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Vector-Valued Analytic Functions Having Vector-Valued Tempered Distributions as Boundary Values

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Abstract: Vector-valued analytic functions in $\mathbb{C}^n$, which are known to have vector-valued tempered distributional boundary values, are shown to be in the Hardy space $H^p,\ 1 \le p < 2$, if the boundary value is in the vector-valued $L^p,\ 1 \le p < 2$, functions. The analysis of this paper extends the analysis of a previous paper that considered the cases for $2 \le p \le \infty$. Thus, with the addition of the results of this paper, the considered problems are proved for all $p,\ 1 \le p \le \infty$.

Keywords: Hardy space; vector-valued analytic function; distribution

MSC: 2A07; 32A35; 46F05; 46F20; 32A26

1. Introduction

Historically, the analysis of tempered distributions as boundary values of analytic functions has found applications in mathematical physics, in the study of quantum field theory. An important reference in this study is Streater and Wightman [1]. In field theory, the "vacuum expectation values" are tempered distributions that are boundary values in the tempered distribution topology of analytic functions, with the analytic functions being Fourier–Laplace transforms. In addition, a field theory can be recovered from its "vacuum expectation values" [1] (Chapter 3). A similar field theory analysis is contained in the work by Simon [2].

Of particular interest with respect to the contents of this paper is the work of Raina [3] in mathematical physics. In [3], Raina considered analytic functions in the upper half plane that satisfied a pointwise growth condition associated with the analytic functions that have tempered distributions as boundary value when $Im(z) \to 0+$. The important mathematical result in [3] showed that if the tempered distributional boundary value was an element of $L^p(\mathbb{R}^1),\ 1 \le p \le \infty$, then the analytic function was in the Hardy space $H^p,\ 1 \le p \le \infty$ of analytic functions in the upper half plane. A converse result was proved. Raina described the importance of the results of this type concerning tempered distributional boundary values and the Hardy spaces $H^p,\ 1 \le p \le \infty$, which, in mathematical physics, are associated with "form factor bounds", including the use of Hardy spaces in general in related topics in mathematical physics. Several associated references are given in [3]. Importantly, the tempered distributions are used in the analysis of the mathematical physics in [1–3].

The results in [3] have led the author to consider the results of the type in [3] for higher dimensions and for the analytic and L^p functions being both scalar-valued and vector-valued. We have also desired to obtain representations of the analytic functions involved in terms of Fourier–Laplace transforms, Cauchy integrals, and Poisson integrals. Further, we have desired to obtain new results concerning both the scalar-valued and vector-valued Hardy functions in higher dimensions, including the growth properties of these functions.

Given our desires expressed in the previous paragraph, we first considered the scalar-valued case in [4] where we obtained the pointwise growth of scalar-valued H^p functions on tubes in $\mathbb{C}^n$. In [4], we considered scalar-valued analytic functions on tubes in $\mathbb{C}^n$ that had a

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specified pointwise growth, leading to the existence of tempered distributions as boundary values, and showed that if these boundary values were a L^p function, $1 \leq p \leq \infty$, then the scalar-valued analytic function was in H^p.Related results for other spaces of distributions were obtained in [4].

Continuing to the vector-valued case and building upon the results of [4], in [5] we considered vector-valued analytic functions in tube domains in $\mathbb{C}^n$ that have pointwise growth, leading to the existence of vector-valued tempered distributions as boundary values, and proved that if the boundary value is a vector-valued $L^p,\ 2 \leq p \leq \infty$ function then the analytic function must be in the Hardy space $H^p,\ 2 \leq p \leq \infty$. We obtained integral representations of the analytic functions and obtained pointwise growth of vector-valued H^p functions in tubes, $1 \leq p \leq \infty$.

In [6], we considered vector-valued analytic functions in tube domains without a defining pointwise growth so that any boundary value would be considered to be in the Schwartz vector-valued $\mathcal{D}'$ space. We showed that if the analytic functions obtained a distributional boundary value in the vector-valued distribution $\mathcal{D}'$ sense with the boundary value being a vector-valued function in $L^p,\ 1 \leq p \leq \infty$, then the analytic function is in the vector-valued Hardy space. We obtained a Poisson integral representation of the analytic functions in this case.

The cases for $1 \leq p < 2$ in the setting of [5] as described above are missing from our analysis at this point. That is, we desire to consider vector-valued analytic functions in tube domains that have specified pointwise growth that leads to the existence of vector-valued tempered distributions as boundary values. We then desire to prove that if the boundary value is a vector-valued $L^p,\ 1 \leq p < 2$ function, then the analytic function is in the vector-valued $H^p,\ 1 \leq p < 2$, space. This additional analysis is desirable in order to obtain the appropriate extension of the important Raina results to all of $1 \leq p \leq \infty$ in our generalized setting. Thus, the analysis in this paper concerns the values of p in $1 \leq p < 2$.

