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Analysis of the Zagreb Indices over the Weakly Zero-Divisor Graph of the Ring $\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s$

Nadeem ur Rehman [1,*,†], Amal S. Alali [2,†], Shabir Ahmad Mir [1,†] and Mohd Nazim [3,†]

1 Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India; gl9941@myamu.ac.in or mirshabir967@gmail.com
2 Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; asalali@pnu.edu.sa
3 School of Basic and Applied Sciences, Faculty of Science and Technology, JSPM University, Pune 412207, India; mnazim1882@gmail.com or mohd.nazim@jspmuni.ac.in
* Correspondence: nu.rehman.mm@amu.ac.in
† These authors contributed equally to this work.

Abstract: Let R be a commutative ring with identity, and $Z(R)$ be the set of zero-divisors of R. The weakly zero-divisor graph of R denoted by $W\Gamma(R)$ is an undirected (simple) graph with vertex set $Z(R)^*$, and two distinct vertices x and y are adjacent, if and only if there exist $r \in ann(x)$ and $s \in ann(y)$, such that $rs = 0$. Importantly, it is worth noting that $W\Gamma(R)$ contains the zero-divisor graph $\Gamma(R)$ as a subgraph. It is known that graph theory applications play crucial roles in different areas one of which is chemical graph theory that deals with the applications of graph theory to solve molecular problems. Analyzing Zagreb indices in chemical graph theory provides numerical descriptors for molecular structures, aiding in property prediction and drug design. These indices find applications in QSAR modeling and chemical informatics, contributing to efficient compound screening and optimization. They are essential tools for advancing pharmaceutical and material science research. This research article focuses on the basic properties of the weakly zero-divisor graph of the ring $\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s$, denoted by $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$, where p, t, and s are prime numbers that may not necessarily be distinct and greater than 2. Moreover, this study includes the examination of various indices and coindices such as the first and second Zagreb indices and coindices, as well as the first and second multiplicative Zagreb indices and coindices of $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$.

Keywords: weakly zero-divisor graph; topological indices; commutative rings; graph parameters

MSC: 05C10; 05C12; 05C25

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1. Introduction

The exploration of graphs linked with algebraic structures has gained significant momentum as a rapidly evolving field. A key focus lies in the classification of graphs corresponding to algebraic structures and the reciprocal endeavor of attributing algebraic properties to these graphs. Researchers are particularly intrigued by unraveling the intricate interplay between the inherent algebraic characteristics of well-known entities and the graph-theoretic attributes exhibited by analogous graphs. A specific instance of this is the zero-divisor graph associated with a commutative ring R, which bears a unity element denoted by 1 (distinct from 0). This graph, denoted as $\Gamma(R)$, is constructed by representing the elements of the ring R as its vertices. Notably, the connectivity between these vertices is established based on a distinct rule: two vertices, namely u and v, are deemed adjacent within the graph solely when the product of these elements, uv, equals zero. This conceptual framework creates an intriguing bridge between the algebraic nature of the ring R and the graph-theoretic structure of $\Gamma(R)$, exemplifying the broader trend of intertwining algebra and graph theory. This concept was originally introduced by Beck [1], subsequently modified by Anderson and Livingston [2] to only consider the nonzero zero-divisors of R as vertices. This graph-theoretical concept provides a way to study the properties of commutative rings and their zero-divisors, with potential applications in diverse fields such as algebraic geometry, algebraic topology, and coding theory. The zero-divisor graph of R have been extensively studied by various researchers, as seen in several literature contributions cited in references [3–11]. Several research studies have explored the characteristics of zero-divisor graphs over finite commutative rings. Sharma et al. [12] focused on the adjacency matrices of these graphs for the ring $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. The contributions of Akgüneş et al. [13], Aykaç and Akgüneş (2020), and Akgüneş and Nacaroğlu (2018) have investigated the graph-theoretical properties of zero-divisor graphs. These graphs are constructed from different rings, such as $\mathbb{Z}_p \times \mathbb{Z}_q$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$, and $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, where p, q, and r are primes. These studies have analyzed various fundamental features of these zero-divisor graphs, including their graph-theoretical properties, topological indices, and other structural characteristics. These investigations provide insights into the behavior and properties of these graphs and their relationship to the underlying algebraic structures. Mohammad et al. [14] proposed a novel concept known as the weakly zero-divisor graph, denoted by $W\Gamma(R)$. This graph is defined based on the nonzero zero-divisors of a ring R, where each vertex of the graph corresponds to a nonzero zero-divisor of R. The graph is constructed such that two vertices u and v are adjacent, if and only if there exist elements $r \in ann(u)^*$ and $s \in ann(v)^*$, where $ann(u)$ represents the set of elements in R that annihilate u. The graph $\Gamma(R)$ is a subgraph of $W\Gamma(R)$ that covers the entire graph. It is noteworthy that this concept provides a new way to study the properties of rings and the relationships between their zero-divisors.

Chemical graph theory is an active area of research in recent times. In this field, researchers study the structures of chemical compounds with the aid of graph theory and mathematics. The most important concept in chemical graph theory is topological indices. Topological indices are numeric values that are associated with graphs that remain invariant under graph automorphisms. In fact, topological indices in graph theory are numerical measures that capture the structural and connectivity properties of graphs. They have a wide range of applications in the field of chemistry, biology, and medicine. In particular, they help in the study of certain physical characteristics of chemical compounds such as the boiling point, ash point, density, stability, and many more. They also serve as powerful tools to execute biological network analysis, and also to determine the physical features and the chemical reactions associated with various drugs without having to carry out the actual experiment, for instance, see [15,16].

Taking motivation from the above cited work, in this research article, we examine the characteristics of the weakly zero-divisor graph of the ring $\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s$, using p, t, and s as prime numbers that are not necessarily distinct and greater than 2. We focus on analyzing its degree sequence, irregularity index, and maximum and minimum degree. Additionally, we explore various topological indices, including the first and second Zagreb indices, first and second Zagreb coindices, first and second multiplicative Zagreb indices, and first and second multiplicative Zagreb coindices index of $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$. Before delving into this, it is essential to revisit some fundamental graph theory definitions. By establishing a solid foundation in graph theory, we can then seamlessly delve into our investigation, leveraging these innovative graph lenses to unveil previously hidden structural characteristics within finite commutative rings.

2. Preliminaries

Let $G = (V, E)$ denote a simple graph with $V(G)$ as the vertex set and $E(G)$ as the edge set. $|V(G)|$ and $|E(G)|$ are referred to be the order and size of the graph, respectively. P_n and C_n denotes the path and cycle on n vertices. K_n and $K_{m,n}$ depicts the complete graph and complete bipartite graph respectively. If uv is an edge of G, then u and v are said to be adjacent in G, and we write $u \sim v$; otherwise $u \nsim v$. Given a graph $G(V, E)$ with $v \in V(G)$, then $deg(v)$ denotes the *degree* of vertex v in G and is defined as the number of vertices adjacent to it. The designations for the lowest and highest degrees are $\delta(G)$ and $\Delta(G)$, respectively. The distance between two vertices $u, v \in V(G)$, denoted by $d(u,v)$, is the number of edges on the shortest path between them in G. The degree sequence of an undirected graph denoted by $DS(G)$ is the list of degrees of all the vertices of the graph. Usually, we list the degrees in non-increasing order, that is from the largest degree to the smallest degree. The degree sequence is a graph invariant; so, isomorphic graphs have the same degree sequence. Also, the irregularity index $t(G)$ of a graph represents the number of different values present in the degree sequence of the graph. In other words, it counts how many unique degrees are assigned to the vertices of the graph. For any undefined notations and terminology, we refer the reader to [17].

Zagreb indices are topological indices that provide information about the degree-based structural characteristics of a graph. Some of the Zagreb indices are defined as follows:

Definition 1 ([18])**.** *The Zagreb group indices of a graph G denoted by $M_1(G)$ (first Zagreb index) and $M_2(G)$ (second Zagreb index) is defined as:*

$$M_1(G) = \sum_{u \in V(G)} d^2(u)$$

and

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v),$$

where $d(u)$ and $d(v)$ stand for the degrees of the distinct vertices u and v, respectively.

