axioms

Article

Lax Extensions of Conical I-Semifilter Monads

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Abstract: For a quantale I, the unit interval endowed with a continuous triangular norm, we introduce the canonical, op-canonical and Kleisli extensions of the conical I-semifilter monad to I-Rel. It is proved that the op-canonical extension coincides with the Kleisli extension.

Keywords: lax extension; conical I-semifilter monad; Kleisli extension

MSC: 18C15; 18D20

Citation: Zhang, G.; Zhang, S.-Q. Lax Extensions of Conical I-Semifilter Monads. *Axioms* **2023**, *12*, 1034. https://doi.org/10.3390/axioms12111034

Academic Editor: Salvador Hernández

Received: 19 September 2023
Revised: 25 October 2023
Accepted: 3 November 2023
Published: 5 November 2023

1. Introduction and Preliminaries

Monoidal topology [1] provides a unification of settings to describe some important mathematical structures as $(\mathbb{T}, \mathsf{Q}, \hat{\mathbb{T}})$-algebras (lax algebras for short) in which Q is a quantale and $\mathbb{T}$ is a monad on Set with a lax extension $\hat{\mathbb{T}}$ to the category Q-Rel of sets and Q-relations.

Examples include:

- Metric spaces can be described as $(\mathbb{I}, \mathsf{P}_+, \overline{\mathbb{I}})$-algebras [2].
- Topological spaces can be characterized as $(\beta, 2, \overline{\beta})$-algebras [3,4].
- Approach spaces [5] can be viewed as $(\beta, \mathsf{P}_+, \overline{\beta})$-algebras [6].

Here, 2 denotes the two-element quantale, $\mathsf{P}_+ = ([0, \infty]^{\mathrm{op}}, +, 0)$ is the Lawvere quantale, $\mathbb{I}$ is the identity monad with the identity extension, and β is the ultrafilter monad with the Barr extensions $\overline{\beta}$ to 2-Rel (Rel for short) and I-Rel, respectively.

To study many-valued topologies within the monoidal topology framework, it is of importance to determine the counterpart of the filter monad in the many-valued context and investigate its lax extensions. Extensive studies have been conducted to develop many-valued filter monads and their lax extensions, including the $\mathfrak{B}$-valued filter monad [7], the $\top$-filter monad with its Kleisli extension to Rel [8], and the saturated prefilter monad with its Kleisli extension to Rel [9]. The lax algebras for the latter two are both CNS spaces, which are a kind of many-value topological spaces introduced in [10].

Lax extensions offer rich topological structures. For example, as demonstrated in [11], there are two lax extensions of the filter monad $\mathbb{F}$ to Q-Rel : the canonical one $\hat{\mathbb{F}}$ and the op-canonical one $\check{\mathbb{F}}$. When $\mathsf{Q} = 2$, the lax algebras with respect to the canonical extension are closure spaces, while those associated with the op-canonical extension are topological spaces. When $\mathsf{Q} = \mathsf{P}_+$, the lax algebras with respect to the canonical extension are closeness spaces, while those for the op-canonical extension are approach spaces.

The approach adopted in this paper is motivated by an observation that the filter monad is the discrete restriction of two composite monads on Ord : up-set-ideal monad IdeUp and the down-set-filter monad FilDn. Furthermore, the canonical (op-canonical) lax extension of the filter monad can be induced from the lax extension of IdeUp (FilDn) to Dist.

In Section 2, we introduce the composite monads $\mathsf{CP}^{\dagger}$ and $\mathsf{C}^{\dagger}\mathsf{P}$ and show that the discrete restriction of them are the conical I-semifilter monad [12], where C is the monad of I-distributors generated by a forward Cauchy net that plays the role of the ordered-ideal monad Ide. The canonical and op-canonical extensions of the conical I-semifilter monad to I-Rel are also presented in this section. Section 3 focuses on the Kleisli extension of the conical I-semifilter monad to I-Rel. The lax algebras for the Kleisli extension to I-Rel are same to those for the Kleisli extension to Rel.

In the remainder of this section, we introduce the many-valued context in which we work, including the quantale I, I-relations and I-categories.

1.1. Monads

A monad on a category $\mathcal{A}$ is a triple $\mathbb{T} = (T, m, e)$, where $T\colon \mathcal{A} \to \mathcal{A}$ is an endfunctor and $m\colon T^2 \to T, e\colon \mathrm{id}_{\mathcal{A}} \to T$ are natural transformations such that

$$m \cdot eT = m \cdot Te = \mathrm{id}_{\mathcal{A}} \quad \text{and} \quad m \cdot mT = m \cdot Tm.$$

Sometimes, we simply write T for (T, m, e) if no confusion arises.

Given two monads $\mathbb{T} = (T, m, e)$ and $\mathbb{S} = (S, n, d)$, a morphism $\sigma\colon \mathbb{T} \to \mathbb{S}$ of monads is a natural transformation $\sigma\colon T \to S$ such that

$$d = \sigma \cdot e \quad \text{and} \quad \sigma \cdot m = n \cdot (\sigma * \sigma),$$

where $*$ is the horizontal composition of natural transformations.

We let (T, m, e) be a monad on $\mathcal{A}$. A submonad of (T, m, e) is a monad (S, n, d) with a monad morphism $i\colon (S, n, d) \to (T, m, e)$ such that every component i_X is monic. In this case, $i\colon S \to T$ is called the inclusion transformation. To keep notations simple, we write (S, m, e) for submonad (S, n, d).

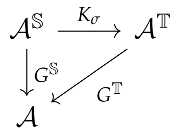
Given monad $\mathbb{T} = (T, m, e)$ on $\mathcal{A}$, an Eilenberg–Moore algebra for $\mathbb{T}$ ($\mathbb{T}$-algebra for short) is a pair (X, a) consisting of an $\mathcal{A}$-object X and an $\mathcal{A}$-morphism $a\colon TX \to X$ subject to the following:

$$a \cdot e_X = 1_X \quad \text{and} \quad a \cdot m_X = a \cdot Ta.$$

(TX, m_X) is obviously a $\mathbb{T}$-algebra, which is called the free $\mathbb{T}$-algebra on X.

A $\mathbb{T}$-homomorphism $f\colon (X, a) \to (X', a')$ of $\mathbb{T}$-algebras is an $\mathcal{A}$-morphism $f\colon X \to X'$ such that $a' \cdot Tf = f \cdot a$. $\mathbb{T}$-algebras and $\mathbb{T}$-homomorphisms assemble into a category $\mathcal{A}^{\mathbb{T}}$ which is called the Eilenberg–Moore category of $\mathbb{T}$.

Given a monad morphism $\sigma\colon \mathbb{S} \to \mathbb{T}$, there exists a functor $K_\sigma\colon \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}^{\mathbb{S}}$ induced by σ, which is identical on morphisms, sends the $\mathbb{T}$-algebra (X, a) to the $\mathbb{S}$-algebra $(X, a \cdot \sigma_X)$, and makes the diagram



commute, where $G^{\mathbb{T}}, G^{\mathbb{S}}$ are forgetful functors.

For more information on monads, we refer to [13,14]. Monads are useful for encoding general algebraic structures. The monograph by Plotkin [15] offers a comprehensive exploration of the algebraic aspects of database theory. Therefore, further research on the application of monads in the theory of databases is warranted.