2. Definitions and Notation

All notation and definitions needed in this paper are the same as described or referred to in [5]. We mention and refer to several of the most frequently used definitions and notations here.

$\mathcal{B}$ will denote a Banach space, $\mathcal{H}$ will denote a Hilbert space, $\mathcal{N}$ will denote the norm of the specified Banach or Hilbert space, and Θ will denote the zero vector of the specified Banach or Hilbert space. $C \subset \mathbb{R}^n$ is a cone with a vertex at $\overline{0} = (0, 0, ..., 0)$ in $\mathbb{R}^n$ if $y \in C$ implies $\lambda y \in C$ for all $\lambda > 0$. The intersection of a cone C with the unit sphere $|y| = 1$ is the projection of C and is denoted $pr(C)$. A cone C' such that $pr(\overline{C'}) \subset pr(C)$ is a compact subcone of C. The dual cone C^* of C is defined as $C^* = \{t \in \mathbb{R}^n :< t, y > \geq\ 0 \text{ for all } y \in C\}$. An open convex cone that does not contain any entire straight line is called a regular cone. Let $v = (v_1, v_2, ..., v_n)$ be any of the 2^n n-tuples whose entries are 0 or 1. The 2^n n-rants $C_v = \{y \in \mathbb{R}^n : (-1)^{v_j} y_j > 0,\ j = 1, 2, ..., n\}$ are examples of regular cones that will be useful in this paper.

The $L^p(\mathbb{R}^n, \mathcal{B})$ functions, $1 \leq p \leq \infty$, with values in $\mathcal{B}$ and their norms $|\mathbf{h}|_p$, the Schwartz test spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}^{(m)}(\mathbb{R}^n), m \in \mathbb{N}$, and the spaces of tempered vector-valued distributions with values in $\mathcal{B}$, $\mathcal{S}'(\mathbb{R}^n, \mathcal{B})$ and $\mathcal{S}^{(m)'}(\mathbb{R}^n, \mathcal{B})$, are all noted in ([5], Section 2). The reference for the $L^p(\mathbb{R}^n, \mathcal{B})$ functions is Dunford and Schwartz [7]. The references for vector-valued distributions are Schwartz [8,9].

The Fourier transform on $\mathcal{S}'(\mathbb{R}^n, \mathcal{B})$ and on $L^1(\mathbb{R}^n)$ or $L^1(\mathbb{R}^n, \mathcal{B})$ is given in [5] (Section 2). The Fourier transform of $U \in \mathcal{S}'(\mathbb{R}^n, \mathcal{B})$ comes from [8], and will be denoted $\mathcal{F}[U]$, with the inverse Fourier transform being denoted $\mathcal{F}^{-1}[U]$. Similarly, all Fourier (inverse Fourier) transforms on scalar-valued or vector-valued functions will be denoted $\mathcal{F}[\phi(t); x]$ or $\widehat{\phi}$ $(\mathcal{F}^{-1}[\phi(t); x])$. Of particular importance in this paper are the Fourier and inverse Fourier transforms on the vector-valued L^2 functions; the results that we need for these functions are discussed and proved in [10] (Section 1.8). As stated in this reference and referenced in [10] (Section 1.11), the Plancherel theory is not valid for vector-valued functions except when $\mathcal{B} = \mathcal{H}$, a Hilbert space. That is, in order for the Fourier transform $\mathcal{F}$ to be an isomorphism of $L^2(\mathbb{R}^n, \mathcal{B})$ onto itself with the Parseval identity $|\widehat{\mathbf{f}}|_2 = |\mathbf{f}|_2$ holding, it is necessary and sufficient that $\mathcal{B} = \mathcal{H}$, a Hilbert space; this fact comes from Kwapień [11]. The Plancherel theory is complete in the $L^2(\mathbb{R}^n, \mathcal{H})$ setting in that the inverse Fourier transform is the inverse mapping of the Fourier transform with $\mathcal{F}^{-1}\mathcal{F} = I = \mathcal{F}\mathcal{F}^{-1}$, with I being the identity mapping. As stated in [10] (Section 1.8), the Plancherel theory stated there is valid for functions of several variables with values in Hilbert space. In the analysis of this paper, we need the Plancherel theory holding on $L^2(\mathbb{R}^n, \mathcal{B})$, and thus where needed we take $\mathcal{B} = \mathcal{H}$, a Hilbert space.