Definition 2 ([19])**.** *The Zagreb group of coindices of a graph G denoted by $\overline{M_1}(G)$ (first Zagreb coindex) and $\overline{M_2}(G)$ (second Zagreb coindex) is defined as:*

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} [d(u) + d(v)]$$

and

$$\overline{M_2}(G) = \sum_{uv \notin E(G)} [d(u)d(v)],$$

where $d(u)$ and $d(v)$ stand for the degrees of the distinct vertices u and u, respectively.

Definition 3 ([20])**.** *The Multiplicative Zagreb group of indices of a graph G denoted by $\prod_1(G)$ (first multiplicative Zagreb index) and $\prod_2(G)$ (second multiplicative Zagreb index) is defined as:*

$$\prod_1(G) = \prod_{u \in V(G)} d(u)^2$$

and

$$\prod_2(G) = \prod_{uv \in E(G)} [d(u)d(v)],$$

where $d(u)$ and $d(v)$ stand for the degree of the distinct vertices u and v, respectively.

Definition 4 ([21])**.** *The Multiplicative Zagreb group of coindices of a graph G denoted by $\overline{\prod}_1(G)$ (first multiplicative Zagreb coindex) and $\overline{\prod}_2(G)$ (second multiplicative Zagreb coindex) is defined as:*

$$\overline{\prod}_1(G) = \prod_{uv \notin E(G)} [d(u) + d(v)]$$

and

$$\overline{\prod}_2(G) = \prod_{uv \notin E(G)} [d(u)d(v)],$$

where $d(u)$ and $d(v)$ denote the degree of the distinct vertices u and v_1, respectively.

This research article aims to investigate various properties and some topological indices related to this graph. In the Section 1, the article discusses the fundamental properties of the weakly zero-divisor graph of the ring $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$, including the adjacency relations between vertices, the degree of each vertex, the degree sequence, the maximum and minimum degrees, and the irregularity index. In the Section 2, the research focuses on Zagreb indices and coindices within the context of $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$. Specifically, it explores the first and second Zagreb indices and coindices, as well as the first and second multiplicative Zagreb indices and coindices. The Section 5 contains the conclusions. Overall, the article delves into both the structural properties of the graph and its algebraic characteristics, providing a comprehensive understanding of this specific ring and its associated weakly zero-divisor graph.

3. Properties of $\mathbf{W\Gamma}(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$

This section is dedicated to outlining several foundational properties of $\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$, starting with the definition of adjacent and nonadjacent vertices. Before delving into this, it is necessary to state a lemma that is used in our subsequent discussion.

Lemma 1. *If u and v are adjacent in $\Gamma(R)$, for distinct $u, v \in Z(R)^*$, then u and v are also adjacent in* $\mathrm{W}\Gamma(R)$.

Proof. Suppose that u and v are adjacent in $\Gamma(R)$, for distinct vertices $u, v \in Z(R)^*$. Thus, $uv = 0$, and clearly $u \in ann(v)$ and $v \in ann(u)$. Hence, u and v are also adjacent in $\mathrm{W}\Gamma(R)$. □

Definition 5. *The adjacent and nonadjacent vertices of* $\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ *are as follows:*

- (i) $(x_i, 0, 0) \sim (0, y_j, 0)$, *where* $0 \neq x_i \in \mathbb{Z}_p$ *and* $0 \neq y_j \in \mathbb{Z}_t$,
- (ii) $(x_i, 0, 0) \sim (0, 0, d_k)$, *where* $0 \neq x_i \in \mathbb{Z}_p$ *and* $0 \neq d_k \in \mathbb{Z}_s$,
- (iii) $(x_i, 0, 0) \sim (0, y_j, d_k)$, *where* $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$, *and* $0 \neq d_k \in \mathbb{Z}_s$,
- (iv) $(x_i, 0, 0) \sim (x_i', 0, 0)$, *where* $0 \neq x_i, x_i' \in \mathbb{Z}_p$ *and* $x_i \neq x_i'$,
- (v) $(x_i, 0, 0) \sim (x_i, y_j, 0)$, *where* $0 \neq x_i \in \mathbb{Z}_p$ *and* $0 \neq y_j \in \mathbb{Z}_t$,
- (vi) $(x_i, 0, 0) \sim (x_i, 0, d_k)$, *where* $0 \neq x_i \in \mathbb{Z}_p$ *and* $0 \neq d_k \in \mathbb{Z}_s$,
- (vii) $(0, y_j, 0) \sim (0, 0, d_k)$, *where* $0 \neq y_j \in \mathbb{Z}_t$ *and* $0 \neq d_k \in \mathbb{Z}_s$,
- $(viii)$ $(0, y_j, 0) \sim (x_i, 0, d_k)$, *where* $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$, *and* $0 \neq d_k \in \mathbb{Z}_s$,
- (ix) $(0, y_j, 0) \sim (0, y_j, d_k)$, *where* $0 \neq y_j \in \mathbb{Z}_t$ *and* $0 \neq d_k \in \mathbb{Z}_s$,
- (x) $(0, y_j, 0) \sim (x_i, y_j, 0)$, *where* $0 \neq x_i \in \mathbb{Z}_p$ *and* $0 \neq y_j \in \mathbb{Z}_t$,
- (xi) $(0, y_j, 0) \sim (0, y_j', 0)$, *where* $0 \neq y_j, y_j' \in \mathbb{Z}_t$ *and* $y_j \neq y_j'$,
- (xii) $(0, 0, d_k) \sim (x_i, y_j, 0)$, *where* $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$, *and* $0 \neq d_k \in \mathbb{Z}_s$,
- $(xiii)$ $(0, 0, d_k) \sim (x_i, 0, d_k)$, *where* $0 \neq x_i \in \mathbb{Z}_p$ *and* $0 \neq d_k \in \mathbb{Z}_s$,
- (xiv) $(0, 0, d_k) \sim (0, y_j, d_k)$, *where* $0 \neq y_j \in \mathbb{Z}_t$ *and* $0 \neq d_k \in \mathbb{Z}_s$,
- (xv) $(0, 0, d_k) \sim (0, 0, d_k')$, *where* $0 \neq d_k, d_k' \in \mathbb{Z}_s$ *and* $d_k \neq d_k'$,
- (xvi) $(x_i, 0, d_k) \sim (x_i, y_j, 0)$, *where* $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$, *and* $0 \neq d_k \in \mathbb{Z}_s$,
- $(xvii)$ $(x_i, 0, d_k) \sim (0, y_j, d_k)$, *where* $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$, *and* $0 \neq d_k \in \mathbb{Z}_s$,
- $(xviii)$ $(x_i, y_j, 0) \sim (0, y_j, d_k)$, *where* $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$, *and* $0 \neq d_k \in \mathbb{Z}_s$,
- (xix) $(x_i, y_j, 0) \nsim (x_i', y_j', 0)$, *where* $x_i \neq x_i'$ *or* $y_j \neq y_j'$, *with* $0 \neq x_i, x_i' \in \mathbb{Z}_p$ *and* $0 \neq y_j, y_j' \in \mathbb{Z}_t$,
- (xx) $(x_i, 0, d_k) \nsim (x_i', 0, d_k')$, *where* $x_i \neq x_i'$ *or* $d_k \neq d_k'$, *with* $0 \neq x_i, x_i' \in \mathbb{Z}_p$ *and* $0 \neq d_k, d_k' \in \mathbb{Z}_s$,
- (xxi) $(0, y_j, d_k) \nsim (0, y_j', d_k')$, *where* $y_j \neq y_j'$ *or* $d_k \neq d_k'$, *with* $0 \neq y_j, y_j' \in \mathbb{Z}_t$ *and* $0 \neq d_k$, $d_k' \in \mathbb{Z}_s$.

The following result shows the calculation of the degree for each vertex in the graph $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$. This means that the number of edges connected to each possible vertex in the graph has been determined.

