axioms

Article

A Time-Fractional Differential Inequality of Sobolev Type on an Annulus

Amal Alshabanat [1], Eman Almoalim [1], Mohamed Jleli [2] and Bessem Samet [2,*]

[1] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, Riyadh 11671, Saudi Arabia; aealshabanat@pnu.edu.sa (A.A.); eoalmoalim@pnu.edu.sa (E.A.)
[2] Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia; jleli@ksu.edu.sa
* Correspondence: bsamet@ksu.edu.sa

Abstract: Several phenomena from natural sciences can be described by partial differential equations of Sobolev-type. On the other hand, it was shown that in many cases, the use of fractional derivatives provides a more realistic model than the use of standard derivatives. The goal of this paper is to study the nonexistence of weak solutions to a time-fractional differential inequality of Sobolev-type. Namely, we give sufficient conditions for the nonexistence or equivalently necessary conditions for the existence. Our method makes use of the nonlinear capacity method, which consists in making an appropriate choice of test functions in the weak formulation of the problem. This technique has been employed in previous papers for some classes of time-fractional differential inequalities of Sobolev-type posed on the whole space $\mathbb{R}^N$. The originality of this work is that the considered problem is posed on an annulus domain, which leads to some difficulties concerning the choice of adequate test functions.

Keywords: differential inequalities of Sobolev-type; Caputo fractional derivative; weak solution; nonexistence

MSC: 35A01; 26A33; 35B44

check for updates

Citation: Alshabanat, A.; Almoalim, E.; Jleli, M.; Samet, B. A Time-Fractional Differential Inequality of Sobolev Type on an Annulus. *Axioms* **2023**, *12*, 993. https://doi.org/10.3390/axioms12100993

Academic Editor: Hari Mohan Srivastava

Received: 13 September 2023
Revised: 15 October 2023
Accepted: 19 October 2023
Published: 20 October 2023

1. Introduction

The issue of nonexistence of solutions is one of the important branches of the theory of partial differential equations. One of the important applications of this issue is the study of the large time behavior of solutions to evolution equations. The objective of this paper is to establish nonexistence theorems for a time-fractional differential inequality of Sobolev-type posed on an annulus domain of $\mathbb{R}^N$. Sobolev-type equations are widely used in the description of various phenomena from natural sciences such as plasma physics, nonstationary processes in crystalline semiconductors, filtration theory, and thermodynamics (see, e.g., [1–5]). In [6], the issue of nonexistence of solutions has been investigated for some classes of Sobolev-type inequalities posed on the whole space $\mathbb{R}^N$. Let us briefly recall some of the obtained results in [6]. Consider the differential inequality

$$-\Delta(\partial_t u) \geq |u|^q, \quad t > 0,\, x \in \mathbb{R}^N, \tag{1}$$

where $u = u(t,x)$ and $\partial_t = \frac{\partial}{\partial t}$. It was proven that, if $u(0,\cdot) \in L^1(\mathbb{R}^N)$, then for all $q > 1$, (1) possesses no nontrivial weak solution. Next, consider the problem

$$-\Delta(\partial_t u) - \Delta u \geq |u|^q, \quad t > 0,\, x \in \mathbb{R}^N. \tag{2}$$

Always under the assumption $u(0,\cdot) \in L^1(\mathbb{R}^N)$, it was proven that:

(i) If $N = 1$ or $N = 2$, then for all $q > 1$, (2) possesses no nontrivial weak solution;

(ii) If $N \geq 3$, then for all $1 < q < \frac{N}{N-2}$, (2) possesses no nontrivial weak solution.

The third problem considered in [6] is the following:

$$-\partial_t(\Delta u + \lambda |u|^p u) - \Delta u \geq |u|^q, \quad t > 0,\, x \in \mathbb{R}^N, \tag{3}$$

where $p \geq 0$, $q > 1$ $p+1 < q$ and $\lambda \neq 0$. Under the assumption

$$u(0,\cdot) \in L^1(\mathbb{R}^N),\ |u(0,\cdot)|^p u(0,\cdot) \in L^1(\mathbb{R}^N),\ \lambda \int_{\mathbb{R}^N} |u(0,x)|^p u(0,x)\, dx \leq 0,$$

it was shown that, if

$$1 < q < \frac{N+\gamma}{N+\gamma-2},$$

where

$$\gamma = \frac{2(q-1-p)}{q-1},$$

then the same conclusion as for the above problems holds. Other related results can be found in [1,7–10] (see also the references therein).

On the other hand, it was observed that in many situations, fractional derivatives provide more realistic models than standard derivatives. This observation motivated the study of fractional partial differential equations in various directions. In particular, several studies related to the nonexistence of solutions have been conducted (see, e.g., [11–17] and the references therein). For instance, in [11], the authors extended some of the obtained results in [6] to the fractional case. We recall below some of the obtained results in [11]. Consider the time-fractional differential inequality

$$-\Delta(\partial_{t^\varrho} u) \geq |u|^q, \quad t > 0,\, x \in \mathbb{R}^N, \tag{4}$$

where $u = u(t,x)$, $0 < \varrho < 1$ and $\partial_{t^\varrho} u$ is the partial derivative of order ϱ of u with respect to the variable t in the Caputo sense. Observe that (4) is a fractional version of (1). It was proven that for suitable initial data, no weak solution to (4) exists for all $q > 1$. The next problem considered in [11] is a time–space fractional version of (2), namely

$$-\Delta(\partial_{t^\varrho} u) + (-\Delta)^\sigma u \geq |u|^q, \quad t > 0,\, x \in \mathbb{R}^N, \tag{5}$$

where $0 < \varrho < 1$, $0 < \sigma < 1$ and $(-\Delta)^\sigma$ is the fractional Laplacian of order σ (with respect to x). It was established that for suitable initial data, if

$$1 < q < 1 + \frac{2\sigma}{2+\varrho-2\sigma},$$

then (5) possesses no weak solution.