Associated with the Fourier transform on vector-valued functions with values in Banach space is the concept of Banach space of type p, $1 \leq p \leq 2$, discussed in [12] (Section 6). We note that every Banach space has Fourier type 1 and leave pursuit of this concept of Fourier type to the interested reader.

Let B be an open subset of $\mathbb{R}^n$. The Hardy space $H^p(T^B, \mathcal{B})$, $0 < p < \infty$, consists of those analytic functions $\mathbf{f}(z)$ on the tube $T^B = \mathbb{R}^n + iB \subset \mathbb{C}^n$ with values in a Banach space $\mathcal{B}$ such that

$$\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{f}(x+iy)))^p dx \ \leq \ M, \ y \in B,$$

where $z = x + iy \in T^B$ and the constant $M > 0$ is independent of $y \in B$; the usual modification is made for the case $p = \infty$.

Let C be an open convex cone in $\mathbb{R}^n$. $\mathcal{E}(\mathbb{R}^n)$ will denote the set of all infinitely differentiable complex valued functions on $\mathbb{R}^n$. We define the function $d_y(t) \in \mathcal{E}(\mathbb{R}^n)$, $t \in \mathbb{R}^n$, $y \in C$, as in [5] (Section 2).

We define and state known results concerning the Cauchy and Poisson kernel functions corresponding to tubes $T^B = \mathbb{R}^n + iC \subset \mathbb{C}^n$. Let C be a regular cone in $\mathbb{R}^n$ and C^* be the corresponding dual cone of C. The Cauchy kernel corresponding to T^C is

$$K(z-t) = \int_{C^*} e^{2\pi i \langle z-t, \eta \rangle} d\eta, \ t \in \mathbb{R}^n, \ z \in T^C,$$

where C^* is the dual cone of C as noted. The Poisson kernel corresponding to T^C is

$$Q(z;t) = \frac{K(z-t)\overline{K(z-t)}}{K(2iy)} = \frac{|K(z-t)|^2}{K(2iy)}, \ t \in \mathbb{R}^n, \ z \in T^C.$$

Referring to [13] (Chapters 1 and 4) for details, we know for $z \in T^C$ that $K(z - \cdot) \in \mathcal{D}(*, L^p) \subset \mathcal{D}_{L^p}$, $1 < p \leq \infty$; and $Q(z; \cdot) \in \mathcal{D}(*, L^p) \subset \mathcal{D}_{L^p}$, $1 \leq p \leq \infty$, where $*$ is Beurling (M_p) or Roumieu $\{M_p\}$. These ultradifferentiable functions are contained in the Schwartz space $\mathcal{D}_{L^p} = \mathcal{D}(L^p, \mathbb{R}^n)$. We also use the results [4] (Lemmas 3.1 and 3.2). Because of the combined properties of the Cauchy and Poisson kernels from [13,14], we know that the Cauchy and Poisson integrals

$$\int_{\mathbb{R}^n} \mathbf{h}(t) K(z-t) dt \quad \text{and} \quad \int_{\mathbb{R}^n} \mathbf{h}(t) Q(z;t) dt, \ z \in T^C,$$

are well defined for $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{B})$, $1 \leq p < \infty$, and $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{B})$, $1 \leq p \leq \infty$, respectively, where $\mathcal{B}$ is a Banach space.

We use [5] (Lemma 3.4) several times in this paper. For convenience to the reader, we state this result here to conclude this section. Throughout $N(\overline{0}, r)$ denotes the closed ball of radius $r > 0$ centered at $\overline{0} \in \mathbb{R}^n$.

Theorem 1. *Let $\boldsymbol{f}$ be analytic in $T^C = \mathbb{R}^n + iC$ with values in a Banach space $\mathcal{B}$, where C is a regular cone in $\mathbb{R}^n$, and have the Poisson integral representation*

$$\boldsymbol{f}(z) = \int_{\mathbb{R}^n} \boldsymbol{h}(t) Q(z;t) dt, \ z \in T^C,$$

for $\boldsymbol{h} \in L^p(\mathbb{R}^n, \mathcal{B}),\ 1 \leq p \leq \infty$. *We have* $\boldsymbol{f} \in H^p(T^C, \mathcal{B}),\ 1 \leq p \leq \infty$. *For* $p = \infty$, $\boldsymbol{f}(x+iy) \to \boldsymbol{h}(x)$ *in the weak-star topology of* $L^\infty(\mathbb{R}^n, \mathcal{B})$ *as* $y \to \overline{0},\ y \in C$; *for* $1 \leq p < \infty$, $\boldsymbol{f}(x+iy) \to \boldsymbol{h}(x),\ x \in \mathbb{R}^n$, *in* $L^p(\mathbb{R}^n, \mathcal{B})$ *as* $y \to \overline{0},\ y \in C$; *for* $1 < p \leq 2$