Theorem 1. *The graph* $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ *exhibits vertex degrees that are given by*
(1) $deg(x_i, 0, 0) = pt + ts + sp - p - t - s - 1$, *where* $0 \neq x_i \in \mathbb{Z}_p$,
(2) $deg(0, y_j, 0) = pt + ts + sp - p - t - s - 1$, *where* $0 \neq y_j \in \mathbb{Z}_t$,
(3) $deg(0, 0, d_k) = pt + ts + sp - p - t - s - 1$, *where* $0 \neq d_k \in \mathbb{Z}_s$,
(4) $deg(x_i, y_j, 0) = ps + ts - s - 1$, *where* $0 \neq x_i \in \mathbb{Z}_p$ *and* $0 \neq y_j \in \mathbb{Z}_t$,
(5) $deg(0, y_j, d_k) = ps + pt - p - 1$, *where* $0 \neq y_j \in \mathbb{Z}_t$ *and* $0 \neq d_k \in \mathbb{Z}_s$,
(6) $deg(x_i, 0, d_k) = pt + ts - t - 1$, *where* $0 \neq x_i \in \mathbb{Z}_p$ *and* $0 \neq d_k \in \mathbb{Z}_s$.

Proof. (1) We have $(x_i, 0, 0) \cdot (0, y_j, d_k) = (0, 0, 0)$ for $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$ and $0 \neq d_k \in \mathbb{Z}_s$, which implies $(x_i, 0, 0) \sim (0, y_j, d_k)$ in $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$. So, by Lemma 1, $(x_i, 0, 0) \sim (0, y_j, d_k)$ in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$. Similarly, for $0 \neq x_i \in \mathbb{Z}_p$ and $0 \neq y_j \in \mathbb{Z}_t$, $(x_i, 0, 0) \sim (0, y_j, 0)$ in $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ as $(x_i, 0, 0) \cdot (0, y_j, 0) = (0, 0, 0)$. Thus, by Lemma 1, $(x_i, 0, 0) \sim (0, y_j, 0)$ in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$. Also, $(x_i, 0, 0) \sim (0, 0, d_k)$ as $(x_i, 0, 0) \cdot (0, 0, d_k) = (0, 0, 0)$ for $0 \neq x_i \in \mathbb{Z}_p$ and $0 \neq d_k \in \mathbb{Z}_s$; then, by Lemma 1, $(x_i, 0, 0) \sim (0, 0, d_k)$ in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$. Further, $(x_i, 0, 0) \sim (x_i, 0, d_k)$ because there exists $x = (0, 0, 1) \in ann(x_i, 0, 0)^*$ and $y = (0, 1, 0) \in ann(x_i, 0, d_k)^*$, such that $xy = 0$. Moreover, $(x_i, 0, 0) \sim (x_i, y_j, 0)$, as there exist $x = (0, 1, 0) \in ann(x_i, 0, 0)^*$ and $y = (0, 0, 1) \in ann(x_i, y_j, 0)^*$, such that $xy = 0$. Again, $(x_i, 0, 0) \sim (x_i{}', 0, 0)$, $0 \neq x_i, x_i^{'} \in \mathbb{Z}_p$, and $x_i \neq x_i^{'}$, since there exist $x = (0, 1, 0) \in ann(x_i, 0, 0)^*$ and $y = (0, 0, 1) \in ann(x_i^{'}, 0, 0)^*$, such that $xy = 0$. Hence, the degree of the vertex $(x_i, 0, 0)$ becomes

$$\begin{aligned} deg(x_i, 0, 0) = &\ (t-1)(s-1) + (t-1) + (s-1) + (p-1)(s-1) + (p-1)(t-1) \\ &+ (p-2) \\ = &\ pt + ts + sp - p - t - s - 1. \end{aligned}$$

(2) $(0, y_j, 0) \sim (x_i, 0, 0)$ in $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ as $(0, y_j, 0) \cdot (x_i, 0, 0) = (0, 0, 0)$ for $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$. Therefore, they are adjacent in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ by Lemma 1. Similarly, for $0 \neq y_j \in \mathbb{Z}_t$ and $0 \neq d_k \in \mathbb{Z}_s$, $(0, y_j, 0) \sim (0, 0, d_k)$, since $(0, y_j, 0) \cdot (0, 0, d_k) = (0, 0, 0)$. Hence, by Lemma 1, $(0, y_j, 0) \sim (0, 0, d_k)$ in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$. Also, $(0, y_j, 0) \sim (x_i, 0, d_k)$, as $(0, y_j, 0) \cdot (x_i, 0, d_k) = (0, 0, 0)$ for $0 \neq x_i \in \mathbb{Z}_p$ and $0 \neq y_j \in \mathbb{Z}_t$, $0 \neq d_k \in \mathbb{Z}_s$. Therefore, $(0, y_j, 0) \sim (x_i, 0, d_k)$ in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ by Lemma 1. Further, $(0, y_j, 0) \sim (0, y_j, d_k)$, because there exist $x = (0, 0, 1) \in ann(0, y_j, 0)^*$ and $y = (1, 0, 0) \in ann(0, y_j, d_k)^*$, such that $xy = 0$. Moreover, $(0, y_j, 0) \sim (x_i, y_j, 0)$, since there exist $x = (1, 0, 0) \in ann(0, y_j, 0)^*$ and $y = (0, 0, 1) \in ann(x_i, y_j, 0)^*$, such that $xy = 0$. Finally, $(0, y_j, 0) \sim (0, y_j^{'}, 0)$, because there exist $x = (1, 0, 0) \in ann(0, y_j, 0)^*$ and $y = (0, 0, 1) \in ann(0, y_j{}', 0)^*$, such that $xy = 0$, for $0 \neq y_j, y_j^{'} \in \mathbb{Z}_t$ and $y_j \neq y_j^{'}$. Hence, the degree of the vertex $(0, y_j, 0)$ becomes

$$\begin{aligned} deg(0, y_j, 0) = &\ (p-1) + (s-1) + (p-1)(s-1) + (t-1)(s-1) + (p-1)(t-1) \\ &+ (t-2) \\ = &\ pt + ts + sp - p - t - s - 1. \end{aligned}$$

(3) We have $(0, 0, d_k) \sim (x_i, 0, 0)$, because $(0, 0, d_k) \cdot (x_i, 0, 0) = (0, 0, 0)$ for $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq d_k \in \mathbb{Z}_s$. Therefore, they are adjacent in both $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ and $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ by Lemma 1. Similarly, $(0, 0, d_k) \cdot (0, y_j, 0) = (0, 0, 0)$ for $0 \neq y_j \in \mathbb{Z}_t$ and $0 \neq d_k \in \mathbb{Z}_s$; so, $(0, 0, d_k) \sim (0, y_j, 0)$ in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$, by Lemma 1. Also, $(0, 0, d_k) \sim (x_i, y_j, 0)$, for $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$, and $0 \neq d_k \in \mathbb{Z}_s$, because there exist $x = (1, 0, 0) \in ann(0, 0, d_k)^*$ and $y = (0, 1, 0) \in ann(x_i, y_j, 0)^*$, such that $xy = 0$. Further, for $0 \neq x_i \in \mathbb{Z}_p$ and $0 \neq d_k \in \mathbb{Z}_s$, $(0, 0, d_k) \sim (x_i, 0, d_k)$, since there exist $x = (1, 0, 0) \in ann(0, 0, d_k)^*$ and

$y = (0,1,0) \in ann(x_i, 0, d_k)^*$, such that $xy = 0$. Similarly $(0,0,d_k) \sim (0, y_j, d_k)$ as there exists $x = (0,1,0) \in ann(0,0,d_k)^*$ and $y = (1,0,0) \in ann(0, y_j, d_k)^*$ such that $xy = 0$. Finally, $(0,0,d_k) \sim (0,0,d_k^{'})$ for $0 \neq d_k, d_k^{'} \in \mathbb{Z}_s$ and $d_k \neq d_k^{'}$, because there exist $x = (0,1,0) \in ann(0,0,d_k)^*$ and $y = (1,0,0) \in ann(0,0,d_k^{'})^*$, such that $xy = 0$. This implies that the degree of the vertex $(0,0,d_k)$ becomes

$$\begin{aligned} deg(0,0,d_k) = & \ (p-1) + (t-1) + (p-1)(t-1) + (p-1)(s-1) + (t-1)(s-1) \\ & + (s-2) \\ = & \ pt + ts + sp - p - t - s - 1. \end{aligned}$$

