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Existence of Multiple Weak Solutions to a Discrete Fractional Boundary Value Problem

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Abstract: The existence of at least three weak solutions to a discrete fractional boundary value problem containing a p-Laplacian operator and subject to perturbations is proved using variational methods. Some applications of the main results are presented. The results obtained generalize some recent results on both discrete fractional boundary value problems and p-Laplacian boundary value problems. Examples illustrating the results are given.

Keywords: three solutions; fractional discrete; p-Laplacian; variational methods

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1. Introduction

The aim of this paper is to establish the existence of at least three weak solutions to the fractional discrete boundary value problem

$$\begin{cases} {}_{T+1}\nabla_k^{\alpha}({}_k\nabla_0^{\alpha}(u(k))) + {}_k\nabla_0^{\alpha}({}_{T+1}\nabla_k^{\alpha}(u(k))) + \varphi_p(u(k)) \\ \qquad\qquad = \lambda f(k,u(k)) + \mu g(k,u(k)), \\ u(0) = u(T+1) = 0, \end{cases} \tag{1}$$

for any $k \in [1,T]_{\mathbb{N}_0}$, where $0 < \alpha < 1$, $\lambda > 0$ and $\mu \geq 0$ are parameters, ${}_k\nabla_0^{\alpha}$is the left nabla discrete fractional difference, and ${}_{T+1}\nabla_k^{\alpha}$ is the right nabla discrete fractionl difference. Here, f, g : $[1,T]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, φ_p is the p-Laplacian operator defined as $\varphi_p(s) = |s|^{p-2}s$ with $1 < p < \infty$.

Fractional differential equations have become an area of great interest in recent years. This is due to both the intensive development of the theory of fractional calculus itself as well as the applications of such problems in various scientific and social scientific fields; see, for example, Refs. [1–6] and the references therein.

A considerable number of boundary value-type problems and problems involving numerical simulations can be formulated as special cases of nonlinear algebraic systems. For this reason, in recent years, many authors have developed various methods and techniques, such as fixed points theorems or upper and lower solutions methods, to study discrete problems. In this paper, we are interested in investigating nonlinear discrete boundary value problems by using a variational approach; for recent contributions, see [7–10] and the references therein.

Nonlinear boundary value problems involving p-Laplacian operators occur in various physical phenomena including non-Newtonian fluids, reaction-diffusion models, petroleum extraction, flows through porous media, etc. Thus, the study of such problems and their generalizations have attracted research mathematicians in recent years (e.g., [11–13]).

While p-Laplacian boundary value problems for ordinary differential equations, finite difference equations, and dynamic equations on time scales have been studied extensively, there are relatively few results on discrete fractional p-Laplacian boundary value problems involving Caputo fractional differences. For example, Lv [14] used Schaefer's fixed point theorem to obtain the existence of solutions to a discrete fractional boundary value problem with a p-Laplacian operator. Heidarkhani and Moradi [15] used variational methods to obtain the existence of at least one solution to the problem (1) in the case where $\mu = 0$. Heidarkhani, Moradi, and Afrouzi [16] applied variational methods to obtain the existence of infinitely many solutions to (1) again in the case where $\mu = 0$.

Motivated by the above observations, in the present paper we use the critical point theorems obtained in [17,18] to obtain two results that ensure the existence of at least three weak solutions to the problem (1). In particular, in Theorem 4 we require that the primitive F of the function f is p-sublinear at infinity and satisfies some other local growth conditions. In Theorem 5, we require a sign condition on the function f and a growth condition on F in a bounded interval, but no asymptotic condition on f at infinity; we obtain that for every non-negative continuous function g, there exist at least three non-negative weak solutions that are uniformly bounded. We then apply our theorems to some special cases and illustrate our results with examples. Compared to previously known results in the literature, our required conditions are new.

In Section 2, we recall some basic definitions and the main tools to be used in the proofs. Section 3 is devoted to our main results and their applications.

2. Materials and Methods: Preliminary Notions

Our main tools are the two following three critical points theorems. In the first one, the coercivity of a certain functional is required, and in the second one, a suitable sign condition is needed.

Theorem 1 ([18] Theorem 3.6)**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact and such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{v} \in X$, with $r < \Phi(\overline{v})$, such that:*

(a_1) $\dfrac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$;

(a_2) *For each $\lambda \in \Lambda_r := \left(\dfrac{\Phi(\overline{v})}{\Psi(\overline{v})}, \dfrac{r}{\sup_{\Phi(u)\le r} \Psi(u)} \right)$, the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2 ([19] Theorem 2.2)**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive, and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact and such that:*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
2. *For each $\lambda > 0$ and all $u_1, u_2 \in X$ that are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \ge 0$ and $\Psi(u_2) \ge 0$, we have*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \ge 0.$$

Assume that there are two positive constants r_1 and r_2 and $\overline{v} \in X$ with $2r_1 < \Phi(\overline{v}) < \dfrac{r_2}{2}$ such that:

(b_1) $\dfrac{\sup_{u\in\Phi^{-1}((-\infty,r_1))} \Psi(u)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$;

(b_2) $\dfrac{\sup_{u\in\Phi^{-1}((-\infty,r_2))}\Psi(u)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$.

Then, for each

$$\lambda \in \left(\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}((-\infty,r_1))}\Psi(u)}, \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}((-\infty,r_2))}\Psi(u)}\right\}\right),$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in $\Phi^{-1}((-\infty, r_2))$.

Theorems 1 and 2 have been successfully used to ensure the existence of at least three solutions for perturbed boundary value problems in the papers [19,20]. Next, we will introduce several basic definitions, notations, and lemmas to be used in this paper.

Definition 1 ([21])**.**

(i) *Let m be a natural number; then the m rising factorial of t (t to the m rising) is defined as*

$$t^{\overline{m}} = \prod_{k=0}^{m-1}(t+k), \quad m \in \mathbb{N}, \quad where \quad t^{\overline{0}} = 1.$$

(ii) *For any real number α, the α rising function (t to the α rising) i s defined as*

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}$$

where $t \in \mathbb{R}\setminus\{\dots,-2,-1,0\}$ and $0^{\overline{\alpha}} = 0$.

In what follows, for $a, b \in N$, we will use the notation:

$$\mathbb{N}_a = \{a, a+1, a+2, \dots\} \quad \text{and} \quad {}_b\mathbb{N} = \{\dots, b-2, b-1, b\}.$$

Definition 2. *Let $\alpha \in (0,1)$ and f be defined on $\mathbb{N}_{a-1} \cap {}_{b+1}\mathbb{N}$ with $a < b$. Then the left nabla discrete Caputo fractional difference is defined by*

$$({}_k^C\nabla_{a-1}^{\alpha} f)(k) = \frac{1}{\Gamma(1-\alpha)}\sum_{s=a}^{k}\nabla_s f(s)(k-\rho(s))^{\overline{-\alpha}}, \; k \in \mathbb{N}_a, \tag{2}$$

and the right nabla discrete Caputo fractional difference by

$$({}_{b+1}^{C}\nabla_k^{\alpha} f)(k) = \frac{1}{\Gamma(1-\alpha)}\sum_{s=k}^{b}(-\Delta_s f)(s)(s-\rho(k))^{\overline{-\alpha}}, \; k \in {}_b\mathbb{N}, \tag{3}$$

where ρ is the backwards operator $\rho(k) = k-1$.

Definition 3. *Let $\alpha \in (0,1)$ and f be defined on $\mathbb{N}_{a-1} \cap {}_{b+1}\mathbb{N}$ with $a < b$. The left and right nabla discrete Riemann fractional differences are defined by*

$$({}_k^R\nabla_{a-1}^{\alpha} f)(k) = \frac{1}{\Gamma(1-\alpha)}\nabla_k\sum_{s=a}^{k} f(s)(k-\rho(s))^{\overline{-\alpha}} = \frac{1}{\Gamma(-\alpha)}\sum_{s=a}^{k} f(s)(k-\rho(s))^{\overline{-\alpha-1}}, \; k \in \mathbb{N}_a,$$

and

$$({}_{b+1}^{R}\nabla_k^{\alpha} f)(k) = \frac{1}{\Gamma(1-\alpha)}(-\Delta_k)\sum_{s=k}^{b}(f(s))(s-\rho(k))^{\overline{-\alpha}} = \frac{1}{\Gamma(-\alpha)}\sum_{s=k}^{b}(f(s))(s-\rho(k))^{\overline{-\alpha-1}}, \; k \in {}_b\mathbb{N},$$

respectively, where again $\rho(k) = k-1$.