$$\mathcal{N}(\boldsymbol{f}(x+iy)) \leq M(C')|\boldsymbol{h}|_p|y|^{-n/p},\ z = x+iy \in T^{C'},$$

for all compact subcones $C' \subset C$, $M(C')$ *being a constant depending on* $C' \subset C$ *and not on* $y \in C'$, *while*

$$\mathcal{N}(\boldsymbol{f}(x+iy)) \leq M_y|\boldsymbol{h}|_p|y|^{-n/p},\ z = x+iy \in T^C,$$

where M_y *is a constant depending on* $y \in C$; *and for* $2 < p < \infty$

$$\begin{aligned}&\mathcal{N}(\boldsymbol{f}(x+iy)) \leq M(C',r)|\boldsymbol{h}|_p,\\ &z = x+iy \in T(C',r) = \{z = x+iy : x \in \mathbb{R}^n,\ y \in (C' \setminus (C' \cap N(\overline{0},r)))\},\end{aligned}$$

for all compact subcones $C' \subset C$ *and all* $r > 0$, $M(C',r)$ *being a constant depending on* $C' \subset C$ *and on* $r > 0$, *but not on* $y \in (C' \setminus (C' \cap N(\overline{0},r)))$, *while*

$$\mathcal{N}(\boldsymbol{f}(x+iy)) \leq M_y|\boldsymbol{h}|_p,\ z = x+iy \in T^C,$$

where M_y *is a constant depending on* $y \in C$.

3. Tempered Distributional Boundary Values

Let C be an open convex cone in $\mathbb{R}^n$ and $T^C = \mathbb{R}^n + iC \subset \mathbb{C}^n$. We denote the set of analytic functions on T^C with values in a Banach space $\mathcal{B}$ by $\mathcal{A}(T^C, \mathcal{B})$. As above, $N(\overline{0}, r)$ denotes the closed ball about $\overline{0} \in \mathbb{R}^n$ of radius $r > 0$.

In [5] (Theorem 4.1), we have stated the following result which we need here.

Theorem 2. *Let* C *be an open convex cone. Let* $\boldsymbol{f} \in \mathcal{A}(T^C, \mathcal{B})$. *For every compact subcone* $C' \subset C$ *and every* $r > 0$, *let*

$$\begin{aligned}&\mathcal{N}(\boldsymbol{f}(x+iy)) \leq M(C',r)(1+|x|)^R|y|^{-k},\\ &z = x+iy \in T(C',r) = \mathbb{R}^n + i(C' \setminus (C' \cap N(\overline{0},r))),\end{aligned} \tag{1}$$

where $M(C',r)$ *is a constant depending on* $C' \subset C$ *and on* r, R *is a nonnegative integer,* k *is an integer greater than* 1, *and neither* R *nor* k *depend on* C' *or* r. *There exists a positive integer* m *and a unique element* $U \in \mathcal{S}^{(m)'}(\mathbb{R}^n, \mathcal{B}) \subset \mathcal{S}'(\mathbb{R}^n, \mathcal{B})$ *such that*

$$\lim_{y \to \overline{0}, y \in C} \mathcal{N}(\langle \boldsymbol{f}(x+iy), \phi(x)\rangle - \langle U, \phi\rangle) \ = \ 0,\ \phi \in \mathcal{S}^{(m)}(\mathbb{R}^n). \tag{2}$$

In Theorem 2, and in the remainder of this paper, by $y \to \overline{0},\ y \in C$, we mean that $y \to \overline{0},\ y \in C' \subset C$, for every compact subcone C' of C.

In [5] (Theorem 4.4), we proved for C, a regular cone, and the boundary value U in Theorem 2 being a function $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{H}),\ 2 \leq p \leq \infty$, that the analytic function $\mathbf{f}$ in Theorem 2 is, in fact, in $H^p(T^C, \mathcal{H}),\ 2 \leq p \leq \infty$. In [5], we were not able to obtain this result for the cases $1 \leq p < 2$. We now have a proof for the cases $1 \leq p < 2$, and we obtain the result [5] (Theorem 4.4) for the cases $1 \leq p < 2$ here.

