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Two Convergence Results for Inexact Orbits of Nonexpansive Operators in Metric Spaces with Graphs

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Abstract: In this work we show that if iterates of a nonexpansive self-mapping of a complete metric with a graph converge uniformly on a subset of the space, then this convergence is stable under the presence of small computational errors.

Keywords: graph; fixed point; iterate; metric space

MSC: 47H09; 47H10; 54E50

1. Introduction

The starting point of the fixed point theory is Banach's seminal paper [1], where it was established that a strict contraction has a fixed point. Since then, various interesting and important results were obtained in this area of research [2–20], which includes the investigation of common fixed points and variational inequalities and their applications [21–30].

In [6], a map acting on a space equipped with a complete metric was considered. Under the assumptions that the map is uniformly continuous on bounded subsets of the space and that all its exact iterates converge uniformly on bounded subsets of the space, it was shown that this convergence is stable under the presence of sufficiently small computational errors. In this work, we generalize this result for nonexpansive self-mapping of a complete metric space with a graph. Note that this class of mappings have recently been discussed in [12,31–39].

Suppose that (X, ρ) is a space equipped with a metric ρ. Denote by $\mathbf{N}$ the set of all natural numbers, by R the set of all real numbers, and by R_+ the set of all positive real numbers. For each $h \in X$ and each nonempty set $C \subset X$, put

$$\rho(h, C) := \inf\{\rho(h, \xi) : \ \xi \in C\}.$$

For each $h \in X$ and each $\Delta \in R_+$, set

$$B(h, \Delta) := \{\xi \in X : \ \rho(h, \xi) \leq \Delta\}.$$

For every map $A{:}X \to X$, set $A^0(v) = v$ for all $v \in X$, $A^1 = A$ and $A^{i+1} = A \circ A^i$ for every nonnegative integer i.

We say that a map $T : X \to X$ is a strict contraction if there is $\lambda \in (0, 1)$, for which

$$\rho(T(u), T(z)) \leq \lambda \rho(u, z)$$

for each $u, z \in X$.

Banach's theorem [1] implies that T possesses a unique point $x_T \in X$, such that

$$T(x_T) = x_T$$

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Citation: Zaslavski, A.J. Two Convergence Results for Inexact Orbits of Nonexpansive Operators in Metric Spaces with Graphs. *Axioms* **2023**, *12*, 999. https://doi.org/10.3390/axioms12100999

Academic Editor: Xueliang Li

Received: 24 September 2023
Revised: 17 October 2023
Accepted: 19 October 2023
Published: 23 October 2023

and that for each $u \in X$,

$$\lim_{i\to\infty} T^i(u) = x_T.$$

Moreover, it is known that this convergence is uniform on all bounded sets.

In [18], A. M. Ostrowski studied the influence of small errors on the convergence of iterates of the strict contraction T, and showed that every sequence $\{u_i\}_{i=0}^{\infty} \subset X$ satisfying

$$\sum_{i=0}^{\infty} \rho(u_{i+1}, T(u_i)) < \infty$$

converges to the fixed point x_T of the map T. In other words, every sequence of inexact iterates of the strict contraction with summable errors converges to its fixed point.

The next step in this direction was performed in [5], where a different approach was used. In that paper, we considered a map $T : X \to X$, which is merely nonexpansive. In other words,

$$\rho(T(u), T(v)) \le \rho(u, v)$$

for all $u, v \in X$. We assumed that for each $u \in X$, the sequence $\{T^n(u)\}_{n=1}^{\infty}$ converges in (X, ρ), and showed that every sequence $\{u_i\}_{i=0}^{\infty} \subset X$ satisfying

$$\sum_{i=0}^{\infty} \rho(u_{i+1}, T(u_i)) < \infty$$

converges to a fixed point of the map T. In other words, if every sequence of exact iterates of the nonexpansive map T converges, then every sequence of inexact iterates of T with summable errors converges to its fixed point too.

This result is an important generalization of the result of [18], since for most of nonexpansive mappings (in the sense of Baire category), all exact iterates converge [40]. The result of [5] mentioned above has numerous applications [22–24,28].

