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On Mond–Weir-Type Robust Duality for a Class of Uncertain Fractional Optimization Problems

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Abstract: This article is focused on the investigation of Mond–Weir-type robust duality for a class of semi-infinite multi-objective fractional optimization with uncertainty in the constraint functions. We first establish a Mond–Weir-type robust dual problem for this fractional optimization problem. Then, by combining a new robust-type subdifferential constraint qualification condition and a generalized convex-inclusion assumption, we present robust ε-quasi-weak and strong duality properties between this uncertain fractional optimization and its uncertain Mond–Weir-type robust dual problem. Moreover, we also investigate robust ε-quasi converse-like duality properties between them.

Keywords: fractional optimization; robust duality; constraint qualification condition

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1. Introduction

Let T be a nonempty infinite index set. Suppose that $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$, and $h_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$. Let us consider the semi-infinite optimization problem:

$$(\mathrm{MP}) \qquad \begin{cases} \mathrm{Min}_{\mathbb{R}^p_+} \ (f_1(x), \ldots, f_p(x)) \\ s.t. \quad h_t(x) \leq 0, \forall t \in T, \\ \qquad\quad x \in \mathbb{R}^n. \end{cases}$$

The study of optimization problem (MP) is a very interesting topic and has been considered extensively by many scholars from different points of view, see [1–13]. However, most semi-infinite optimization models of real-world problems are contaminated by prediction errors or asymmetry knowledge. Thus, it is necessary to consider semi-infinite optimization problems under uncertain data. This optimization problem (MP) with uncertainty can be captured by

$$(\mathrm{UMP}) \qquad \begin{cases} \mathrm{Min}_{\mathbb{R}^p_+} \ (f_1(x), \ldots, f_p(x)) \\ s.t. \quad h_t(x, v_t) \leq 0, \forall t \in T, \\ \qquad\quad x \in \mathbb{R}^n. \end{cases}$$

Here, $h_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $t \in T$, are given functions, v_t, $t \in T$, are uncertain parameters which belongs to compact sets $\mathcal{V}_t \subseteq \mathbb{R}^q$.

As we know, robust optimization [14–16] is an useful approach to solve optimization problems with uncertainty. Following robust optimization methodology, we usually associate UMP with its robust counterpart

$$(\text{RMP}) \qquad \begin{cases} \text{Min}_{\mathbb{R}^p_+} \ (f_1(x), \ldots, f_p(x)) \\ s.t. \quad h_t(x, v_t) \leq 0, \forall v_t \in \mathcal{V}_t, t \in T, \\ \qquad\ \ x \in \mathbb{R}^n. \end{cases}$$

Recently, following robust optimization methodology, many interesting results devoted to (UMP) and its generalizations have been obtained from several different perspectives. By using scalarizing methods and robust optimization, Lee and Lee [17] establish necessary optimality theorems for robust weakly and properly efficient solutions of a multi-objective optimization problem with uncertainty. By virtue of a new concept of generalized convexity and robust type constraint qualification conditions, Chen et al. [18] give some optimality conditions and duality results for an uncertain nonconvex and nonsmooth multi-objective optimization problem. Guo and Yu [19] obtain optimality conditions for robust approximate quasi-weakly efficient solutions for uncertain multi-objective convex optimization problems. By combining robust optimization and scalarization technique, Sun et al. [20] give some new characterizations of Wolfe type robust approximate duality and saddle point theorems for a nonsmooth robust multi-objective optimization problem. Sun et al. [21] investigate optimality conditions for robust ϵ-quasi efficient solutions of a class of uncertain semi-infinite multi-objective optimization under some tools of non-smooth analysis and a new modified scalarization technique. In addition, nonsmooth robust ϵ-duality properties and ϵ-quasi saddle point theorems are also established. New results on optimality and duality results for uncertain multiobjective polynomial optimization problems are given in [22]. By using tangential subdifferential and robust optimization, Liu et al. [23] obtained some characterizations of robust optimal solution sets for nonconvex uncertain semi-infinite optimization problems.

On the other hand, the fractional multi-objective optimization problem is an important subclass of multi-objective optimization problems. In the last decades, a wide variety of interesting works devoted to fractional multi-objective optimization problems and its generalizations have been given, see, for example, [24–33]. We observe that there are some papers devoted to the study of uncertain fractional multi-objective optimization problems under a robust optimization approach. In [34], the authors study approximate optimality conditions and Wolfe-type robust approximate duality of robust approximate weakly efficient solutions for uncertain fractional multi-objective optimization problems. Li et al. [35] establish optimality theorems and robust duality properties for minimax convex–concave fractional optimization problems with uncertainty. Antczak [36] establish a new parametric approach for robust approximate quasi-efficient solutions of robust fractional multi-objective optimization problems. Feng and Sun [37] obtain some new results for robust weakly ϵ-efficient solutions for an uncertain fractional multi-objective semi-infinite optimization by employing conjugate analysis. Very recently, by employing robust limiting constraint qualification conditions and generalized convexity assumptions, Thuy and Su [38] consider optimality conditions and duality results for nonsmooth fractional multi-objective semi-infinite optimization problems with uncertain data.

In this paper, our main concern is to give new duality results of robust ϵ-quasi-efficient solutions for fractional multi-objective semi-infinite optimization problems (UFP, for brevity) with uncertainty appearing in the constraint functions. We first introduce the robust counterpart model (RFP, for brevity) for UFP. Then, with the help of a robust-type subdifferential constraint qualification, we present a necessary approximate optimality condition for robust ϵ-quasi-efficient solutions for (UFP). Subsequently, we introduce a Mond–Weir-type robust approximate dual problem of (UFP) based on the obtained necessary optimality conditions. Then, we investigate robust weak, strong and converse-like duality results between them under a new assumption of generalized convex-inclusion for Lipschitz functions.

This paper is organized as follows. In Section 2, we first recall some basic concepts in nonsmooth analysis and present approximate optimality results for robust ϵ-quasi-efficient solutions of (UFP). In Section 3, we introduce a Mond–Weir-type robust approximate dual problem for (UFP), and establish the robust ϵ-quasi duality results between them. As a special case, we also deal with robust ϵ-quasi duality results of the uncertain multi-objective optimization problem (UMP) and its robust approximate dual problem.