(4) For $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$, and $0 \neq d_k \in \mathbb{Z}_s$, we have $(x_i, y_j, 0) \sim (0,0,d_k)$ as $(x_i, y_j, 0) \cdot (0,0,d_k) = (0,0,0)$. Therefore, by Lemma 1, $(x_i, y_j, 0) \sim (0,0,d_k)$ in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$. Further, $(x_i, y_j, 0) \sim (x_i, 0, 0)$, because there exist $x = (0,0,1) \in ann(x_i, y_j, 0)^*$ and $y = (0,1,0) \in ann(x_i, 0, 0)^*$, such that $xy = 0$. Similarly, $(x_i, y_j, 0) \sim (0, y_j, 0)$, since there exist $x = (0,0,1) \in ann(x_i, y_j, 0)^*$ and $y = (1,0,0) \in ann(0, y_j, 0)^*$, such that $xy = 0$. Also, $(x_i, y_j, 0) \sim (x_i, 0, d_k)$ in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$, because there exist $x = (0,0,1) \in ann(x_i, y_j, 0)^*$ and $y = (0,1,0) \in ann(x_i, 0, d_k)^*$, such that $xy = 0$. Finally, $(x_i, y_j, 0) \sim (0, y_j, d_k)$, since there exist $x = (0,0,1) \in ann(x_i, y_j, 0)^*$ and $y = (1,0,0) \in ann(0, y_j, d_k)^*$, such that $xy = 0$. Hence, the degree of the vertex $(x_i, y_j, 0)$ becomes

$$\begin{aligned} deg(x_i, y_j, 0) = & \ (s-1) + (p-1) + (t-1) + (p-1)(s-1) + (t-1)(s-1) \\ = & \ ps + ts - s - 1. \end{aligned}$$

(5) For $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$, and $0 \neq d_k \in \mathbb{Z}_s$, we have $(0, y_j, d_k) \sim (x_i, 0, 0)$ as $(0, y_j, d_k) \cdot (x_i, 0, 0) = (0,0,0)$. Therefore, by Lemma 1, adjacency is also followed in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$. Also, $(0, y_j, d_k) \sim (0, y_j, 0)$, because there exist $x = (1,0,0) \in ann(0, y_j, d_k)^*$ and $y = (0,0,1) \in ann(0, y_j, 0)^*$, such that $xy = 0$. Similarly, $(0, y_j, d_k) \sim (0,0,d_k)$, since there exist $x = (1,0,0) \in ann(0, y_j, d_k)^*$ and $y = (0,1,0) \in ann(0,0,d_k)^*$, such that $xy = 0$. Further, $(0, y_j, d_k) \sim (x_i, 0, d_k)$ in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$, since there exist $x = (1,0,0) \in ann(0, y_j, d_k)^*$ and $y = (0,1,0) \in ann(x_i, 0, d_k)^*$, such that $xy = 0$. Finally, $(0, y_j, d_k) \sim (x_i, y_j, 0)$ in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$, because there exist $x = (1,0,0) \in ann(0, y_j, d_k)^*$ and $y = (0,0,1) \in ann(0, y_j, d_k)^*$, such that $xy = 0$. Hence, the degree of the vertex $(0, y_j, d_k)$ becomes

$$\begin{aligned} deg(0, y_j, d_k) = & \ (p-1) + (t-1) + (s-1) + (p-1)(s-1) + (p-1)(t-1) \\ = & \ ps + pt - p - 1. \end{aligned}$$

(6) For $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$, and $0 \neq d_k \in \mathbb{Z}_s$, we have $(x_i, 0, d_k) \sim (x_i, 0, 0)$ in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$, since there exist $x = (0,1,0) \in ann(x_i, 0, d_k)^*$ and $y = (0,0,1) \in ann(x_i, 0, 0)^*$, such that $xy = 0$. Also, $(x_i, 0, d_k) \sim (0, y_j, 0)$ as $(x_i, 0, d_k) \cdot (0, y_j, 0) = (0,0,0)$. So, by using Lemma 1, $(x_i, 0, d_k) \sim (0, y_j, 0)$ in $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$. Further, $(x_i, 0, d_k) \sim (0,0,d_k)$, because there exist $x = (0,1,0) \in ann(x_i, 0, d_k)^*$ and $y = (1,0,0) \in ann(0,0,d_k)^*$, such that $xy = 0$. Moreover, $(x_i, 0, d_k) \sim (x_i, y_j, 0)$, since there exist $x = (0,1,0) \in ann(x_i, 0, d_k)^*$ and $y = (0,0,1) \in ann(x_i, y_j, 0)^*$, such that $xy = 0$. Finally, $(x_i, 0, d_k) \sim (0, y_j, d_k)$, as there exist $x = (0,1,0) \in ann(x_i, 0, d_k)^*$ and $y = (0,0,1) \in ann(0, y_j, d_k)^*$, such that $xy = 0$. Hence, the degree of the vertex $(x_i, 0, d_k)$ becomes

$$\begin{aligned} deg(x_i, 0, d_k) = & \ (p-1) + (t-1) + (s-1) + (p-1)(t-1) + (t-1)(s-1) \\ = & \ pt + ts - t - 1. \end{aligned}$$

□

Theorem 2. *The maximum degree of the graph* $\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ *is given by*

$$\Delta(\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = pt + ts + sp - p - t - s - 1.$$

Proof. In view of Theorem 1, we have $deg(x_i, 0, 0) = deg(0, y_j, 0) = deg(0, 0, d_k) = pt + ts + sp - p - t - s - 1$. Also, $deg(x_i, y_j, 0) = ps + ts - s - 1$, $deg(0, y_j, d_k) = ps + pt - p - 1$, and $deg(x_i, 0, d_k) = pt + ts - t - 1$, where $0 \neq a \in \mathbb{Z}_p$, $0 \neq b \in \mathbb{Z}_t$ and $0 \neq c \in \mathbb{Z}_s$. Hence, the maximum degree of the graph $\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ is

$$\Delta(\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = pt + ts + sp - p - t - s - 1.$$

□

Example 1. *The maximum degree of the graph* $\mathrm{W}\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ *is given by* $\Delta(\mathrm{W}\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 3 \times 3 + 3 \times 3 + 3 \times 3 - 3 - 3 - 3 - 1 = 17$. *The verification is also shown in Example 3.*

Theorem 3. *The minimum degree of the graph* $\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ *is given by*

$$\delta(\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = min\{ps + ts - s - 1, ps + pt - p - 1, pt + ts - t - 1\}.$$

Proof. By employing Theorem 1, we obtain $deg(x_i, y_j, 0) = ps + ts - s - 1$, $deg(0, y_j, d_k) = ps + pt - p - 1$, and $deg(x_i, 0, d_k) = pt + ts - t - 1$, where $0 \neq x_i \in \mathbb{Z}_p$, $0 \neq y_j \in \mathbb{Z}_t$ and $0 \neq d_k \in \mathbb{Z}_s$. Hence, the minimum degree of the graph $\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ is

$$\delta(\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = min\{ps + ts - s - 1, ps + pt - p - 1, pt + ts - t - 1\}.$$

□

Example 2. *The minimum degree of the graph* $\mathrm{W}\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ *is given by* $\delta(\mathrm{W}\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = min\{3 \times 3 + 3 \times 3 - 3 - 1, 3 \times 3 + 3 \times 3 - 3 - 1, 3 \times 3 + 3 \times 3 - 3 - 1\} = 14$. *The result is also verified by Example 3.*