For example, if $(X, \|\cdot\|)$ is a Banach space, $\rho(u, v) = \|u - v\|$ for all $u, v \in X$, for each $v \in X$, the sequence $\{T^n(v)\}_{n=1}^{\infty}$ converges in the norm topology, $z_0 \in X$, $\{\beta_k\}_{k=0}^{\infty} \subset (0, \infty)$ satisfies

$$\sum_{k=0}^{\infty} \beta_k < \infty,$$

$\{u_k\}_{k=0}^{\infty} \subset X$ satisfies

$$\sup\{\|u_k\|\}_{k=0}^{\infty} < \infty$$

and if for any $k \in \mathbf{N} \cup \{0\}$,

$$z_{k+1} = T(z_k + \beta_k u_k),$$

then the sequence $\{z_k\}_{k=0}^{\infty}$ converges in the norm topology of X. We can choose the bounded sequence $\{u_k\}_{k=0}^{\infty}$ such that the sequence $\{g(z_k)\}_{k=1}^{\infty}$ is decreasing where g is a given objective function.

It should be mentioned that if the map T is a strict contraction, then its exact iterates converge to its unique fixed point uniformly on bounded sets in a complete metric space X. Moreover, this uniform convergence holds for most of nonexpansive mappings (in the sense of Baire category) [40]. It turns out that the uniform convergence of iterates of nonexpansive mappings on bounded sets is stable under small errors that are not necessarily summable. The first result in this direction was obtained in [6]. Note that the results of this kind were obtained for operators acting on metric space without graphs. In this paper, we show that if iterates of a nonexpansive self-mapping of a complete metric with a graph converge uniformly on a subset of the space to some set, then this convergence is stable under the presence of small computational errors.

Recall (X, ρ) is a metric space. Let G be a graph for which $V(G) \subset X$ is the set of all its vertices and the set $E(G) \subset X \times X$ is the set of all its edges. We identify the graph G with $(V(G), E(G))$.

Fix $\theta \in X$.

Let $T : X \to X$ be a mapping and that the following assumption holds:

(A) For each $\xi, \eta \in X$ satisfying $(\xi, \eta) \in E(G)$ the relations

$$(T(\xi), T(\eta)) \in E(G) \text{ and } \rho(T(\xi), T(\eta)) \leq \rho(\xi, \eta)$$

are valid.

2. The First Main Result

Theorem 1. *Assume that $F,\ X_0 \subset X$ are nonempty subsets of X, $r_0 > 0$,*

$$\cup\{B(x, r_0) :\ x \in F\} \subset X_0 \tag{1}$$

and that

$$\lim_{n\to\infty} \rho(T^n(x), F) = 0$$

uniformly on X_0. Let $\epsilon \in (0, r_0)$. Then, there exists $n_0 \in \mathbf{N}$, such that for each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfying

$$x_0 \in X_0$$

and

$$\rho(T(x_i), x_{i+1}) \leq (8n_0)^{-1}\epsilon, \tag{2}$$

$$(T(x_i), x_{i+1}) \in E(G) \tag{3}$$

for each integer $i \in \mathbf{N} \cup \{0\}$ the inequality $\rho(x_i, F) \leq \epsilon$ holds for each integer $i \geq n_0$.

Proof. By our assumptions, there is $n_0 \in \mathbf{N}$, such that the following property holds:

(a) For each $x \in X_0$, the relation $\rho(T^i(x), F) < \epsilon/8$ is valid for all $i \in \mathbf{N} \cap [n_0, \infty)$.

Assume that $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies

$$x_0 \in X_0$$

and that for each $i \in \mathbf{N} \cup \{0\}$, Equations (2) and (3) hold. Set

$$\delta = (8n_0)^{-1}\epsilon. \tag{4}$$

Let $p \in \mathbf{N} \cup \{0\}$ and $q \in \mathbf{N}$. By (2), (3) and assumption (A), for each $i, j \in \mathbf{N} \cup \{0\}$,

$$(T(x_{p+i}), x_{p+i+1}) \in E(G), \tag{5}$$

$$\rho(x_{p+i+1}, T(x_{p+i})) \leq \delta, \tag{6}$$

$$(T^{j+1}(x_{p+i}), T^j(x_{p+i+1})) \in E(G). \tag{7}$$

It follows from (5), (6) and assumption (A) that

$$\rho(T^{j+1}(x_{p+i}), T^j(x_{p+i+1})) \leq \rho(T(x_{p+i}), x_{p+i+1})) \leq \delta. \tag{8}$$