Power-Enriched Monads

The powerset monad $\mathbb{P}$ is given by the covariant powerset functor $P\colon \mathsf{Set} \to \mathsf{Set}$ and two natural transformations:

$$\{-\}_X\colon X \to PX,\ x \mapsto \{x\},$$
$$\bigcup\nolimits_X\colon P^2X \to PX,\ \mathfrak{A} \mapsto \bigcup \mathfrak{A}.$$

The Eilenberg–Moore category of the powerset monad is isomorphic to the category Sup of complete lattices and sup-maps.

We consider monad $\mathbb{T}$ on Set equipped with monad morphism $\sigma\colon \mathbb{P} \to \mathbb{T}$. By the functor $K_\sigma\colon \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}^{\mathbb{P}}$, every $\mathbb{T}$-algebra (X, a) carries an order making X a complete lattice, and every morphism of $\mathbb{T}$-algebras is a sup-map. In particular, endowed with the order induced by the free $\mathbb{T}$-algebra structure on X, every set TX becomes a complete lattice.

If, for any sets X, Y, the map

$$(-)^{\mathbb{T}}\colon \mathsf{Set}(X, TY) \to \mathsf{Set}(TX, TY), \quad f \mapsto m_Y \cdot Tf$$

is monotone, where the hom-sets $\mathsf{Set}(-, TY)$ are ordered pointwise, then we refer to $(\mathbb{T}, \sigma)$ as a power-enriched monad. Morphism $\sigma\colon (\mathbb{T}, \sigma_1) \to (\mathbb{S}, \sigma_1')$ of power-enriched monads is monad morphism $\sigma\colon \mathbb{T} \to \mathbb{S}$ such that $\sigma_1' = \sigma \cdot \sigma_1$.

1.2. I*-Settings*

1.2.1. Continuous Triangular Norms

A triangular norm [16] (t-norm for short) is a binary operation & on the unit interval I subject to the following:

- & is associated;
- & is commutative;
- $a\&(-)$ is monotone for any $a \in I$;
- $a\&1 = a$ for any $a \in I$.

A t-norm & is called continuous if map $\&\colon I^2 \to I$ is continuous with respect to the standard topologies. We denote by $\mathsf{I} = (I, \&, 1)$ the unit interval I endowed with a continuous t-norm &.

Example 1. *There are three basic continuous t-norms.*

(1) *The Gödel t-norm* $a\&b = a \wedge b$;
(2) *The product t-norm* $a\&b = a \times b$;
(3) *The Łukasiewicz t-norm* $a\&b = \max\{0, a + b - 1\}$.

For each $a \in I$, since $a\&(-)\colon I \to I$ preserves arbitrary joints, then there exists a map $a \to (-)\colon I \to I$ which is right adjoint to $a\&(-)$ and is determined by

$$a\&b \le c \iff b \le a \to c.$$

A continuous t-norm is said to satisfy condition (S); if it satisfies that, for each $a \in (0, 1]$, map $a \to (-)$ is continuous on the interval $[0, a)$.

The following proposition includes some basic properties of continuous t-norms.

Proposition 1. *For any* $a, b, c \in I$ *and* $\{a_i\}_i \subset I$,

(1) $a\&(a \to b) \le b$;
(2) $1 \to a = a$;
(3) $a \to b = 1 \iff a \le b$;
(4) $(a\&b) \to c = a \to (b \to c)$;
(5) $a \to (\bigwedge_i a_i) = \bigwedge_i (a \to a_i)$;
(6) $(\bigvee_i a_i) \to a = \bigwedge_i (a_i \to a)$.

The reasons why we work with the particular quantale I include:

i. Some important many-valued topological structures are considered as topologies valued in $\mathsf{I} = (I, \&, 1)$ with & being certain t-norms. For example, fuzzy topologies can be seen as topologies valued in $(I, \wedge, 1)$, and since $(I, \times, 1)$ is isomorphic to the Lawvere quantale P_+, approach spaces can be considered as topological spaces valued in $(I, \times, 1)$.

ii. Many results about topologies valued in Q rely on the structure of Q; due to the celebrated ordinal sum decomposition theorem [16,17], the structure of I is clear.

1.2.2. I-Relations

An I-relation $r\colon X \nrightarrow Y$ is a map $r\colon X \times Y \to I$. The composition of $r\colon X \nrightarrow Y, s\colon Y \nrightarrow Z$ is an I-relation $(s \cdot r)\colon X \nrightarrow Z$ given by

$$(s \cdot r)(x,z) = \bigvee_{y \in Y} r(x,y) \& s(y,z).$$

Sets and I-relations assemble into a category

$$\text{I-Rel}.$$

Since the composition of I-relations preserves arbitrary joins in each variable, for each $r\colon X \nrightarrow Y$ and set Z, there are two maps $(-) \multimapinv r\colon \text{I-Rel}(X,Z) \to \text{I-Rel}(Y,Z)$ and $r \multimap (-)\colon \text{I-Rel}(Z,Y) \to \text{I-Rel}(Z,X)$ determined by

$$r \cdot t \leq s \iff t \leq s \multimapinv r;$$
$$t' \cdot r \leq s \iff t' \leq r \multimap s$$

for any $t \in \text{I-Rel}(Y,Z)$ and $t' \in \text{I-Rel}(Z,X)$.

For each $r\colon X \nrightarrow Y$, there is an I-relation $r^{\mathrm{op}}\colon Y \nrightarrow X$ given by $r^{\mathrm{op}}(y,x) = r(x,y)$. For each map $f\colon X \to Y$, graph $f_\circ\colon X \nrightarrow Y$ of f is given by

$$f_\circ(x,y) = \begin{cases} 1, & f(x) = y; \\ 0, & f(x) \neq y. \end{cases}$$

And the cograph f° of f is given by $f^\circ = (f_\circ)^{\mathrm{op}}$. There are two functors:

$$(-)_\circ\colon \mathsf{Set} \to \text{I-Rel} \quad \text{and} \quad (-)^\circ\colon \mathsf{Set} \to \text{I-Rel}^{\mathrm{op}}.$$

1.2.3. Lax Extensions to I-Rel

We let (T,m,e) be a monad on Set. A lax extension [18] of (T,m,e) to I-Rel is a triple $\hat{\mathbb{T}} = (\hat{T},m,e)$, where $\hat{T}$ is given by a family of maps

$$\hat{T}_{X,Y}\colon \text{I-Rel}(X,Y) \to \text{I-Rel}(TX,TY)$$

subject to the following conditions:

(1) Every $\hat{T}_{X,Y}$ is monotone;
(2) $\hat{T}r \cdot \hat{T}s \leq \hat{T}(r \cdot s)$;
(3) $(Tf)_\circ \leq \hat{T}(f_\circ)$ and $(Tf)^\circ \leq \hat{T}(f^\circ)$;
(4) $s \cdot e_X^\circ \leq e_Y^\circ \cdot \hat{T}s$;
(5) $\hat{T}\hat{T}s \cdot m_X^\circ \leq m_Y^\circ \cdot \hat{T}s$

for any sets X,Y,Z, I-relations $s\colon X \nrightarrow Y, r\colon Y \nrightarrow Z$ and every map $f\colon X \to Y$.

Morphism $\sigma\colon (\hat{S},n,d) \to (\hat{T},m,e)$ of lax extensions is a monad morphism $\sigma\colon (S,n,d) \to (T,m,e)$ such that $\hat{S}r \leq (\sigma_Y)^\circ \cdot \hat{T}r \cdot (\sigma_X)_\circ$ for any I-relation $r\colon X \nrightarrow Y$.