Theorem 4. *The degree sequence and irregularity index for* $\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$, *where* p, t, *and* s *are distinct prime numbers greater than* 2, *is given by*

$$\begin{aligned} DS(\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = \Big\{ & \underbrace{pt + ts + ps - s - p - t - 1,}_{(p-1) times} \ \underbrace{pt + ts + ps - s - p - t - 1,}_{(t-1) times} \\ & \underbrace{pt + ts + ps - s - p - t - 1,}_{(s-1) times} \ \underbrace{ps + ts - s - 1,}_{(p-1)(t-1) times} \\ & \underbrace{ps + pt - p - 1,}_{(t-1)(s-1) times} \ \underbrace{pt + ts - t - 1}_{(p-1)(s-1) times} \Big\}, \end{aligned}$$

and

$$t(\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = 4.$$

Proof. From Theorem 1, we see that $deg(x_i, 0, 0) = pt + ts + ps - p - t - s - 1$, and the vertices of type $(x_i, 0, 0)$ are $p - 1$, where $0 \neq x_i \in \mathbb{Z}_p$. Also, $deg(0, y_j, 0) = pt + ts + ps - p - t - s - 1$, and the number of these types of vertices are $t - 1$, for $0 \neq y_j \in \mathbb{Z}_t$. Moreover, the vertices of the form $(0, 0, d_k)$ are $s - 1$ having degree $pt + ts + ps - p - t - s - 1$, where $0 \neq d_k \in \mathbb{Z}_s$.

Again, by Theorem 1, for $0 \neq x_i \in \mathbb{Z}_p$ and $0 \neq y_j \in \mathbb{Z}_t$, $deg(x_i, y_j, 0) = ps + ts - s - 1$, and the number of these type of vertices is $(p - 1)(t - 1)$. Similarly, for $0 \neq y_j \in \mathbb{Z}_t$ and $0 \neq d_k \in \mathbb{Z}_s$, $deg(0, y_j, d_k) = ps + pt - p - 1$, and the number of vertices of the form $(0, y_j, d_k)$ is $(t-1)(s-1)$. Moreover, the number of vertices of the type $(x_i, 0, d_k)$ is $(p-1)(s-1)$ and $deg(x_i, 0, d_k) = pt + ts - t - 1$, where $0 \neq x_i \in \mathbb{Z}_p$ and $0 \neq d_k \in \mathbb{Z}_s$; hence, we obtain the desired result. Also, it is clear that the irregularity index of the graph $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ is 4. $\square$

Remark 1. *$t(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = 2$ when the prime numbers $p, t,$ and s are equal.*

The following example shows the degree sequence, maximum and minimum degree, and irregularity index of $W\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$, which is helpful for the verification of our results.

Example 3. $W\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ *is shown in Figure 1, and some properties of* $W\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ *are as follows:*

(1) $\Delta(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 17$,
(2) $\delta(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 14$,
(3) $DS(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = \{17, 17, 17, 17, 17, 17, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14\}$,
(4) $t(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 2$.

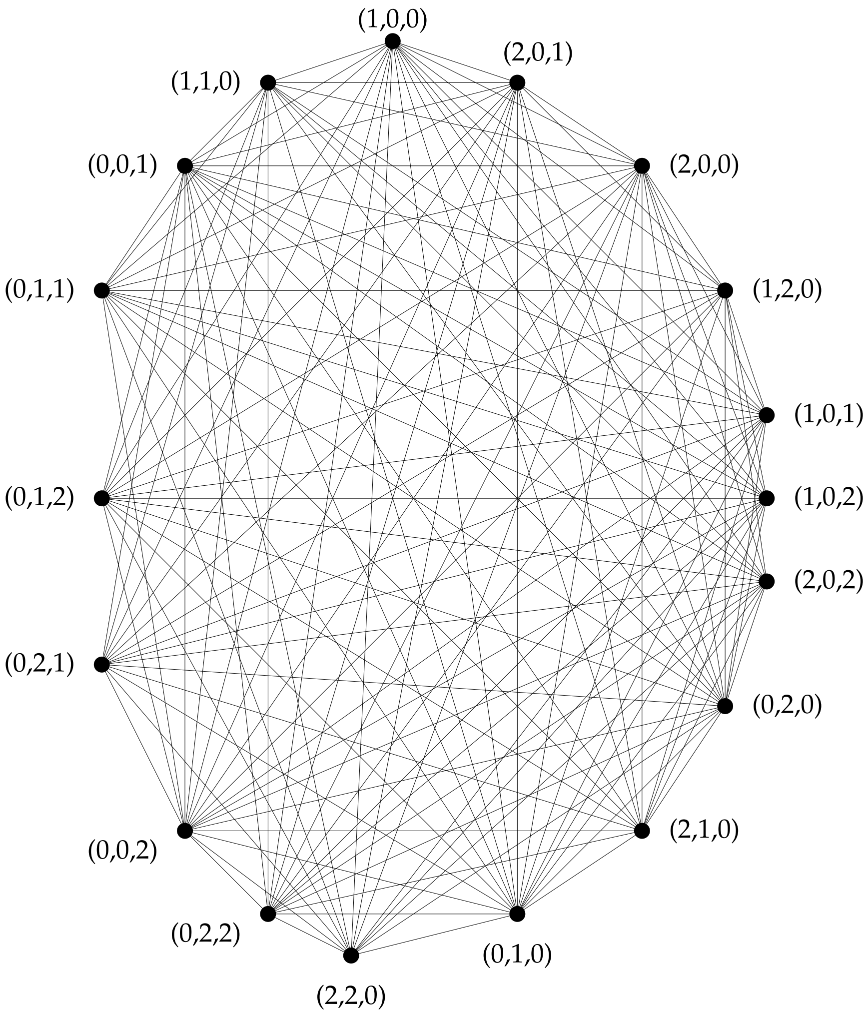

Figure 1. $W\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$.

4. Some Topological Indices of $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$

In the subsequent section, we delve into distinct attributes of the graph $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$, known as topological indices. The numerical value $Top(G)$, referred to as the topological index of a graph G, remains constant for all graphs H that exhibit isomorphism with G. In essence, the topological index serves as an unchanging attribute of a graph, maintaining its consistency across isomorphic instances. We discuss some of the specific topological indices associated with the graph $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$.

Theorem 5. *The first Zagreb index of the graph* $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ *is given by*

$$\begin{aligned} M_1(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = & \sum_{\substack{i,j,k \in \{p,t,s\} \\ i \neq j \neq k}} (i-1)(k-1)(ij+jk-j-1)^2 \\ & + (pt+ts+ps-p-t-s-1)^2(p+t+s-3). \end{aligned}$$

Proof. From Theorem 4 and Definition 1, we have

$$\begin{aligned} M_1(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = & \sum_{u \in V(G)} d^2(u) \\ = & \ (p-1)(pt+ts+ps-p-t-s-1)^2 \\ & + (t-1)(pt+ts+ps-p-t-s-1)^2 \\ & + (s-1)(pt+ts+ps-p-t-s-1)^2 \\ & + (p-1)(t-1)(ps+ts-s-1)^2 \\ & + (t-1)(s-1)(ps+pt-p-1)^2 \\ & + (s-1)(p-1)(pt+ts-t-1)^2 \\ = & \sum_{\substack{i,j,k \in \{p,t,s\} \\ i \neq j \neq k}} (i-1)(k-1)(ij+jk-j-1)^2 \\ & + (pt+ts+sp-p-t-s-1)^2(p+t+s-3). \end{aligned}$$

□

Theorem 6. *The second Zagreb index of the graph* $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ *is given by*

$$M_2(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = A_1^3 - 2A_1^2 + A_0A_1(A_2 + A_4 + A_5) + A_2A_3 + A_4A_5,$$

where

$$A_0 = p+t+s-3, A_1 = pt+ts+sp-p-t-s-1,$$
$$A_2 = (ps+pt-p-1)(t-1)(s-1),$$
$$A_3 = 2p^2ts - p^2t - t^2s - s^2t - sp^2 + pts^2 + pt^2s - 6pts - 3pt - 4ts - ps - 2p - 2,$$
$$A_4 = (ps+ts-s-1)(p-1)(t-1), A_5 = (pt+ts-t-1)(p-1)(s-1).$$