In view of (8),

$$\rho(x_{p+q}, T^q(x_p)) \leq \sum_{i=0}^{q-1} \rho(T^{q-i-1}(x_{p+i+1}), T^{q-i}(x_{p+i})) \leq q\delta.$$

Thus, for each $p \in \mathbf{N} \cup \{0\}$ and each $q \in \mathbf{N}$,

$$\rho(x_{p+q}, T^q(x_p)) \leq q\delta. \tag{9}$$

In view of property (a),

$$\rho(T^i(x_0), F) < \epsilon/8 \text{ for each integer } i \geq n_0. \tag{10}$$

By (4), (9) and (10), for each integer $k \in [n_0, \ldots, 4n_0]$,

$$\begin{aligned}\rho(x_k, F) &= \rho(x_k, T^k(x_0)) + \rho(T^k(x_0), F) \\ &< k\delta + \epsilon/8 \leq 4n_0\epsilon(8n_0)^{-1} + 8^{-1}\epsilon < \epsilon < r_0.\end{aligned} \tag{11}$$

Assume that $s \in \mathbf{N} \cup (n_0, \infty)$ is an integer and that

$$\rho(x_s, F) > \epsilon. \tag{12}$$

In view of (11) and (12),

$$s > 4n_0. \tag{13}$$

By (11)–(13), we may assume, without loss of generality, that

$$\rho(x_i, F) \leq \epsilon,\ i = n_0, \ldots, s-1. \tag{14}$$

By (13) and (14),

$$\rho(x_{s-n_0}, F) \leq \epsilon. \tag{15}$$

Equations (1) and (15), and $\epsilon < r_0$ imply that

$$x_{s-n_0} \in X_0. \tag{16}$$

Property (a) and (16) imply that

$$\rho(T^{n_0}(x_{s-n_0}), F) < \epsilon/8. \tag{17}$$

By (4), (9) and (13),

$$\rho(x_s, T^{n_0}(x_{s-n_0})) \leq n_0\delta < \epsilon/8.$$

In view of (17) and the relation above,

$$\rho(x_s, F) \leq \rho(x_s, T^{n_0}(x_{s-n_0})) + \rho(T^{n_0}(x_{s-n_0}), F) < \epsilon.$$

This contradicts (12) and completes the proof of Theorem 1. □

3. The Second Result

Theorem 2. *Assume that $F \subset X_0 \subset X$ are nonempty subsets of X, $r_0 > 0$,*

$$\cup\{B(x, r_0) :\ x \in F\} \subset X_0, \tag{18}$$

$$\lim_{n\to\infty} \rho(T^n(x), F) = 0 \tag{19}$$

uniformly on X_0 and that the following assumption holds:

(a) For each $x \in \cup_{i=0}^{\infty} T^i(X_0)$ and each $z \in B(x, r_0)$ there exists $\xi \in X$, such that

$$(x, \xi),\ (z, \xi) \in E(G)$$

and

$$\rho(x, \xi) \leq c_0\rho(z, x).$$

Let $\epsilon \in (0, r_0)$. Then, there exists $n_0 \in \mathbf{N}$, such that for each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfying

$$x_0 \in X_0$$

and

$$\rho(T(x_i), x_{i+1}) \le (32n_0)^{-1}(4c_0+2)^{-4n_0}\epsilon \tag{20}$$

for each integer $i \ge 0$ the inequality $\rho(x_i, F) \le \epsilon$ holds for each integer $i \ge n_0$.

Proof. By (19), there exists $n_0 \in \mathbf{N}$, such that the following property holds:

(b) For each $x \in X_0$, the relation $\rho(T^i(x), F) < \epsilon/8$ is true for all integers $i \ge n_0$.

Set

$$\delta = (32n_0)^{-1}(4c_0+2)^{-4n_0}\epsilon. \tag{21}$$

Assume that $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies

$$x_0 \in X_0$$

and that for each integer $i \ge 0$, Equation (20) holds. Assume that $p \in \mathbf{N} \cup \{0\}$, $i \in \mathbf{N}$,