To obtain [5] (Theorem 4.4) for $1 \leq p < 2$, we follow some of the structure of [5] by first proving our result for the case that the cone C is a n-rant cone C_v or is contained in a n-rant cone and then using this case to obtain the general result for the cone C being any regular cone. Because $1 \leq p < 2$, here the details of our proof in the case $C \subseteq C_v$ in

Theorem 3 below are different in many instances than those of [5] (Theorems 4.2 and 4.3). The values of the functions and distributions in the remainder of this section will be in Hilbert space $\mathcal{H}$ because of the need for the Fourier transform properties on $L^2(\mathbb{R}^n, \mathcal{H})$, as described in Section 2 above.

We give an outline of the proof of Theorem 3 for the benefit of the reader. Given the assumed function $\mathbf{f}(z)$ in Theorem 3, we will divide it by a structured analytic function $X_\epsilon(z),\ z \in T^C,\ \epsilon > 0$, and put $\mathbf{g}_\epsilon(z) = \mathbf{f}(z)/X_\epsilon(z),\ z \in T^C,\ \epsilon > 0$. $\mathbf{g}_\epsilon(z)$ is represented as the Fourier transform involving a function $\mathbf{G}_\epsilon(t)$, which has support in C^*. $\mathbf{g}_\epsilon(x+iy)$ is shown to have boundary value $\mathcal{F}[\mathbf{G}_\epsilon]$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ as $y \to \overline{0},\ y \in C$, and then is shown to equal the Cauchy integral and the Poisson integral of a function involving the boundary value $\mathbf{h}$ of $\mathbf{f}(x+iy)$. After establishing some important limit analysis, we proceed to prove that $\mathbf{f}(z)$ equals the Poisson integral of the boundary value $\mathbf{h}$, which will then yield the conclusions of Theorem 3.

Theorem 3. *Let C be an open convex cone which is contained in or is any of the 2^n n-rants $C_v \subset \mathbb{R}^n$. Let $\mathcal{H}$ be a Hilbert space. Let $\boldsymbol{f} \in \mathcal{A}(T^C, \mathcal{H})$ and satisfy (1). Let the unique boundary value U of Theorem 2 be $\boldsymbol{h} \in L^p(\mathbb{R}^n, \mathcal{H}),\ 1 \le p < 2$. We have $\boldsymbol{f} \in H^p(T^C, \mathcal{H}),\ 1 \le p < 2$, and*

$$\boldsymbol{f}(z) = \int_{\mathbb{R}^n} \boldsymbol{h}(t) Q(z;t)\,dt,\ z \in T^C. \tag{3}$$

Proof. As noted above, the proof has a structure similar to that of [5] (Theorems 4.2 and 4.3), but many details are different. We refer to [5] (Theorems 4.2 and 4.3) where appropriate. Put $\mathbf{g}_\epsilon(z) = \mathbf{f}(z)/X_\epsilon(z),\ z \in T^C,\ \epsilon > 0$, where

$$X_\epsilon(z) = \prod_{j=1}^{n} (1 - i\epsilon(-1)^{v_j} z_j)^{R+n+2},\ \epsilon > 0.$$

$\mathbf{g}_\epsilon(z)$ satisfies (1). (By Theorem 2, there is a unique $U_\epsilon \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ such that (2) holds for $\mathbf{g}_\epsilon(z)$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$, a fact that we use later in this proof.) By the same analysis as in the proof of [5] (Theorem 4.2), we obtain [5, (15)] here; that is,

$$\begin{aligned} &\mathcal{N}(\mathbf{g}_\epsilon(z)) \le M'(C', r, \epsilon)(1 + |z|)^{-n-2}, \\ &z \in T(C', r) = \mathbb{R}^n + i(C' \setminus (C' \cap N(\overline{0}, r))), \end{aligned} \tag{4}$$

for all compact subcones $C' \subset C \subseteq C_v$ and all $r > 0$ where $M'(C', r, \epsilon)$ is a constant. Put

$$\mathbf{G}_\epsilon(t) = \int_{\mathbb{R}^n} \mathbf{g}_\epsilon(x+iy) e^{-2\pi i \langle x+iy, t\rangle}\,dx,\ y \in C,\ t \in \mathbb{R}^n. \tag{5}$$

Using (4), the same proof as in the proof of [5] (Theorem 4.2) yields that $\mathbf{G}_\epsilon(t)$ is a continuous function of $t \in \mathbb{R}^n$ for $y \in C$ and $\epsilon > 0$, is independent of $y \in C$, and has support in C^*, the dual cone of C. □

