|\frac{\{2M_2+4(N_2\tau(\tau-1)+2\nu(1-2\tau))\}\eta - L_2^2}{2\eta M_2}\right|,$$

L_2, M_2, and N_2 are as in (32), (33), and (34), respectively.

Corollary 6 would yield the following if $\eta = \tau = 1$.

Corollary 7. *If the function s of the form (1) $\in \mathfrak{Q}^{1}_{\sigma_1}(1, \nu, \varphi)$ and $\delta \in \mathbb{R}$, then*

$$|d_2| \leq \frac{\varrho\sqrt{2}}{\sqrt{\varrho(1-\nu)+2}},$$
$$|d_3| \leq \begin{cases} \frac{2\varrho}{3(\nu+1)} & ; 0 < \varrho < \frac{2}{3(\nu+1)} \\ \frac{2\varrho}{3(\nu+1)} + \left(1 - \frac{2}{3\varrho(\nu+1)}\right)\frac{2\varrho^2}{\varrho(1-\nu)+2} & ; \varrho \geq \frac{2}{3(\nu+1)}, \end{cases}$$

and

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{2\varrho}{3(\nu+1)} & ; |1-\delta| < \frac{1-\nu}{3(\nu+1)} \\ \frac{\varrho|1-\delta|}{1-\nu} & ; |1-\delta| \geq \frac{1-\nu}{3(\nu+1)}. \end{cases}$$

Remark 7. *Letting $\nu = 0$ in Corollary 7, we obtain Corollary 2 of Tang et al. [19]. The estimate obtained here for $|d_3|$ is more accurate when compared to that in Theorem 2 of Srivastava et al. [9].*

4. The Class $\mathfrak{X}^{\tau}_{\sigma_\kappa}(\eta, \nu, \xi)$

If $\varphi(\varsigma) = 1 + 2(1-\xi)\varsigma + 2(1-\xi)\varsigma^2 + \cdots = \frac{1+(1-2\xi)\varsigma}{1-\varsigma}$ in Definition 1, then we have the subset of all $s \in \sigma_\kappa$ satisfying

$$\Re\left[\left(\nu\frac{\{(\varsigma s'(\varsigma))'\}^{\tau}}{s'(\varsigma)} + (1-\nu)(s'(\varsigma))^{\tau} - 1\right)\frac{1}{\eta} + 1\right] > \xi$$

and

$$\Re\left[\left(\nu\frac{\{(\varkappa g'(\varkappa))'\}^{\tau}}{g'(\varkappa)} + (1-\nu)(g'(\varkappa))^{\tau} - 1\right)\frac{1}{\eta} + 1\right] > \xi,$$

where $g = s^{-1}$, $\eta \in \mathbb{C}^*$, $0 \leq \xi < 1$, $\tau \geq 1$, and $0 \leq \nu < 1$. We denote this set by $\mathfrak{X}^{\tau}_{\sigma_\kappa}(\eta, \nu, \xi) = \mathfrak{P}^{\tau}_{\sigma_\kappa}\left(\eta, \nu, \left(\frac{1+(1-2\xi)\varsigma}{1-\varsigma}\right)\right)$.

Remark 8. *(i). The family* $\mathfrak{X}_{\sigma_\kappa}^{\tau}(\eta,0,\xi)\equiv\mathscr{E}_{\sigma_\kappa}^{\tau}(\eta,\xi)$, $\tau\geq1$, $0\leq\xi<1$, *and was studied in [24].*
(ii). $\mathfrak{X}_{\sigma_\kappa}^{1}(\eta,\nu,\xi)\equiv\mathscr{F}_{\sigma_\kappa}(\eta,\nu,\xi)$ *is a set of all* $s\in\sigma_\kappa$ *satisfying*

$$\Re\left[\left(\nu\frac{(\varsigma s'(\varsigma))'}{s'(\varsigma)}+(1-\nu)s'(\varsigma)-1\right)\frac{1}{\eta}+1\right]>\xi$$

and its inverse $g=s^{-1}$ *satisfies*

$$\Re\left[\left(\nu\frac{(\varkappa g'(\varkappa))'}{g'(\varkappa)}+(1-\nu)g'(\varkappa)-1\right)\frac{1}{\eta}+1\right]>\xi,$$

where $\eta\in\mathbb{C}^*, 0\leq\xi<1$, *and* $0\leq\nu<1$.