We point out that all the studied problems in [6,11] are posed on the whole space $\mathbb{R}^N$. Furthermore, in both references, the used approach is based on the nonlinear capacity method that consists in making an appropriate choice of test functions in the weak formulation of the problem. For more details about this method, we refer to [18,19]. Notice that in the case of $\mathbb{R}^N$, the considered test functions involve cut-off functions of the form

$$\xi_R(x) = \xi\left(R^{-2}|x|^2\right), \quad x \in \mathbb{R}^N, \tag{6}$$

where R is sufficiently large, $\xi \in C^\infty([0,\infty))$, $0 \leq \xi \leq 1$, $\xi = 1$ if $|x| < \frac{1}{2}$ and $\xi = 0$ if $|x| > 1$. One of the important properties of this function is that

$$\lim_{R\to\infty} \xi_R(x) = 1.$$

It is natural to ask whether it is possible to establish nonexistence theorems for time-fractional Sobolev-type inequalities posed on a bounded domain. Observe that in this case, the application of the nonlinear capacity method presents some difficulties. Namely, if the domain is bounded, functions of the form (6) cannot be used. Motivated by this fact, a time-fractional Sobolev-type inequality on an annulus domain is considered and the issue of nonexistence is investigated using the nonlinear capacity method with an adequate choice of test functions.

The considered problem and its weak formulation are given in Section 2. In Section 3, we state our main result and a special case is studied in detail. The proof of the main result is presented in Section 4.

2. Problem Formulation

Let $0 < \eta_1 < \eta_2$, $\mathcal{D} = \{x \in \mathbb{R}^N : \eta_1 < |x| < \eta_2\}$, $N \geq 3$ and $\Sigma_k = \{x \in \mathbb{R}^N : |x| = \eta_k\}$, $k = 1, 2$. We investigate the time-fractional Sobolev-type inequality

$$\begin{cases} -\Delta(\partial_{t^\varrho} u) - \imath \partial_{t^\varsigma}(|u|^m u) - \Delta u \geq \jmath(t)|u|^r & \text{in } \mathbb{R}_+ \times \mathcal{D}, \\ u(0, \cdot) = f & \text{in } \mathcal{D}, \\ u = 0 & \text{on } \Sigma_2, \end{cases} \tag{7}$$

where $\mathbb{R}_+ = (0, \infty)$, $u = u(t, x)$, $\imath \neq 0$, $m \geq 0$, $r > m + 1$, $0 < \varrho, \varsigma < 1$, $\jmath > 0$ almost everywhere and $f = f(x)$. Here, for any $0 < \varkappa < 1$, $\partial_{t^\varkappa} u$ is the partial derivative of order $\varkappa$ of u with respect to the variable t in the Caputo sense, that is,

$$\partial_{t^\varkappa} u(t, x) = [\Gamma(1 - \varkappa)]^{-1} \int_0^t (t - z)^{-\varkappa} \partial_t u(z, x)\, dz.$$

Namely, our goal is to derive sufficient conditions ensuring that (7) has no weak solution (or equivalently, necessary conditions for the existence of weak solutions).

Problem (7) is a fractional version of

$$-\Delta(\partial_t u) - \imath \partial_t(|u|^m u) - \Delta u \geq \jmath(t)|u|^r. \tag{8}$$

The above problem belongs to the class of differential inequalities of Sobolev-type, and includes various physics problems as special cases. For instance, if $\imath = -1$, $m = 0$, $\jmath \equiv \tau \in \mathbb{R}$, $r = 3$ and $u \geq 0$, (8) (with equality) reduces to the semiconductor equation that was derived in [7].

In order to define weak solutions to (7), let us recall some basic definitions from fractional calculus (see [20] for more details). Let $w_j = w_j(t, x) : [0, S] \times \Omega \to \mathbb{R}$, $S > 0$, $j = 1, 2$, be two continuous functions, where Ω is a subset of $\mathbb{R}^N$. For $\varkappa > 0$, let

$$I_0^\varkappa w_j(t, x) = C_\varkappa \int_0^t (t - z)^{\varkappa - 1} w_j(z, x)\, dz$$

and

$$I_S^\varkappa w_j(t, x) = C_\varkappa \int_t^S (z - t)^{\varkappa - 1} w_j(z, x)\, dz,$$

where $C_\varkappa = [\Gamma(\varkappa)]^{-1}$ and Γ is the Gamma function.

We have the following property:

$$\int_0^S I_0^\varkappa w_1(s, x)\, w_2(s, x)\, ds = \int_0^S w_1(s, x)\, I_S^\varkappa w_2(s, x)\, ds \tag{9}$$

for all $x \in \Omega$.