For any compact subcone $C' \subset C$, any $r > 0$, and any $\epsilon > 0$ Equation (4) yields

$$\mathcal{N}(\mathbf{G}_\epsilon(t)) \le M''(C', r, \epsilon) e^{2\pi\langle y,t\rangle},\ t \in \mathbb{R}^n,\ y \in (C' \setminus (C' \cap N(\overline{0}, r))), \tag{6}$$

from which $e^{-2\pi\langle y,t\rangle}\mathbf{G}_\epsilon(t) \in L^p(\mathbb{R}^n, \mathcal{H})$ for $y \in C$ and for all $p,\ 1 \le p < \infty$, by ([5], Lemma 3.1). From Equation (5), $e^{-2\pi\langle y,t\rangle}\mathbf{G}_\epsilon(t) = \mathcal{F}^{-1}[\mathbf{g}_\epsilon(x+iy); t],\ y \in C$, with the transform holding in both the $L^1(\mathbb{R}^n, \mathcal{H})$ and $L^2(\mathbb{R}^n, \mathcal{H})$ cases, and in $L^2(\mathbb{R}^n, \mathcal{H})$:

$$\mathbf{g}_\epsilon(x+iy) = \mathcal{F}[e^{-2\pi\langle y,t\rangle}\mathbf{G}_\epsilon(t); x],\ z = x+iy \in T^C. \tag{7}$$

From the properties of $\mathbf{G}_\epsilon$, the Fourier transform in (7) is in both the $L^1(\mathbb{R}^n, \mathcal{H})$ and $L^2(\mathbb{R}^n, \mathcal{H})$ cases, and (7) becomes

$$\mathbf{g}_\epsilon(x+iy) = \int_{\mathbb{R}^n} \mathbf{G}_\epsilon(t) e^{2\pi i \langle x+iy, t\rangle} dt, \; z = x+iy \in T^C. \tag{8}$$

Both $\mathbf{G}_\epsilon(t)$ and $e^{-2\pi\langle y,t\rangle}\mathbf{G}_\epsilon(t),\; y \in C$, are elements of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$, and $\mathbf{g}_\epsilon(x+iy) \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H}),\; y \in C$. Thus, $\mathbf{g}_\epsilon(x+iy) = \mathcal{F}[e^{-2\pi\langle y,t\rangle}\mathbf{G}_\epsilon(t); x],\; z = x+iy \in T^C$, in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ now. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi = \mathcal{F}[\phi; \cdot]$. We have

$$\begin{aligned} &\langle \mathbf{g}_\epsilon(x+iy), \phi(x)\rangle = \langle e^{-2\pi\langle y,t\rangle}\mathbf{G}_\epsilon(t), \psi(t)\rangle \\ &\to \langle \mathbf{G}_\epsilon(t), \psi(t)\rangle = \langle \mathcal{F}[\mathbf{G}_\epsilon], \phi\rangle \end{aligned} \tag{9}$$

as $y \to \overline{0},\; y \in C$. As noted above, by Theorem 2, there is a unique $\mathbf{U}_\epsilon \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ such that $\mathbf{g}_\epsilon(x+iy) \to \mathbf{U}_\epsilon$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ as $y \to \overline{0},\; y \in C$; hence, $\mathcal{F}[\mathbf{G}_\epsilon] = \mathbf{U}_\epsilon$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ and $\mathbf{G}_\epsilon = \mathcal{F}^{-1}[\mathbf{U}_\epsilon] \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$.

Since $\mathbf{g}_\epsilon(z) = \mathbf{f}(z)/X_\epsilon(z),\; z \in T^C,\; \epsilon > 0$, we have $\mathbf{f}(z) = \mathbf{g}_\epsilon(z) X_\epsilon(z),\; z \in T^C$. By hypothesis, $\mathbf{f}(x+iy)$ has boundary value $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{H})$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ as $y \to \overline{0},\; y \in C$, and $X_\epsilon(x+iy)\mathbf{g}_\epsilon(x+iy) \to X_\epsilon \mathcal{F}[\mathbf{G}_\epsilon] = X_\epsilon \mathbf{U}_\epsilon$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ as $y \to \overline{0},\; y \in C$. Thus, $X_\epsilon \mathbf{U}_\epsilon = \mathbf{h}$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H}),\; \epsilon > 0$. For $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \frac{\mathbf{h}(x)}{X_\epsilon(x)}, \phi(x)\rangle = \langle \mathbf{h}(x), \frac{\phi(x)}{X_\epsilon(x)}\rangle = \langle \mathbf{U}_\epsilon, \phi\rangle$$