For example, let $f(k) = 1$ be defined on $\mathbb{N}_{a-1} \cap {}_{b+1}\mathbb{N}$; then from (2) and (3), we have

$$ {}^{C}_{b+1}\nabla^{\alpha}_{k} 1 = {}^{C}_{k}\nabla^{\alpha}_{a-1} 1 = 0, \quad k \in \mathbb{N}_a \cap {}_b\mathbb{N}. \tag{4} $$

The relationships between the left and right nabla Caputo and Riemann fractional differences are as follows:

$$ ({}^{C}_{k}\nabla^{\alpha}_{a-1} f)(k) = ({}^{R}_{k}\nabla^{\alpha}_{a-1} f)(k) - \frac{(k-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(a-1), \tag{5} $$

$$ ({}^{C}_{b+1}\nabla^{\alpha}_{k} f)(k) = ({}^{R}_{b+1}\nabla^{\alpha}_{k} f)(k) - \frac{(b+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(b+1). \tag{6} $$

Thus, by (4)–(6), for any $k \in \mathbb{N}_a \cap {}_b\mathbb{N}$, we have

$$ {}^{R}_{b+1}\nabla^{\alpha}_{k} 1 = \frac{(b+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}, \quad {}^{R}_{k}\nabla^{\alpha}_{a-1} 1 = \frac{(k-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}. $$

Concerning the domains of the fractional differences, we see that (i) the left nabla fractional difference ${}_{a-1}\nabla^{\alpha}_{k}$ map functions defined on ${}_{a-1}\mathbb{N}$ to functions defined on ${}_a\mathbb{N}$, and (ii) the right nabla fractional difference ${}_k\nabla^{\alpha}_{b+1}$ maps functions defined on ${}_{b+1}\mathbb{N}$ to functions defined on ${}_b\mathbb{N}$. It can be shown that as $\alpha \to 0$, ${}_a\nabla^{\alpha}_{k}(f(k)) \to f(t)$, and as $\alpha \to 1$, ${}_a\nabla^{\alpha}_{k}(f(k)) \to \nabla f(t)$. We note that for $0 < \alpha < 1$, the nabla Riemann and Caputo fractional differences agree for functions that vanish at the endpoints, that is, if $f(a-1) = 0 = f(b+1)$ (see [22,23]), which is our situation here. For $0 < \alpha < 1$, these follow from (5) and (6). So, for convenience, in the future, we will use the symbol ∇^{α} instead of ${}^{R}\nabla^{\alpha}$ or ${}^{C}\nabla^{\alpha}$.

Next, we present a summation by parts formula for this new discrete fractional calculus.

Theorem 3 ([24] Theorem 4.4 (Integration by parts for fractional differences))**.** *For functions f and g defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$, with $a < b$ and $0 < \alpha < 1$, we have*

$$ \sum_{k=a}^{b} f(k)({}_k\nabla^{\alpha}_{a-1} g)(k) = \sum_{k=a}^{b} g(k)({}_{b+1}\nabla^{\alpha}_{k} f)(k). $$

Similarly,

$$ \sum_{k=a}^{b} f(k)({}_{b+1}\nabla^{\alpha}_{k} g)(k) = \sum_{k=a}^{b} g(k)({}_k\nabla^{\alpha}_{a-1} f)(k). $$

In order to give a variational formulation for the problem (1), we define the finite T-dimensional Banach space

$$ W = \left\{ u : [0, T+1]_{\mathbb{N}_0} \to \mathbb{R} : u(0) = u(T+1) = 0 \right\}, $$

equipped with the norm

$$ \|u\| = \left(\sum_{k=1}^{T} |u(k)|^2 \right)^{\frac{1}{2}}. $$

The next lemma is obvious.

Lemma 1. *For every $0 < \alpha < 1$ and $u \in W$, we have*

$$ \|u\|_{\infty} = \max_{k \in [1,T]_{\mathbb{N}_0}} |u(k)| \le \|u\|. \tag{7} $$

Corresponding to the functions f and g, we define the functions $F, G : [1, T]_{N_0} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x,t) = \int_0^t f(x,\xi)d\xi \text{ and } G(x,t) = \int_0^t g(x,\xi)d\xi$$

for all $(x,t) \in [1,T]_{N_0} \times \mathbb{R}$. For all $u \in W$, consider the functionals Φ, Ψ, and I_λ defined by

$$\Phi(u) = \frac{1}{2}\sum_{k=1}^{T} |({}_k\nabla_0^\alpha u)(k)|^2 + |({}_{T+1}\nabla_k^\alpha u)(k)|^2 + \frac{1}{p}\sum_{k=1}^{T} |u(k)|^p, \tag{8}$$

$$\Psi(u) = \sum_{k=1}^{T} F(k,u(k)) + \frac{\mu}{\lambda}\sum_{k=1}^{T} G(k,u(k)), \tag{9}$$

and $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$.

Definition 4. *By a weak solution to the BVP (1), we mean any function $u \in W$ such that*

$$\sum_{k=1}^{T} ({}_k\nabla_0^\alpha u(k))({}_k\nabla_0^\alpha v(k)) + ({}_{T+1}\nabla_k^\alpha u(k))({}_{T+1}\nabla_k^\alpha v(k)) + \sum_{k=1}^{T} |u(k)|^{p-2}u(k)v(k)$$
$$- \lambda\sum_{k=1}^{T} f(k,u(k))v(k) - \mu\sum_{k=1}^{T} g(k,u(k))v(k) = 0$$

for every $v \in W$.

Our next lemma clarifies the relationship between critical points of the functional I_λ and a weak solution to our problem.

Lemma 2. *A function $u \in W$ is a critical point of I_λ if and only if u is a solution to (1).*

Proof. If $u \in W$ be a critical point of I_λ, then for every $v \in W$, we have

$$\sum_{k=1}^{T} ({}_k\nabla_0^\alpha u(k))({}_k\nabla_0^\alpha v(k)) + ({}_{T+1}\nabla_k^\alpha u(k))({}_{T+1}\nabla_k^\alpha v(k)) + \sum_{k=1}^{T} |u(k)|^{p-2}u(k)v(k)$$
$$- \lambda\sum_{k=1}^{T} f(k,u(k))v(k) - \mu\sum_{k=1}^{T} g(k,u(k))v(k) = 0.$$

Bearing in mind that $v \in W$ is arbitrary, we have that for some $\bar{u} \in W$,

$${}_{T+1}\nabla_k^\alpha({}_k\nabla_0^\alpha(\bar{u}(k))) + {}_k\nabla_0^\alpha({}_{T+1}\nabla_k^\alpha(\bar{u}(k))) + |\bar{u}(k)|^{p-2}\bar{u}(k)$$
$$- \lambda f(k,\bar{u}(k)) - \mu\sum_{k=1}^{T} g(k,\bar{u}(k)) = 0$$

for every $k \in [1,T]_{\mathbb{N}}$. Therefore, $\bar{u}$ is a weak solution to the problem (1). Hence, every critical point of the functional I_λ in W is a weak solution to the problem (1).

On the other hand, if $\bar{u}$ is a weak solution to the problem (1), then arguing in the reverse order, completes the proof. □

The following lemma helps us satisfy an important assumption in Theorems 1 and 2.

Lemma 3. *Let $S : W \to W^*$ be the operator defined by*

$$S(u)(v) = \sum_{k=1}^{T} ({}_k\nabla_0^\alpha u(k))({}_k\nabla_0^\alpha v(k)) + ({}_{T+1}\nabla_k^\alpha u(k))({}_{T+1}\nabla_k^\alpha v(k))$$

$$+\sum_{k=1}^{T} |u(k)|^{p-2} u(k) v(k)$$

for every u, $v \in W$. Then, S admits a continuous inverse on W^*.