If $0 < \varkappa < 1$, the partial derivative of order $\varkappa$ of w_j with respect to the variable t in the Caputo sense is given by

$$\partial_{t^\varkappa} w_j(t,x) = I_0^{1-\varkappa}\partial_t w_j(t,x) = C_{1-\varkappa}\int_0^t (t-z)^{-\varkappa}\partial_t w_j(z,x)\,dz.$$

For all $S > 0$, let $\mathcal{D}_S = [0,S] \times \widetilde{\mathcal{D}}$, where

$$\widetilde{\mathcal{D}} = \left\{x \in \mathbb{R}^N : \eta_1 < |x| \le \eta_2\right\}$$

and

$$\Psi_S = \left\{\psi \in C^3(\mathcal{D}_S) : \psi \ge 0, \operatorname{supp}(\psi) \subset\subset \mathcal{D}_S, \psi|_{\Sigma_2} = 0\right\}.$$

Definition 1. *Let $0 < \eta_1 < \eta_2$, $\iota \neq 0$, $m \ge 0$, $r > m+1$, $0 < \varrho, \varsigma < 1$, $\jmath = \jmath(t) > 0$ almost everywhere and $f \in L^{m+1}_{\mathrm{loc}}(\widetilde{\mathcal{D}})$. A function $u \in L^{m+1}_{\mathrm{loc}}([0,\infty) \times \widetilde{\mathcal{D}}) \cap L^r_{\mathrm{loc}}([0,\infty) \times \widetilde{\mathcal{D}}, \jmath\, dt\, dx)$ is a weak solution to (7) if, for every $S > 0$, it holds that*

$$\begin{aligned} &\int_{\mathcal{D}_S} |u|^r \jmath\, \psi\, dx\, dt - \int_{\mathcal{D}} \left(f I_S^{1-\varrho}\Delta\psi(0,x) + \iota |f|^m f I_S^{1-\varsigma}\psi(0,x)\right) dx \\ &\le -\int_{\mathcal{D}_S} u\Delta\psi\, dx\, dt + \int_{\mathcal{D}_S}\left(u\partial_t\left(I_S^{1-\varrho}\Delta\psi\right) + \iota|u|^m u \partial_t\left(I_S^{1-\varsigma}\psi\right)\right) dx\, dt \end{aligned} \tag{10}$$

for all $\psi \in \Psi_S$.

We denote $u \in \mathcal{WS}$ to say that u is a weak solution to (7). Making use of (9) and integrating by parts, we can show that, if u is a regular solution to (7), then $u \in \mathcal{WS}$.

3. Main Result

We now give our main result.

Theorem 1. *Let $N \ge 3$, $0 < \eta_1 < \eta_2$, $m \ge 0$, $r > m+1$, $\jmath = \jmath(t) > 0$ almost everywhere and $\jmath^{\frac{-1}{r-1}}, \jmath^{-\frac{m+1}{r-m-1}} \in L^1_{\mathrm{loc}}([0,\infty))$. Let $f \in L^{m+1}(\mathcal{D})$ and*

$$\iota \int_{\mathcal{D}} f(x)|f(x)|^m \left(\eta_2^{N-2}|x|^{2-N} - 1\right) dx < 0. \tag{11}$$

If $0 < \varsigma < \varrho < 1$ and

$$\liminf_{k\to\infty} k^{\frac{r+1}{r-1}+\zeta(\varsigma-1)}\int_0^{k^\zeta} \jmath^{\frac{-1}{r-1}}(t)\,dt + k^{\zeta\left(\frac{-r+(m+1)(1-\varsigma)}{r-m-1}\right)}\int_0^{k^\zeta} \jmath^{-\frac{m+1}{r-m-1}}(t)\,dt = 0 \tag{12}$$

for some $\zeta \gg 1$, then $\mathcal{WS} = \emptyset$.

Assume now that

$$\jmath(s) = s^d,$$

where

$$d(m+1) < r - m - 1.$$

In this case, we obtain

$$\int_0^{k^\zeta} \jmath^{\frac{-1}{r-1}}(t)\,dt = Ck^{\frac{\zeta(r-1-d)}{r-1}}$$

and

$$\int_0^{k^\zeta} \jmath^{-\frac{m+1}{r-m-1}}(t)\,dt = Ck^{\frac{\zeta(r-(m+1)(d+1))}{r-m-1}},$$

which yield

$$k^{\frac{r+1}{r-1}+\zeta(\varsigma-1)}\int_0^{k^\zeta} \jmath^{\frac{-1}{r-1}}(t)\,dt + k^{\zeta\left(\frac{-r+(m+1)(1-\varsigma)}{r-m-1}\right)}\int_0^{k^\zeta} \jmath^{-\frac{m+1}{r-m-1}}(t)\,dt \le C\left(k^{R_1}+k^{R_2}\right),$$

where

$$R_1 = \frac{\zeta(\varsigma(r-1)-d)+r+1}{r-1}$$

and

$$R_2 = \frac{-\zeta(m+1)(\sigma+d)}{r-m-1}.$$

Observe that (12) holds if, for some $\zeta \gg 1$, one has $R_i < 0$, $i = 1, 2$. On the other hand, $R_1 < 0$ if and only if

$$\zeta(\varsigma(r-1)-d) < -(r+1) < 0.$$

So, for $d > \varsigma(r-1)$, taking

$$\zeta > \frac{r+1}{d-\varsigma(r-1)},$$

we obtain $R_1 < 0$. Furthermore, since $\varsigma(r-1) > -\varsigma$, then $d > \varsigma(r-1)$ implies that $d > -\varsigma$, and then $R_2 < 0$. Consequently, if

$$\varsigma(r-1) < d < \frac{r-m-1}{m+1},$$

then (12) holds. But in order to ensure that the set of d satisfying the above condition is nonempty, we should impose that $\varsigma(r-1) < \frac{r-m-1}{m+1}$, that is,

$$\varsigma < \frac{1}{m+1},\ \ r > \frac{(m+1)(1-\varsigma)}{1-(m+1)\varsigma}\ (\ge m+1).$$

Therefore, by Theorem 1, the following result holds.