2. Mathematical Preliminaries

In this paper, let us recall some concepts and preliminary results [39,40]. Let $\mathbb{R}^p$ be the p-dimensional Euclidean space. We use the notation $\|\cdot\|$ for the Euclidean norm for $\mathbb{R}^p$. The nonnegative orthant of $\mathbb{R}^p$ is defined by $\mathbb{R}^p_+ := \{x = (x_1, \ldots, x_n) \mid x_k \geq 0, k = 1, \ldots, n\}$. We always use the symbol $\langle \cdot, \cdot \rangle$ for the inner product in $\mathbb{R}^p$. The closed unit ball of $\mathbb{R}^p$ is denoted by $\mathbb{B}^*$. For a nonempty infinite index set T, the linear space $\mathbb{R}^{(T)}$[41] is denoted by

$$\mathbb{R}^{(T)} := \{\gamma_T = (\gamma_t)_{t \in T} \mid \gamma_t = 0 \text{ for all } t \in T \text{ except for finitely many } \gamma_t \neq 0\}.$$

Let $\mathbb{R}^{(T)}_+$ be the nonnegative cone of $\mathbb{R}^{(T)}$, i.e.,

$$\mathbb{R}^{(T)}_+ := \{\gamma_T \in \mathbb{R}^{(T)} \mid \gamma_t \geq 0, \forall t \in T\}.$$

Let $\phi : \mathbb{R}^p \to \mathbb{R}$ be a locally Lipschitz function. The Clarke generalized directional derivative of ϕ at $x \in \mathbb{R}^p$ in the direction $d \in \mathbb{R}^p$ is defined by

$$\phi^c(x; d) := \limsup_{y \to x, t \downarrow 0} \frac{\phi(y + td) - \phi(y)}{t}.$$

The one-sided directional derivative of ϕ at $x \in \mathbb{R}^p$ in direction $d \in \mathbb{R}^p$ is defined by

$$\phi'(x; d) := \lim_{t \downarrow 0} \frac{\phi(x + td) - \phi(x)}{t}.$$

We say that ϕ is quasidifferentiable at $x \in \mathbb{R}^p$ iff, for each $d \in \mathbb{R}^n$, $\varphi'(x; d)$ exists and $\varphi'(x; d) = \varphi^c(x; d)$. The Clarke subdifferential $\partial^c \phi(x)$ of ϕ at $x \in \mathbb{R}^p$ is defined by

$$\partial^c \phi(x) := \{\xi^* \in \mathbb{R}^p \mid \phi^c(x; d) \geq \langle \xi^*, d \rangle, \forall d \in \mathbb{R}^p\}.$$

Obviously,

$$\phi^c(x; d) = \sup_{\xi \in \partial^c \phi(x)} \langle \xi, d \rangle, \ \ \forall d \in \mathbb{R}^n.$$

On the other hand, if $\phi : \mathbb{R}^p \to \mathbb{R}$ is a convex function, $\partial^c \phi(x)$ coincides with the convex subdifferential $\partial \phi(x)$, that is

$$\partial \phi(x) := \{\xi^* \in \mathbb{R}^p \mid \phi(y) - \phi(x) \geq \langle \xi^*, y - x \rangle, \forall y \in \mathbb{R}^p\}.$$

Let $\Omega \subseteq \mathbb{R}^p$ be a nonempty subset. The Clarke normal cone to Ω at $x \in \Omega$ is defined by

$$N^c(\Omega, x) := \{\xi \in \mathbb{R}^p \mid \langle \xi^*, w \rangle \leq 0, \forall w \in T_\Omega(x)\}.$$

Here, $T_\Omega(x)$ is the Clarke tangent cone to Ω at $x \in \Omega$. Clearly, if $\Omega \subseteq \mathbb{R}^n$ is a nonempty closed convex set, $N^c(\Omega, x)$ becomes the following normal cone:

$$N(\Omega, x) := \{\xi^* \in \mathbb{R}^p \mid \langle \xi^*, y - x \rangle \leq 0, \forall y \in \Omega\}.$$

In what follows, let $f_i, g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$, and $h_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$. We consider the following fractional multi-objective optimization problem

$$(\text{FP}) \qquad \begin{cases} \text{Min}_{\mathbb{R}^p_+} \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)} \right) \\ s.t. \ \ h_t(x) \leq 0, \forall t \in T, \\ \qquad x \in \mathbb{R}^n. \end{cases}$$

The fractional optimization problem (FP) under uncertain data in the constraint functions becomes

$$\text{(UFP)} \qquad \begin{cases} \operatorname{Min}_{\mathbb{R}^p_+} \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)} \right) \\ s.t. \quad h_t(x, v_t) \le 0, \forall t \in T, \\ \qquad\quad x \in \mathbb{R}^n. \end{cases}$$

Here $h_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$. $v_t \in \mathcal{V}_t \subseteq \mathbb{R}^q$, $t \in T$ are uncertain parameters.

For (UFP), we consider its robust counterpart, namely

$$\text{(RFP)} \qquad \begin{cases} \operatorname{Min}_{\mathbb{R}^p_+} \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)} \right) \\ s.t. \quad h_t(x, v_t) \le 0, \forall v_t \in \mathcal{V}_t, t \in T, \\ \qquad\quad x \in \mathbb{R}^n. \end{cases}$$

In this paper, without special statements, let f_i, $i = 1, \ldots, p$, be locally Lipschitz functions with $f_i(x) \ge 0, \forall x \in \mathbb{R}^n$, and g_i, $i = 1, \ldots, p$, be locally Lipschitz functions with $g_i(x) > 0, \forall x \in \mathbb{R}^n$.

Now, we give the following important notations, which will be used later in this paper.

Definition 1. *For (UFP). We say that $\mathcal{F}$ is the robust feasible set of (UFP) iff*

$$\mathcal{F} := \{x \in \mathbb{R}^n \mid h_t(x, v_t) \le 0, \forall v_t \in \mathcal{V}_t, t \in T\}.$$

Now, we consider the concept of robust ϵ-quasi efficient solution for (UFP). We refer the readers to [19,21,37] for other kinds of robust approximate efficient solutions.