Allowing $\varphi(\varsigma)=\frac{1+(1-2\xi)\varsigma}{1-\varsigma}$, $0\leq\xi<1$, in Theorem 1, we obtain

Corollary 8. *Let the function s of the form (3) belong to the class* $\mathfrak{X}_{\sigma_\kappa}^{\tau}(\eta,\nu,\xi)$ *and* $\delta\in\mathbb{R}$. *Then,*

$$|d_{\kappa+1}|\leq\frac{(1-\xi)2|\eta|}{\sqrt{|\{(1+\kappa)M+(1+\kappa)^2[N\tau(\tau-1)+(1-(1+\kappa)\tau)2\nu]\}(1-\xi)\eta-L^2|+L^2}},$$

$$|d_{2\kappa+1}|\leq$$

$$\begin{cases}\frac{(1-\xi)2|\eta|}{M} & ;\ 1-\frac{L^2}{(1+\kappa)M|\eta|}<\xi<1\\ \frac{(1-\xi)2|\eta|}{M}+\left(1+\kappa-\frac{L^2}{(1-\xi)M|\eta|}\right)\frac{2(1-\xi)^2|\eta|^2}{|\{(1+\kappa)M+[N\tau(\tau-1)+(1-(1+\kappa)\tau)2\nu](1+\kappa)^2\}(1-\xi)\eta-L^2|+L^2} & \\ & ;\ 0\leq\xi\leq1-\frac{L^2}{(1+\kappa)M|\eta|}\end{cases}$$

and

$$|d_{2\kappa+1}-\delta d_{\kappa+1}^2|\leq\begin{cases}\frac{(1-\xi)2|\eta|}{M} & ;\ |\kappa-2\delta+1|<J_6\\ \frac{2(1-\xi)^2|\eta|^2|\kappa-2\delta+1|}{|\{(1+\kappa)M+[N\tau(\tau-1)+(1-(1+\kappa)\tau)2\nu](1+\kappa)^2\}(1-\xi)\eta-L^2|} & ;\ |\kappa-2\delta+1|\geq J_6,\end{cases}$$

where

$$J_6=\left|\frac{\{(1+\kappa)M+(1+\kappa)^2[N\tau(\tau-1)+(1-(1+\kappa)\tau)2\nu]\}(1-\xi)\eta-L^2}{M(1-\xi)\eta}\right|.$$

L, M, and N are as in (13), (14), and (15), respectively.

Remark 9. *We obtain Corollary 7 of Aldawish et al. [24] if* $\nu=0$ *in Corollary 8. In addition, we obtain Corollary 11 of Swamy et al. [22] when* $\eta=\tau=1$.

Corollary 9. *Let the function s of the form (3) belong to the class* $\mathscr{F}_{\sigma_\kappa}^{\tau}(\eta,\nu,\xi)$ *and* $\delta\in\mathbb{R}$. *Then,*

$$|d_{\kappa+1}|\leq\frac{(1-\xi)2|\eta|}{\sqrt{|\{(1+\kappa)M_1-2\nu\kappa(1+\kappa)^2\}(1-\xi)\eta-L_1^2|+L_1^2}},$$

$$|d_{2\kappa+1}|\leq$$

$$\begin{cases}\frac{(1-\xi)2|\eta|}{M_1} & ;\ 1-\frac{L_1^2}{(\kappa+1)M_1|\eta|}<\xi<1\\ \frac{(1-\xi)2|\eta|}{M_1}+\left(1+\kappa-\frac{L_1^2}{(1-\xi)M_1|\eta|}\right)\frac{2(1-\xi)^2|\eta|^2}{|\{(1+\kappa)M_1-2\nu\kappa(1+\kappa)^2\}\eta(1-\xi)-L_1^2|+L_1^2} & \\ & ;\ 0\leq\xi\leq1-\frac{L_1^2}{(1+\kappa)M_1|\eta|}\end{cases}$$

and

$$|d_{2\kappa+1}-\delta d_{\kappa+1}^2|\leq\begin{cases}\frac{2|\eta|(1-\xi)}{M_1} & ;\ |\kappa-2\delta+1|<J_7\\ \frac{2|\eta|^2(1-\xi)^2|\kappa-2\delta+1|}{|\{(1+\kappa)M_1-2\nu\kappa(1+\kappa)^2\}(1-\xi)\eta-L_1^2|} & ;\ |\kappa-2\delta+1|\geq J_7,\end{cases}$$

where

$$J_7 = \left| \frac{\{(1+\kappa)M_1 - 2\nu\kappa(1+\kappa)^2\}(1-\xi)\eta - L_1^2}{M_1(1-\xi)\eta} \right|,$$

L_1 and M_1 are as in (30) and (31), respectively.