Corollary 1. *Let $N \ge 3$, $0 < \eta_1 < \eta_2$, $m \ge 0$ and $\jmath(t) = t^d$. Let $f \in L^{m+1}(\mathcal{D})$ satisfy (11). If $0 < \varsigma < \min\left\{\varrho, \frac{1}{m+1}\right\}$, $\varrho < 1$ and*

$$r > \frac{(m+1)(1-\varsigma)}{1-(m+1)\varsigma},\ \ \varsigma(r-1) < d < \frac{r-m-1}{m+1},$$

then $\mathcal{WS} = \emptyset$.

An example illustrating the above result is given below.

Example 1. *Consider the problem*

$$\begin{cases} -\Delta(\partial_{t^{1/2}}u) - \partial_{t^{1/8}}(|u|u) - \Delta u \ge t^{1/4}|u|^r, & t>0,\ \eta_1<|x|<\eta_2, \\ u(0,x) = -|x|^\alpha, & \eta_1<|x|<\eta_2, \\ u = 0 \quad \text{on } \Sigma_2, \end{cases} \tag{13}$$

where $\alpha \in \mathbb{R}$ and $r > 2$. Problem (13) is a special case of (7) with

$$m=1,\ \varrho = \frac{1}{2},\ \varsigma = \frac{1}{8},\ \iota = 1,\ \jmath(t) = t^d,\ d = \frac{1}{4},\ f(x) = -|x|^\alpha.$$

Clearly, the function f belongs to $L^2(\mathcal{D})$ and satisfies (11). Moreover, we have

$$0 < \varsigma = \frac{1}{8} < \frac{1}{2} = \min\left\{\varrho, \frac{1}{m+1}\right\},\ \varrho = \frac{1}{2} < 1.$$

On the other hand,

$$r > \frac{(m+1)(1-\varsigma)}{1-(m+1)\varsigma}$$

is equivalent to

$$r > \frac{7}{3}$$

and

$$\varsigma(r-1) < d < \frac{r-m-1}{m+1}$$

is equivalent to

$$\frac{5}{2} < r < 3.$$

Hence, by Corollary 1, we deduce that, if $\frac{5}{2} < r < 3$, *then* (13) *possesses no weak solution.*

4. Proof of the Main Result

This section is devoted to the proof of Theorem 1. We first fix some notations:

- C, C_i: positive constants independent of S and k (their values are not necessarily the same from one inequality to another);
- C_τ, $\tau > 0$: positive constant that depends only on τ but not on S or k;
- $\ell \gg 1$, $\ell \in \mathbb{R}$: $\ell > 0$ is sufficient large.

The following preliminary results will be useful in the proof.

4.1. Preliminaries

Let $0 < \eta_1 < \eta_2$ and $N \geq 3$.
Let

$$v(x) = \eta_2^{N-2}|x|^{2-N} - 1, \quad x \in \widetilde{\mathcal{D}}. \tag{14}$$

The proof of the following lemma is immediate.

Lemma 1. *We have*

$$v \geq 0, \ \Delta v = 0, \ v|_{\Sigma_2} = 0.$$

For $k \gg 1$, let $\{\gamma_k\}$ be a sequence of smooth functions defined in $\widetilde{\mathcal{D}}$ and satisfy

$$\begin{gathered} 0 \leq \gamma_k \leq 1, \ \operatorname{supp}(\gamma_k) \subset\subset \left\{ x \in \mathbb{R}^N : \eta_1 + \frac{1}{2k} \leq |x| \leq \eta_2 \right\}; \\ \gamma_k = 1 \text{ if } \eta_1 + \frac{1}{k} \leq |x| \leq \eta_2, \ |\nabla \gamma_k| \leq Ck, \ |\Delta \gamma_k| \leq Ck^2. \end{gathered} \tag{15}$$

For the existence of such functions, see, e.g., [21].

For $q \gg 1$, we also consider the sequence of functions $\{\alpha_k\}$ defined in $\widetilde{\mathcal{D}}$ by

$$\alpha_k(x) = v(x)\gamma_k^q(x).$$

Lemma 2. *Let* $r > 1$. *We have*

$$v(x) \leq C, \qquad \eta_1 + \frac{1}{2k} \leq |x| \leq \eta_2, \tag{16}$$

$$v(x) \geq C, \qquad \eta_1 + \frac{1}{2k} \leq |x| \leq \eta_1 + \frac{1}{k} \tag{17}$$

and

$$\int_{\eta_1 + \frac{1}{2k} \leq |x| \leq \eta_1 + \frac{1}{k}} \alpha_k^{\frac{-1}{r-1}}(x) |\Delta \alpha_k(x)|^{\frac{r}{r-1}} \, dx \leq Ck^{\frac{r+1}{r-1}}. \tag{18}$$

Proof. The estimates (16) and (17) are obvious. Let $\eta_1 + \frac{1}{2k} \leq |x| \leq \eta_1 + \frac{1}{k}$. By Lemma 1, we have

$$\begin{aligned}\Delta\alpha_k(x) &= v(x)\Delta(\gamma_k^q)(x) + \gamma_k^q(x)\Delta v(x) + 2\nabla v(x)\cdot\nabla(\gamma_k^q)(x)\\ &= v(x)\Delta(\gamma_k^q)(x) + 2\nabla v(x)\cdot\nabla(\gamma_k^q)(x),\end{aligned} \tag{19}$$

which implies by (15) and (16) that

$$\begin{aligned}|\Delta\alpha_k(x)| &\leq C\Big(\gamma_k^{q-2}(x)|\nabla\gamma_k(x)|^2 + \gamma_k^{q-1}(x)|\Delta\gamma_k(x)| + |\nabla v(x)|\gamma_k^{q-1}(x)|\nabla\gamma_k(x)|\Big)\\ &\leq C\Big(k^2+k^2+k\Big)\gamma_k^{q-2}(x)\\ &\leq Ck^2\gamma_k^{q-2}(x).\end{aligned} \tag{20}$$