We let $\sigma\colon \mathbb{S} \to \mathbb{T}$ be a monad morphism and $\hat{\mathbb{T}}$ a lax extension of $\mathbb{T}$ to I-Rel. There is a lax extension of $\mathbb{S}$ given by

$$\hat{S}r = (\sigma_Y)^\circ \cdot \hat{T}r \cdot (\sigma_X)_\circ$$

for any I-relation $r\colon X \nrightarrow Y$. This lax extension $\hat{\mathbb{S}}$ is called the initial extension of $\mathbb{S}$ induced by σ.

1.2.4. I-Categories

An I-category [2,19] is a pair (X, r) consisting of a set X and a transitive and reflexive I-relation r, that is,

$$r(x,y)\& r(y,z) \leq r(x,z) \quad \text{and} \quad r(x,x) = 1$$

for all $x, y, z \in X$. For convenience, we simply use X to denote an I-category (X, r) and use $X(-,-)$ to denote $r(-,-)$.

For every I-category X, the I-relation $X^{\mathrm{op}}(x,y) = X(y,x)$ also gives an I-category, which is called the dual of X.

Example 2. (1) *The singleton* $\{*\}$ *set endowed with* $(\mathrm{id})_\circ$ *is obviously an* I*-category.*
(2) *The set* I^X *can be made an* I*-category via*

$$\mathrm{sub}_X(\mu,\nu) = \bigwedge_{x\in X} \mu(x) \to \nu(x).$$

An I-functor $f\colon X \to Y$ is a map $f\colon X \to Y$ between I-categories such that

$$X(x,y) \leq Y(f(x), f(y))$$

for all $x, y \in X$. If the converse of the above inequality also holds, we refer to this I-functor as fully faithful. I-functors $f\colon X \to Y, g\colon Y \to X$ are called an adjunction $f \dashv g$ if

$$Y(f(x),y) = X(x, g(y))$$

for any $x \in X, y \in Y$. In this case, we say f is left adjoint to g.

Example 3. *Given an* I*-relation* $r\colon X \nrightarrow Y$*, there is an adjunction* $r_\vee \dashv r_\wedge$*, in which* $r_\wedge, r_\vee$ *are given by*

$$r_\wedge\colon I^X \to I^Y, \mu \mapsto \bigwedge_{x\in X} r(x,-) \to \mu(x);$$
$$r_\vee\colon I^Y \to I^X, \nu \mapsto \bigvee_{y\in Y} r(-,y)\&\nu(y).$$

I-categories and I-functors assemble into a category

$$\textsf{I-Cat}.$$

The forgetful functor $o\colon \textsf{I-Cat} \to \textsf{Set}$ admits a left adjoint:

$$d\colon \textsf{Set} \to \textsf{I-Cat}, \quad X \mapsto (X, 1_X^\circ).$$

A locally small category is ordered if every hom-set carries an order such that the composition maps are monotone. A functor $F\colon \mathcal{A} \to \mathcal{B}$ between ordered categories is called a 2-functor if every $F_{A,B}\colon \mathcal{A}(A,B) \to \mathcal{B}(FA, FB)$ is monotone. A monad on an ordered category is called a 2-monad if the endfunctor is a 2-functor.

The underlying order of an I-category X is given by

$$x \leq_X y \iff X(x,y) = 1.$$

An I-category X is called separated if its underlying order is a partial order. I-Cat is an ordered category with I-Cat(X, Y) carrying the pointwise order.

Given an I-category X and $p \in I, x \in X$, the tensor of (p, x) is an element $p \otimes x$ of X such that $X(p \otimes x, -) = p \to X(x, -)$; the cotensor of (p, x) is an element $p \rightarrowtail x$ of X such that $X(-, p \rightarrowtail x) = p \to X(-, x)$.

An I-category X is called tensored (cotensored) if it fulfills that the tensor $p \otimes x$ (cotensor $p \rightarrowtail x$) exists for all $p \in I, x \in X$.

Proposition 2 ([20]). *The following statements are equivalent:*

(1) *X is tensored, $(X, \leq_X)$ is complete, and*

$$X(\bigvee_i x_i, y) = \bigwedge_i X(x_i, y)$$

for all $\{x_i\}_i \subset X, y \in X$;

(2) *X is cotensored, $(X, \leq_X)$ is complete, and*

$$X(x, \bigwedge_i y_i) = \bigwedge_i X(x, y_i)$$

for all $\{y_i\}_i \subset X, x \in X$.

An I-category is called complete if it satisfies the equivalent conditions stated above. For a complete I-category, we have $p \otimes (-) \dashv p \rightarrowtail (-)$.

Example 4. *The I-category (I^X, sub_X) is complete and separated. For any $p \in I, \mu \in I^X$, the cotensor of (p, μ) is given by $p \to \mu$.*

The following proposition is useful in ensuring the existence of adjunctions.

Proposition 3 ([20]). *We let $f\colon X \to Y, g\colon Y \to X$ be I-functors between I-categories. Then, $f \dashv g$ is an adjunction if and only if $f \dashv g\colon (Y, \leq_Y) \to (X, \leq_X)$ is an adjunction.*

2. The Lax Extensions from the Laxly Extended Monads on I-Cat

2.1. I-*Distributors*

Given two I-categories, X and Y, an I-distributor [2] $r\colon X \nrightarrow Y$ is an I-relation such that

$$r \cdot X \leq r \quad \text{and} \quad Y \cdot r \leq r.$$

If an I-distributor $r\colon X \nrightarrow Y$ is dummy in one variable, that is $X = \{*\}$ or $Y = \{*\}$, then we simply write $r(x)$ for $r(x, *)$ or $r(*, x)$. I-categories and I-distributors give rise to an ordered category

$$\mathsf{I\text{-}Dist}.$$

The forgetful functor $\mathsf{o}\colon \mathsf{I\text{-}Dist} \to \mathsf{I\text{-}Rel}$ admits a left adjoint:

$$\mathsf{d}\colon \mathsf{I\text{-}Rel} \to \mathsf{I\text{-}Dist}, \quad \mathsf{d}X = (X, 1_X^\circ), \quad r \mapsto r.$$

There are two 2-functors $(-)_*\colon \mathsf{I\text{-}Cat} \to \mathsf{I\text{-}Dist}^{\mathrm{co}}$ and $(-)^*\colon \mathsf{I\text{-}Cat} \to \mathsf{I\text{-}Dist}^{\mathrm{op}}$ defined on objects and morphisms by

$$(X)_* = X, \quad (f\colon X \to Y) \mapsto (f_* = (Y \cdot f_\circ)\colon X \nrightarrow Y);$$
$$(X)^* = X, \quad (f\colon X \to Y) \mapsto (f^* = (f^\circ \cdot Y)\colon Y \nrightarrow X).$$

We denote the set of I-distributors from an I-category X to $\{*\}$ by $\mathsf{P}X$. Then, the set $\mathsf{P}X$ can be made an I-category via

$$\mathsf{P}X(\mu, \nu) = \nu \multimapinv \mu = \mathrm{sub}_X(\mu, \nu).$$

Furthermore, P can be made a 2-functor from $\mathsf{I\text{-}Dist}^{\mathrm{op}}$ to $\mathsf{I\text{-}Cat}$ via

$$(r\colon X \nrightarrow Y) \mapsto (\mathsf{P}(r)\colon \mu \mapsto \mu \cdot r).$$

It is routine to check that $(-)^*$ is left adjoint to P. The induced 2-monad $(\mathsf{P}, \mathsf{s}, \mathsf{y})$ on I-Cat is called the presheaf monad.