If we let $\kappa = 1$ in Corollary 8, then we have

Corollary 10. *Let the function s of the form (1) belong to the class $\mathfrak{X}^{\tau}_{\sigma_1}(\eta, \nu, \xi)$ and $\delta \in \mathbb{R}$. Then,*

$$|d_2| \leq \frac{2(1-\xi)|\eta|}{\sqrt{|\{2M_2+4(N_2\tau(\tau-1)+2\nu(1-2\tau))\}(1-\xi)\eta - L_2^2| + L_2^2}},$$

$$|d_3| \leq \begin{cases} \frac{2(1-\xi)|\eta|}{M_2} & ;\ 1 - \frac{L_2^2}{2M_2|\eta|} < \xi < 1 \\ \frac{2(1-\xi)|\eta|}{M_2} + \left(2 - \frac{L_2^2}{(1-\xi)M_2|\eta|}\right) \frac{2(1-\xi)^2|\eta|^2}{|\{2M_2+4(N\tau(\tau-1)+2\nu(1-2\tau))\}(1-\xi)\eta - L_2^2| + L_2^2} \\ & ;\ 0 \leq \xi \leq 1 - \frac{L_2^2}{2M_2|\eta|} \end{cases}$$

and

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{2(1-\xi)|\eta|}{3(\nu+1)} & ;\ |1-\delta| < J_8 \\ \frac{2(1-\xi)^2|\eta|^2|1-\delta|}{|\{2M_2+4(N_2\tau(\tau-1)+2\nu(1-2\tau))\}(1-\xi)\eta - L_2^2|} & ;\ |1-\delta| \geq J_8, \end{cases}$$

where

$$J_8 = \left| \frac{\{2M_2 + 4(N\tau(\tau-1) + 2\nu(1-2\tau))\}(1-\xi)\eta - L_2^2}{2M_2(1-\xi)\eta} \right|,$$

L_2, M_2, and N_2 are as in (32), (33), and (34), respectively.

If $\eta = \tau = 1$ in Corollary 10, then we obtain

Corollary 11. *If $s \in \mathfrak{X}^{1}_{\sigma_1}(1, \nu, \varphi)$ is of the form (1) and $\delta \in \mathbb{R}$, then*

$$|d_2| \leq \frac{\sqrt{2}(1-\xi)}{\sqrt{|(3-\nu)(1-\xi)-2| + 2}},$$

$$|d_3| \leq \begin{cases} \frac{2(1-\xi)}{3(1+\nu)} & ;\ \frac{3\nu+1}{3(1+\nu)} < \xi < 1 \\ \frac{2(1-\xi)}{3(1+\nu)} + \left(1 - \frac{2}{3(1-\xi)(1+\nu)}\right) \frac{2(1-\xi)^2}{|(1-\xi)(3-\nu)-2| + 2} & ;\ 0 \leq \xi \leq \frac{3\nu+1}{3(1+\nu)}, \end{cases}$$

and

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{2(1-\xi)}{3(1+\nu)} & ;\ |1-\delta| < \left| \frac{(1-\xi)(3-\nu)-2}{3(1-\xi)(1+\nu)} \right| \\ \frac{2|1-\delta|(1-\xi)^2}{|(1-\xi)(3-\nu)-2|} & ;\ |1-\delta| \geq \left| \frac{(3-\nu)(1-\xi)-2}{3(1-\xi)(1+\nu)} \right|, \end{cases}$$

Remark 10. *Putting $\nu = 0$ in Corollary 11, we obtain Corollary 3 of Tang et al. [19]. The estimates obtained here for $|d_2|$ and $|d_3|$ are more accurate when compared to those estimates of Theorem 2 in [9].*