Proof. By the use of Definitions 1 and 5, we have

$$\begin{aligned} M_2(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = & \sum_{(i) \in E(G)} d(x_i,0,0)d(0,y_j,0) \\ & + \sum_{(ii) \in E(G)} d(x_i,0,0)d(0,0,d_k) \\ & + \sum_{(iii) \in E(G)} d(x_i,0,0)d(0,y_j,d_k) \\ & + \sum_{(v) \in E(G)} d(x_i,0,0)d(x_i,y_j,0) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{(vi)\in E(G)} d(x_i,0,0)d(x_i,0,d_k)\\
&+\sum_{(iv)\in E(G),\ x_i\neq x_i^{'}} d(x_i,0,0)d(x_i^{'},0,0)\\
&+\sum_{(vii)\in E(G)} d(0,y_j,0)d(0,0,d_k)\\
&+\sum_{(x)\in E(G)} d(0,y_j,0)d(x_i,y_j,0)\\
&+\sum_{(ix)\in E(G)} d(0,y_j,0)d(0,y_j,d_k)\\
&+\sum_{(xiii)\in E(G)} d(0,y_j,0)d(x_i,0,d_k)\\
&+\sum_{(xi)\in E(G),\ y_j\neq y_j^{'}} d(0,y_j,0)d(0,y_j^{'},0)\\
&+\sum_{(xiii)\in E(G)} d(0,0,d_k)d(x_i,0,d_k)\\
&+\sum_{(xiv)\in E(G)} d(0,0,d_k)d(0,y_j,d_k)\\
&+\sum_{(xii)\in E(G)} d(0,0,d_k)d(x_i,y_j,0)\\
&+\sum_{(xv)\in E(G),\ d_k\neq d_k^{'}} d(0,0,d_k)d(0,0,d_k^{'})\\
&+\sum_{(xvi)\in E(G)} d(x_i,0,d_k)d(x_i,y_j,0)\\
&+\sum_{(xvii)\in E(G)} d(x_i,0,d_k)d(0,y_j,d_k)\\
&+\sum_{(xviii)\in E(G)} d(x_i,y_j,0)d(0,y_j,d_k).
\end{aligned}
$$

By applying Theorem 1, we obtain

$$
\begin{aligned}
M_2(\mathrm{W}\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s)) =\ &(pt+ts+sp-p-t-s-1)(p-1)\\
&(pt+ts+sp-p-t-s-1)(t-1)\\
&+(pt+ts+sp-p-t-s-1)(p-1)\\
&(pt+ts+sp-p-t-s-1)(s-1)\\
&+(pt+ts+sp-p-t-s-1)(p-1)\\
&(ps+pt-p-1)(t-1)(s-1)\\
&+(pt+ts+sp-p-t-s-1)(p-1)\\
&(ps+ts-s-1)(p-1)(t-1)\\
&+(pt+ts+sp-p-t-s-1)(p-1)\\
&(pt+ts-t-1)(p-1)(s-1)\\
&+(pt+ts+sp-p-t-s-1)1\\
&(pt+ts+sp-p-t-s-1)(p-2)\\
&+(pt+ts+sp-p-t-s-1)(t-1)\\
&(pt+ts+sp-p-t-s-1)(s-1)\\
&+(pt+ts+sp-p-t-s-1)(t-1)\\
&(ps+ts-s-1)(p-1)(t-1)
\end{aligned}
$$

$$
\begin{aligned}
&+(pt+ts+sp-p-t-s-1)(t-1)\\
&(ps+pt-p-1)(t-1)(s-1)\\
&+(pt+ts+sp-p-t-s-1)(t-1)\\
&(pt+ts-t-1)(p-1)(s-1)\\
&+(pt+ts+sp-p-t-s-1)1\\
&(pt+ts+sp-p-t-s-1)(t-2)\\
&+(pt+ts+sp-p-t-s-1)(s-1)\\
&(pt+ts-t-1)(p-1)(s-1)\\
&+(pt+ts+sp-p-t-s-1)(s-1)\\
&(ps+pt-p-1)(t-1)(s-1)\\
&+(pt+ts+sp-p-t-s-1)(s-1)\\
&(ps+ts-s-1)(p-1)(t-1)\\
&+(pt+ts+sp-p-t-s-1)1\\
&(pt+ts+sp-p-t-s-1)(s-2)\\
&+(pt+ts-t-1)(p-1)(s-1)(ps+ts-s-1)(p-1)(t-1)\\
&+(pt+ts-t-1)(p-1)(s-1)(ps+pt-p-1)(t-1)(s-1)\\
&+(ps+ts-s-1)(p-1)(t-1)(ps+pt-p-1)(t-1)(s-1).
\end{aligned}
$$

On solving and taking

$$
\begin{gathered}
A_0=p+t+s-3, A_1=pt+ts+sp-p-t-s-1,\\
A_2=(ps+pt-p-1)(t-1)(s-1),\\
A_3=2p^2ts-p^2t-t^2s-s^2t-sp^2+pts^2+pt^2s-6pts-3pt-4ts-ps-2p-2,\\
A_4=(ps+ts-s-1)(p-1)(t-1), A_5=(pt+ts-t-1)(p-1)(s-1),
\end{gathered}
$$

we obtain

$$
M_2(\mathrm{W}\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s))=A_1^3-2A_1^2+A_0A_1(A_2+A_4+A_5)+A_2A_3+A_4A_5.
$$

□

Theorem 7. *The first Zagreb coindex of the graph* $\mathrm{W}\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s)$ *is given as*

$$
\overline{M_1}(\mathrm{W}\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s))=2(P+Q+R),
$$

where $P=(p-1)(t-1)(ps+ts-s-1)$, $Q=(t-1)(s-1)(ps+pt-p-1)$, *and* $R=(s-1)(p-1)(pt+ts-t-1)$.

Proof. From Definition 5, we have, $(x_i,y_j,0)\nsim(x_i',y_j',0)$, $(0,y_j,d_k)\nsim(0,y_j',d_k')$, and $(x_i,0,d_k)\nsim(x_i',0,d_k')$. Therefore, using Definition 2, we obtain

$$
\begin{aligned}
\overline{M}_1(\mathrm{W}\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s))=&\sum_{(x_i,y_j,0)\nsim(x_i',y_j',0),\ x_i\neq x_i'\ \ or\ \ y_j\neq y_j'}[d(x_i,y_j,0)+d(x_i',y_j',0)]\\
&+\sum_{(0,y_j,d_k)\nsim(0,y_j',d_k'),\ y_j\neq y_j'\ \ or\ \ d_k\neq d_k'}[d(0,y_j,d_k)+d(0,y_j',d_k')]\\
&+\sum_{(x_i,0,d_k)\nsim(x_i',0,d_k'),\ x_i\neq x_i'\ \ or\ \ d_k\neq d_k'}[d(x_i,0,d_k)+d(x_i{}',0,d_k{}')].
\end{aligned}
$$

Using Theorem 4, we have

$$\begin{aligned}\overline{M}_1(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = {} & (ps+ts-s-1)(p-1)(t-1)\\ & + (ps+ts-s-1)(p-1)(t-1)\\ & + (ps+pt-p-1)(t-1)(s-1)\\ & + (ps+pt-p-1)(t-1)(s-1)\\ & + (pt+ts-t-1)(p-1)(s-1)\\ & + (pt+ts-t-1)(p-1)(s-1).\end{aligned}$$

On taking $P = (p-1)(t-1)(ps+ts-s-1)$, $Q = (t-1)(s-1)(ps+pt-p-1)$, $R = (s-1)(p-1)(pt+ts-t-1)$, we obtain

$$\overline{M}_1(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = 2(P+Q+R).$$

This completes the proof. □

Theorem 8. *The second Zagreb coindex of the graph* $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ *is given by*

$$\overline{M_2}(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = (A^2+B^2+C^2),$$

where $A = (p-1)(t-1)(ps+ts-s-1), B = (t-1)(s-1)(ps+pt-p-1)$, *and* $C = (s-1)(p-1)(pt+ts-t-1)$.