Similarly, taking the I-distributors of type $\{*\} \nrightarrow X$ also gives rise to a 2-functor $\mathsf{P}^\dagger\colon \text{I-Dist}^{\text{co}} \to \text{I-Cat}$:

$$X \mapsto \mathsf{P}^\dagger X, \quad (r\colon X \nrightarrow Y) \mapsto (\mathsf{P}^\dagger(r)\colon \mu \mapsto r \cdot \mu),$$

in which

$$\mathsf{P}^\dagger X(\mu, \nu) = \nu \multimap \mu = \text{sub}_X^{\text{op}}(\mu, \nu)$$

for any $\mu, \nu \in \mathsf{P}^\dagger X$. The functor $(-)_*$ is left adjoint to $\mathsf{P}^\dagger$. The induced 2-monad $(\mathsf{P}^\dagger, \mathsf{s}^\dagger, \mathsf{y}^\dagger)$ on I-Cat is called the copresheaf monad.

The following lemmas present some basic properties of I-distributors.

Lemma 1 (Yoneda Lemma)**.** *For any $\nu \in \mathsf{P}^\dagger X, \mu \in \mathsf{P}X$, we have*

$$(\mathsf{y}_X)_*(-, \mu) = \mu \quad \text{and} \quad ({\mathsf{y}^\dagger}_X)^*(\nu, -) = \nu.$$

Lemma 2. *We let $f\colon X \to Y, g\colon Z \to Y$ be I-functors. For any $\mu \in \mathsf{P}Z, \nu \in \mathsf{P}^\dagger X, \phi \in \mathsf{P}^\dagger\mathsf{P}X$, and $\psi \in \mathsf{P}\mathsf{P}^\dagger Z$, we have the following statements:*

(1) $(\mathsf{P}g)^* \cdot (\mathsf{P}f)_*(-, \mu) = \mathsf{y}_{\mathsf{P}X}(\mu \cdot g^* \cdot f_*)$;
(2) $(\mathsf{P}^\dagger g)^* \cdot (\mathsf{P}^\dagger f)_*(\nu, -) = {\mathsf{y}^\dagger}_{\mathsf{P}^\dagger Z}(g^* \cdot f_* \cdot \nu)$;
(3) $(\mathsf{P}g)^* \cdot (\mathsf{P}f)_* \cdot \phi = \phi(- \cdot g^* \cdot f_*)$;
(4) $\psi \cdot (\mathsf{P}^\dagger g)^* \cdot (\mathsf{P}^\dagger f)_* = \psi(g^* \cdot f_* \cdot -)$.

2.2. Composite Monads on I-Cat

We let $\mathbb{T} = (T, m, e)$ and $\mathbb{S} = (S, n, d)$ be monads. A distributive law of $\mathbb{T}$ over $\mathbb{S}$ is a natural transformation $\sigma\colon TS \to ST$ subject to some conditions. A composite monad of $\mathbb{T}$ and $\mathbb{S}$ is a monad $(ST, \mathfrak{m}, d * e)$ such that $Se\colon S \to ST, dT\colon T \to ST$ are monad morphisms and $\mathfrak{m}$ satisfies that $\mathfrak{m} \cdot (SedT) = \text{id}_{ST}$. A distributive law σ yields a composite monad

$$(ST, (n * m) \cdot S\sigma T, d * e).$$

This correspondence is bijective. Details can be found in [21].

A saturated class of weights is a submonad A of the presheaf monad P. It is easy to check that it also offers a submonad $\mathsf{A}^\dagger$ of $\mathsf{P}^\dagger$ by $\mathsf{A}^\dagger X = (\mathsf{A}X^{\text{op}})^{\text{op}}$ for any X.

A distributive law $\sigma\colon \mathsf{P}^\dagger\mathsf{A} \to \mathsf{A}^\dagger\mathsf{P}$ of $\mathsf{P}^\dagger$ over A also offers a distributive law of P over $\mathsf{A}^\dagger$ whose components are given by

$$\sigma'_X\colon \mathsf{P}\mathsf{A}^\dagger X = (\mathsf{P}^\dagger\mathsf{A}X^{\text{op}})^{\text{op}} \xrightarrow{\sigma_{X^{\text{op}}}} (\mathsf{A}\mathsf{P}^\dagger X^{\text{op}})^{\text{op}} = \mathsf{A}^\dagger\mathsf{P}X.$$

One example of distributive laws is that the copresheaf monad distributes over the presheaf monad.

Proposition 4 ([22])**.** *There is a distributive law of $\mathsf{P}^\dagger$ over P, which offers the double presheaf 2-monad $\mathsf{P}\mathsf{P}^\dagger$ on I-Cat.*

We let X be an I-category. A forward Cauchy net [23] on X is a net $\{x_i\}_{i\in D}$ such that

$$\bigvee_{i \in D} \bigwedge_{k \geq j \geq i} X(x_j, x_k) = 1.$$

A forward Cauchy net generates an I-distributor $\mu\colon X \nrightarrow \{*\}$:

$$\mu = \bigvee_{i \in D} \bigwedge_{j \geq i} X(-, x_j).$$

Example 5. *A directed set D of $(X, \leq_X)$ is a forward Cauchy net $\{x_i\}_{i\in D}$ on X. The* I*-distributor generated by D is*
$$\bigvee_{d\in D} X(-,d).$$

We denote by $\mathsf{C}X$ the set of all I-distributors $\mu\colon X \nrightarrow \{*\}$ generated by forward Cauchy nets. The proof of that C is a saturated class of weights can be found in [24]. The following lemma offers a characterization of $\mathsf{C}X$ when X is complete and separated.

Lemma 3 (Proposition 4.8 in [25])**.** *We let X be a complete separated* I*-category. For every $\phi \in \mathsf{C}X$, we have that $D = \{x \in X \mid \phi(x) = 1\}$ is a directed set on $(X, \leq_X)$ and*
$$\phi = \bigvee_{d\in D} X(-,d).$$

The existence of a distributive law of $\mathsf{P}^\dagger$ over C depends on the structure of quantale I.

Proposition 5 (Theorem 6.4 in [25])**.** *There is a distributive law of $\mathsf{P}^\dagger$ over C if and only if the continuous t-norm satisfies the condition (S).*

In the remainder of this paper, we always assume that the continuous t-norm & satisfies the condition (S).

2.3. The Lax Extensions of Composite Monads to I-Dist

We let (T, m, e) be a 2-monad on I-Cat. A lax extension of (T, m, e) to I-Dist is a family of maps
$$\hat{T}_{X,Y}\colon \mathsf{I}\text{-Dist}(X,Y) \to \mathsf{I}\text{-Dist}(TX,TY)$$
subject to the following conditions:

(1) Every $\hat{T}_{X,Y}$ is monotone;
(2) $\hat{T}r \cdot \hat{T}s \leq \hat{T}(r\cdot s)$;
(3) $(Tf)_* \leq \hat{T}(f_*)$ and $(Tf)^* \leq \hat{T}f^*$;
(4) $s \cdot e_X^* \leq e_Y^* \cdot \hat{T}s$;
(5) $\hat{T}\hat{T}s \cdot m_X^* \leq m_Y^* \cdot \hat{T}s$

for any I-categories X, Y, Z, distributors $s\colon X \nrightarrow Y, r\colon Y \nrightarrow Z$ and every I-functor $f\colon X \to Y$.