Here, $\cdot$ is the inner product in $\mathbb{R}^N$. Then, by (17), we obtain (since $0 \leq \gamma_k \leq 1$)

$$\alpha_k^{\frac{-1}{r-1}}(x)|\Delta\alpha_k(x)|^{\frac{r}{r-1}} \leq Ck^{\frac{2r}{r-1}}\gamma_k^{q-\frac{2r}{r-1}}(x) \leq Ck^{\frac{2r}{r-1}}.$$

We finally integrate the above inequality over $\eta_1 + \frac{1}{2k} \leq |x| \leq \eta_1 + \frac{1}{k}$ to obtain (18). □

For all $S > 0$, let

$$\theta(s) = S^{-q}(S-s)^q, \quad 0 \leq s \leq S.$$

The following result can be found in [22].

Lemma 3. *Let $0 < \tau < 1 < j$. For all $0 \leq s \leq S$, we have*

$$I_S^{1-\tau}\theta(s) = C_\tau S^{-q}(S-s)^{q+1-\tau} \tag{21}$$

and

$$\theta^{\frac{-1}{j-1}}(s)\left|\left(I_S^{1-\tau}\theta\right)'(s)\right|^{\frac{j}{j-1}} \leq CS^{-q}(S-s)^{q-\frac{\tau j}{j-1}}. \tag{22}$$

4.2. Proof of Theorem 1

Suppose that $u \in \mathcal{WS}$. For $q, S, k \gg 1$, let

$$\psi(t,x) = \theta(t)\alpha_k(x), \quad (t,x) \in \mathcal{D}_S,$$

where θ and α_k are the functions defined in the previous section. By the properties of α_k and θ, we can show that $\psi \in \Psi_S$. Hence, by (10), it holds that

$$\begin{aligned}&\int_{\mathcal{D}_S}|u|^r\jmath\,\psi\,dx\,dt - \int_{\mathcal{D}}\Big(f I_S^{1-\varrho}\Delta\psi(0,x) + \iota|f|^m f I_S^{1-\varsigma}\psi(0,x)\Big)\,dx\\ &\leq \int_{\mathcal{D}_S}|u||\Delta\psi|\,dx\,dt + \int_{\mathcal{D}_S}|u|\left|\partial_t\left(I_S^{1-\varrho}\Delta\psi\right)\right|dx\,dt + |\iota|\int_{\mathcal{D}_S}|u|^{m+1}\left|\partial_t\left(I_S^{1-\varsigma}\psi\right)\right|dx\,dt.\end{aligned} \tag{23}$$

Young's inequality gives us that

$$\begin{aligned}\int_{\mathcal{D}_S}|u||\Delta\psi|\,dx\,dt &= \int_{\mathcal{D}_S}\left(|u|\jmath^{\frac{1}{r}}\psi^{\frac{1}{r}}\right)\left(\jmath^{\frac{-1}{r}}\psi^{\frac{-1}{r}}|\Delta\psi|\right)dx\,dt\\ &\leq \frac{1}{3}\int_{\mathcal{D}_S}|u|^r\jmath\,\psi\,dx\,dt + C\int_{\mathrm{supp}(\Delta\psi)}\psi^{\frac{-1}{r-1}}\jmath^{\frac{-1}{r-1}}|\Delta\psi|^{\frac{r}{r-1}}\,dx\,dt.\end{aligned} \tag{24}$$

Similarly, we obtain

$$\begin{aligned}&\int_{\mathcal{D}_S}|u|\left|\partial_t\left(I_S^{1-\varrho}\Delta\psi\right)\right|dx\,dt\\ &\leq \frac{1}{3}\int_{\mathcal{D}_S}|u|^r\jmath\,\psi\,dx\,dt + C\int_{\mathrm{supp}\left(\partial_t\left(I_S^{1-\varrho}\Delta\psi\right)\right)}\psi^{\frac{-1}{r-1}}\jmath^{\frac{-1}{r-1}}\left|\partial_t\left(I_S^{1-\varrho}\Delta\psi\right)\right|^{\frac{r}{r-1}}dx\,dt\end{aligned} \tag{25}$$

and

$$\int_{\mathcal{D}_S} |u|^{m+1} \left| \partial_t \left(I_S^{1-\varsigma} \psi \right) \right| dx\, dt \\ \leq \frac{1}{3} \int_{\mathcal{D}_S} |u|^r \jmath\, \psi\, dx\, dt + C \int_{\mathrm{supp}\left(\partial_t \left(I_S^{1-\varsigma}\psi\right)\right)} \psi^{-\frac{m+1}{r-m-1}} \jmath^{-\frac{m+1}{r-m-1}} \left| \partial_t \left(I_S^{1-\varsigma} \psi \right) \right|^{\frac{r}{r-m-1}} dx\, dt. \tag{26}$$

Then, (23)–(26) yield

$$\chi_f := -\int_{\mathcal{D}} \left(f I_S^{1-\varrho} \Delta\psi(0,x) + \iota |f|^m f I_S^{1-\varsigma} \psi(0,x) \right) dx \leq C \sum_{j=1}^{3} \Xi_j, \tag{27}$$

where

$$\begin{aligned} \Xi_1 &= \int_{\mathrm{supp}(\Delta\psi)} \psi^{\frac{-1}{r-1}} \jmath^{\frac{-1}{r-1}} |\Delta\psi|^{\frac{r}{r-1}} \, dx\, dt, \\ \Xi_2 &= \int_{\mathrm{supp}\left(\partial_t\left(I_S^{1-\varrho}\Delta\psi\right)\right)} \psi^{\frac{-1}{r-1}} \jmath^{\frac{-1}{r-1}} \left| \partial_t \left(I_S^{1-\varrho} \Delta\psi \right) \right|^{\frac{r}{r-1}} dx\, dt, \\ \Xi_3 &= \int_{\mathrm{supp}\left(\partial_t\left(I_S^{1-\varsigma}\psi\right)\right)} \psi^{-\frac{m+1}{r-m-1}} \jmath^{-\frac{m+1}{r-m-1}} \left| \partial_t \left(I_S^{1-\varsigma} \psi \right) \right|^{\frac{r}{r-m-1}} dx\, dt. \end{aligned}$$