5. Conclusions

In this paper, a new class $\mathfrak{P}^{\tau}_{\sigma_\kappa}(\eta, \nu, \varphi)$ is explored and the upper bounds of $|d_{\kappa+1}|$, $|d_{2\kappa+1}|$, and $|d_{2\kappa+1} - \delta d^2_{\kappa+1}|$ $\delta \in \Re$, are estimated for elements in $\mathfrak{P}^{\tau}_{\sigma_\kappa}(\eta, \nu, \varphi)$. Two special cases $\mathfrak{Q}^{\varrho}_{\sigma_\kappa}(\eta, \nu, \tau) = \mathfrak{P}^{\tau}_{\sigma_\kappa}(\eta, \nu, \left(\frac{1+\varsigma}{1-\varsigma}\right)^{\varrho}), 0 < \varrho \leq 1$ and $\mathfrak{X}^{\xi}_{\sigma_\kappa}(\eta, \nu, \tau) = \mathfrak{P}^{\tau}_{\sigma_\kappa}(\eta, \nu, \frac{1+(1-2\xi)\varsigma}{1-\varsigma})$, $0 \leq \xi < 1$, have been considered. In addition, we have uncovered pertinent links to previous results and given a few observations. This paper could inspire researchers towards further investigations using the (i) integro-differential operator [31], (ii) q-differential operator [32], (iii) q-integral operator [33], and (iv) Hohlov operator [34].