Theorem 1 (Theorem 8.5 in [26])**.** *We let $\mathbb{T}$ be a 2-monad on* I-Cat*. Then,*
$$\hat{T}r = (T\overleftarrow{r})^* \cdot (T\mathsf{y}_X)_*\colon TX \nrightarrow TY$$
defines a lax extension of $\mathbb{T}$ to I-Dist*, where $\overleftarrow{r}\colon Y \to \mathsf{P}X, y \mapsto r(-,y)$.*

We let A be a saturated class of weights and assume that there is a distributive law $\sigma\colon \mathsf{P}^\dagger\mathsf{A} \to \mathsf{A}\mathsf{P}^\dagger$. Then, by Theorem 1, there are lax extensions of the monad $\mathsf{A}\mathsf{P}^\dagger$ and $\mathsf{A}^\dagger\mathsf{P}$ given by
$$\overline{\mathsf{A}\mathsf{P}^\dagger}r = (\mathsf{A}\mathsf{P}^\dagger\overleftarrow{r})^* \cdot (\mathsf{A}\mathsf{P}^\dagger\mathsf{y}_X)_*;$$
$$\overline{\mathsf{A}^\dagger\mathsf{P}}r = (\mathsf{A}^\dagger\mathsf{P}\overleftarrow{r})^* \cdot (\mathsf{A}^\dagger\mathsf{P}\mathsf{y}_X)_*.$$

In [27], Lai and Tholen introduced a functor Γ which maps monads (T, m, e) on I-Cat with a lax extension $\hat{T}$ to I-Dist to monads on Set with a lax extension to I-Rel :
$$\Gamma(T, m, e) = (oTd, omd \cdot oT\epsilon Td, oed),$$
$$\Gamma(\hat{T})r = \mathsf{o}\hat{T}\mathsf{d}(r),$$

in which ϵ is the counit of the adjunction $d \dashv o$.

It is routine to check that $\Gamma(\mathsf{A}^\dagger\mathsf{P}) = \Gamma(\mathsf{AP}^\dagger)$. We denote this monad by $(\mathsf{U_A}, \mathsf{n}, \mathsf{d})$.

For the lax extensions, using Lemma 2, we can compute as follows: for any I-relation $r\colon X \nrightarrow Y, \phi, \in \mathsf{AP}^\dagger X, \phi' \in \mathsf{AP}^\dagger Y, \psi \in \mathsf{A}^\dagger\mathsf{P}X$, and $\psi' \in \mathsf{A}^\dagger\mathsf{P}X$,

$$\begin{aligned}\overline{\mathsf{AP}^\dagger} r(\phi, \phi') &= ((\mathsf{AP}^\dagger \overleftarrow{r})^* \cdot (\mathsf{AP}^\dagger \mathsf{y}_X)_*)(\phi, \phi') \\ &= \mathsf{PP}^\dagger X(\phi, \phi' \cdot (\mathsf{P}^\dagger \overleftarrow{r})^* \cdot (\mathsf{P}^\dagger \mathsf{y}_X)_*) \\ &= \mathsf{PP}^\dagger X(\phi, \phi'(\overleftarrow{r}^* \cdot \mathsf{y}_{X*} \cdot -))\end{aligned}$$

and

$$\begin{aligned}\overline{\mathsf{A}^\dagger\mathsf{P}} r(\psi, \psi') &= ((\mathsf{A}^\dagger\mathsf{P}\mathsf{y}_X)^* \cdot (\mathsf{A}^\dagger\mathsf{P}\overleftarrow{r})_*)(\psi, \psi') \\ &= \mathsf{P}^\dagger\mathsf{P}Y((\mathsf{P}\overleftarrow{r})^* \cdot (\mathsf{P}\mathsf{y}_X)_* \cdot \psi, \psi') \\ &= \mathsf{P}^\dagger\mathsf{P}Y(\psi(- \cdot \overleftarrow{r}^* \cdot \mathsf{y}_{X*}), \psi').\end{aligned}$$

Thus, we obtain the following result.

Proposition 6. *We let* $\mathsf{AP}^\dagger$ *be a composite monad. There are two lax extensions of the monad* $(\mathsf{U_A}, \mathsf{n}, \mathsf{d})$:

$$\widehat{\mathsf{U_A}} r(\phi, \psi) = \bigwedge_{\mu \in I^X} \phi(\mu) \to \psi((r^{\mathrm{op}})_\vee(\mu)), \qquad \text{(canonical)}$$

$$\widecheck{\mathsf{U_A}} r(\phi, \psi) = \bigwedge_{\nu \in I^Y} \psi(\nu) \to \phi(r_\vee(\nu)), \qquad \text{(op-canonical)}$$

where $r\colon X \nrightarrow Y$ *is an* I*-relation,* $\phi \in \mathsf{U_A}X, \psi \in \mathsf{U_A}Y$.

2.4. The Conical I-Semifilter Monad

A conical I-semifilter [12] on set X is a function $\phi\colon I^X \to I$ subject to the following:

(F1) $\phi(1_X) = 1$;
(F2) $\phi(\mu \wedge \nu) = \phi(\mu) \wedge \phi(\nu)$;
(F3) $\mathrm{sub}_X(\mu, \nu) \le \phi(\mu) \to \phi(\nu)$;
(F4) $\phi = \bigvee_{\phi(\xi)=1} \mathrm{sub}_X(\xi, -)$.

Proposition 7. *The elements of* $\mathsf{CP}^\dagger dX$ *are exactly the conical* I*-semifilters.*

Proof. Given a conical I-semifilter ϕ on X, it follows from (F2) that $\{\mu \mid \phi(\mu) = 1\}$ is a directed set of $\mathsf{P}^\dagger dX$; hence, by (F4), we have $\phi \in \mathsf{CP}^\dagger dX$.

We let $\phi \in \mathsf{CP}^\dagger dX$. Since $\mathsf{P}^\dagger dX$ is separated and complete, by Lemma 3, it holds that

$$\phi = \bigvee_{\phi(\nu)=1} \mathsf{P}^\dagger dX(-, \nu) = \bigvee_{\phi(\nu)=1} \mathrm{sub}_X(\nu, -).$$

Hence, (F1), (F3) and (F4) are obvious. For (F2),

$$\phi(\mu_1 \wedge \mu_2) = \bigvee_{\phi(\nu)=1} (\mathsf{P}^\dagger dX(\mu_1, \nu) \wedge \mathsf{P}^\dagger dX(\mu_2, \nu)) = \phi(\mu_1) \wedge \phi(\mu_2),$$

the last equality holds because $\{\nu \mid \phi(\nu) = 1\}$ is directed. □

For every set X, $o(\mathsf{y} * \mathsf{y}^\dagger)_{dX}$ maps $x \in X$ to $\mathsf{P}^\dagger dX(-, \mathsf{y}_{dX}(x)) = (-)(x)$; $(o(\mathsf{s} * \mathsf{s}^\dagger)d \cdot o\mathsf{C}\sigma\mathsf{P}^\dagger d \cdot o\mathsf{CP}^\dagger\epsilon\mathsf{CP}^\dagger d)_X$ maps $\Phi \in \mathsf{U_C}^2 X$ to the conical I-semifilter

$$\phi\colon \mathsf{P}^\dagger dX \to I, \mu \mapsto \Phi(\mu^\sharp),$$

where $\mu^\sharp$ belongs to $\mathsf{P}^\dagger do\mathsf{CP}^\dagger dX$ and maps every $\psi \in do\mathsf{CP}^\dagger dX$ to $\psi(\mu)$. Therefore, the monad $(\mathsf{U_C}, \mathsf{n}, \mathsf{d})$ is exactly the conical I-semifilter monad in [12]. We adopt the notation from [12] and denote $(\mathsf{U_C}, \mathsf{n}, \mathsf{d})$ by $(\mathsf{CSF}, \mathsf{n}, \mathsf{d})$.