On the other hand, in view of (15), (18) and (19), we have

$$\begin{aligned} \Xi_1 &= \int_0^S \int_{\eta_1 + \frac{1}{2k} < |x| < \eta_1 + \frac{1}{k}} \theta(t) \alpha_k^{\frac{-1}{r-1}}(x) |\Delta\alpha_k(x)|^{\frac{r}{r-1}} \jmath^{\frac{-1}{r-1}}(t)\, dx\, dt \\ &= \left(\int_{\eta_1 + \frac{1}{2k} < |x| < \eta_1 + \frac{1}{k}} \alpha_k^{\frac{-1}{r-1}}(x) |\Delta\alpha_k(x)|^{\frac{r}{r-1}}\, dx \right) \left(\int_0^S \theta(t) \jmath^{\frac{-1}{r-1}}(t)\, dt \right) \\ &\leq C k^{\frac{r+1}{r-1}} \int_0^S \jmath^{\frac{-1}{r-1}}(t)\, dt. \end{aligned} \tag{28}$$

We also have, by (18) and (22),

$$\begin{aligned} \Xi_2 &= \int_0^S \int_{\eta_1 + \frac{1}{2k} < |x| < \eta_1 + \frac{1}{k}} \theta^{\frac{-1}{r-1}}(t) \left| \left(I_S^{1-\varrho}\theta \right)'(t) \right|^{\frac{r}{r-1}} \alpha_k^{\frac{-1}{r-1}}(x) |\Delta\alpha_k(x)|^{\frac{r}{r-1}} \jmath^{\frac{-1}{r-1}}(t)\, dx\, dt \\ &= \left(\int_{\eta_1 + \frac{1}{2k} < |x| < \eta_1 + \frac{1}{k}} \alpha_k^{\frac{-1}{r-1}}(x) |\Delta\alpha_k(x)|^{\frac{r}{r-1}}\, dx \right) \left(\int_0^S \theta^{\frac{-1}{r-1}}(t) \left| \left(I_S^{1-\varrho}\theta \right)'(t) \right|^{\frac{r}{r-1}} \jmath^{\frac{-1}{r-1}}(t)\, dt \right) \\ &\leq C k^{\frac{r+1}{r-1}} S^{\frac{-\varrho r}{r-1}} \int_0^S \jmath^{\frac{-1}{r-1}}(t)\, dt. \end{aligned} \tag{29}$$

By (15), (16) and (22), we have

$$\begin{aligned} \Xi_3 &= \int_0^S \int_{\eta_1 + \frac{1}{2k} < |x| < \eta_2} \theta^{-\frac{m+1}{r-m-1}}(t) \left| \left(I_S^{1-\varsigma}\theta \right)'(t) \right|^{\frac{r}{r-m-1}} \alpha_k(x) \jmath^{-\frac{m+1}{r-m-1}}(t)\, dx\, dt \\ &= \left(\int_{\eta_1 + \frac{1}{2k} < |x| < \eta_2} \alpha_k(x)\, dx \right) \left(\int_0^S \theta^{-\frac{m+1}{r-m-1}}(t) \left| \left(I_S^{1-\varsigma}\theta \right)'(t) \right|^{\frac{r}{r-m-1}} \jmath^{-\frac{m+1}{r-m-1}}(t)\, dt \right) \\ &\leq C S^{\frac{-\varsigma r}{r-m-1}} \int_0^S \jmath^{-\frac{m+1}{r-m-1}}(t)\, dt. \end{aligned} \tag{30}$$

Furthermore, by (15) and (21), we obtain

$$\begin{aligned}\chi_f = &-C_\varrho S^{1-\varrho}\int_{\eta_1+\frac{1}{2k}<|x|<\eta_1+\frac{1}{k}} f(x)\Delta\alpha_k(x)\,dx\\ &- C_\varsigma \iota S^{1-\varsigma}\int_{\eta_1+\frac{1}{2k}<|x|<\eta_2} f(x)|f(x)|^m\alpha_k(x)\,dx.\end{aligned} \tag{31}$$

Next, in view of (27)–(31), it holds that

$$\begin{aligned}&- C_\varsigma \iota S^{1-\varsigma}\int_{\eta_1+\frac{1}{2k}<|x|<\eta_2} f(x)|f(x)|^m\alpha_k(x)\,dx\\ &\leq C\left(k^{\frac{r+1}{r-1}}\int_0^S \jmath^{\frac{-1}{r-1}}(t)\,dt + k^{\frac{r+1}{r-1}}S^{\frac{-\varrho r}{r-1}}\int_0^S \jmath^{\frac{-1}{r-1}}(t)\,dt + S^{\frac{-\varsigma r}{r-m-1}}\int_0^S \jmath^{-\frac{m+1}{r-m-1}}(t)\,dt\right)\\ &\quad + C_\varrho S^{1-\varrho}\int_{\eta_1+\frac{1}{2k}<|x|<\eta_1+\frac{1}{k}} f(x)\Delta\alpha_k(x)\,dx,\end{aligned}$$