$$x_p \in X_0,\ i \le 4n_0 \tag{22}$$

and that

$$\rho(x_{p+i}, T^i(x_p)) \le \delta(4c_0+2)^{i-1}. \tag{23}$$

(In view of (20), Equation (23) holds for $i = 1$.) By (21)–(23),

$$\rho(x_{i+p}, T^i(x_p)) \le r_0/4. \tag{24}$$

Property (a) and Equations (22) and (24) imply that there exists

$$\xi_i \in X$$

such that

$$(x_{i+p}, \xi_i),\ (T^i(x_p), \xi_i) \in E(G) \tag{25}$$

and

$$\rho(T^i(x_p), \xi_i) \le c_0\rho(x_{i+p}, T^i(x_p)). \tag{26}$$

By (25) and (26),

$$\begin{aligned}\rho(\xi_i, x_{i+p}) &\le \rho(x_{i+p}, T^i(x_p)) + \rho(T^i(x_p), \xi_i)\\ &\le (c_0+1)\rho(x_{i+p}, T^i(x_p)).\end{aligned} \tag{27}$$

Assumption (A) and Equations (25) and (26) imply that

$$(T(x_{i+p}), T(\xi_i)),\ (T^{i+1}(x_p), T(\xi_i)) \in E(G), \tag{28}$$

$$\rho(T(x_{i+p}), T(\xi_i)) \le \rho(x_{i+p}, \xi_i), \tag{29}$$

$$\rho(T^{i+1}(x_p), T(\xi_i)) \le \rho(T^i(x_p), \xi_i). \tag{30}$$

By (26), (27), (29) and (30),

$$\begin{aligned}\rho(T^{i+1}x_p, T(x_{i+p})) &\le \rho(T^{i+1}x_p, T(\xi_i)) + \rho(T(\xi_i), T(x_{i+p}))\\ &\le \rho(T^i(x_p), \xi_i) + \rho(x_{i+p}, \xi_i)\\ &\le c_0\rho(x_{i+p}, T^i(x_p)) + (c_0+1)\rho(x_{i+p}, T^i(x_p))\\ &\le (2c_0+1)\rho(x_{i+p}, T^i(x_p)).\end{aligned} \tag{31}$$

By (20), (23) and (31),

$$\begin{aligned}\rho(x_{i+1+p}, T^{i+1}(x_p)) &\le \rho(x_{i+1+p}, T(x_{i+p})) + \rho(T(x_{i+p}), T^{i+1}(x_p))\\ &\le \delta + (2c_0+1)\rho(x_{i+p}, T^i(x_p))\end{aligned}$$

$$\leq \delta + \delta(4c_0+2)^{i-1}(2c_0+1) \leq \delta(4c_0+2)^i.$$

Hence, we showed by induction that for all $i = 1, \dots, 4n_0$, (23) is true.

Therefore, the following property holds:

(c) If $p \in \mathbf{N} \cup \{0\}$ and

$$x_p \in X_0,$$

then (23) holds for every $i \in \{1, \dots, 4n_0\}$.

In view of (21) and property (c) with $p = 0$, for every $i \in \{1, \dots, 4n_0\}$,

$$\rho(x_i, T^i(x_0)) \leq \delta(4c_0+2)^{i-1} \leq \delta(4c_0+2)^{4n_0-1} < \epsilon/8 \leq r_0/8. \quad (32)$$

Property (b) implies that

$$\rho(T^i(x_0), F) < \epsilon/8, \ i \in \mathbf{N} \cup [n_0, \infty). \quad (33)$$

It follows from (32) and (33) that for each integer $k \in \{n_0, \dots, 4n_0\}$,

$$\rho(x_k, F) \leq \rho(x_k, T^k(x_0)) + \rho(T^k(x_0), F) \leq \epsilon/8 + \epsilon/8 < \epsilon < r_0. \quad (34)$$

Assume that $s \in \mathbf{N} \cap (n_0, \infty)$ and that

$$\rho(x_s, F) > \epsilon. \quad (35)$$

By (34) and (35),

$$s > 4n_0. \quad (36)$$

By (34)–(36), we may assume, without loss of generality, that

$$\rho(x_i, F) \leq \epsilon, \ i = n_0, \dots, s-1. \quad (37)$$

By (37),

$$\rho(x_{s-n_0}, F) \leq \epsilon. \quad (38)$$

Property (b) and Equations (18) and (38) imply that

$$x_{s-n_0} \in X_0 \quad (39)$$

and

$$\rho(T^{n_0}(x_{s-n_0}), F) < \epsilon/8. \quad (40)$$

In view of (21) and (39) and property (c) with $p = s - n_0$,

$$\rho(x_s, T^{n_0}(x_{s-n_0})) \leq \delta(4c_0+2)^{n_0-1} < \epsilon/8. \quad (41)$$