Corollary 1. *There are two lax extensions of the conical* I*-semifilter monad* $(\mathsf{CSF}, \mathsf{n}, \mathsf{d})$*:*

$$\widehat{\mathsf{CSF}}r(\phi, \psi) = \bigwedge_{\mu \in I^X} \phi(\mu) \to \psi((r^{\mathrm{op}})_\vee(\mu)), \qquad \text{(canonical)}$$

$$\widetilde{\mathsf{CSF}}r(\phi, \psi) = \bigwedge_{\nu \in I^Y} \psi(\nu) \to \phi(r_\vee(\nu)), \qquad \text{(op-canonical)}$$

where $r\colon X \nrightarrow Y$ *is an* I*-relation,* $\phi \in \mathsf{CSF}X, \psi \in \mathsf{CSF}Y$*.*

Remark 1. *Here, we prove that the continuous t-norm satisfies the condition (S) is a sufficient condition for conical* I*-semifilters to give rise to a monad. In fact, it is also a necessary condition; see [12].*

3. The Kleisli Extensions of $(\mathsf{U_A}, \mathsf{n}, \mathsf{d})$

3.1. The I-Powerset Monad

For each set X, we let $P_\mathsf{I} X = I^X$. Then, P_I can be made a functor from $\mathsf{I}\text{-}\mathsf{Rel}^{\mathrm{op}}$ to Set by letting

$$P_\mathsf{I}(r)(\mu) = r_\vee(\mu) = \bigvee_{y \in Y} \mu(y) \& r(-, y)$$

for each I-relation $r\colon X \nrightarrow Y$ and $\mu \in I^Y$. It is routine to check that $(-)^\circ$ is left adjoint to P_I. The induced monad is called the I-powerset monad and is denoted by $\mathbb{P}_\mathsf{I} = (P_\mathsf{I}, \mathsf{m}, \mathsf{e})$. We spell it out here: for any maps $f\colon X \to Y$ and $\mu \in P_\mathsf{I} X$,

$$P_\mathsf{I}(f)(\mu)\colon y \mapsto \bigvee_{f(x)=y} \mu(x),$$

$$\mathsf{e}_X\colon x \mapsto 1_x,$$

$$\mathsf{m}_X\colon \phi \mapsto \bigvee_{\mu \in P_\mathsf{I} X} \phi(\mu) \& \mu,$$

where 1_A is defined as $1_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$ and 1_x denotes $1_{\{x\}}$.

It is easy to check that the I-powerset monad is power-enriched by

$$\theta_X\colon PX \to P_\mathsf{I} X, \quad A \mapsto 1_A.$$

It also holds that $\mathbb{P}_\mathsf{I} = \Gamma(\mathsf{P}, \mathsf{s}, \mathsf{y}) = \Gamma(\mathsf{P}^\dagger, \mathsf{s}^\dagger, \mathsf{y}^\dagger)$.

3.2. I-Power-Enriched Monads

An I-power-enriched monad is a pair $(\mathbb{T}, \sigma)$ composed of a monad (T, m, e) on Set and a monad morphism $\sigma\colon \mathbb{P}_\mathsf{I} \to \mathbb{T}$ such that $(\mathbb{T}, \sigma \cdot \theta)$ is a power-enriched monad. A morphism $\sigma\colon (\mathbb{T}, \sigma_1) \to (\mathbb{S}, \sigma_2)$ of I-power-enriched monads is a monad morphism $\sigma\colon \mathbb{T} \to \mathbb{S}$ such that $\sigma_2 = \sigma \cdot \sigma_1$.

We let $\mathsf{AP}^\dagger$ be a composite monad. Since there is a monad morphism $\mathsf{yP}^\dagger\colon \mathsf{P}^\dagger \to \mathsf{AP}^\dagger$, by applying the functor Γ, we obtain the following Proposition.

Proposition 8. *The monad* $(\mathsf{U_A}, \mathsf{n}, \mathsf{d})$ *is* I*-power-enriched by* κ *whose components are given by*

$$\kappa_X\colon P_\mathsf{I} X \to \mathsf{U_A} X, \quad \mu \mapsto \mathrm{sub}_X(\mu, -).$$

An I-action in Sup is a complete lattice X endowed with a map $-\otimes-\colon I\times X\to X$ subject to the following: for any $p,q\in I$ and $x\in X$

(1) $p\otimes-$ and $-\otimes x$ are sup-maps;
(2) $(p\& q)\otimes x=p\otimes(q\otimes x)$ and $1\otimes x=x$.

A morphism of I-actions is a sup-map $f\colon X\to Y$ such that $p\otimes_Y f(x)=f(p\otimes_X x)$ for any $p\in I$ and $x\in X$. I-actions in Sup and their morphisms assemble into a category $\mathsf{Sup}^{\mathsf{I}}$.

It is shown in [28] that $\mathsf{Sup}^{\mathsf{I}}$ is isomorphic to the Eilenberg–Moore category of the I-powerset monad and there exists a functor $\Lambda\colon \mathsf{Set}^{\mathbb{P}_{\mathsf{I}}}\to\mathsf{I}\text{-}\mathsf{Cat}$.

Explicitly, we let (X,a) be a $\mathbb{P}_{\mathsf{I}}$-algebra; by functor $K_\theta\colon \mathsf{Set}^{\mathbb{P}_{\mathsf{I}}}\to\mathsf{Set}^{\mathbb{P}}$, X can be made a complete lattice. The I-action on X in Sup is given by

$$-\otimes-\colon I\times X\to X,\quad (p,x)\mapsto a(p\& 1_x).$$

Conversely, an I-action $(X,-\otimes-)$ yields a $\mathbb{P}_{\mathsf{I}}$-algebra structure as follows:

$$a\colon P_{\mathsf{I}}X\to X,\quad \mu\mapsto\bigvee_x \mu(x)\otimes x.$$

The functor Λ maps a $\mathbb{P}_{\mathsf{I}}$-algebra (X,a) to

$$\Lambda(X,a)(x,y)=a^{\dashv}(y)(x),$$

where $a\dashv a^{\dashv}\colon (X,\leq_X)\to(P_{\mathsf{I}}X,\leq_{P_{\mathsf{I}}X})$ is an adjunction. Furthermore, we have the following proposition.

Proposition 9. *Every* I*-category* $\Lambda(X,a)$ *is complete.*

Proof. For every $p\in I$, since $p\otimes-$ and a are sup-maps, we have the following adjunctions:

$$X\ \underset{p\otimes-}{\overset{p\rightarrowtail-}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}}\ X\ \underset{a}{\overset{a^{\dashv}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}}\ P_{\mathsf{I}}X.$$

To show X is cotensored by $\rightarrowtail$, we can follow these steps:

$$\begin{aligned}\mu\leq p\to a^{\dashv}(x) &\iff p\&\mu\leq a^{\dashv}(x)\\ &\iff a(p\&\mu)\leq x\\ &\iff \bigvee_t (p\&\mu(t))\otimes t\leq x\\ &\iff p\otimes\left(\bigvee_t \mu(t)\otimes t\right)\leq x\\ &\iff p\otimes a(\mu)\leq x\\ &\iff a(\mu)\leq p\rightarrowtail x\\ &\iff \mu\leq a^{\dashv}(p\rightarrowtail x).\quad\square\end{aligned}$$

Thus, the tensor of $\Lambda(X,a)$ is given by its I-action, the cotensor is given by the right adjoint of its I-action. That is the reason why we use the same notations.