Proof. From Definition 5, we have, $(x_i, y_j, 0) \nsim (x_i', y_j', 0)$, $(0, y_j, d_k) \nsim (0, y_j', d_k')$, and $(x_i, 0, d_k) \nsim (x_i', 0, d_k')$. Therefore, using Definition 2, we have

$$\begin{aligned}\overline{M}_1(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = {} & \sum_{(x_i,y_j,0)\nsim(x_i',y_j',0),\ x_i\neq x_i' \ \ or \ \ y_j\neq y_j',} [d(x_i,y_j,0)\, d(x_i',y_j',0)]\\ & + \sum_{(0,y_j,d_k)\nsim(0,y_j',d_k'),\ y_j\neq y_j' \ \ or \ \ d_k\neq d_k'} [d(0,y_j,d_k)\, d(0,y_j',d_k')]\\ & + \sum_{(x_i,0,d_k)\nsim(x_i',0,d_k'),\ x_i\neq x_i' \ \ or \ \ d_k\neq d_k',} [d(x_i,0,d_k)\, d(x_i',0,d_k')].\end{aligned}$$

Using Theorem 4, we have

$$\begin{aligned}\overline{M}_1(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = {} & (ps+ts-s-1)(p-1)(t-1)\ (ps+ts-s-1)(p-1)(t-1)\\ & + (ps+pt-p-1)(t-1)(s-1)\ (ps+pt-p-1)(t-1)(s-1)\\ & + (pt+ts-t-1)(p-1)(s-1)\ (pt+ts-t-1)(p-1)(s-1).\end{aligned}$$

On taking $A = (p-1)(t-1)(ps+ts-s-1), B = (t-1)(s-1)(ps+pt-p-1)$ and $C = (s-1)(p-1)(pt+ts-t-1)$, we obtain

$$\overline{M_2}(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = (A^2+B^2+C^2).$$

Hence, the proof is complete. □

Theorem 9. *The first multiplicative Zagreb index of the graph* $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ *is given by*

$$\begin{aligned}\prod\nolimits_1(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = {} & (p-1)^3(t-1)^3(s-1)^3(pt+ts+sp-p-t-s-1)^6\\ & (ps+ts-s-1)^2(ps+pt-p-1)^2(pt+ts-t-1)^2.\end{aligned}$$

Proof. From Theorem 4 and Definition 3, we have

$$\begin{aligned}
\prod\nolimits_1(\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) &= \prod_{v \in V(G)} d(v)^2 \\
&= \quad (pt + ts + sp - p - t - s - 1)^2 (p-1) \\
&\quad \times (pt + ts + sp - p - t - s - 1)^2 (t-1) \\
&\quad \times (pt + ts + sp - p - t - s - 1)^2 (s-1) \\
&\quad \times (ps + ts - s - 1)^2 (p-1)(t-1) \\
&\quad \times (ps + pt - p - 1)^2 (t-1)(s-1) \\
&\quad \times (pt + ts - t - 1)^2 (s-1)(p-1) \\
&= (p-1)^3 (t-1)^3 (s-1)^3 (pt + ts + sp - p - t - s - 1)^6 \\
&\quad (ps + ts - s - 1)^2 (ps + pt - p - 1)^2 (pt + ts - t - 1)^2.
\end{aligned}$$

□

Theorem 10. *The second multiplicative Zagreb index of the graph* $\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ *is given as*

$$\begin{aligned}
\prod\nolimits_2(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) &= (p-1)^{15}(t-1)^{15}(s-1)^{15}(p-2)(t-2)(s-2) \\
&\quad (pt + ts + sp - p - t - s - 1)^{15}(ps + pt - p - 1)^5 \\
&\quad (ps + ts - s - 1)^5 (pt + ts - t - 1)^5.
\end{aligned}$$

Proof. From Definition 3, we have

$$\begin{aligned}
\prod\nolimits_2(\mathrm{W}\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = &\prod_{(i) \in E(G)} d(x_i, 0, 0) d(0, y_j, 0) \\
&\times \prod_{(ii) \in E(G)} d(x_i, 0, 0) d(0, 0, d_k) \\
&\times \prod_{(iii) \in E(G)} d(x_i, 0, 0) d(0, y_j, d_k) \\
&\times \prod_{(v) \in E(G)} d(x_i, 0, 0) d(x_i, y_j, 0) \\
&\times \prod_{(vi) \in E(G)} d(x_i, 0, 0) d(x_i, 0, d_k) \\
&\times \prod_{(iv) \in E(G),\, x_i \neq x_i'} d(x_i, 0, 0) d(x_i', 0, 0) \\
&\times \prod_{(vii) \in E(G)} d(0, y_j, 0) d(0, 0, d_k) \\
&\times \prod_{(x) \in E(G)} d(0, y_j, 0) d(x_i, y_j, 0) \\
&\times \prod_{(ix) \in E(G)} d(0, y_j, 0) d(0, y_j, d_k) \\
&\times \prod_{(viii) \in E(G)} d(0, y_j, 0) d(x_i, 0, d_k) \\
&\times \prod_{(xi) \in E(G),\, y_j \neq y_j'} d(0, y_j, 0) d(0, y_j', 0)
\end{aligned}$$

$$\begin{aligned}
&\times \prod_{(xiii)\in E(G)} d(0,0,d_k)d(x_i,0,d_k) \\
&\times \prod_{(xiv)\in E(G)} d(0,0,d_k)d(0,y_j,d_k) \\
&\times \prod_{(xii)\in E(G)} d(0,0,d_k)d(x_i,y_j,0) \\
&\times \prod_{(xv)\in E(G),\ d_k\neq d_k'} d(0,0,d_k)d(0,0,d_k') \\
&\times \prod_{(xvi)\in E(G)} d(x_i,0,d_k)d(x_i,y_j,0) \\
&\times \prod_{(xvii)\in E(G)} d(x_i,0,d_k)d(0,y_j,d_k) \\
&\times \prod_{(xviii)\in E(G)} d(x_i,y_j,0)d(0,y_j,d_k).
\end{aligned}$$

By applying Theorem 1, we obtain

$$\begin{aligned}
\prod\nolimits_2(\mathrm{W}\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s)) = {} & (pt+ts+ps-p-t-s-1)(p-1) \\
& (pt+ts+ps-p-t-s-1)(t-1) \\
& \times (pt+ts+ps-p-t-s-1)(p-1) \\
& (pt+ts+ps-p-t-s-1)(s-1) \\
& \times (pt+ts+ps-p-t-s-1)(p-1) \\
& (ps+pt-p-1)(t-1)(s-1) \\
& \times (pt+ts+ps-p-t-s-1)(p-1) \\
& (ps+ts-s-1)(p-1)(t-1) \\
& \times (pt+ts+ps-p-t-s-1)(p-1) \\
& (pt+ts-t-1)(p-1)(s-1) \\
& \times (pt+ts+ps-p-t-s-1)1 \\
& (pt+ts+ps-p-t-s-1)(p-2) \\
& \times (pt+ts+ps-p-t-s-1)(t-1) \\
& (pt+ts+ps-p-t-s-1)(s-1) \\
& \times (pt+ts+ps-p-t-s-1)(t-1) \\
& (ps+ts-s-1)(p-1)(t-1) \\
& \times (pt+ts+ps-p-t-s-1)(t-1) \\
& (ps+pt-p-1)(t-1)(s-1) \\
& \times (pt+ts+ps-p-t-s-1)(t-1) \\
& (pt+ts-t-1)(p-1)(s-1) \\
& \times (pt+ts+ps-p-t-s-1)1 \\
& (pt+ts+ps-p-t-s-1)(t-2) \\
& \times (pt+ts+ps-p-t-s-1)(s-1) \\
& (pt+ts-t-1)(p-1)(s-1) \\
& \times (pt+ts+ps-p-t-s-1)(s-1) \\
& (ps+pt-p-1)(t-1)(s-1) \\
& \times (pt+ts+ps-p-t-s-1)(s-1)
\end{aligned}$$

$$
\begin{aligned}
&(ps+ts-s-1)(p-1)(t-1)\\
&\times(pt+ts+ps-p-t-s-1)1\\
&(pt+ts+ps-p-t-s-1)(s-2)\\
&\times(pt+ts-t-1)(p-1)(s-1)(ps+ts-s-1)(p-1)(t-1)\\
&\times(pt+ts-t-1)(p-1)(s-1)(ps+pt-p-1)(t-1)(s-1)\\
&\times(ps+ts-s-1)(p-1)(t-1)(ps+pt-p-1)(t-1)(s-1).
\end{aligned}
$$