Proof. Now

$$S(u)(u) = \sum_{k=1}^{T} \left({}_k\nabla_0^\alpha u(k)\right)^2 + \left({}_{T+1}\nabla_k^\alpha u(k)\right)^2 + \sum_{k=1}^{T} |u(k)|^p \geq (T+1)^{\frac{p(p-2)}{4}} \|u\|^p, \quad (10)$$

so S is coercive. Also,

$$\langle S(u) - S(v), u - v \rangle = \sum_{k=1}^{T} \left({}_k\nabla_0^\alpha (u(k) - v(k))\right)^2 + \left({}_{T+1}\nabla_k^\alpha (u(k) - v(k)\right)^2 \sum_{k=1}^{T} |(u(k) - v(k))|^p$$
$$\geq (T+1)^{\frac{p(p-2)}{4}} \|u - v\|^p > 0 \quad (11)$$

for every u, $v \in W$. Hence, S is strictly increasing. Moreover, since W is reflexive, for $u_n \to u$ strongly in W as $n \to \infty$, we have $S(u_n) \to S(u)$ weakly in W^* as $n \to \infty$. Hence, S is demicontinuous, so by [25] (Theorem 26.A(d)), the inverse operator S^{-1} exists and it is continuous.

Let e_n be a sequence in W^* such that $e_n \to e$ strongly in W^* as $n \to \infty$. Let u_n, $u \in W$ be such that $S^{-1}(e_n) = u_n$ and $S^{-1}(e) = u$. Taking into account the fact that S is coercive, we see that the sequence u_n is bounded in the reflexive space W. For a suitable subsequence, once again called u_n, we have $u_n \to \hat{u}$ weakly for some $\hat{u} \in W$. This implies

$$\langle S(u_n) - S(u), u_n - \hat{u} \rangle = \langle e_n - e, u_n - \hat{u} \rangle = 0.$$

Since $u_n \to \hat{u}$ weakly in W and $S(u_n) \to S(\hat{u})$ strongly in W^*, we have $u_n \to \hat{u}$ strongly in W. Since S is continuous, $S(\hat{u}) = S(u)$. Hence, taking into account that S is an injection, we have $u = \hat{u}$. □

Now set

$$G^\theta := \sum_{k=1}^{T} \max_{|\xi| \leq \theta} G(k, \xi) \quad \text{for all } \theta > 0$$

and

$$G_\sigma := T \inf_{[1,T]_{N_0} \times \mathbb{R}} G(k, \xi) \quad \text{for all } \sigma > 0.$$

If g is sign-changing, then clearly $G^\theta \geq 0$ and $G_\sigma \leq 0$.

3. Results

We are ready to present our main existence results.

Fix two positive constants θ and σ such that

$$\frac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p}}{\sum_{k=1}^{T} F(k, \sigma)} < \frac{(T+1)^{\frac{p(p-2)}{4}} \theta^p}{p \sum_{k=1}^{T} \max_{|x| \leq \theta} F(k, x)},$$

choose

$$\lambda \in \Lambda := \left(\frac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p}}{\sum_{k=1}^{T} F(k, \sigma)}, \frac{(T+1)^{\frac{p(p-2)}{4}} \theta^p}{p \sum_{k=1}^{T} \max_{|x| \leq \theta} F(k, x)} \right),$$

set $\delta_{\lambda,g} =$

$$\min\left\{\frac{(T+1)^{\frac{p(p-2)}{4}}\theta^p-\lambda p\sum_{k=1}^{T}\max_{|x|\le\theta}F(k,x)}{pG^\theta},\left|\frac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2+\frac{T\sigma^p}{p}-\lambda\sum_{k=1}^{T}F(k,\sigma)}{\min\{0,G_\sigma\}}\right|\right\},$$

and

$$\overline{\delta}_{\lambda,g}:=\min\left\{\delta_{\lambda,g},\ \frac{1}{\max\left\{0,\frac{pT}{(T+1)^{\frac{p(p-2)}{4}}}\limsup_{|x|\to+\infty}\frac{\sup_{k\in[1,T]_{N_0}}G(k,x)}{x^p}\right\}}\right\}. \tag{12}$$

Here we mean $\gamma/0=+\infty$, so that, for example, $\overline{\delta}_{\lambda,g}=+\infty$ if

$$\limsup_{|x|\to+\infty}\frac{\sup_{k\in[1,T]_{N_0}}G(k,x)}{x^p}\le 0$$

and $G_\sigma=G^\theta=0$.

Our first existence result is given in the following theorem.

Theorem 4. *Assume that there exist positive constants θ and σ with*

$$\theta<\sqrt[p]{\frac{p}{(T+1)^{\frac{p(p-2)}{4}}}\left(\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2+\frac{T\sigma^p}{p}\right)}, \tag{13}$$

such that

(A_1) $$\frac{\sum_{k=1}^{T}\max_{|x|\le\theta}F(k,x)}{\theta^p}<\frac{(T+1)^{\frac{p(p-2)}{4}}}{\frac{p\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2+T\sigma^p}\sum_{k=1}^{T}F(k,\sigma);$$

(A_2) $T\limsup_{|x|\to\infty}\frac{F(k,x)}{x^p}<\Theta$ *uniformly with respect to $k\in[1,T]_{N_0}$, where*

$$\Theta:=\frac{\sum_{k=1}^{T}\max_{|x|\le\theta}F(k,x)}{\frac{(T+1)^{\frac{p(p-2)}{4}}}{p}\theta^p}.$$

Then, for each $\lambda\in\Lambda$ and for every continuous function $g:[1,T]_{N_0}\times\mathbb{R}\to\mathbb{R}$ satisfying

$$\limsup_{|x|\to+\infty}\frac{\sup_{k\in[1,T]_{N_0}}G(k,x)}{x^p}<+\infty,$$

there exists $\overline{\delta}_{\lambda,g}>0$ given by (12) such that, for each $\mu\in[0,\overline{\delta}_{\lambda,g})$, the problem (1) admits at least three distinct weak solutions in W.

Proof. Fix λ, g, and μ as in the conclusion of the theorem, and consider the functionals Φ, Ψ, and I_λ as given in (8) and (9). We first wish to prove that the functionals Φ and Ψ satisfy the basic conditions in Theorem 1.

Since W is compactly embedded in $(C^0([1,T]_{N_0}),\mathbb{R})$, it is well known that Ψ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in W$ is the functional $\Psi'(u) \in W^*$ given by

$$\Psi'(u)(v) = \sum_{k=1}^{T} f(k,u(k))v(k) + \frac{\mu}{\lambda}\sum_{k=1}^{T} g(k,u(k))v(k)$$

for every $v \in W$, and Ψ is sequentially weakly upper semicontinuous. Moreover, Φ is Gâteaux differentiable with Gâteaux derivative at the point $u \in W$ given by

$$\begin{aligned}\Phi'(u)(v) = &\sum_{k=1}^{T} ({}_k\nabla_0^\alpha u(k))({}_k\nabla_0^\alpha v(k)) + ({}_{T+1}\nabla_k^\alpha u(k))({}_{T+1}\nabla_k^\alpha v(k)) \\ &+ \sum_{k=1}^{T} |u(k)|^{p-2}u(k)v(k)\end{aligned}$$

for every $v \in W$. In addition, from the definition of Φ, we see that it is sequentially weakly lower semicontinuous and strongly continuous. For every $u \in W$, we have (see (10) and (11))

$$\frac{1}{p}(T+1)^{\frac{p(p-2)}{4}}\|u\|^p \le \Phi(u) \le 2T(T+1)\|u\|^2 + \frac{1}{p}(T+1)^{\frac{2-p}{2}}\|u\|^p. \tag{14}$$

Using the first inequality in (14), it follows that $\lim_{\|u\|\to+\infty}\Phi(u) = +\infty$, i.e., Φ is coercive. Lemma 3 shows that Φ' admits a continuous inverse on W^*. Therefore, the regularity assumptions on Φ and Ψ required in Theorem 1 are satisfied. We also note that I_λ is a $C^1(W,\mathbb{R})$ functional and the critical points of I_λ are weak solutions to the problem (1).