Definition 2. *Let $\epsilon \in \mathbb{R}^p_+ \backslash \{0\}$. $\bar{x} \in \mathcal{F}$ is a robust ϵ-quasi efficient solution of* (UFP) *if there is not $x \in \mathcal{F}$, such that*

$$\frac{f_i(x)}{g_i(x)} \le \frac{f_i(\bar{x})}{g_i(\bar{x})} - \epsilon_i \|x - \bar{x}\|, \ \textit{for all } i = 1, \ldots, p,$$

and

$$\frac{f_j(x)}{g_j(x)} < \frac{f_j(\bar{x})}{g_j(\bar{x})} - \epsilon_j \|x - \bar{x}\|, \ \textit{for some } j \in \{1, \ldots, p\}.$$

Remark 1. *Note that $g_i \equiv 1$, the concept of robust ε-quasi efficient solution of (UFP) deduces to the robust ε-quasi efficient solution of (UMP), i.e., there is not $x \in \mathcal{F}$, such that*

$$f_i(x) \le f_i(\bar{x}) - \epsilon_i \|x - \bar{x}\|, \ \textit{for all } i = 1, \ldots, p,$$

and

$$f_j(x) < f_j(\bar{x}) - \epsilon_j \|x - \bar{x}\|, \ \textit{for some } j \in \{1, \ldots, p\}.$$

For more details, see [20,21,42].

Definition 3 ([43] (Definition 3.2))**.** *Consider (UFP). We say that the robust-type subdifferential constraint qualification condition* RSCQ *holds at $\bar{x} \in \mathcal{F}$, iff*

$$N^c(\mathcal{F}, \bar{x}) \subseteq \bigcup_{\substack{\lambda_T \in T(\bar{x}), \\ v_T \in \mathcal{V}_T}} \left[\sum_{t \in T} \lambda_t \partial^c_x h_t(\bar{x}, v_t) \right],$$

where $T(\bar{x}) = \left\{\lambda_T \in \mathbb{R}^{(T)}_+ \mid \lambda_t h_t(\bar{x}, v_t) = 0, \forall v_t \in \mathcal{V}_t, t \in T\right\}$.

Next, we recall the following necessary optimality conditions for robust ϵ-quasi-efficient solutions for (UFP) under the RSCQ. For convenience, let $\epsilon := (\epsilon_1, \ldots, \epsilon_p) \in \mathbb{R}^p_+\setminus\{0\}$.

Proposition 1 ([44] (Theorem 1))**.** *Let* $\epsilon \in \mathbb{R}^p_+\setminus\{0\}$. *Assume that* (RSCQ) *holds at* $\bar{x} \in \mathcal{F}$. *If* $\bar{x}$ *is a robust* ϵ*-quasi-efficient solution of* (UFP), *then there exist* $\bar{\eta}_t \geq 0$, *and* $\bar{v}_t \in V_t$, $t \in T$, *such that*

$$0 \in \sum_{i=1}^{p} \partial^c f_i(\bar{x}) + \sum_{i=1}^{p} \phi_i(\bar{x})\partial^c(-g_i)(\bar{x}) + \sum_{t\in T} \bar{\eta}_t \partial^c h_t(\cdot, \bar{v}_t)(\bar{x}) + 2\sum_{i=1}^{p} \varepsilon_i g_i(\bar{x})\mathbb{B}^*, \tag{1}$$

and

$$\bar{\eta}_t h_t(\bar{x}, \bar{v}_t) = 0, \forall t \in T. \tag{2}$$

Here, $\phi_i(\cdot) = \frac{f_i(\bar{x})}{g_i(\bar{x})} - \epsilon_i\| \cdot - \bar{x}\|$, $i = 1, \ldots, p$.

Remark 2. *Proposition 1 extends* [45] *(Theorem 3.1) from the case of scalar optimization to the multi-objective setting.*

In the case that $g_i \equiv 1$, the following result can be easily obtained by Proposition 1.

Proposition 2. *Let* $\epsilon \in \mathbb{R}^p_+\setminus\{0\}$. *Assume that* (RSCQ) *holds at* $\bar{x} \in \mathcal{F}$. *If* $\bar{x}$ *is a robust* ϵ*-quasi-efficient solution of* (UMP), *then there exist* $\bar{\eta}_t \geq 0$, *and* $\bar{v}_t \in V_t$, $t \in T$, *such that*

$$0 \in \sum_{i=1}^{p} \partial^c f_i(\bar{x}) + \sum_{t\in T} \bar{\eta}_t \partial^c h_t(\cdot, \bar{v}_t)(\bar{x}) + 2\sum_{i=1}^{p} \varepsilon_i \mathbb{B}^*, \tag{3}$$

and

$$\bar{\eta}_t h_t(\bar{x}, \bar{v}_t) = 0, \forall t \in T. \tag{4}$$

3. Main Results

In this section, based on the optimality conditions obtained in Proposition 1, we establish a robust *Mond-Weir-type* approximate dual problem for (UMFP), and then investigate robust duality properties between them. Here, we only consider their robust ϵ-quasi-efficient solutions. For the sake of convenience in the sequel, we set $f := (f_1, \ldots, f_p)$, $g := (g_1, \ldots, g_p)$, $h_T := (h_t)_{t\in T}$, $\eta_T := (\eta_t)_t \in \mathbb{R}^{(T)}_+$, $\mathcal{V}_T := \prod_{t\in T} \mathcal{V}_t$, and $v_T := (v_t)_{t\in T} \in \mathcal{V}_T$.

Let $y \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}^p_+\setminus\{0\}$. For given $v_t \in \mathcal{V}_t$, $t \in T$, the *Mond-Weir-type* uncertain approximate dual problem (UFD) of (UFP) is

$$(\text{UFD}) \quad \begin{cases} \text{Max}_{\mathbb{R}^p_+} \left(\frac{f_1(y)}{g_1(y)}, \ldots, \frac{f_p(y)}{g_p(y)}\right) \\ \text{s.t.} \quad 0 \in \sum\limits_{i=1}^{p} \partial^c f_i(y) + \sum\limits_{i=1}^{p} \frac{f_i(y)}{g_i(y)}\partial^c(-g_i)(y) + \sum\limits_{t\in T} \eta_t \partial^c h_t(\cdot, v_t)(y) + 2\sum\limits_{i=1}^{p} \epsilon_i g_i(y)\mathbb{B}^*, \\ \quad \eta_t h_t(y, v_t) \geq 0, t \in T, \\ \quad y \in \mathbb{R}^n, \epsilon_i \geq 0, i = 1, \ldots, p, \eta_t \geq 0, t \in T. \end{cases}$$