and $\mathbf{U}_\epsilon = \mathbf{h}(x)/X_\epsilon(x) \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. For $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{H}),\; 1 \le p < 2$, we have $\mathbf{h}/X_\epsilon \in L^1(\mathbb{R}^n, \mathcal{H})$, since $1/X_\epsilon \in L^q(\mathbb{R}^n)$ for all $q,\; 1 \le q \le \infty$. We put $\mathbf{H}_\epsilon = \mathcal{F}^{-1}[\mathbf{h}(x)/X_\epsilon(x); \cdot]$; thus $\mathbf{H}_\epsilon \in L^\infty(\mathbb{R}^n, \mathcal{H})$. Since $\mathrm{supp}(\mathbf{G}_\epsilon) \subseteq C^*$, then $\mathrm{supp}(\mathbf{H}_\epsilon) \subseteq C^*$ almost everywhere. Recalling the function $d_y(t)$ defined in Section 2, we have $d_y(t)e^{2\pi i\langle z,t\rangle} \in \mathcal{S}(\mathbb{R}^n),\; z \in T^C$. For $z \in T^C$

$$\begin{aligned} &\int_{C^*} \mathbf{G}_\epsilon(t) e^{2\pi i\langle z,t\rangle} dt = \langle \mathbf{U}_\epsilon, \mathcal{F}^{-1}[d_y(t)e^{2\pi i\langle z,t\rangle}; \eta]\rangle \\ &= \langle \frac{\mathbf{h}(\eta)}{X_\epsilon(\eta)}, \mathcal{F}^{-1}[d_y(t)e^{2\pi i\langle z,t\rangle}; \eta]\rangle = \int_{C^*} \mathbf{H}_\epsilon(t) e^{2\pi i\langle z,t\rangle} dt \end{aligned} \tag{10}$$

with $\mathbf{H}_\epsilon \in L^\infty(\mathbb{R}^n, \mathcal{H})$. From [4] (Lemma 2.1), $I_{C^*}(t)e^{2\pi i\langle z,t\rangle} \in L^p(\mathbb{R}^n)$ for all $p,\; 1 \le p \le \infty$, for $z \in T^C$ where $I_{C^*}(t)$ is the characteristic function of C^*, and the integral on the right of (10) is convergent, since $\mathbf{H}_\epsilon \in L^\infty(\mathbb{R}^n, \mathcal{H})$ and $\mathrm{supp}(\mathbf{H}_\epsilon) \subseteq C^*$ almost everywhere. From (8), (10), and the fact that $\mathrm{supp}(\mathbf{G}_\epsilon) \subseteq C^*$, we have for $z \in T^C$

$$\begin{aligned} &\mathbf{g}_\epsilon(x+iy) = \langle \mathbf{H}_\epsilon(t), I_{C^*}(t)e^{2\pi i\langle z,t\rangle}\rangle \\ &= \langle \mathcal{F}^{-1}[\mathbf{h}(x)/X_\epsilon(x); t], I_{C^*}(t)e^{2\pi i\langle z,t\rangle}\rangle = \int_{\mathbb{R}^n} \frac{\mathbf{h}(\eta)}{X_\epsilon(\eta)} K(z-\eta) d\eta. \end{aligned} \tag{11}$$

We proceed to construct a Poisson integral representation for $\mathbf{g}_\epsilon(z)$ in addition to the Cauchy integral representation in (11). Let w be an arbitrary point of T^C. Using [4] (Lemma 3.2), we have for $z \in T^C$ that $K(z+w)\mathbf{g}_\epsilon(z) = K(z+w)\mathbf{f}(z)/X_\epsilon(z)$ is analytic in $z \in T^C$ and satisfies the growth (1) of $\mathbf{f}(z)$. Further,

$$\lim_{y \to \overline{0}, y \in C} K(x+iy+w)\mathbf{g}_\epsilon(x+iy) = K(x+w)\mathbf{U}_\epsilon = \frac{K(x+w)\mathbf{h}(x)}{X_\epsilon(x)}$$

in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ with $K(x+w)\mathbf{h}(x)/X_\epsilon(x) \in L^p(\mathbb{R}^n, \mathcal{H}),\ 1 \le p < 2$, since both $K(x+w)$ and $1/X_\epsilon(x)$ are bounded for $x \in \mathbb{R}^n$. The same proof leading to (11) applied to $K(z+w)\mathbf{g}_\epsilon(z),\ z \in T^C$, yields

$$K(z+w)\mathbf{g}_\epsilon(z) = \int_{\mathbb{R}^n} \frac{\mathbf{h}(t)}{X_\epsilon(t)} K(t+w)K(z-t)dt,\ z \in T^C. \tag{12}$$