Choose

$$r := \frac{(T+1)^{\frac{p(p-2)}{4}}}{p}\theta^p$$

and set

$$w_\sigma(k) = \begin{cases}\sigma, & k \in [1,T]_{N_0},\\ 0, & k \in 0, T+1.\end{cases}$$

Clearly, $w_\sigma \in W$. Since w_σ vanishes at the endpoints, its nabla Riemann and Caputo fractional differences coincide. Hence, for any $k \in \mathbb{N}_1 \cap {}_T\mathbb{N}$, we have

$$({}_{T+1}\nabla_k^\alpha w_\sigma)(k) = ({}_{T+1}^{R}\nabla_k^\alpha w_\sigma)(k) = ({}_{T+1}^{C}\nabla_k^\alpha w_\sigma)(k) = \frac{\sigma(T+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)},$$

and

$$({}_k\nabla_0^\alpha w_\sigma)(k) = ({}_k^R\nabla_0^\alpha w_\sigma)(k) = ({}_k^C\nabla_0^\alpha w_\sigma)(k) = \frac{\sigma(k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}.$$

Thus,

$$\begin{aligned}\Phi(w_\sigma) &= \frac{1}{2}\sum_{k=1}^{T}|({}_k\nabla_0^\alpha w_\sigma)(k)|^2 + |({}_{T+1}\nabla_k^\alpha w_\sigma)(k)|^2 + \frac{1}{p}\sum_{k=1}^{T}|w_\sigma(k)|^p \\ &= \frac{1}{2}\sum_{k=1}^{T}\left|\frac{\sigma(k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}\right|^2 + \left|\frac{\sigma(T+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}\right|^2 + \frac{T\sigma^p}{p} \\ &= \frac{\sigma^2}{2(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2 + |(T+1-k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p} \\ &= \frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p}.\end{aligned} \tag{15}$$

We also have

$$\Psi(w_\sigma) = \sum_{k=1}^{T} F(k, w_\sigma(k)) = \sum_{k=1}^{T} F(k, \sigma). \tag{16}$$

Then, from condition (13), we see that $0 < r < \Phi(w_\sigma)$. From the definition of Φ, and in view of (7) and (14), for every $r > 0$,

$$\begin{aligned}
\Phi^{-1}(-\infty, r) &= \{u \in W : \Phi(u) < r\} \\
&\subseteq \left\{u \in W : \|u\|^p \leq \frac{pr}{(T+1)^{\frac{p(p-2)}{4}}}\right\} \subseteq \left\{u \in W : \|u\|_\infty^p \leq \frac{pr}{(T+1)^{\frac{p(p-2)}{4}}}\right\} \\
&= \left\{u \in W : \|u\|_\infty^p \leq \theta^p\right\},
\end{aligned}$$

and it follows that

$$\sup_{\Phi(u)<r} \Psi(u) = \sup_{\Phi(u)<r} \sum_{k=1}^{T} F(k, u(k)) \leq \sum_{k=1}^{T} \max_{|x|\leq\theta} F(k, x).$$

for every $u \in X$ such that $\Phi(u) \leq r$. Therefore,

$$\frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r]} \Psi(u)}{r} \leq \frac{p}{(T+1)^{\frac{p(p-2)}{4}}} \frac{\sum\limits_{k=1}^{T} \max\limits_{|x|\leq\theta} F(k, x)}{\theta^p}, \tag{17}$$

and

$$\begin{aligned}
\frac{\Psi(w_\sigma)}{\Phi(w_\sigma)} &= \frac{\sum\limits_{k=1}^{T} F(k, w_\sigma(k)) + \frac{\mu}{\lambda} \sum\limits_{k=1}^{T} G(k, w_\sigma(k))}{\frac{\sigma^2}{(\Gamma(1-\alpha))^2} \sum\limits_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p}} \\
&\geq \frac{\sum\limits_{k=1}^{T} F(k, \sigma) + \frac{\mu}{\lambda} G_\sigma}{\frac{\sigma^2}{(\Gamma(1-\alpha))^2} \sum\limits_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p}}.
\end{aligned} \tag{18}$$

Since $\mu < \delta_{\lambda,g}$, we have

$$\mu < \frac{(T+1)^{\frac{p(p-2)}{4}} \theta^p - \lambda p \sum\limits_{k=1}^{T} \max\limits_{|x|\leq\theta} F(k, x)}{pG^\theta},$$

which implies

$$\frac{p}{(T+1)^{\frac{p(p-2)}{4}}} \frac{\sum\limits_{k=1}^{T} \max\limits_{|x|\leq\theta} F(k, x) + \frac{\mu}{\lambda} G^\theta}{\theta^p} < \frac{1}{\lambda}.$$

Also,

$$\mu < \frac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2} \sum\limits_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p} - \lambda \sum\limits_{k=1}^{T} F(k, \sigma)}{G_\sigma},$$

so

$$\frac{\sum_{k=1}^{T} F(k,\sigma) + \frac{\mu}{\lambda} G_\sigma}{\frac{(T+1)^{\frac{p(p-2)}{4}}}{p}\theta^p} > \frac{1}{\lambda}.$$

Therefore,

$$\frac{p}{(T+1)^{\frac{p(p-2)}{4}}} \frac{\sum_{k=1}^{T} \max_{|x|\leq\theta} F(k,x) + \frac{\mu}{\lambda} G^\theta}{\theta^p} < \frac{1}{\lambda} < \frac{\sum_{k=1}^{T} F(k,\sigma) + \frac{\mu}{\lambda} G_\sigma}{\frac{(T+1)^{\frac{p(p-2)}{4}}}{p}\theta^p}. \tag{19}$$

Hence, from (17)–(19), condition (a_1) of Theorem 1 is satisfied.

Finally, since $\mu < \overline{\delta}_{\lambda,g}$, we can fix $l > 0$ such that

$$\limsup_{|x|\to+\infty} \frac{\sup_{k\in[1,T]_{N_0}} G(k,x)}{x^p} < l$$

and $\mu l < \frac{(T+1)^{\frac{p(p-2)}{4}}}{pT}$. Therefore, there exists a function $\varrho \in \mathbb{R}$ such that

$$G(k,x) \leq lx^p + \varrho, \tag{20}$$

for every $(k,x) \in [1,T]_{N_0} \times \mathbb{R}$. Now fix $0 < \gamma < \frac{1}{\lambda\Theta_1}\left(\frac{(T+1)^{\frac{p(p-2)}{4}}}{p} - \mu l T\right)$. From (A_2), there is a function $\tau \in \mathbb{R}$ such that

$$\frac{T}{\Theta_1} F(k,x) \leq \gamma|x|^p + \tau, \tag{21}$$

for every $(k,x) \in [1,T]_{N_0} \times \mathbb{R}$. In view of (7), from (20) and (21), for each $u \in W$, we have

$$\begin{aligned} I_\lambda(u) = \Phi(u) - \lambda\Psi(u) &\geq \frac{1}{p}(T+1)^{\frac{p(p-2)}{4}} \|u\|^p - \lambda \sum_{k=1}^{T} [F(k,u(k)) + \frac{\mu}{\lambda} G(k,u(k))] \\ &\geq \frac{1}{p}(T+1)^{\frac{p(p-2)}{4}} \|u\|^p - \frac{\lambda\Theta_1(\gamma \sum_{k=1}^{T} |u(k)|^p + \tau)}{T} - \mu(l \sum_{k=1}^{T} |u(k)|^p + \varrho) \\ &\geq \left(\frac{1}{p}(T+1)^{\frac{p(p-2)}{4}} - \lambda\Theta_1\gamma - \mu l T\right) \|u\|^p - \frac{\lambda\Theta_1\tau}{T} - \mu\varrho, \end{aligned}$$

and so

$$\lim_{\|u\|\to+\infty} \Phi(u) - \lambda\Psi(u) = +\infty.$$

That is, the functional I_λ is coercive. From (17)–(19) we also have

$$\lambda \in \left(\frac{\Phi(w_\sigma)}{\Psi(w_\sigma)}, \frac{r}{\sup_{\Phi(u)\leq r} \Psi(u)}\right),$$

and so condition (a_2) of Theorem 1 holds. Theorem 1 then assures the existence of three critical points for the functional I_λ that correspond to solutions to the problem (1). This completes the proof of the theorem. □

We next present two variants of Theorem 4. Instead of an asymptotic condition on the function g, in the first result, the functions f and g are assumed to be non-negative. In the second one, the function g is taken to be non-negative.