The optimistic counterpart of (UFD) is defined by

$$(\text{OFD}) \quad \begin{cases} \text{Max}_{\mathbb{R}^p_+} \left(\frac{f_1(y)}{g_1(y)}, \ldots, \frac{f_p(y)}{g_p(y)}\right) \\ \text{s.t.} \quad 0 \in \sum\limits_{i=1}^{p} \partial^c f_i(y) + \sum\limits_{i=1}^{p} \frac{f_i(y)}{g_i(y)}\partial^c(-g_i)(y) + \sum\limits_{t\in T} \eta_t \partial^c h_t(\cdot, v_t)(y) + 2\sum\limits_{i=1}^{p} \epsilon_i g_i(y)\mathbb{B}^*, \\ \quad \eta_t h_t(y, v_t) \geq 0, t \in T, \\ \quad y \in \mathbb{R}^n, \epsilon_i \geq 0, i = 1, \ldots, p, \eta_t \geq 0, v_t \in \mathcal{V}_t, t \in T. \end{cases}$$

Here, the maximization is also over all the parameters $v_t \in \mathcal{V}_t$, $t \in T$. The feasible set of (OFD) is defined as

$$\widehat{\mathcal{F}} := \Bigg\{ (y, \eta_T, v_T) \in \mathbb{R}^n \times \mathbb{R}_+^{(T)} \times \mathcal{V}_T \,\Big|\, 0 \in \sum_{i=1}^p \partial^c f_i(y) + \sum_{i=1}^p \frac{f_i(y)}{g_i(y)} \partial^c(-g_i)(y) + \sum_{t \in T} \eta_t \partial^c h_t(\cdot, v_t)(y)$$
$$+ 2\sum_{i=1}^p \epsilon_i g_i(y) \mathbb{B}^*, \eta_t h_t(y, v_t) \geq 0, t \in T \Bigg\}.$$

Remark 3. (i) *Obviously, if* $g_i(x) \equiv 1, i = 1, \ldots, p$, (UFD) *becomes the following conventional* Mond-Weir-type *uncertain approximate dual problem of* (UMP)

$$\text{(UMD)} \quad \begin{cases} \text{Max}_{\mathbb{R}_+^p} \ (f_1(y), \ldots, f_p(y)) \\ s.t. \ \ 0 \in \sum\limits_{i=1}^p \partial^c f_i(y) + \sum\limits_{t \in T} \eta_t \partial^c h_t(\cdot, v_t)(y) + 2\sum\limits_{i=1}^p \epsilon_i \mathbb{B}^*, \\ \qquad \eta_t h_t(y, v_t) \geq 0, t \in T, \\ \qquad y \in \mathbb{R}^n, \epsilon_i \geq 0, i = 1, \ldots, p, \eta_t \geq 0, t \in T. \end{cases}$$

and (OFD) *becomes the following Mond-Weir-type optimistic dual problem of* (UMP)

$$\text{(OMD)} \quad \begin{cases} \text{Max}_{\mathbb{R}_+^p} \ (f_1(y), \ldots, f_p(y)) \\ s.t. \ \ 0 \in \sum\limits_{i=1}^p \partial^c f_i(y) + \sum\limits_{t \in T} \eta_t \partial^c h_t(\cdot, v_t)(y) + 2\sum\limits_{i=1}^p \epsilon_i \mathbb{B}^*, \\ \qquad \eta_t h_t(y, v_t) \geq 0, t \in T, \\ \qquad y \in \mathbb{R}^n, \epsilon_i \geq 0, i = 1, \ldots, p, \eta_t \geq 0, v_t \in \mathcal{V}_t, t \in T. \end{cases}$$

Here, we denote the feasible set of (OMD) *by*

$$\overline{\mathcal{F}} := \Bigg\{ (y, \eta_T, v_T) \in \mathbb{R}^n \times \mathbb{R}_+^{(T)} \times \mathcal{V}_T \,\Big|\, 0 \in \sum_{i=1}^p \partial^c f_i(y) + \sum_{t \in T} \eta_t \partial^c h_t(\cdot, v_t)(y)$$
$$+ 2\sum_{i=1}^p \epsilon_i g_i(y) \mathbb{B}^*, \eta_t h_t(y, v_t) \geq 0, t \in T \Bigg\}.$$

(ii) *In the case that* $\epsilon = 0$ *and there is no uncertainty in the constraint functions. Then,* (UFP) *becomes* (FP), *and* (OMD) *collapses to*

$$\begin{cases} \text{Max}_{\mathbb{R}_+^p} \ \left(\frac{f_1(y)}{g_1(y)}, \ldots, \frac{f_p(y)}{g_p(y)} \right) \\ s.t. \ \ 0 \in \sum\limits_{i=1}^p \partial f_i(y) + \sum\limits_{i=1}^p \frac{f_i(y)}{g_i(y)} \partial^c(-g_i)(y) + \sum\limits_{t \in T} \eta_t \partial^c h_t(y), \\ \qquad \eta_t h_t(y) \geq 0, t \in T, \\ \qquad y \in \mathbb{R}^n, \eta_t \geq 0, t \in T. \end{cases}$$

Now, similar to Definition 2, we introduce robust ϵ-quasi efficient solutions for (UFD).

Definition 4. *Let* $\varepsilon \in \mathbb{R}_+^p \backslash \{0\}$. $(\bar{y}, \bar{\eta}_T, \bar{v}_T) \in \widehat{\mathcal{F}}$ *is said to be a robust* ε*-quasi efficient solution of* (UFD), *iff it is an* ε*-quasi efficient solution of* (OFD), *i.e., there is no* $(y, \eta_T, v_T) \in \widehat{\mathcal{F}}$, *such that*

$$\frac{f_i(y)}{g_i(y)} \geq \frac{f_i(\bar{y})}{g_i(\bar{y})} + \epsilon_i \|y - \bar{y}\|, \ \text{for all } i = 1, \ldots, p,$$

and

$$\frac{f_j(y)}{g_j(y)} > \frac{f_j(\bar{y})}{g_j(\bar{y})} + \epsilon_j\|y-\bar{y}\|, \text{ for some } j \in \{1,\dots,p\}.$$

Remark 4. *In particular, if $g_i \equiv 1$, the concept of robust ε-quasi efficient solution of* (UFD) *deduces to the robust ε-quasi efficient solution of* (UMD)*, i.e., there is no $(y,\eta_T,v_T) \in \overline{\mathcal{F}}$, such that*

$$f_i(y) \geq f_i(\bar{y}) + \epsilon_i\|y-\bar{y}\|, \text{ for all } i = 1,\dots,p,$$

and

$$f_j(y) > f_j(\bar{y}) + \epsilon_j\|y-\bar{y}\|, \text{ for some } j \in \{1,\dots,p\}.$$

In order to give robust duality relations for (UFP) and (UFD), we introduce the new definition of generalized convex-inclusion for Lipschitz functions, which is inspired by [32] (Definition 3.4) and [21] (Definition 3.3).