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References

1. Guney, H.O.; Murugusundaramoorthy, G. New classes of pseudo-type bi-univalent fumctions. *Rev. Real Acad. Cienc. Exactas, Fis. Nat. Ser. A Mat.* **2020**, *114*, 65. [CrossRef]
2. Babalola, K.O. On λ-pseudo-starlke functions. *J. Class. Anal.* **2012**, 3, 137–147.
3. Al-Amiri, H.S.; Reade, M.O. On a linear combination of some expressions in the theory of univalent functions. *Monatsh. Math.* **1975**, *80*, 257–264. [CrossRef]
4. Sukhjit, S.; Sushma, G. On a problem of H. S. Almiri and M. O. Reade. *Demonstr. Math.* **2005**, *38*, 303–311.
5. Duren, P.L. *Univalent Functions*; Springer: New York, NY, USA, 1983.
6. Lewin, M. On a coefficient problem for bi-univalent functions. *Proc. Am. Math. Soc.* **1967**, *18*, 63–68. [CrossRef]
7. Tan, D.L. Coefficient estimates for bi-univalent functions. *Chin. Ann. Math. Ser. A* **1984**, *5*, 559–568.
8. Brannan, D.A.; Taha, T.S. On some classes of bi-univalent functions. *Math. Anal. Appl.* **1985**, *3*, 18–21.
9. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses analytic and bi-univalent functions. *Appl. Math. Lett.* **2010**, *23*, 1188–1192. [CrossRef]
10. Srivastava, H.M.; Gaboury, S.; Ghanim, F. Coefficients estimate for some general subclasses of analytic and bi-univalent functions. *Afr. Mat.* **2017**, *28*, 693–706. [CrossRef]
11. Frasin, B.A.; Aouf, M.K. New subclass of bi-univalent functons. *Appl. Math. Lett.* **2011**, *24*, 1569–1573. [CrossRef]
12. Deniz, E. Certain subclasses of bi-univalent functions satisfying subordinate conditions. *J. Class. Anal.* **2013**, *2*, 49–60. [CrossRef]
13. Tang, H.; Deng, G.; Li, S. Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions. *J. Ineq. Appl.* **2013**, *317*, 10. [CrossRef]
14. Frasin, B.A. Coefficient bounds for certain classes of bi-univalent functions. *Hacet. J. Math. Stat.* **2014**, *43*, 383–389. [CrossRef]
15. Srivastava, H.M.; Sivasubramanian, S.; Sivakumar, R. Initial coefficients estimate for some subclasses of m-fold symmetric bi-univalent functions. *Tbilisi Math J.* **2014**, *7*, 1–10. [CrossRef]
16. Srivastava, H.M.; Gaboury, S.; Ghanim, F. Initial coefficients estimate for some subclasses of m-fold symmetric bi-univalent functions. *Acta Math. Sci. Ser. B*. **2016**, *36*, 863–971. [CrossRef]
17. Srivastava, H.M.; Zireh, A.; Hajiparvaneh, S. Coefficients estimate for some subclasses of m-fold symmetric bi-univalent functions. *Filomat* **2018**, *32*, 3143–3153. [CrossRef]
18. Srivastava, H.M.; Wanas, A.K. Initial Maclaurin coefficients bounds for new subclasses of m-fold symmetric bi-univalent functions defined by a linear combination. *Kyungpook Math. J.* **2019**, *59*, 493–503.
19. Tang, H.; Srivastava, H.M.; Sivasubramanian, S.; Gurusamy, P. Fekete-Szegö functional problems of m-fold symmetric bi-univalent functions. *J. Math. Ineq.* **2016**, *10*, 1063–1092. [CrossRef]
20. Wanas, A.K.; Páll-Szabó, A.O. Coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions. *Stud. Univ. Babes-Bolai Math.* **2021**, *66*, 659–666. [CrossRef]
21. Fekete, M.; Szegö, G. Eine Bemerkung Über Ungerade Schlichte Funktionen. *J. Lond. Math. Soc.* **1933**, *89*, 85–89. [CrossRef]
22. Swamy, S.R.; Frasin, B.A.; Aldawish, I. Fekete-Szegö functional problem for a special family of m-fold symmetric bi-univalent functions. *Mathematics* **2022**, *10*, 1165. [CrossRef]
23. Swamy S.R.; Cotîrlă, L.-I. On τ-pseudo-ν-convex κ-fold symmetric bi-univalent function family. *Symmetry* **2022**, *14*, 1972. [CrossRef]
24. Aldawish, I.; Swamy, S.R.; Frasin, B.A. A special family of m-fold symmetric bi-univalent functions satisfying subordination condition. *Fractal Fract.* **2022**, *6*, 271. [CrossRef]
25. Breaz, D.; Cotîrlă, L.-I. The study of the new classes of m-fold symmetric bi-univalent Functions. *Mathematics* **2022**, *10*, 75. [CrossRef]
26. Oros, G.I.; Cotîrlă, L.-I. Coefficient Estimates and the Fekete–Szegö problem for new classes of m-fold symmetric bi-univalent functions. *Mathematics* **2022**, *10*, 129. [CrossRef]
27. Shehab, N.H.; Juma, A.R.S. Coefficient bounds of m-fold symmetric bi-univalent functions for certain subclasses. *Int. J. Nonlinear Anal. Appl.* **2021**, *12*, 71–82. [CrossRef]
28. Sabir, P.O.; Srivastava, H.M.; Atshan, W.G.; Mohammed, P.O.; Chorfi, N.; Miguel, V.-C.A family of holomorphic and m-fold symmetric bi-univalent functions endowed with coefficient estimate problems. *Mathematics* **2023**, *11*, 3970. [CrossRef]
29. Pommerenke, C. *Univalent Functions*; Vandenhoeck and Ruprecht: Gottingen, Germany, 1975.

30. Peng, Z.-G.; Han, Q.-Q. On the coefficients of several classes of bi-univalent functions. *Acta. Math. Sci. Ser. B Engl. Ed.* **2014**, 34, 228–240. [CrossRef]
31. Páll-Szabó, Á.O.; Oros, G.I. Coefficient related studies for new classes of bi-univalent functions. *Mathematics* **2020**, *8*, 1110. [CrossRef]
32. Srivastava, H.M.; Mostafa, A.O.; Aouf, M.K.; Zayed, H.M. Basic and fractional q-calculus and associated Fekete-Szegö problem for p-valently q-starlike functions and p-valently q-convex functions of complex order. *Miskolc Math. Notes* **2019**, *20*, 489–509. [CrossRef]
33. Khan, B.; Srivastava, H.M.; Tahir, M.; Darus, M.; Ahmad, Q.Z.; Khan, N. Applications of a certain q-integral operator to he subclasses of analytic and bi-uivalent functions. *AIMS Math.* **2020**, *6*, 1024–1039. [CrossRef]
34. Murugusundaramoorthy, G.; Cotîrlă, L.-I. Bi-univalent functions of complex order defined by Hohlov operator associated with legendrae polynomial. *AIMS Math.* **2022**, *7*, 8733–8750. [CrossRef]