For the first theorem, let us fix positive constants θ_1, θ_2, and σ such that

$$\frac{3}{2}\frac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2+\frac{T\sigma^p}{p}}{\sum_{k=1}^{T}F(k,\sigma)} < \frac{(T+1)^{\frac{p(p-2)}{4}}}{p}\min\left\{\frac{\theta_1^p}{\sum_{k=1}^{T}\max_{|x|\le\theta_1}F(k,x)}, \frac{\theta_2^p}{2\sum_{k=1}^{T}\max_{|x|\le\theta_2}F(k,x)}\right\},$$

and take

$$\lambda\in\Lambda' := \left(\frac{3}{2}\frac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2+\frac{T\sigma^p}{p}}{\sum_{k=1}^{T}F(k,\sigma)}, \frac{(T+1)^{\frac{p(p-2)}{4}}}{p}\min\left\{\frac{\theta_1^p}{\sum_{k=1}^{T}\max_{|x|\le\theta_1}F(k,x)}, \frac{\theta_2^p}{2\sum_{k=1}^{T}\max_{|x|\le\theta_2}F(k,x)}\right\}\right).$$

Theorem 5. *Assume that there exist three positive constants θ_1, θ_2, and σ with*

$$\sqrt[p]{2}\theta_1 < \sqrt[p]{\frac{p}{(T+1)^{\frac{p(p-2)}{4}}}\left(\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2+\frac{T\sigma^p}{p}\right)} < \frac{\theta_2}{\sqrt[p]{2}}, \tag{22}$$

such that

(B_1) $f(k,x)\ge 0$ *for each* $(k,x)\in[1,T]_{N_0}\times\mathbb{R}$;

(B_2) $$\max\left\{\frac{\sum_{k=1}^{T}\max_{|x|\le\theta_1}F(k,x)}{\theta_1^p}, \frac{2\sum_{k=1}^{T}\max_{|x|\le\theta_2}F(k,x)}{\theta_2^p}\right\} < \frac{2}{3}\frac{(T+1)^{\frac{p(p-2)}{4}}}{\frac{p\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2+T\sigma^p}\sum_{k=1}^{T}F(k,\sigma).$$

Then, for each $\lambda\in\Lambda'$ and for every non-negative continuous function $g:[1,T]_{N_0}\times\mathbb{R}\to\mathbb{R}$, there exists $\delta^*_{\lambda,g}>0$ given by

$$\min\left\{\frac{(T+1)^{\frac{p(p-2)}{4}}\theta_1^p-p\lambda\sum_{k=1}^{T}\max_{|x|\le\theta_1}F(k,x)}{pG^{\theta_1}}, \frac{(T+1)^{\frac{p(p-2)}{4}}\theta_2^p-2p\lambda\sum_{k=1}^{T}\max_{|x|\le\theta_2}F(k,x)}{2pG^{\theta_2}}\right\}$$

such that, for each $\mu \in [0, \delta^*_{\lambda,g})$*, problem* (1) *admits at least three distinct non-negative weak solutions* u_i*,* $i = 1,2,3$*, such that*

$$0 \le u_i(k) < \theta_2, \text{ for all } k \in [1,T]_{N_0},\ i = 1,2,3.$$

Proof. In order to apply Theorem 2, we consider the auxiliary problem

$$\begin{cases} {}_{T+1}\nabla^{\alpha}_{k}({}_{k}\nabla^{\alpha}_{0}(u(k))) + {}_{k}\nabla^{\alpha}_{0}({}_{T+1}\nabla^{\alpha}_{k}(u(k))) + \varphi_p(u(k)) = \lambda \hat{f}(k,u(k)) + \mu g(k,u(k)), \\ u(0) = u(T+1) = 0, \end{cases} \tag{23}$$

where $\hat{f} : [1,T]_{N_0} \times \mathbb{R} \to \mathbb{R}$ is the continuous function defined by

$$\hat{f}(k,\xi) = \begin{cases} f(k,0), & \text{if } \xi < -\theta_2, \\ f(k,\xi), & \text{if } -\theta_2 \le \xi \le \theta_2, \\ f(k,\theta_2), & \text{if } \xi > \theta_2. \end{cases}$$

If any solution to the problem (1) satisfies the condition $-\theta_2 \le u(k) \le \theta_2$ for every $k \in [1,T]_{N_0}$, then any weak solution to problem (23) is also a weak solution to (1). Therefore, it suffices to show that our conclusion holds for (1).

Fix λ, g, and μ as in the conclusion of the theorem and take Φ and Ψ as in (8) and (9). We note that as before, the regularity assumptions of Theorem 2 on Φ and Ψ are satisfied. We need to show that conditions (b_1) and (b_2) hold.

To this end, we choose

$$w_\sigma(k) = \begin{cases} \sigma, & k \in [1,T]_{N_0}, \\ 0, & k \in 0, T+1, \end{cases}$$

$$r_1 := \frac{(T+1)^{\frac{p(p-2)}{4}}}{p}\theta_1^p, \quad \text{and} \quad r_2 := \frac{(T+1)^{\frac{p(p-2)}{4}}}{p}\theta_2^p.$$

In view of (22), we see that $2r_1 < \Phi(w_\sigma) < \frac{r_2}{2}$. Since $\mu < \delta^*_{\lambda,g}$ and $G_\sigma \ge 0$, taking (15) into account, we have

$$\begin{aligned} \frac{\sup\limits_{u \in \Phi^{-1}(-\infty,r_1)} \Psi(u)}{r_1} &\le \frac{\sup_{u \in \Phi^{-1}(-\infty,r_1)} \sum\limits_{k=1}^{T} F(k,u(k)) + \sup\limits_{u \in \Phi^{-1}(-\infty,r_1)} \frac{\mu}{\lambda}\sum\limits_{k=1}^{T} G(k,u(k))}{r_1} \\ &\le \frac{\sum\limits_{k=1}^{T} \max\limits_{|x| \le \theta_1} F(k,x) + \frac{\mu}{\lambda} G^{\theta_1}}{\frac{(T+1)^{\frac{p(p-2)}{4}}}{p}\theta_1^p} \\ &< \frac{1}{\lambda} < \frac{2}{3}\frac{\sum\limits_{k=1}^{T} F(k,\sigma) + \frac{\mu}{\lambda}G_\sigma}{\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum\limits_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p}} \\ &\le \frac{2}{3}\frac{\sum\limits_{k=1}^{T} F(k,\sigma) + \frac{\mu}{\lambda}\sum\limits_{k=1}^{T} G(k,w_\sigma(k))}{\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum\limits_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p}} \\ &= \frac{2}{3}\frac{\Psi(w_\sigma)}{\Phi(w_\sigma)} \end{aligned} \tag{24}$$

and

$$\frac{2\sup\limits_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2} = 2\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\sum\limits_{k=1}^{T}F(k,u(k)) + \sup\limits_{u\in\Phi^{-1}(-\infty,r_2)}\frac{\mu}{\lambda}\sum\limits_{k=1}^{T}G(k,u(k))}{r_2}$$
$$\leq 2\frac{\sum\limits_{k=1}^{T}\max\limits_{|x|\leq\theta_2}F(k,x) + \frac{\mu}{\lambda}G^{\theta_2}}{\frac{(T+1)^{\frac{p(p-2)}{4}}}{p}\theta_2^p}$$
$$< \frac{1}{\lambda} < \frac{2}{3}\frac{\sum\limits_{k=1}^{T}F(k,\sigma) + \frac{\mu}{\lambda}G_\sigma}{\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum\limits_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p}}$$
$$\leq \frac{2}{3}\frac{\sum\limits_{k=1}^{T}F(k,\sigma) + \frac{\mu}{\lambda}\sum\limits_{k=1}^{T}G(k,w_\sigma(k))}{\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum\limits_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p}}$$
$$= \frac{2}{3}\frac{\Psi(w_\sigma)}{\Phi(w_\sigma)}.$$

Therefore, conditions (b_1) and (b_2) of Theorem 2 are satisfied.