Definition 5. *Let $\Omega \subseteq \mathbb{R}^n$. $(f,-g,h_T)$ is said to generalized convex-inclusion on Ω at $x \in \Omega$, iff for any $y \in \Omega$, $\xi_i^* \in \partial^c f_i(x)$, $\xi_i^{**} \in \partial^c(-g_i)(x)$, $i = 1,\dots,p$, and $\gamma_t^* \in \partial_x^c h_t(x,v_t)$, $v_t \in \mathcal{V}_t$, $t \in T$, there exists $\omega \in \mathbb{R}^n$, such that*

$$f_i(y) - f_i(x) > \langle \xi_i^*, \omega\rangle, i = 1,\dots,p,$$
$$-g_i(y) + g_i(x) \geq \langle \xi_i^{**}, \omega\rangle, i = 1,\dots,p,$$
$$h_t(y,v_t) - h_t(x,v_t) \geq \langle \gamma_t^*, \omega\rangle, t \in T,$$
$$\langle b^*, \omega\rangle \leq \|y-x\|, \forall b^* \in \mathbb{B}^*,$$

and

$$0 \in \partial^c g_i(y), i = 1,\dots,p.$$

Remark 5. (i) *In the special case that $g_i \equiv 1$, the concept of generalized convex-inclusion reduces to the concept of generalized convexity, i.e., (f,h_T) is generalized convex on Ω at $x \in \Omega$, iff for any $y \in \Omega$, $\xi_i^* \in \partial^c f_i(x)$, $i = 1,\dots,p$, and $\gamma_t^* \in \partial_x^c g_t(x,v_t)$, $v_t \in \mathcal{V}_t$, $t \in T$, there exists $\omega \in \mathbb{R}^n$, such that*

$$f_i(y) - f_i(x) > \langle \xi_i^*, \omega\rangle, i = 1,\dots,p,$$
$$h_t(y,v_t) - h_t(x,v_t) \geq \langle \gamma_t^*, \omega\rangle, t \in T,$$

and

$$\langle b^*, \omega\rangle \leq \|y-x\|, \forall b^* \in \mathbb{B}^*.$$

(ii) *If $g_i \equiv 1$ and there is uncertain data on f_i, $i = 1,\dots,p$, Definition 5 reduces to [21] (Definition 3.3).*

(iii) *If $g_i \equiv 1$ and there is no uncertain data on h_t, $t \in T$, Definition 5 reduces to the concept of generalized convexity-inclusion introduced in [32] (Definition 3.4), i.e., for any $y \in \Omega$, $\xi_i^* \in \partial^c f_i(x)$, $\xi_i^{**} \in \partial^c(-g_i)(x)$, $i = 1,\dots,p$, and $\gamma_t^* \in \partial^c h_t(x)$, $t \in T$, there exists $\omega \in \mathbb{R}^n$, such that*

$$f_i(y) - f_i(x) > \langle \xi_i^*, \omega\rangle, i = 1,\dots,p,$$
$$-g_i(y) + g_i(x) \geq \langle \xi_i^{**}, \omega\rangle, i = 1,\dots,p,$$
$$h_t(y) - h_t(x) \geq \langle \gamma_t^*, \omega\rangle, t \in T,$$
$$\langle b^*, \omega\rangle \leq \|y-x\|, \forall b^* \in \mathbb{B}^*,$$

and

$$0 \in \partial^c g_i(y), i = 1, \dots, p.$$

Note that this concept has been used to establish sufficient optimality conditions for weakly ϵ-quasi-efficient solution for fractional optimization problem. For more details, please see [32] (Theorem 3.5).

Now, we show robust approximate duality properties for (UFP) and (UFD) by showing approximate duality properties between the robust counterpart (RMP) and the optimistic counterpart (OFD). In what follows, we set

$$\omega_1 \preceq \omega_2 \Leftrightarrow \omega_2 - \omega_1 \in \mathbb{R}^p_+ \setminus \{0\}, \ \ \forall \omega_1, \omega_2 \in \mathbb{R}^p,$$

$$\omega_1 \not\preceq \omega_2 \Leftrightarrow \omega_2 - \omega_1 \notin \mathbb{R}^p_+ \setminus \{0\}, \ \ \forall \omega_1, \omega_2 \in \mathbb{R}^p.$$

The following result gives robust ϵ-quasi-weak duality between (UFP) and (UFD).

Theorem 1. *Let $\epsilon \in \mathbb{R}^p_+ \setminus \{0\}$. Suppose that $x \in \mathcal{F}$ and $(y, \eta_T, v_T) \in \widehat{\mathcal{F}}$. If $(f, -g, h_T)$ is generalized convex-inclusion on $\mathbb{R}^n$ at $y \in \mathbb{R}^n$, then,*

$$\left(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_p(x)}{g_p(x)}\right) \not\preceq \left(\frac{f_1(y)}{g_1(y)} - 2\epsilon_1\|x - y\|, \dots, \frac{f_p(y)}{g_p(y)} - 2\epsilon_p\|x - y\|\right).$$

Proof. Suppose to the contrary that

$$\left(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_p(x)}{g_p(x)}\right) \preceq \left(\frac{f_1(y)}{g_1(y)} - 2\epsilon_1\|y - x\|, \dots, \frac{f_p(y)}{g_p(y)} - 2\epsilon_p\|y - x\|\right).$$

Then,

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(y)}{g_i(y)} - 2\epsilon_i\|y - x\|, \ \text{for all } i = 1, \dots, p, \tag{5}$$

and

$$\frac{f_i(x)}{g_i(x)} < \frac{f_j(y)}{g_j(y)} - 2\epsilon_j\|y - x\|, \ \text{for some } j \in \{1, \dots, p\}. \tag{6}$$