On solving, we obtain

$$
\begin{aligned}
\prod\nolimits_2(W\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s)) = {} & (p-1)^{15}(t-1)^{15}(s-1)^{15}(p-2)(t-2)(s-2)\\
&(pt+ts+ps-p-t-s-1)^{15}(ps+pt-p-1)^5\\
&(ps+ts-s-1)^5(pt+ts-t-1)^5.
\end{aligned}
$$

□

Theorem 11. *The first multiplicative Zagreb coindices index of the graph* $W\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s)$ *is given by*

$$\overline{\prod}_1(W\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s)) = 2(P\times Q\times R),$$

where $P=(ps+ts-s-1)(p-1)(t-1)$, $Q=(ps+pt-p-1)(t-1)(s-1)$, *and* $R=(pt+ts-t-1)(p-1)(s-1)$.

Proof. From Definition 5, clearly the nonadjacent vertices are $(x_i,y_j,0)\nsim(x_i^{'},y_j^{'},0)$, $(0,y_j,d_k)\nsim(0,y_j^{'},d_k^{'})$, and $(x_i,0,d_k)\nsim(x_i^{'},0,d_k{'})$. Therefore, using Definition 4, we have

$$
\begin{aligned}
\overline{\prod}_1(W\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s)) = {} & \prod_{(x_i,y_j,0)\nsim(x_i^{'},y_j^{'},0),\ x_i\neq x_i^{'}\ \ or\ \ y_j\neq y_j^{'}}[d(x_i,y_j,0)+d(x_i^{'},y_j^{'},0)]\\
&\times\prod_{(0,y_j,d_k)\nsim(0,y_j^{'},d_k^{'}),\ y_j\neq y_j^{'}\ \ or\ \ d_k\neq d_k^{'}}[d(0,y_j,d_k)+d(0,y_j^{'},d_k^{'})]\\
&\times\prod_{(x_i,0,d_k)\nsim(x_i^{'},0,d_k{'}),\ x_i\neq x_i^{'}\ \ or\ \ d_k\neq d_k^{'}}[d(x_i,0,d_k)+d(x_i^{'},0,d_k{'})].
\end{aligned}
$$

Now using Theorem 4, we obtain

$$
\begin{aligned}
\overline{\prod}_1(W\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s)) = {} & [(ps+ts-s-1)(p-1)(t-1)+(ps+ts-s-1)(p-1)(t-1)]\\
&\times[(ps+pt-p-1)(t-1)(s-1)+(ps+pt-p-1)(t-1)(s-1)]\\
&\times[(pt+ts-t-1)(p-1)(s-1)+(pt+ts-t-1)(p-1)(s-1)].
\end{aligned}
$$

On taking $P=(ps+ts-s-1)(p-1)(t-1)$, $Q=(ps+pt-p-1)(t-1)(s-1)$, and $R=(pt+ts-t-1)(p-1)(s-1)$, we obtain

$$\overline{\prod}_1(W\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s)) = 2(P\times Q\times R).$$

□

Theorem 12. *The second multiplicative Zagreb coindices index of the graph* $W\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s)$ *is given by*

$$
\begin{aligned}
\overline{\prod}_2(W\Gamma(\mathbb{Z}_p\times\mathbb{Z}_t\times\mathbb{Z}_s)) \quad = \quad & (p-1)^4(t-1)^4(s-1)^4(ps+ts-s-1)^2(ps+pt-p-1)^2\\
&(pt+ts-t-1)^2.
\end{aligned}
$$

Proof. From Definition 5, it is clear that the nonadjacent vertices of $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$ are $(x_i, y_j, 0) \nsim (x_i^{'}, y_j^{'}, 0)$, $(0, y_j, d_k) \nsim (0, y_j^{'}, d_k^{'})$, and $(x_i, 0, d_k) \nsim (x_i^{'}, 0, d_k{'})$. Therefore, using Definition 4, we have

$$\begin{aligned}\overline{\prod}_2(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = & \prod_{(x_i,y_j,0)\nsim(x_i^{'},y_j^{'},0),\ x_i \neq x_i^{'}\ \ or\ \ y_j \neq y_j^{'}} [d(x_i, y_j, 0)\ d(x_i^{'}, y_j^{'}, 0)] \\ & \times \prod_{(0,y_j,d_k)\nsim(0,y_j^{'},d_k^{'}),\ y_j \neq y_j^{'}, d_k \neq d_k{'}} [d(0, y_j, d_k)\ d(0, y_j^{'}, d_k^{'})] \\ & \times \prod_{(x_i,0,d_k)\nsim(x_i^{'},0,d_k{'}),\ x_i \neq x_i^{'}\ \ or\ \ d_k \neq d_k^{'}} [d(x_i, 0, d_k)\ d(x_i^{'}, 0, d_k{'})].\end{aligned}$$

Now using Theorem 4, we obtain

$$\begin{aligned}\overline{\prod}_2(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = & [(ps + ts - s - 1)(p-1)(t-1) \times (ps + ts - s - 1)(p-1)(t-1)] \\ & \times[(ps + pt - p - 1)(t-1)(s-1) \times (ps + pt - p - 1)(t-1)(s-1)] \\ & \times[(pt + ts - t - 1)(p-1)(s-1) \times (pt + ts - t - 1)(p-1)(s-1)].\end{aligned}$$

On solving, we obtain

$$\begin{aligned}\overline{\prod}_2(W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)) = & (p-1)^4(t-1)^4(s-1)^4(ps + ts - s - 1)^2 \\ & (ps + pt - p - 1)^2(pt + ts - t - 1)^2.\end{aligned}$$

Hence, the proof is complete. □

Remark 2. *From Example 3, the graph structures like* $W\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$*, are similar to an important class of carbon compounds in chemistry called fullerenes. Fullerenes are the compounds in which carbon atoms are arranged in a cage-like or spherical structure. The study of fullerenes is closely related to mathematical concepts such as topology and geometry. The unique structure of fullerenes with carbon atoms forming polygons on the surface of a sphere is of great interest for mathematicians. We can analyze some mathematical properties such as topological indices of these complicated compounds by these formulae, and our results are more general than the previous studies.*

5. Conclusions

The study has delved into the examination of various graph indices and coindices within this unique mathematical structure, including the first and second Zagreb indices and coindices, as well as the first and second multiplicative Zagreb indices and coindices. This research is significant for several reasons: it provides valuable insights into the structure and properties of the weakly zero-divisor graph, extending our understanding of this mathematical concept in the context of ring theory. The study reinforces the connections between ring theory and graph theory, highlighting the usefulness of graph theory in representing and analyzing algebraic structures. By examining Zagreb indices and coindices, this research offers a quantitative approach to characterizing the topology of $W\Gamma(\mathbb{Z}_p \times \mathbb{Z}_t \times \mathbb{Z}_s)$. Such numerical descriptors have practical applications in various fields, including chemistry and molecular research. The work discussed here can potentially find applications in chemical graph theory, where graph theory concepts are employed to solve molecular problems. The study of Zagreb indices can aid in predicting molecular properties, contributing to drug design, compound screening, and material science research. The approach taken in this research is versatile, as it can be applied to a wide range of ring structures, offering potential extensions and applications in diverse mathematical and scientific contexts.

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