Finally, to show that $I_\lambda = \Phi - \lambda\Psi$ satisfies condition 2 of Theorem 2, let u_1 and u_2 be two local minima of I_λ. Then u_1 and u_2 are critical points of $\Phi - \lambda\Psi$, and so they are weak solutions to the problem (1). We want to prove that they are non-negative, so let u_0 be a weak solution to (1). Arguing by contradiction, assume that the set $\mathcal{A} = \{k \in [1,T]_{N_0} : u_0(k) < 0\}$ is non-empty and of positive measure. Set $\bar{v}(k) = \min\{0, u_0(k)\}$ for all $k \in [1,T]_{N_0}$. Clearly, $\bar{v}s. \in W$ and

$$\sum_{k=1}^{T}({}_k\nabla_0^\alpha u_0(k))({}_k\nabla_0^\alpha \bar{v}(k)) + ({}_{T+1}\nabla_k^\alpha u_0(k))({}_{T+1}\nabla_k^\alpha \bar{v}(k)) + \frac{1}{p}\sum_{k=1}^{T}|u_0(k)|^{p-2}u_0(k)\bar{v}(k)$$
$$- \lambda\sum_{k=1}^{T}f(k,u_0(k))\bar{v}(k) - \mu\sum_{k=1}^{T}g(k,u_0(k))\bar{v}(k) = 0.$$

Thus, from our sign assumptions,

$$0 \leq (T+1)^{\frac{p(p-2)}{4}}\|u\|_{\mathcal{A}}^p \leq \sum_{\mathcal{A}}({}_k\nabla_0^\alpha u_0(k))^2 + ({}_{T+1}\nabla_k^\alpha u_0(k))^2 + \frac{1}{p}\sum_{\mathcal{A}}|u_0(k)|^p$$
$$= \lambda\sum_{\mathcal{A}}f(k,u_0(k))u_0(k) + \mu\sum_{\mathcal{A}}g(k,u_0(k))u_0(k) \leq 0.$$

Hence, $\|u_0\|_{\mathcal{A}} = 0$, which is a contradiction, and so $u_1(k) \geq 0$ and $u_2(k) \geq 0$ for every $k \in [1,T]_{N_0}$. It follows that $su_1 + (1-s)u_2 \geq 0$ for all $s \in [0,1]$, and

$$(\lambda f + \mu g)(x, su_1 + (1-s)u_2) \geq 0.$$

and so $\Psi(su_1 + (1-s)u_2) \geq 0$ for every $s \in [0,1]$. From Theorem 2, for every

$$\lambda \in \left(\frac{3}{2}\frac{\Phi(w)}{\Psi(w)}, \min\left\{\frac{r_1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}, \frac{r_2/2}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right),$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points that in turn are solutions to the problem (1). This proves the theorem. $\square$

Remark 1. *In Theorems 4 and 5, if either $f(k,0) \neq 0$ for some $k \in [1,T]_{N_0}$ or $g(k,0) \neq 0$ for some $k \in [1,T]_{N_0}$, then the solutions obtained from the theorems are non-trivial.*

Remark 2. *If, in Theorem 4, $f(\cdot,x)$ and $g(\cdot,x)$ are odd functions in x, then we are guaranteed the existence of at least five distinct weak solutions. The reason for this is that if u is a nontrivial weak solution, then $-u$ is a weak solution since satisfies the equation*

$$\sum_{k=1}^{T}({}_k\nabla_0^\alpha(-u(k)))({}_k\nabla_0^\alpha v(k)) + ({}_{T+1}\nabla_k^\alpha(-u(k)))({}_{T+1}\nabla_k^\alpha v(k))$$
$$+\sum_{k=1}^{T}|(-u(k))|^{p-2}(-u(k))v(k) - \lambda\sum_{k=1}^{T}f(k,(-u(k)))v(k) - \mu\sum_{k=1}^{T}g(k,(-u(k)))v(k) = 0$$

for every $v \in W$.

Remark 3. *If we consider the autonomous case of (1) (i.e., the functions f and g do not explicitly depend on k), namely,*

$$\begin{cases} {}_{T+1}\nabla_k^\alpha({}_k\nabla_0^\alpha(u(k))) +_k \nabla_0^\alpha({}_{T+1}\nabla_k^\alpha(u(k))) + \varphi_p(u(k)) = \lambda f(u(k)) + \mu g(u(k)), \\ u(0) = u(T+1) = 0, \end{cases} \tag{25}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are non-negative, continuous, and not identically zero functions, then putting $F(x) = \int_0^x f(\xi)d\xi$, for each $x \in \mathbb{R}$, in Theorem 4 the conditions (A_1) and (A_2) take the form

$(\widehat{A}_1)$ $$\frac{\max_{|x|\le\theta}F(x)}{\theta^p} < \frac{(T+1)^{\frac{p(p-2)}{4}}}{\frac{p\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2 + T\sigma^p}F(\sigma);$$

$(\widehat{A}_2)$ $T\limsup_{|x|\to\infty}\frac{F(x)}{x^p} < \Theta$ *where*

$$\Theta := \frac{T\max_{|x|\le\theta}F(x)}{\frac{(T+1)^{\frac{p(p-2)}{4}}}{p}\theta^p},$$

respectively. In addition,

$$\Lambda := \left(\frac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p}}{TF(\sigma)}, \frac{(T+1)^{\frac{p(p-2)}{4}}\theta^p}{pT\max_{|x|\le\theta}F(x)}\right),$$

and $\hat{\delta}_{\lambda,g} =$

$$\min\left\{\frac{(T+1)^{\frac{p(p-2)}{4}}\theta^p - \lambda pT\max_{|x|\le\theta}F(x)}{pG^\theta}, \left|\frac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p} - \lambda TF(\sigma)}{\min\{0, G_\sigma\}}\right|\right\}.$$

In this case, condition (B_2) in Theorem 5 takes the form

$(\widehat{B}_2)$ $$\max\left\{\frac{\max_{|x|\le\theta_1}F(x)}{\theta_1^p}, \frac{2\max_{|x|\le\theta_2}F(x)}{\theta_2^p}\right\} < \frac{2}{3}\frac{(T+1)^{\frac{p(p-2)}{4}}}{\frac{p\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2 + T\sigma^p}F(\sigma).$$

Moreover,

$$\Lambda' := \left(\frac{3}{2} \frac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p}}{TF(k)}, \right.$$

$$\left. \frac{(T+1)^{\frac{p(p-2)}{4}}}{Tp} \min\left\{ \frac{\theta_1^p}{\max_{|x|\leq\theta_1} F(x)}, \frac{\theta_2^p}{2\max_{|x|\leq\theta_2} F(x)} \right\} \right)$$

and

$$\delta^*_{\lambda,g} = \min\left\{ \frac{(T+1)^{\frac{p(p-2)}{4}} \theta_1^p - p\lambda T \max\limits_{|x|\leq\theta_1} F(x)}{pG^{\theta_1}}, \frac{(T+1)^{\frac{p(p-2)}{4}} \theta_2^p - 2p\lambda T \max_{|x|\leq\theta_2} F(x)}{2pG^{\theta_2}} \right\}.$$

As a special case of Theorem 4, we have the following theorem in which the functions f and g are autonomous.

Theorem 6. *Assume that*

$$\liminf_{x\to 0} \frac{F(x)}{x^p} = \limsup_{x\to+\infty} \frac{F(x)}{x^p} = 0. \tag{26}$$

Then, there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$\limsup_{x\to+\infty} \frac{G(x)}{|x|^p} < +\infty,$$

there exists $\delta > 0$ such that, for each $\mu \in [0, \delta)$, the problem (25) admits at least three distinct solutions.