On the other hand, note that $(y, \eta_T, v_T) \in \widehat{\mathcal{F}}$. Then, $y \in \mathbb{R}^n$, $\eta_t \geq 0$, $v_t \in \mathcal{V}_t$, $t \in T$, and

$$0 \in \sum_{i=1}^p \partial^c f_i(y) + \sum_{i=1}^p \frac{f_i(y)}{g_i(y)} \partial^c(-g_i)(y) + \sum_{t \in T} \eta_t \partial^c h_t(\cdot, v_t)(y) + 2\sum_{i=1}^p \epsilon_i g_i(y) \mathbb{B}^*, \tag{7}$$

and

$$\eta_t h_t(y, v_t) \geq 0, t \in T. \tag{8}$$

By (5), there exist $\xi_i^* \in \partial^c f_i(y)$, $\xi_i^{**} \in \partial^c(-g_i)(y)$, $i = 1, \dots, p$, $\zeta_t^* \in \partial^c h_t(\cdot, v_t)(y)$, $t \in T$, and $b^* \in \mathbb{B}^*$, such that

$$\sum_{i=1}^p \xi_i^* + \sum_{i=1}^p \frac{f_i(y)}{g_i(y)} \xi_i^{**} + \sum_{t \in T} \eta_t \zeta_t^* + 2\sum_{i=1}^p \epsilon_i g_i(y) b^* = 0. \tag{9}$$

Since $(f, -g, h_T)$ is generalized convex-inclusion on $\mathbb{R}^n$ at $y \in \mathbb{R}^n$, we have for such $\xi_i^* \in \partial^c f_i(y)$, $\xi_i^{**} \in \partial^c(-g_i)(y)$, $i = 1, \dots, p$, and $\zeta_t^* \in \partial^c h_t(\cdot, v_t)(y)$, $t \in T$, there exists $\vartheta \in \mathbb{R}^n$, such that

$$f_i(x) - f_i(y) > \langle \xi_i^*, \vartheta \rangle, \ \ i = 1, \dots, p,$$

$$-g_i(x) + g_i(y) \geq \langle \xi_i^{**}, \vartheta \rangle, \ \ i = 1, \ldots, p,$$
$$h_t(x, v_t) - h_t(y, v_t) \geq \langle \zeta_t^*, \vartheta \rangle, \ \ t \in T,$$
$$\langle b^*, \vartheta \rangle \leq \|x - y\|, \forall b^* \in \mathbb{B}^*,$$

and

$$0 \in \partial^c g_i(y), i = 1, \ldots, p.$$

Together with (7)–(9), these follow that

$$\begin{aligned}
&\sum_{i=1}^{p}\left(f_i(x) - \frac{f_i(y)}{g_i(y)}g_i(x) + 2\epsilon_i g_i(y)\|y - x\|\right) \\
&> \sum_{i=1}^{p}\left(f_i(y) + \langle \xi_i^*, \vartheta \rangle - \frac{f_i(y)}{g_i(y)}g_i(y) + \frac{f_i(y)}{g_i(y)}\langle \xi_i^{**}, \vartheta \rangle + 2\epsilon_i g_i(y)\langle b^*, \vartheta \rangle\right) \\
&= \left\langle \sum_{i=1}^{p} \xi_i^* + \sum_{i=1}^{p} \frac{f_i(y)}{g_i(y)}\xi_i^{**} + 2\sum_{i=1}^{p} \epsilon_i g_i(y) b^*, \vartheta \right\rangle \\
&= -\left\langle \sum_{i=1}^{p} \eta_t \zeta_t^*, \vartheta \right\rangle \\
&\geq -\sum_{t \in T} \eta_t h_t(x, v_t) + \sum_{t \in T} \eta_t h_t(y, v_t).
\end{aligned}$$

Together with $\eta_t h_t(x, v_t) \leq 0, \forall x \in F$, and $\eta_t h_t(y, v_t \geq 0$, we have

$$\sum_{i=1}^{p}\left(f_i(x) - \frac{f_i(y)}{g_i(y)}g_i(x) + 2\epsilon_i g_i(y)\|y - x\|\right) > 0.$$

Then, there exists $i_0 \in \{1, \ldots, p\}$, such that

$$f_{i_0}(x) - \frac{f_{i_0}(y)}{g_{i_0}(y)}g_{i_0}(x) + 2\epsilon_{i_0} g_{i_0}(y)\|y - x\| > 0,$$

which follows that

$$\frac{f_{i_0}(x)}{g_{i_0}(x)} - \frac{f_{i_0}(y)}{g_{i_0}(y)} + 2\epsilon_{i_0}\frac{g_{i_0}(y)}{g_{i_0}(x)}\|y - x\| > 0. \tag{10}$$

Moreover, it follows from $0 \in \partial^c g_i(y), i = 1, \ldots, p$, that

$$g_{i_0}(x) \geq g_{i_0}(y). \tag{11}$$

Together with (10) and (11) , we have

$$\frac{f_{i_0}(x)}{g_{i_0}(x)} - \frac{f_{i_0}(y)}{g_{i_0}(y)} + 2\epsilon_{i_0}\|y - x\| > 0.$$

which is a contradiction to (5) and (6). Thus, the conclusion holds. □

Now, we give the following example to justify the importance of the assumption of generalized convex-inclusion in Theorem 1.

Example 1. *Let* $\mathcal{V}_t := [1 - t, 1 + t]$, $t \in T := \left[0, \frac{1}{2}\right]$. *Let* $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ *and* $g_t : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, t \in T$, *be defined by*

$$f_1(x) = f_2(x) := \frac{1}{2}|x| + \frac{1}{6}x^3, g_1(x) = g_2(x) := |x| + 1,$$

and

$$h_t(x, v_t) := tx^2 - tx - 2v_t,$$

where $x \in \mathbb{R}$ *and* $v_t \in \mathcal{V}_t$, $t \in T$. *Then,* (UFP) *becomes*

$$\begin{cases} \mathrm{Min}_{\mathbb{R}^2_+} \left(\frac{\frac{1}{2}|x| + \frac{1}{6}x^3}{|x|+1}, \frac{\frac{1}{2}|x| + \frac{1}{6}x^3}{|x|+1} \right) \\ s.t. \quad tx^2 - tx - 2v_t \le 0, \forall t \in \left[0, \frac{1}{2}\right], \\ \qquad\; x \in \mathbb{R}, \end{cases}$$

and (RFP) *becomes*

$$\begin{cases} \mathrm{Min}_{\mathbb{R}^2_+} \left(\frac{\frac{1}{2}|x| + \frac{1}{6}x^3}{|x|+1}, \frac{\frac{1}{2}|x| + \frac{1}{6}x^3}{|x|+1} \right) \\ s.t. \quad tx^2 - tx - 2v_t \le 0, \forall v_t \in [1-t, 1+t], t \in \left[0, \frac{1}{2}\right], \\ \qquad\; x \in \mathbb{R}. \end{cases}$$