Example 6. *For a composite monad* $\mathsf{AP}^{\dagger}$*, since* $(\mathsf{U_A}X,\mathsf{n}_X\cdot\kappa_{\mathsf{U_A}X})=K_\kappa(\mathsf{U_A}X,\mathsf{n}_X)$ *is a* $\mathbb{P}_{\mathsf{I}}$*-algebra,* $\mathsf{U_A}X$ *can be made a complete* I*-category via*

$$\mathsf{U_A}(\phi,\psi)=(\mathsf{n}_X\cdot\kappa_{\mathsf{U_A}X})^{\dashv}(\psi)(\phi)=\mathrm{sub}_{I^X}(\psi,\phi)=\bigwedge_{\mu\in I^X}\psi(\mu)\to\phi(\mu).$$

The tensor of (p, ϕ) *in* $\mathsf{U_A}X$ *is given by*

$$(\mathsf{n}_X \cdot \kappa_{\mathsf{U_A}X})(p\&1_\phi) = \bigwedge_{\psi \in \mathsf{U_A}X} (p\&1_\phi(\psi) \to \psi) = p \to \phi.$$

3.3. Kleisli Extensions

Given an I-power-enriched category $(\mathbb{T}, \sigma)$, for any I-relations $r\colon X \to Y$, the composite $\mathbb{P}_\mathsf{I}$-homomorphism

$$(TY, m_Y) \xrightarrow{T(\sigma_X \cdot r^\flat)} (T^2X, m_{TX}) \xrightarrow{m_X} (TX, m_X)$$

offers an I-functor $r^\sigma\colon TY \to TX$, where $r^\flat\colon Y \to P_\mathsf{I}X, y \mapsto r(-, y)$.

According to Section 4.5 in [18], there is a lax extension $\hat{\mathbb{T}}$ of $\mathbb{T}$ to I-Rel named the Kleisli extension, which is given by

$$\hat{T}r(\phi, \psi) = TX(\phi, r^\sigma(\psi))$$

for any $\phi \in TX, \psi \in TY$ and every I-relation $r\colon X \nrightarrow Y$.

Proposition 10. *For a composite monad* $\mathsf{AP}^\dagger$*, the Kleisli extension of* $(\mathsf{U_A}, \mathsf{n}, \mathsf{e})$ *is given by*

$$\overline{\mathsf{U_A}}r(\phi, \psi) = \mathsf{U_A}X(\phi, r^\kappa(\psi)) = \bigwedge_{\mu \in I^X} \psi(r_\wedge(\mu)) \to \phi(\mu),$$

where $r\colon X \nrightarrow Y$ *is an* I*-relation,* $\phi \in \mathsf{U_A}X, \psi \in \mathsf{U_A}Y$.

Theorem 2. *For the monad* $\mathsf{U_P}$*, the op-canonical extension to* I-Rel *coincides with the Kleisli extension to* I-Rel.

Proof. For any I-relation $r\colon X \nrightarrow Y$ and $\phi \in \mathsf{U_P}X$, by Lemma 2, the I-distributor

$$* \overset{\phi}{\longrightarrow\!\!\!\!\!\circ\;} \mathsf{P}dX \overset{(\mathsf{Py}_{dX})_*}{\longrightarrow\!\!\!\!\!\circ\;} \mathsf{P}^2dX \overset{(\mathsf{P}\overleftarrow{dr})^*}{\longrightarrow\!\!\!\!\!\circ\;} \mathsf{P}dY$$

is given by $\phi(- \cdot (\overleftarrow{r})^* \cdot (\mathsf{y}_{dX})_*) = \phi(r_\vee(-))$. Thus, mapping ϕ to $\phi(r_\vee(-))$ is an I-functor $f\colon \mathsf{U_P}X \to \mathsf{U_P}Y$.

To show the op-canonical extension to I-Rel coincides with the Kleisli extension to I-Rel, by Proposition 3, it suffices to show that $f \dashv r^\kappa\colon (\mathsf{U_P}Y, \leq_{\mathsf{U_P}Y}) \to (\mathsf{U_P}X, \leq_{\mathsf{U_P}X})$ is an adjunction. For any $\chi \in \mathsf{U_P}X, \psi \in \mathsf{U_P}Y$, since $r_\vee \dashv r_\wedge$ we have

$$(r^\kappa \cdot f)(\chi) = \chi \cdot r_\vee \cdot r_\wedge \geq_{\mathsf{U_P}X} \chi \quad \text{and} \quad (f \cdot r^\kappa)(\psi) = \psi \cdot r_\wedge \cdot r_\vee \leq_{\mathsf{U_P}Y} \psi.$$

This completes the proof. □

Since

$$\widetilde{\mathsf{CSF}}r(\phi, \psi) = \widetilde{\mathsf{U_P}}r\big(i_X(\phi), i_Y(\psi)\big) \quad \text{and} \quad \overline{\mathsf{CSF}}r(\phi, \psi) = \overline{\mathsf{U_P}}r\big(i_X(\phi), i_Y(\psi)\big)$$

for any $\phi \in \mathsf{CSF}X, \psi \in \mathsf{CSF}Y, r\colon X \nrightarrow Y$, where $i\colon \mathsf{CSF} \to \mathsf{U_P}$ is the inclusion transformation, we have the following corollary.

Corollary 2. *For the conical* I*-semifilter monad, the op-canonical extension to* I-Rel *coincides with the Kleisli extension to* I-Rel.

Proposition 11. *We let $\lambda\colon(\mathbb{S},\sigma)\to(\mathbb{T},\sigma')$ be a morphism of* I*-power-enriched monads. Then, λ is a morphism of the Kleisli extensions to* I-Rel. *Furthermore, every component $\lambda_X\colon SX\to TX$ is fully faithful if and only if the initial extension of $\mathbb{S}$ induced by λ is the Kleisli extension of $\mathbb{S}$.*

Proof. We denote $\mathbb{T}=(T,m,e)$ and $\mathbb{S}=(S,n,d)$. By the commutative diagram

$$\begin{array}{ccccc} S^2X & & \xrightarrow{\quad n_X \quad} & & SX \\ {\scriptstyle S(\lambda_X)}\downarrow & \searrow^{(\lambda*\lambda)_X} & & & \downarrow{\scriptstyle \lambda_X} \\ STX & \xrightarrow{\lambda_{TX}} & T^2X & \xrightarrow{m_X} & TX, \end{array}$$

$\lambda_X\colon(SX,n_X)\to(TX,m_X\cdot\lambda_{TX})$ is an $\mathbb{S}$-homomorphism; hence, it is an I-functor:

$$\hat{S}r(\alpha,\beta)=SX(\alpha,r^{\sigma}(\beta))\leq TX\big(\lambda_X(\alpha),\lambda_X(r^{\sigma}(\beta))\big).$$