that is,

$$\begin{aligned}&- C_\varsigma \iota \int_{\eta_1+\frac{1}{2k}<|x|<\eta_2} f(x)|f(x)|^m\alpha_k(x)\,dx\\ &\leq C\left(k^{\frac{r+1}{r-1}}S^{\varsigma-1}\int_0^S \jmath^{\frac{-1}{r-1}}(t)\,dt + k^{\frac{r+1}{r-1}}S^{\frac{-\varrho r}{r-1}+\varsigma-1}\int_0^S \jmath^{\frac{-1}{r-1}}(t)\,dt + S^{\frac{-\varsigma r}{r-m-1}+\varsigma-1}\int_0^S \jmath^{-\frac{m+1}{r-m-1}}(t)\,dt\right)\\ &\quad + C_\varrho S^{\varsigma-\varrho}\int_{\eta_1+\frac{1}{2k}<|x|<\eta_1+\frac{1}{k}} f(x)\Delta\alpha_k(x)\,dx.\end{aligned}$$

Since

$$\varsigma-1 > \frac{-\varrho r}{r-1}+\varsigma-1,$$

the above estimate yields

$$\begin{aligned}&- C_\varsigma \iota \int_{\eta_1+\frac{1}{2k}<|x|<\eta_2} f(x)|f(x)|^m\alpha_k(x)\,dx\\ &\leq C\left(k^{\frac{r+1}{r-1}}S^{\varsigma-1}\int_0^S \jmath^{\frac{-1}{r-1}}(t)\,dt + +S^{\frac{-\varsigma r}{r-m-1}+\varsigma-1}\int_0^S \jmath^{-\frac{m+1}{r-m-1}}(t)\,dt\right)\\ &\quad + C_\varrho S^{\varsigma-\varrho}\int_{\eta_1+\frac{1}{2k}<|x|<\eta_1+\frac{1}{k}} f(x)\Delta\alpha_k(x)\,dx.\end{aligned}$$

On the other hand, using (20) and the fact that $f\in L^1(\mathcal{D})$, we obtain

$$\begin{aligned}\int_{\eta_1+\frac{1}{2k}<|x|<\eta_1+\frac{1}{k}} f(x)\Delta\alpha_k(x)\,dx &\leq \int_{\eta_1+\frac{1}{2k}<|x|<\eta_1+\frac{1}{k}} |f(x)||\Delta\alpha_k(x)|\,dx\\ &\leq Ck^2\int_{\eta_1+\frac{1}{2k}<|x|<\eta_1+\frac{1}{k}} |f(x)|\,dx\\ &\leq Ck^2\int_{\mathcal{D}}|f(x)|\,dx.\end{aligned}$$

Consequently, it holds that

$$\begin{aligned}&- C_\varsigma \iota \int_{\eta_1+\frac{1}{2k}<|x|<\eta_2} f(x)|f(x)|^m\alpha_k(x)\,dx\\ &\leq C\left(k^{\frac{r+1}{r-1}}S^{\varsigma-1}\int_0^S \jmath^{\frac{-1}{r-1}}(t)\,dt + S^{\frac{-r+(m+1)(1-\varsigma)}{r-m-1}}\int_0^S \jmath^{-\frac{m+1}{r-m-1}}(t)\,dt + k^2S^{\varsigma-\varrho}\int_{\mathcal{D}}|f(x)|\,dx\right).\end{aligned}$$

Taking $S=k^\zeta$, where

$$\zeta > \frac{2}{\varrho-\varsigma},$$

(notice that $\varrho > \varsigma$), we obtain

$$
\begin{aligned}
&- C_\varsigma \iota \int_{\eta_1+\frac{1}{2k}<|x|<\eta_2} f(x)|f(x)|^m \alpha_k(x)\,dx \\
&\leq C\left(k^{\frac{r+1}{r-1}+\zeta(\varsigma-1)} \int_0^{k^\zeta} \jmath^{\frac{-1}{r-1}}(t)\,dt + k^{\zeta\left(\frac{-r+(m+1)(1-\varsigma)}{r-m-1}\right)} \int_0^{k^\zeta} \jmath^{-\frac{m+1}{r-m-1}}(t)\,dt + k^{2+(\varsigma-\varrho)\zeta} \int_{\mathcal{D}} |f(x)|\,dx \right).
\end{aligned}
$$

Taking the infimum limit as $k \to \infty$ in the above inequality, using (15) and using the fact that $f \in L^{m+1}(\mathcal{D})$, we obtain

$$
\begin{aligned}
&- C_\varsigma \iota \int_{\mathcal{D}} f(x)|f(x)|^m v(x)\,dx \\
&\leq \liminf_{k\to\infty} k^{\frac{r+1}{r-1}+\zeta(\varsigma-1)} \int_0^{k^\zeta} \jmath^{\frac{-1}{r-1}}(t)\,dt + k^{\zeta\left(\frac{-r+(m+1)(1-\varsigma)}{r-m-1}\right)} \int_0^{k^\zeta} \jmath^{-\frac{m+1}{r-m-1}}(t)\,dt.
\end{aligned}
$$

In particular, for $\zeta \gg 1$ satisfying (12), we obtain

$$
\iota \int_{\mathcal{D}} f(x)|f(x)|^m v(x)\,dx \geq 0,
$$

which is a contradiction with (11). Consequently, $\mathcal{WS} = \emptyset$. □

5. Conclusions

We studied problem (7) that belongs to the class of time-fractional differential inequalities of Sobolev-type. By means of the nonlinear capacity methods, a sufficient condition under which $\mathcal{WS} = \emptyset$ is obtained (see Theorem 1). We next studied in detail the particular case $\jmath(s) = s^d$ (see Corollary 1).