Obviously, $\mathcal{F} = [-1, 2]$. *Let us consider* $\bar{x} := -1 \in \mathcal{F}$. *Then,*

$$\left(\frac{f_1(\bar{x})}{g_1(\bar{x})}, \frac{f_2(\bar{x})}{g_2(\bar{x})} \right) = \left(\frac{1}{6}, \frac{1}{6} \right).$$

Now, consider the dual problem (UFD). *In this setting,* (OFD) *becomes*

$$\begin{cases} \mathrm{Max}_{\mathbb{R}^2_+} \left(\frac{f_1(y)}{g_1(y)}, \frac{f_2(y)}{g_2(y)} \right) \\ s.t. \quad 0 \in \partial^c f_1(y) + \partial^c f_2(y) + \frac{f_1(y)}{g_1(y)} \partial^c(-g_1)(y) + \frac{f_2(y)}{g_2(y)} \partial^c(-g_2)(y) \\ \qquad + \sum_{t \in T} \eta_t \partial^c h_t(\cdot, v_t)(y) + 2\epsilon_1 g_1(y) \mathbb{B}^* + 2\epsilon_2 g_2(y) \mathbb{B}^*, \\ \qquad \eta_t h_t(y, v_t) \ge 0, t \in \left[0, \frac{1}{2}\right], \\ \qquad y \in \mathbb{R}, \epsilon_1 \ge 0, \epsilon_2 \ge 0, \eta_t \ge 0, v_t \in [1-t, 1+t], t \in \left[0, \frac{1}{2}\right]. \end{cases}$$

Clearly, for any $y \in \mathbb{R}$ *and* $v_T \in \mathcal{V}_T$, *we have*

$$\partial^c f_1(y) = \partial^c f_2(y) = \left[\frac{1}{2}y^2 - \frac{1}{2}, \frac{1}{2}y^2 + \frac{1}{2} \right],$$

$$\partial^c(-g_1)(y) = \partial^c(-g_2)(y) = [-1, 1],$$

and

$$\partial^c h_t(\cdot, v_t)(y) = \{2ty - t\}, \forall t \in T.$$

By selecting $\bar{y} := 1$, $\bar{\eta}_t := 0$, *and* $\bar{v}_t := -t$, *we have*

$$\begin{aligned} &\partial^c f_1(\bar{y}) + \partial^c f_2(\bar{y}) + \frac{f_1(\bar{y})}{g_1(\bar{y})} \partial^c(-g_1)(\bar{y}) + \frac{f_2(\bar{y})}{g_2(\bar{y})} \partial^c(-g_2)(\bar{y}) \\ &\quad + \sum_{t \in T} \bar{\eta}_t \partial^c h_t(\cdot, \bar{v}_t)(\bar{y}) + 2\epsilon_1 g_1(\bar{y}) \mathbb{B}^* + 2\epsilon_2 g_2(\bar{y}) \mathbb{B}^* \\ &= \left[-4\epsilon_1 - 4\epsilon_2 - \frac{1}{3}, 4\epsilon_1 + 4\epsilon_2 + \frac{7}{3} \right], \end{aligned}$$

and

$$\bar{\eta}_t h_t(\bar{y}, \bar{v}_t) \ge 0, t \in \left[0, \frac{1}{2}\right].$$

These mean that $(\bar{y}, \bar{\eta}_T, \bar{v}_T) \in \widehat{\mathcal{F}}$.

Now, take an arbitrarily $\epsilon = (\epsilon_1, \epsilon_2) \in \mathbb{R}^2_+ \setminus \{0\}$ *such that* $\varepsilon_i < \frac{1}{12}, i = 1, 2$. *Clearly,*

$$\left(\frac{f_1(\bar{y})}{g_1(\bar{y})} - 2\epsilon_1\|\bar{x} - \bar{y}\|, \frac{f_2(\bar{y})}{g_2(\bar{y})} - 2\epsilon_2\|\bar{x} - \bar{y}\|\right) = \left(\frac{1}{3} - 2\varepsilon_1, \frac{1}{3} - 2\varepsilon_2\right) \succ \left(\frac{1}{6}, \frac{1}{6}\right) = \left(\frac{f_1(\bar{x})}{g_1(\bar{x})}, \frac{f_2(\bar{x})}{g_2(\bar{x})}\right).$$

Thus, Theorem 1 is not applicable since $(f, -g, h_T)$ *is not generalized convex-inclusion at* $\bar{y}$. *To do this, by choosing* $\bar{\xi}_i := 0 \in \partial^c f_i(\bar{y})$, $i = 1, 2$, *we have*

$$f_i(\bar{x}) - f_i(\bar{y}) = -\frac{2}{3} < 0 = \langle \bar{\xi}_k, \omega \rangle, \forall \omega \in \mathbb{R}.$$

Similarly, we obtain the following robust weak duality between (UMP) and (UMD).

Corollary 1. *Let* $\epsilon \in \mathbb{R}^p_+ \setminus \{0\}$. *Suppose that* $x \in \mathcal{F}$ *and* $(y, \eta_T, v_T) \in \overline{\mathcal{F}}$. *If* (f, h_T) *is generalized convex on* $\mathbb{R}^n$ *at* $y \in \mathbb{R}^n$, *then,*

$$\big(f_1(x), \ldots, f_p(x)\big) \not\preceq \big(f_1(y) - 2\epsilon_1\|x - y\|, \ldots, f_p(y) - 2\epsilon_p\|x - y\|\big).$$

Remark 6. *Clearly, by virtue of Example 1, we can also illustrate that the assumption of generalized convexity imposed in Corollary 1 is indispensable.*

Now, we give robust strong duality results between (UFP) and (UFD).