Proof. Fix $\lambda > \lambda^* := \dfrac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p}}{TF(\sigma)}$ for some $\sigma \geq 0$. From (26), there is a sequence $\{\theta_n\} \subset (0, +\infty)$ such that $\lim\limits_{n\to\infty} \theta_n = 0$ and $\lim_{n\to+\infty} \dfrac{\sup\limits_{|\xi|\leq\theta_n} F(\xi)}{\theta_n^p} = 0$. We then have

$$\lim_{n\to\infty} \frac{\sup\limits_{|\xi|\leq\theta_n} F(\xi)}{\theta_n^p} = \lim_{n\to\infty} \frac{F(\xi_{\theta_n})}{\xi_{\theta_n}^p} \frac{\xi_{\theta_n}^p}{\theta_n^p} = 0,$$

where $F(\xi_{\theta_n}) = \sup\limits_{|\xi|\leq\theta_n} F(\xi)$. Hence, there exists $\overline{\theta} > 0$ such that

$$\frac{\sup_{|x|\leq\overline{\theta}} F(x)}{\overline{\theta}^p} < \min\left\{ \frac{(T+1)^{\frac{p(p-2)}{4}}}{\frac{p\sigma^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + T\sigma^p} F(\sigma), \frac{(T+1)^{\frac{p(p-2)}{4}}}{\lambda pT} \right\}$$

and

$$\overline{\theta} < \sqrt[p]{\frac{p}{(T+1)^{\frac{p(p-2)}{4}}} \left(\frac{\sigma^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^p}{p} \right)}.$$

Applying Theorem 4 proves the theorem. □

The following example illustrates Theorem 6.

Example 1. *Let $T = 2$, $p = 5$, and consider the problem*

$$\begin{cases} {}_3\nabla_k^\alpha({}_k\nabla_0^\alpha(u(k))) + {}_k\nabla_0^\alpha({}_3\nabla_k^\alpha(u(k))) + \varphi_5(u(k)) = \lambda f(u(k)), \\ u(0) = u(3) = 0, \end{cases} \tag{27}$$

where $\alpha = \dfrac{1}{2}$ and

$$f(\xi) = \begin{cases} 6\xi^5, & \text{if } \xi \le 1, \\ \dfrac{6}{\xi} + \sin^2(\xi - 1), & \text{if } \xi > 1. \end{cases}$$

From f, we have

$$F(\xi) = \begin{cases} \xi^6, & \text{if } \xi \le 1, \\ 6\ln(\xi) + \dfrac{1}{2}\xi - \dfrac{1}{4}\sin 2(\xi - 1) + \dfrac{1}{2}, & \text{if } \xi > 1. \end{cases}$$

and

$$\lim_{x \to 0^+} \frac{F(x)}{x^5} = \lim_{x \to +\infty} \frac{F(x)}{x^5} = 0.$$

Taking $\sigma = 1$, we see that all the conditions of Theorem 6 are satisfied. Therefore, for each

$$\lambda > \frac{1}{2(\Gamma(\frac{1}{2}))^2} \sum_{k=1}^{2} |(k)^{\overline{-\frac{1}{2}}}|^2 + \frac{1}{5}$$

and for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$\limsup_{x \to +\infty} \frac{G(x)}{|x|^5} < +\infty,$$

there exists $\delta > 0$ such that, for each $\mu \in [0, \delta)$, problem (27) admits at least three distinct solutions.

The next result is a consequence of Theorem 5. Again here, f and g are independent of k.

Theorem 7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that*

$$\lim_{x \to 0^+} \frac{f(x)}{x^3} = 0$$

and

$$F(100) < \frac{3 \times 10^8 \times (\Gamma(\frac{1}{2}))^2}{12(\Gamma(\frac{1}{2}))^2 + 8(\Gamma(\frac{3}{2}))^2} F(1).$$

Then, for every $\lambda \in \left(\dfrac{9(\Gamma(\frac{1}{2}))^2 + 6(\Gamma(\frac{3}{2}))^2}{4F(1)(\Gamma(\frac{1}{2}))^2}, \dfrac{9 \times 10^8}{16F(100)} \right)$ and for every nonnegative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta)$, (25) admits at least three distinct non-negative solutions.

Proof. Our aim is to employ Theorem 5 by taking $T = 2$, $p = 4$, $\alpha = \dfrac{1}{2}$, $\sigma = 1$, and $\theta_2 = 100$. Simple calculations show that

$$\frac{3}{2}\frac{\dfrac{\sigma^2}{(\Gamma(1-\alpha))^2}\displaystyle\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2+\dfrac{T\sigma^p}{p}}{\displaystyle\sum_{k=1}^{T}F(\sigma)}=\frac{9(\Gamma(\frac{1}{2}))^2+6(\Gamma(\frac{3}{2}))^2}{4F(1)(\Gamma(\frac{1}{2}))^2}$$

and

$$\frac{(T+1)^{\frac{p(p-2)}{4}}}{p}\frac{\theta_2^p}{2\displaystyle\sum_{k=1}^{T}\max_{|x|\leq\theta_2}F(x)}=\frac{9\times 10^8}{16F(100)}.$$

Moreover, since $\lim\limits_{x\to 0^+}\dfrac{f(x)}{x^3}=0$, we have

$$\lim_{x\to 0^+}\frac{\displaystyle\int_0^x f(\xi)d\xi}{x^4}=0.$$

Then there exists a positive constant

$$\theta_1<\sqrt[4]{\frac{12(\Gamma(\frac{1}{2}))^2+8(\Gamma(\frac{3}{2}))^2}{36(\Gamma(\frac{1}{2}))^2}}$$

such that

$$\frac{F(\theta_1)}{\theta_1^4}<\frac{6(\Gamma(\frac{1}{2}))^2F(1)}{6(\Gamma(\frac{1}{2}))^2+4(\Gamma(\frac{3}{2}))^2}$$

and

$$\frac{\theta_1^4}{F(\theta_1)}>\frac{9\times 10^8}{16F(100)}.$$

The conditions of Theorem 5 are satisfied, and this proves the theorem. □

We end this paper by presenting the following versions of Theorems 4 and 5 for the case where $p = 2$.

Theorem 8. *Assume that there exist two positive constants θ and σ with*

$$\theta<\sqrt{2\left(\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2+\frac{T\sigma^2}{2}\right)}$$

such that

$$\frac{\displaystyle\sum_{k=1}^{T}\max_{|x|\leq\theta}F(k,x)}{\theta^2}<\frac{1}{\dfrac{2\sigma^2}{(\Gamma(1-\alpha))^2}\displaystyle\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2+T\sigma^2}\sum_{k=1}^{T}F(k,\sigma)$$

and

$$T \limsup_{|x|\to\infty} \frac{F(k,x)}{x^2} < \Theta \text{ uniformly with respect to } k \in [1,T]_{N_0}, \text{ where}$$

$$\Theta := \frac{2\sum_{k=1}^{T} \max_{|x|\le\theta} F(k,x)}{\theta^2}.$$

Then, for each

$$\lambda \in \left(\frac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^2}{2}}{\sum_{k=1}^{T} F(k,\sigma)}, \frac{\theta^2}{2\sum_{k=1}^{T} \max_{|x|\le\theta} F(k,x)} \right),$$

and for every continuous function $g : [1,T]_{N_0} \times \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\limsup_{|x|\to+\infty} \frac{\sup_{k\in[1,T]_{N_0}} G(k,x)}{x^2} < +\infty,$$

there exists $\overline{\delta}_{\lambda,g} > 0$ *given by*

$$\overline{\delta}_{\lambda,g} := \min\left\{ \delta_{\lambda,g}, \frac{1}{\max\left\{0, 2T \limsup_{|x|\to+\infty} \frac{\sup_{k\in[1,T]_{N_0}} G(k,x)}{x^2}\right\}} \right\}$$

where

$$\delta_{\lambda,g} = \min\left\{ \frac{\theta^2 - 2\lambda\sum_{k=1}^{T} \max_{|x|\le\theta} F(k,x)}{2G^\theta}, \left| \frac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^2}{2} - \lambda\sum_{k=1}^{T} F(k,\sigma)}{\min\{0, G_\sigma\}} \right| \right\}$$

such that, for each $\mu \in [0, \overline{\delta}_{\lambda,g})$, *the problem*

$$\begin{cases} {}_{T+1}\nabla_k^\alpha({}_k\nabla_0^\alpha(u(k))) + {}_k\nabla_0^\alpha({}_{T+1}\nabla_k^\alpha(u(k))) + u(k) = \lambda f(k,u(k)) + \mu g(k,u(k)), \\ u(0) = u(T+1) = 0, \end{cases} \tag{28}$$

admits at least three distinct weak solutions in W.