In this study, we considered the N-dimensional case, where $N \geq 3$. The cases $N = 1$ or $N = 2$ are not investigated. A careful study of such cases would be interesting. Notice that the dimension N is involved in the choice of the function $\psi \in \Psi_S$, and more precisely in the definition of the function v given by (14).

The weight function $\jmath$ involved in (7) depends only on the time-variable t. It would also be interesting to consider the general case, when j depends on t and x, that is, $\jmath = \jmath(t,x)$.

Author Contributions: Conceptualization, M.J. and B.S.; formal analysis, E.A.; investigation, A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research project was funded by the Deanship of Scientific Research, Princess Nourah bint Abdulrahman University, through the Program of Research Project Funding After Publication, grant No. (43- PRFA-P-68).

Data Availability Statement: Data sharing are not applicable to this article, as no datasets were generated or analyzed during the current study.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Al'shin, A.B.; Korpusov, M.O.; Sveshnikov, A.G. Blow-up in nonlinear Sobolev type equations. In *Series in Nonlinear Analysis and Applications*; Walter de Gruyter: Berlin, Germany, 2011; Volume 15, 660p.
2. Zamyshlyaeva, A.A.; Muravyev, A.S. Computational experiment for one mathematical model of ion-acoustic waves. *Bull. South Ural. State Univ. Ser. Math. Program. Comput. Softw.* **2015**, *8*, 127–132. [CrossRef]
3. Zamyshlyaeva, A.A.; Bychkov, E.V.; Tsyplenkova, O.N. Mathematical models based on Boussinesq-Love equation. *Appl. Math. Sci.* **2014**, *8*, 5477–5483. [CrossRef]
4. Sviridyuk, G.A.; Zagrebina, S.A. The Showalter-Sidorov problem as a phenomena of the Sobolev-type equations. *Bull. Irkutsk. State Univ. Ser. Math.* **2010**, *3*, 104–125.
5. Sviridyuk, G.A. A problem for the generalized Boussinesq filtration equations. *Sov. Math.* **1989**, *33*, 62–73.
6. Korpusov, M.; Sveshnikov, A. Application of the nonlinear capacity method to differential inequalities of Sobolev type. *Differ. Equ.* **2009**, *45*, 951–959. [CrossRef]

7. Korpusov, M.; Sveshnikov, A. Blow-up of solutions of abstract Cauchy problems for nonlinear differential-operator equations. *Dokl. Akad. Nauk.* **2005**, *401*, 168–171.
8. Korpusov, M.O. Critical exponents of instantaneous blow-up or local solubility of non-linear equations of Sobolev type. *Izv. RAN. Ser. Mat.* **2015**, *79*, 103–162. [CrossRef]
9. Korpusov, M.O.; Panin, A.A. On the nonextendable solution and blow-up of the solution of the one-dimensional equation of ion-sound waves in a plasma. *Math. Notes* **2017**, *102*, 350–360. [CrossRef]
10. Jleli, M.; Samet, B. Instantaneous blow-up for nonlinear Sobolev type equations with potentials on Riemannian manifolds. *Commun. Pure Appl. Anal.* **2022**, *21*, 2065–2078. [CrossRef]
11. Alsaedi, A.; Alhothuali, M.S.; Ahmad, B.; Kerbal, S.; Kirane, M. Nonlinear fractional differential equations of Sobolev type. *Math. Methods Appl. Sci.* **2014**, *37*, 2009–2016. [CrossRef]
12. Kirane, M.; Laskri, Y.; Tatar, N.E. Critical exponents of Fujita type for certain evolution equations and systems with spatio-temporal fractional derivatives. *J. Math. Anal. Appl.* **2005**, *312*, 488–501. [CrossRef]
13. Kirane, M.; Tatar, N.E. Exponential growth for a fractionally damped wave equation. *Z. Anal. Anwend.* **2003**, *22*, 167–177. [CrossRef]
14. Li, C.; Li, Z. The finite-time blow-up for semilinear fractional diffusion equations with time-Caputo derivative. *J. Nonlinear Sci.* **2022**, *32*, 1–42. [CrossRef]
15. Fino, A.Z.; Kirane, M. Qualitative properties of solutions to a time-space fractional evolution equation. *Quart. Appl. Math.* **2012**, *70*, 133–157. [CrossRef]
16. Agarwal, R.P.; Alhumayan, S.M.; Jleli, M.; Samet, B. Nonexistence of global solutions to higher-order time-fractional evolution inequalities with subcritical degeneracy. *Mathematics* **2021**, *9*, 2765.
17. Zhang, Q.; Sun, H.R.; Li, Y. The nonexistence of global solutions for a time fractional nonlinear Schrödinger equation without gauge invariance. *Appl. Math. Lett.* **2017**, *64*, 119–124.
18. Pokhozaev, S.I. Critical nonlinearities in partial differential equations. *Milan J. Math.* **2009**, *77*, 127–150. [CrossRef]
19. Mitidieri, E.; Pokhozhaev, S.I. A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. *Tr. Mat. Inst. Steklova* **2001**, *234*, 1–384.
20. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach Science Publishers: Amsterdam, The Nertherland, 1993.
21. Jleli, M.; Samet, B. Nonexistence criteria for systems of parabolic inequalities in an annulus. *J. Math. Anal. Appl.* **2022**, *514*, 126352. [CrossRef]
22. Sultan, A.B.; Jleli, M.; Samet, B.; Vetro, C. On the critical behavior for time-fractional pseudo-parabolic-type equations with combined nonlinearities. *Bound. Value Probl.* **2022**, *2022*, 1–20.