Theorem 2. *Let* $\epsilon \in \mathbb{R}^p_+ \setminus \{0\}$. *Assume that* (RSCQ) *holds at* $\bar{x} \in \mathcal{F}$. *Suppose that* $(f, -g, h_T)$ *is generalized convex-inclusion on* $\mathbb{R}^n$ *at* $y \in \mathbb{R}^n$. *If* $\bar{x}$ *is a robust* ϵ*-quasi-efficient solution of* (UFP), *then there exist* $\bar{\eta}_T \in \mathbb{R}^{(T)}_+$ *and* $\bar{v}_T \in \mathcal{V}_T$, *such that* $(\bar{x}, \bar{\eta}_T, \bar{v}_T) \in \widehat{\mathcal{F}}$ *is a robust* 2ϵ*-quasi-efficient solution of* (UFD).

Proof. Assume that $\bar{x} \in \mathcal{F}$ is a robust ϵ-quasi-efficient solution of (UFP). By Theorem 1, there exist $\bar{\eta}_t \geq 0$, and $\bar{v}_t \in \mathcal{V}_t$, $t \in T$, such that

$$0 \in \sum_{i=1}^{p} \partial f_i(\bar{x}) - \sum_{i=1}^{p} \phi_i(\bar{x}) \partial g_i(\bar{x}) + \sum_{t \in T} \bar{\eta}_t \partial h_t(\cdot, \bar{v}_t)(\bar{x}) + 2\sum_{i=1}^{p} \varepsilon_i g_i(\bar{x}) \mathbb{B}^*, \tag{12}$$

and

$$\bar{\eta}_t h_t(\bar{x}, \bar{v}_t) = 0, \forall t \in T. \tag{13}$$

From (12), (13) and $\phi_i(\bar{x}) = \frac{f_i(\bar{x})}{g_i(\bar{x})}$, we have

$$(\bar{x}, \bar{\eta}_T, \bar{v}_T) \in \widehat{\mathcal{F}}.$$

By Theorem 1, for all $(y, \eta_T, v_T) \in \widehat{\mathcal{F}}$, we have

$$\left(\frac{f_1(\bar{x})}{g_1(\bar{x})}, \ldots, \frac{f_p(\bar{x})}{g_p(\bar{x})}\right) \not\preceq \left(\frac{f_1(y)}{g_1(y)} - 2\epsilon_1\|\bar{x} - y\|, \ldots, \frac{f_p(y)}{g_p(y)} - 2\epsilon_p\|\bar{x} - y\|\right).$$

Thus, $(\bar{x}, \bar{\eta}_T, \bar{v}_T)$ is a robust 2ϵ-quasi-efficient solutions of (UFD). Thus, the conclusion holds. □

Remark 7. *In [32] (Theorem 4.2), the authors established duality properties for* ϵ*-quasi-weakly efficient solutions between* (FP) *and its Mond Weir-type dual problem. Therefore, Theorem 2 encompasses [32] (Theorem 4.2), where the corresponding results were given in terms of the similar methods.*

Similarly, we give robust strong duality properties for robust ϵ-quasi efficient solutions between (UMP) and (UMD).

Corollary 2. *Let $\epsilon \in \mathbb{R}^p_+ \backslash \{0\}$. Assume that* (RSCQ) *holds at $\bar{x} \in \mathcal{F}$. Suppose that (f, h_T) is generalized convex on $\mathbb{R}^n$ at $y \in \mathbb{R}^n$. If $\bar{x}$ is a robust ϵ-quasi-efficient solution of* (UMP)*, then there exist $\bar{\eta}_T \in \mathbb{R}^{(T)}_+$ and $\bar{v}_T \in \mathcal{V}_T$, such that $(\bar{x}, \bar{\eta}_T, \bar{v}_T) \in \overline{\mathcal{F}}$ is a robust 2ϵ-quasi-efficient solution of* (UMD).

Now, we give a robust converse-like duality property between (UFP) and (UFD).

Theorem 3. *Let $\varepsilon \in \mathbb{R}^p_+ \backslash \{0\}$ and $(\bar{x}, \bar{\eta}_T, \bar{v}_T) \in \widehat{\mathcal{F}}$. If $(f, -g, h_T)$ is generalized convex-inclusion on $\mathbb{R}^n$ at $\bar{x} \in \mathcal{F}$, then, $\bar{x} \in \mathcal{F}$ is a robust 2ε-quasi efficient solution of* (UMP).

Proof. Sine $(\bar{x}, \bar{\eta}_T, \bar{v}_T) \in \widehat{\mathcal{F}}$ and $(f, -g, h_T)$ is generalized convex-inclusion on $\mathbb{R}^n$ at $\bar{x}$, it follows from Theorem 1 that

$$\left(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_p(x)}{g_p(x)}\right) \not\leq \left(\frac{f_1(\bar{x})}{g_1(\bar{x})} - 2\epsilon_1\|x - \bar{x}\|, \dots, \frac{f_p(\bar{x})}{g_p(\bar{x})} - 2\epsilon_p\|x - \bar{x}|\right), \forall x \in \mathcal{F}.$$

Therefore, $\bar{x} \in \mathcal{F}$ is a robust 2ε-quasi efficient solution of (UFP) and the proof is complete. □

Remark 8. *Note that the converse-like duality result obtained in Theorem 3 extends* [32] *(Theorem 4.4) from the deterministic (i.e., with singleton uncertainty sets) to the robust setting. Moreover, Theorem 3 extends* [43] *(Theorem 4.3) from the scalar case to the multi-objective setting.*

Similarly, we have the following results for (UMP) and (UMD), which has been considered in [21] (Theorem 4.3).

Corollary 3. *Let $\varepsilon \in \mathbb{R}^p_+ \backslash \{0\}$ and $(\bar{x}, \bar{\eta}_T, \bar{v}_T) \in \overline{\mathcal{F}}$. If (f, h_T) is generalized convex on $\mathbb{R}^n$ at $\bar{x} \in \mathcal{F}$, then, $\bar{x} \in \mathcal{F}$ is a robust ε-quasi efficient solution of* (UMP).

4. Conclusions

In this paper, we consider robust ε-quasi-efficient solutions for a class of uncertain fractional optimization problems. By employing robust optimization and the obtained optimality conditions, a Mond–Weir-type robust dual problem for the fractional optimization problem is established. Then, we give robust ε-quasi-weak, strong and converse duality properties between them in terms of generalized convex-inclusion assumptions. We also show that the obtained results extend the corresponding results obtained in [21,32,37].

In the future, similar to [21,43], it is of interest to formulate *Mixed-type* robust approximate dual problem of uncertain fractional optimization problems, and study robust ε-quasi-weak, strong, and converse duality properties between them.

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