By (40) and (41),

$$\rho(x_s, F) \leq \rho(x_s, T^{n_0}(x_{s-n_0})) + \rho(T^{n_0}(x_{s-n_0}), F) < \epsilon.$$

This contradicts (35). The contradiction we have reached completes the proof of Theorem 2. □

Note that Theorems 1 and 2 were obtained for a large class of maps. They cover the case when $E(G) = X \times X$ and the case of monotone nonexpansive mappings [41,42] and they can also be applied for uniformly locally nonexpansive mappings [43].

Example 1. *Theorem 2 was proved under assumption (a). Now, we show that it holds for monotone operators. Assume that* $(Y, \|\cdot\|)$ *is a Banach space ordered by a closed convex cone* Y_+ *(*$u \le v$ *for* $u, v \in Y$ *if and only if* $u - v \in Y_+$*) such that*

$$Y_+ - Y_+ = Y.$$

Then by the Krein-Shmulyan theorem [44], there exists $c_0 > 0$ *such that for each* $y \in Y$*, there exist* $y_1, y_2 \in Y_+$*, such that*

$$y = y_1 - y_2,\ \|y_k\| \le c_0\|y\|,\ k = 1, 2.$$

Let $(u, v) \in E(G)$ *if and only if* $v \ge u$ *and* $\rho(u, v) = \|u - v\|$, $u, v \in Y$.

Assume that $r_0 \in (0, 1]$, $c_0 \ge 1$, $x \in Y$ *and that* $z \in Y$ *satisfies*

$$\|z - x\| \le r_0.$$

Then, there exists $u_1, u_2 \in Y_+$*, such that*

$$z - x = u_2 - u_1,\ \|u_k\| \le c_0\|z - x\|,\ k = 1, 2.$$

We have

$$z = x + u_2 - u_1 \le x + u_2,$$

$$x \le x + u_2.$$

Set

$$\xi = x + u_2.$$

Evidently,

$$\rho(x, \xi) = \|u_2\| \le c_0\|z - x\| = \rho(z, x)c_0.$$

Thus, property (a) of Theorem 2 holds.

4. Extensions

In the sequel, we denote by $\mathrm{Card}(E)$ the cardinality of a set E.

Proposition 1. *Assume that* $F,\ X_0 \subset X$ *are nonempty subsets of* X, $r_0 > 0$,

$$\cup\{B(x, r_0) :\ x \in F\} \subset X_0$$

and that

$$\lim_{n\to\infty} \rho(T^n(x), F) = 0$$

uniformly on X_0*. Let a sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *satisfy*

$$x_0 \in X_0$$

$$(T(x_i), x_{i+1}) \in E(G)$$

for each integer $i \ge 0$ *and*

$$\lim_{i\to\infty} \rho(T(x_i), x_{i+1}) = 0. \tag{42}$$

Then, $\lim_{i\to\infty} \rho(x_i, F) = 0$.

Proof. Let $\epsilon \in (0, r_0)$. By Theorem 1, there exists $n_0 \in \mathbf{N}$, such that the following property holds:

(i) For each sequence $\{y_i\}_{i=0}^{\infty} \subset X$ satisfying

$$y_0 \in X_0,$$

$$\rho(T(y_i), y_{i+1}) \le (8n_0)^{-1}\epsilon, \tag{43}$$

$$(T(y_i), y_{i+1}) \in E(G) \tag{44}$$

for each integer $i \geq 0$, the inequality $\rho(y_i, F) \leq \epsilon$ holds for each integer $i \geq n_0$.

In view of (42), there exists $n_1 \in \mathbf{N}$, such that for each integer $i \in \mathbf{N} \cap [n_1, \infty)$,

$$\rho(T(x_i), x_{i+1}) \leq (8n_0)^{-1}\epsilon.$$

Set

$$y_i = x_{i+n_1},\ i = 0, 1, \ldots, .$$

It is easy to see that (42) and (43) hold for each integer $i \geq 0$. Property (i) implies that for every $i \in \mathbf{N} \cap [n_0, \infty)$,

$$d(x_{i+n_1}, F) = d(y_i, F) \leq \epsilon.$$

Since ϵ is any element of the interval $(0, r_0)$, this completes the proof of Proposition 1. □