By the commutative diagram

$$\begin{array}{ccccccccc} SY & \xrightarrow{S(r^\flat)} & SP_{\mathsf{I}}X & \xrightarrow{S(\sigma_X)} & S^2X & & \xrightarrow{\qquad n_X \qquad} & & SX \\ \downarrow{\scriptstyle\lambda_Y} & & {\scriptstyle\lambda_{P_{\mathsf{I}}X}}\downarrow & \searrow^{(\lambda*\sigma)_X} & \downarrow{\scriptstyle\lambda_{SX}} & \searrow^{(\lambda*\lambda)_X} & & & \downarrow{\scriptstyle\lambda_X} \\ TY & \xrightarrow[T(r^\flat)]{} & TP_{\mathsf{I}}X & \xrightarrow[T(\sigma_X)]{} & TSX & \xrightarrow[T(\lambda_X)]{} & T^2X & \xrightarrow[m_X]{} & TX, \end{array}$$

we have

$$TX\big(\lambda_X(\alpha),\lambda_X(r^{\sigma}(\beta))\big)=TX\big(\lambda_X(\alpha),r^{\sigma'}(\lambda_Y(\beta))\big)=\hat{T}r(\lambda_X(\alpha),\lambda_Y(\beta)).$$

This completes the proof. □

An element of I^X is called bounded if $\bigwedge\mu>0$. A conical I-semifilter ϕ is called bounded if $\phi(\mu)<1$ for any unbounded μ. Conical bounded I-semiflters also give rise to a monad $(\mathsf{ConBSF},\mathsf{n},\mathsf{d})$, and there is a monad morphism $\eta\colon\mathsf{CSF}\to\mathsf{ConBSF}$

$$\eta_X\colon\mathsf{CSF}X\to\mathsf{ConBSF}X,\quad \phi\mapsto\bigvee_{\substack{\phi(\mu)=1\\ \bigwedge\mu>0}}\mathrm{sub}_X(\mu,-);$$

see [12] for details.

Example 7.

(1) *The Kleisli extension of the conical* I*-semifilter monad to* I-Rel *coincides with the initial extension induced by the inclusion transformation $i\colon\mathsf{CSF}\to\mathsf{U}_{\mathsf{P}}$.*

(2) *The conical bounded* I*-semifilter monad is* I*-power-enriched by $\eta\cdot\kappa$, and $\eta\colon(\mathsf{CSF},\kappa)\to(\mathsf{ConBSF},\eta\cdot\kappa)$ is a morphism of* I*-power-enriched monads. Since κ is not fully faithful, the Kleisli extension $\overline{\mathsf{CSF}}$ does not coincide with the initial extension induced by κ.*

3.4. Lax Algebras

Given a lax extension $\hat{\mathbb{T}}$ of $\mathbb{T}$ to I-Rel, a $(\mathbb{T},\mathsf{I},\hat{\mathbb{T}})$-algebra (lax algebra for short) is a pair $(X,a\colon TX\nrightarrow X)$ so that

$$(1_X)_\circ\leq a\cdot(e_X)_\circ\quad\text{and}\quad a\cdot\hat{T}a\leq a\cdot(m_X)_\circ.$$

A morphism $f\colon(X,a)\to(Y,b)$ of lax algebras is a map $f\colon X\to Y$ subject to

$$f_\circ\cdot a\leq b\cdot(Tf)_\circ.$$

Lax algebras and morphisms of lax algebras form a category denoted by

$$(\mathbb{T}, \mathsf{I}, \hat{\mathbb{T}})\text{-}\mathsf{Cat}.$$

When the involved lax extension is clear, we simply write $(\mathbb{T}, \mathsf{I})$-Cat.

Lax extensions $\hat{\mathbb{T}}$ of monad $\mathbb{T}$ to Rel and lax algebras of $(\mathbb{T}, 2, \hat{\mathbb{T}})$ are defined in a manner similar to those of lax extensions to I-Rel and lax algebras of $(\mathbb{T}, \mathsf{I}, \hat{\mathbb{T}})$. Given an I-power-enriched monad $(\mathbb{T}, \sigma)$, it can be extended to Rel via

$$\alpha(\overline{T}r)\psi \iff \phi \leq_{TX} r^{\sigma}(\psi),$$

which is called the Kleisli extension of $\mathbb{T}$ to Rel, where r is a 2-relation and r^{σ} is defined by treating r as the I-relation $r(x, y) = \begin{cases} 1, & x\, r\, y, \\ 0 & otherwise. \end{cases}$

The following proposition affirms that, at the level of lax algebras, there is no distinction between the Kleisli extension to I-Rel and the Kleisli extension to Rel.

Proposition 12 (Proposition 6.1 in [18])**.** *We let (X, σ) be an* I*-power-enriched category. Then, there is an isomorphism*

$$(\mathbb{T}, \mathsf{I})\text{-}\mathsf{Cat} \cong (\mathbb{T}, 2)\text{-}\mathsf{Cat},$$

in which the lax extensions are the Kleisli extensions.

In [9], it is proven that

$$(\mathsf{CSF}, 2, \overline{\mathsf{CSF}})\text{-}\mathsf{Cat} \cong \mathsf{CNS},$$

where CNS is the category of CNS spaces. Therefore, we have the following corollary.

Corollary 3. *There is an isomorphism:*

$$(\mathsf{CSF}, \mathsf{I}, \overline{\mathsf{CSF}})\text{-}\mathsf{Cat} \cong \mathsf{CNS}.$$

When & is the product t-norm, the conical bounded I-semifilter monad is isomorphic to the functional ideal monad, and by [29], we have

$$(\mathsf{ConBSF}, 2, \overline{\mathsf{ConBSF}})\text{-}\mathsf{Cat} \cong \mathsf{App},$$

where App is the category of approach spaces and $\overline{\mathsf{ConBSF}}$ is the Kleisli extension to Rel.

Since $\eta \colon (\mathsf{CSF}, \kappa) \to (\mathsf{ConBSF}, \eta \cdot \kappa)$ is a morphism of the I-power-enriched category, by Theorem 11, it is a morphism of the Kleisli extensions. Hence, it induces an algebraic functor as follows:

Proposition 13. *If & is the product t-norm, there is a functor $A_\kappa \colon \mathsf{CNS} \to \mathsf{App}$:*

$$(X, (-)^{\circ}) \mapsto (X, \mathfrak{A})$$

that maps a CNS space X to the approach space $(X, \mathfrak{A})$, where the bounded approach system $\{\mathfrak{A}(x)\}_{x \in X}$ is given by

$$\mathfrak{A}(x) = \{\mu \in [0, \infty]^X \mid \bigvee_{\substack{\omega^{\circ}(x)=1 \\ \wedge\, \omega > 0}} \mathrm{sub}_X(\omega, e^{-\mu}) = 1\},$$

in which $(-)^{\circ}$ is the interior operator of the CNS space X.

4. Conclusions

In order to find the many-valued version of the filter monad, we begin with the composite monads $\mathsf{CP}^{\dagger}, \mathsf{C}^{\dagger}\mathsf{P}$ on I-Cat and then restrict them to Set to obtain the monad U_{C}.

This Set-based monad U_C is precisely the conical I-semifilter monad. Three lax extensions of the conical I-semifilter monad to I-Rel are presented: the canonical, op-canonical and Kleisli extensions. We prove that the op-canonical extension coincides with the Kleisli extension. Lax algebras of this extension can be described using relations rather than I-relations; hence, they are CNS spaces.

Problem 1. *When considering the canonical extension of the conical* I*-semifilter monad, what are the lax algebras?*

As for the future research direction, exploring the connections between monoidal topology and nonstandard analysis [30,31] is of interest.

Author Contributions: G.Z. and S.-Q.Z. contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Natural Science Foundation of China (Project No. 12271258), the Nanjing University start-up funding for talent research (Grant No. 16002203).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained in the body of the papaer.

Acknowledgments: Our sincere thanks are expressed to the anonymous reviewers for their meticulous reading of the entire manuscript and for providing helpful advice as well as references, which has significantly enhanced the quality of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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