Theorem 9. *Assume that there exist three positive constants* θ_1, θ_2, *and* σ, *with*

$$\sqrt{2}\theta_1 < \sqrt{2\left(\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^2}{2}\right)} < \frac{\theta_2}{\sqrt{2}},$$

such that

$$f(k,x) \ge 0 \text{ for each } (k,x) \in [1,T]_{N_0} \times \mathbb{R}$$

and

$$\max\left\{\frac{\sum_{k=1}^{T}\max_{|x|\leq\theta_1}F(k,x)}{\theta_1^2}, \frac{2\sum_{k=1}^{T}\max_{|x|\leq\theta_2}F(k,x)}{\theta_2^2}\right\} < \frac{2}{3}\frac{1}{\frac{2\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2 + T\sigma^2}\sum_{k=1}^{T}F(k,\sigma).$$

Then, for each

$$\lambda \in \left(\frac{3}{2}\frac{\frac{\sigma^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2 + \frac{T\sigma^2}{2}}{\sum_{k=1}^{T}F(k,\sigma)}, \min\left\{\frac{\theta_1^2}{2\sum_{k=1}^{T}\max_{|x|\leq\theta_1}F(k,x)}, \frac{\theta_2^2}{4\sum_{k=1}^{T}\max_{|x|\leq\theta_2}F(k,x)}\right\}\right),$$

and for every non-negative continuous function $g : [1,T]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$, *there exists* $\delta^*_{\lambda,g} > 0$ *given by*

$$\min\left\{\frac{\theta_1^2 - 2\lambda\sum_{k=1}^{T}\max_{|x|\leq\theta_1}F(k,x)}{2G^{\theta_1}}, \frac{\theta_2^2 - 4\lambda\sum_{k=1}^{T}\max_{|x|\leq\theta_2}F(k,x)}{4G^{\theta_2}}\right\}$$

such that, for each $\mu \in [0, \delta^*_{\lambda,g})$, *the problem* (28) *admits at least three distinct non-negative weak solutions* u_i *for* $i = 1,2,3$, *such that*

$$0 \leq u_i(k) < \theta_2, \text{ for all } k \in [1,T]_{\mathbb{N}_0},\ i = 1,2,3.$$

4. Discussion

In this paper, we used two the critical point theorems [17,18] to obtain two new results that ensure the existence of at least three weak solutions to the problem under discussion, namely, (1). In our first main result, Theorem 4, under modest conditions on the nonlinear functions f and g, we were able to obtain the existence of three solutions to our problem.

Based on this result (Theorem 4), we were able to present some variant results, one of which showed that the three solutions obtained were uniformly bounded. An example illustrates some of the results.

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References

1. Diethelm, K. *The Analysis of Fractional Differential Equation*; Springer: Heidelberg, Germany, 2010.
2. Galewski, M.; Molica Bisci, G. Existence results for one-dimensional fractional equations. *Math. Meth. Appl. Sci.* **2016**, *39*, 1480–1492. [CrossRef]
3. Graef, J.R.; Ho, S.; Kong, L.; Wang, M. A fractional differential equation model for bike share systems. *J. Nonlinear Funct. Anal.* **2019**, *2019*, 1–14.
4. Graef, J.R.; Kong, L.; Ledoan, A.; Wang, M. Stability analysis of a fractional online social network model. *Math. Comput. Simul.* **2020**, *178*, 625–645. [CrossRef]
5. Heidarkhani, S. Multiple solutions for a nonlinear perturbed fractional boundary value problem. *Dyn. Syst. Appl.* **2014**, *23*, 317–331.

6. Kong, L. Existence of solutions to boundary value problems arising from the fractional advection dispersion equation. *Electron. J. Diff. Equ.* **2013**, *2013*, 1–15.
7. Cabada, A.; Iannizzotto, A.; Tersian, S. Multiple solutions for discrete boundary value problem. *J. Math. Anal. Appl.* **2009**, *356*, 418–428. [CrossRef]
8. Galewski, M.; Glkab, S. On the discrete boundary value problem for anisotropic equation. *J. Math. Anal. Appl.* **2012**, *386*, 956–965. [CrossRef]
9. Galewski, M.; Wieteska, R. On the system of anisotropic discrete BVPs. *J. Differ. Equ. Appl.* **2013**, *19*, 1065–1081. [CrossRef]
10. Henderson, J.; Thompson, H.B. Existence of multiple solutions for second order discrete boundary value problems. *Comput. Math. Appl.* **2002**, *43*, 1239–1248. [CrossRef]
11. Faraci, F. Multiple solutions for two nonlinear problems involving the p-Laplacian. *Nonlinear Anal.* **2005**, *63*, e1017–e1029. [CrossRef]
12. Li, W.; He, Z. The applications of sums of ranges of accretive operators to nonlinear equations involving the p-Laplacian operator. *Nonlinear Anal.* **1995**, *24*, 185–193. [CrossRef]
13. Wei, L.; Agarwal, R.P. Existence of solutions to nonlinear Neumann boundary value problems with generalized p-Laplacian operator. *Comput. Math. Appl.* **2008**, *56*, 530–541. [CrossRef]
14. Lv, W. Existence of solutions for discrete fractional boundary value problems with a p-Laplacian operator. *Adv. Differ. Equ.* **2012**, *2012*, 1–10. [CrossRef]
15. Heidarkhani, S.; Moradi, S.; Barilla, D. Existence results for discrete fractional boundary value problems with a p-Laplacian operator via variational methods. *Res. Sq.* 2022, *preprint*.
16. Heidarkhani, S.; Moradi, S.; Afrouzi, G.A. Existence infinitely many solutions for discrete fractional boundary value problem with a p-Laplacian operator. *Res. Sq.* 2022, *preprint*.
17. Bonanno, G.; Candito, P. Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities. *J. Differ. Equ.* **2008**, *244*, 3031–3059. [CrossRef]
18. Bonanno, G.; Marano, S.A. On the structure of the critical set of non-differentiable functions with a weak compactness condition. *Appl. Anal.* **2010**, *89*, 1–10. [CrossRef]
19. D'Aguì, G.; Heidarkhani, S.; Molica Bisci, G. Multiple solutions for a perturbed mixed boundary value problem involving the one-dimensional p-Laplacian. *Electron. J. Qual. Theory Differ. Equ.* **2013**, *2013*, 1–14. [CrossRef]
20. Bonanno, G.; D'Aguì, G. Multiplicity results for a perturbed elliptic Neumann problem. *Abstr. Appl. Anal.* **2010**, *2010*, 564363. [CrossRef]
21. Atici, F.M.; Eloe, P.W. Discrete fractional calculus with the nabla operator. *Electron. J. Qual. Theory Differ. Equ. Spec. Ed.* **2009**, *2009*, 1–12. [CrossRef]
22. Abdeljawad, T. On delta and nabla Caputo fractional differences and dual identities. *Discret. Dyn. Nat. Soc.* **2013**, *2013*, 406910. [CrossRef]
23. Abdeljawad, T. On Riemann and Caputo fractional differences. *Comput. Math Appl.* **2011**, 62, 1602–1611. [CrossRef]
24. Abdeljawad, T.; Atici, F. On the definitions of nabla fractional operators. *Abstr. Appl. Anal.* **2012**, *2012*, 406757. [CrossRef]
25. Zeidler, E. *Nonlinear Functional Analysis and Its Applications, Vol. III*; Springer: New York, NY, USA, 1985.