Proposition 2. *Assume that* $F,\ X_0 \subset X$ *are nonempty subsets of* X, $r_0 > 0$,

$$\cup\{B(x, r_0) :\ x \in F\} \subset X_0,$$

$$T(X_0) \subset X_0 \tag{45}$$

and that

$$\lim_{n\to\infty} \rho(T^n(x), F) = 0$$

uniformly on X_0. *Let a sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *satisfy*

$$x_0 \in X_0$$

$$(T(x_i), x_{i+1}) \in E(G),\ (T^k(x_i), T^k(x_i)) \in E(G) \tag{46}$$

for each $i, k \in \mathbf{N} \cup \{0\}$, *and that for each* $\epsilon > 0$,

$$\lim_{n\to\infty} n^{-1} Card(\{i \in \{0, \ldots, n-1\} :\ \rho(T(x_i), x_{i+1}) \geq \epsilon\}) = 0. \tag{47}$$

Then,

$$\liminf_{i\to\infty} \rho(x_i, F) = 0. \tag{48}$$

Proof. Assume that (48) does not hold. Then, there exists $\epsilon \in (0, r_0)$ and $n_1 \in \mathbf{N}$ such that for each $n \in \mathbf{N} \cap [n_1, \infty)$,

$$\rho(x_i, F) \geq 2\epsilon. \tag{49}$$

By Theorem 1, there exists $n_0 \in \mathbf{N}$, such that the following property holds:

(i) For each sequence $\{y_i\}_{i=0}^{\infty} \subset X$ satisfying

$$y_0 \in X_0,$$

$$\rho(T(y_i), y_{i+1}) \leq (8n_0)^{-1}\epsilon,$$

$$(T(y_i), y_{i+1}) \in E(G)$$

for every $i \in \mathbf{N} \cup \{0\}$ the inequality $\rho(y_i, F) \leq \epsilon$ holds for each integer $i \geq n_0$.

Let $n \in \mathbf{N} \cap [n_0 + n_1, \infty)$. In view of (49),

$$\rho(x_n, F) \geq 2\epsilon. \tag{50}$$

We show that there exists $i \in \{n - n_0, \ldots, n\}$, such that

$$\rho(T(x_i), x_{i+1}) > (8n_0)^{-1}\epsilon. \tag{51}$$

Assume the contrary. Then, (51) does not hold and

$$\rho(T(x_i), x_{i+1}) \leq (8n_0)^{-1}\epsilon,\ i \in \{n - n_0, \ldots, n\}. \tag{52}$$

Set

$$y_i = x_{i+n-n_0},\ i = 0, \ldots, n_0, \tag{53}$$

$$y_{i+1} = T(y_i),\ i \in \mathbf{N} \cap [n_0, \infty). \tag{54}$$

Property (i) and Equations (45), (46) and (52)–(54) imply that for each $i \in \mathbf{N} \cap [n_0, \infty)$,

$$\rho(y_i, F) \leq \epsilon.$$

Together with (53), this implies that

$$\rho(x_n, F) = \rho(y_{n_0}, F) \leq \epsilon.$$

This contradicts (50). The contradiction we have reached proves (51). Thus, we showed that for each $i \in \mathbf{N} \cap [n_0 + n_1, \infty)$,

$$\max\{\rho(T(x_i), x_{i+1}) :\ i = n_0, \ldots, n\} > (8n_0)^{-1}\epsilon.$$

This contradicts (47). The contradiction we have reached proves (48) and Proposition 2 itself. □

Proposition 3. *Assume that $F,\ X_0 \subset X$ are nonempty subsets of X, $r_0 > 0$,*

$$\cup\{B(x, r_0) :\ x \in F\} \subset X_0,$$

$$\lim_{n \to \infty} \rho(T^n(x), F) = 0$$

uniformly on X_0, and that the following assumption holds:

(a) For each $x \in \cup_{i=0}^{\infty} T^i(X_0)$ and each $z \in B(x, r_0)$, there exists $\xi \in X$, such that

$$(x, \xi),\ (z, \xi) \in E(G)$$

and

$$\rho(x, \xi) \leq c_0 \rho(z, x).$$

Let a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfy

$$x_0 \in X_0$$

$$\lim_{i \to \infty} \rho(T(x_i), x_{i+1}) = 0. \tag{55}$$

Then, $\lim_{i \to \infty} \rho(x_i, F) = 0$.

Proof. Let $\epsilon \in (0, r_0)$. By Theorem 2, there exists $n_0 \in \mathbf{N}$ and $\delta > 0$, such that the following property holds:

(b) For each sequence $\{y_i\}_{i=0}^{\infty} \subset X$ satisfying

$$y_0 \in X_0,$$

$$\rho(T(y_i), y_{i+1}) \leq \delta$$

for each integer $i \geq 0$, the inequality $\rho(y_i, F) \leq \epsilon$ holds for each integer $i \geq n_0$.

In view of (55), there exists a natural number n_1, such that for each integer $i \geq n_1$,

$$\rho(T(x_i), x_{i+1}) \leq \delta. \tag{56}$$

Set

$$x_0 \in X_0$$

$$y_i = x_{i+n_1},\ i = 0, 1, \ldots \tag{57}$$

By (56), (57) and property (b), for each $i \in \mathbf{N} \cap [n_0, \infty)$,

$$d(x_{i+n_1}, F) = d(y_i, F) \leq \epsilon.$$

Since ϵ is an arbitrary number of the interval $(0, r_0)$, this completes the proof of Proposition 3. □

Proposition 4. *Assume that $F,\ X_0 \subset X$ are nonempty subsets of X, $r_0 > 0$,*

$$\cup\{B(x, r_0) :\ x \in F\} \subset X_0,$$

$$T(X_0) \subset X_0,$$

$$\lim_{n \to \infty} \rho(T^n(x), F) = 0$$

uniformly on X_0 and that assumption (a) of Proposition 3 holds.

Let a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfy for each $\epsilon > 0$,

$$\lim_{n \to \infty} n^{-1} Card(\{i \in \{0, \ldots, n-1\} :\ \rho(T(x_i), x_{i+1}) \geq \epsilon) = 0. \tag{58}$$

Then,

$$\liminf_{i \to \infty} \rho(x_i, F) = 0. \tag{59}$$

Proof. Assume that (58) does not hold. Then, there exists $\epsilon \in (0, r_0)$ and $n_1 \in \mathbf{N}$, such that for each $i \in \mathbf{N} \cap [n_1, \infty)$,

$$\rho(x_i, F) \geq 2\epsilon. \tag{60}$$

By Theorem 2, there exists $n_0 \in \mathbf{N}$ and $\delta > 0$, such that the following property holds:

(c) For each sequence $\{y_i\}_{i=0}^{\infty} \subset X$ satisfying

$$y_0 \in X_0,$$

$$\rho(T(y_i), y_{i+1}) \leq \delta$$

for each $i \in \mathbf{N} \cup \{0\}$, the inequality $\rho(y_i, F) \leq \epsilon$ holds for each $i \in \mathbf{N} \cap [n_0, \infty)$.

Assume that $n \in \mathbf{N} \cap [n_0 + n_1, \infty)$ is an integer. We show that

$$\max\{\rho(T(x_i), x_{i+1}) :\ i \in \{n - n_0, \ldots, n\}\} > \delta. \tag{61}$$

Assume the contrary. Then,

$$\rho(T(x_i), x_{i+1}) \leq \delta,\ i \in \{n - n_0, \ldots, n\}.$$

Set

$$y_i = x_{i+n-n_0},\ i = 0, \ldots, n_0,$$

$$y_{i+1} = T(y_i) \text{ for each integer } i \geq n_0.$$

Property (c) and the equations above imply that for each $i \in \mathbf{N} \cap [n_0, \infty)$,

$$\epsilon \geq \rho(y_i, F) = \rho(x_{i+n-n_0}, F).$$

This contradicts (59). The contradiction we have reached proves (60). Thus, we showed that for each $i \in \mathbf{N} \cap [n_0 + n_1)$, (60) holds. This contradicts (57). The contradiction we have reached proves (58) and Proposition 7 itself. □

5. Conclusions

In this paper, we show that if iterates of a nonexpansive self-mapping of a complete metric with a graph converge uniformly on a subset of the space, then this convergence is stable under the presence of small computational errors. Our results generalize and extend many results known in the literature. As particular cases, they can be applied to a self-mapping of a complete metric space without graphs and for monotone nonexpansive mapping in ordered Banach spaces. They are important because of the computational errors that are always present in calculations.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflict of interest.

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