For $z = x+iy \in T^C$, we choose $w = -x+iy \in T^C$. Then, (12) combined with (11) becomes

$$\begin{aligned} \mathbf{g}_\epsilon(z) &= \int_{C^*} \mathbf{G}_\epsilon(t) e^{2\pi i \langle z,t\rangle} dt \\ &= \int_{\mathbb{R}^n} \frac{\mathbf{h}(t)}{X_\epsilon(t)} K(z-t)dt = \int_{\mathbb{R}^n} \frac{\mathbf{h}(t)}{X_\epsilon(t)} Q(z;t)dt,\ z \in T^C. \end{aligned} \tag{13}$$

We now present some limited analyses, which we need to analyze the function, the Poisson integral of $\mathbf{h}$, that we will show represents $\mathbf{f}(z)$ and from which the conclusion of the proof of this theorem will follow. Since $|1/X_\epsilon(x)| \le 1,\ x \in \mathbb{R}^n,\ \epsilon > 0$, both $\mathbf{h}$ and $\mathbf{h}/X_\epsilon$ are in $L^p(\mathbb{R}^n, \mathcal{H}),\ 1 \le p < 2$. We have

$$(\mathcal{N}(\frac{\mathbf{h}(x)}{X_\epsilon(x)} - \mathbf{h}(x)))^p \le (\mathcal{N}(\frac{\mathbf{h}(x)}{X_\epsilon(x)}) + \mathcal{N}(\mathbf{h}(x)))^p \le 2^{p+2}(\mathcal{N}(\mathbf{h}(x)))^p$$

with the right side being independent of $\epsilon > 0$. Further,

$$\lim_{\epsilon \to 0+} \mathcal{N}\left(\frac{\mathbf{h}(x)}{X_\epsilon(x)} - \mathbf{h}(x)\right) = \lim_{\epsilon \to 0+} |(1/X_\epsilon(x)) - 1| \mathcal{N}(\mathbf{h}(x)) = 0,\ x \in \mathbb{R}^n.$$

By the Lebesgue dominated convergence theorem

$$\lim_{\epsilon \to 0+} \left| \frac{\mathbf{h}(x)}{X_\epsilon(x)} - \mathbf{h}(x) \right|_p = \lim_{\epsilon \to 0+} \left(\int_{\mathbb{R}^n} \left(\mathcal{N}\left(\frac{\mathbf{h}(x)}{X_\epsilon(x)} - \mathbf{h}(x) \right) \right)^p dx \right)^{1/p} = 0 \tag{14}$$

which proves $\mathbf{h}/X_\epsilon \to \mathbf{h}$ in $L^p(\mathbb{R}^n, \mathcal{H}),\ 1 \le p < 2$, as $\epsilon \to 0+$.

We now define and analyze the function which we desire to be the Poisson integral representation of $\mathbf{f}(z)$, as noted in the preceding paragraph; this function is

$$\mathbf{G}(z) = \int_{\mathbb{R}^b} \mathbf{h}(t) Q(z;t)dt,\ z \in T^C. \tag{15}$$

Let z_o be an arbitrary but fixed point of T^C. Choose the closed neighborhood $N(z_o, \rho) = \{z : |z - z_o| \le \rho,\ \rho > 0\} \subset T^C$ of [5] (Lemma 3.3), and note that [5] (Lemma 3.3) holds for all $p,\ 1 \le p \le \infty$. Let the constant $B(z_o)$ in (16) below be the constant obtained in [5] (Lemma 3.3). Using the Hölder inequality if $1 < p < 2$ and the boundedness of $Q(z;t)$ from the proof of [5] (Lemma 3.3) ([4] (Lemma 3.4)) if $p = 1$ and using (13) and (15), we have

$$\begin{aligned} \mathcal{N}(\mathbf{g}_\epsilon(z) - \mathbf{G}(z)) &= \mathcal{N}\left(\int_{\mathbb{R}^n} \left(\frac{\mathbf{h}(t)}{X_\epsilon(t)} - \mathbf{h}(t) \right) Q(z;t)dt \right) \\ &\le B(z_o) \left(\int_{\mathbb{R}^n} \left(\mathcal{N}\left(\frac{\mathbf{h}(t)}{X_\epsilon(t)} - \mathbf{h}(t) \right) \right)^p dt \right)^{1/p} \\ &= B(z_o) \left| \frac{\mathbf{h}(t)}{X_\epsilon(t)} - \mathbf{h}(t) \right|_p \end{aligned} \tag{16}$$

for $z \in N(z_o, \rho) \subset T^C$. Using (14) and (16) for $1 \leq p < 2$, we have

$$\lim_{\epsilon \to 0+} \mathbf{g}_\epsilon(z) = \mathbf{G}(z)$$

uniformly in $z \in N(z_o, \rho)$. Since $\mathbf{g}_\epsilon(z)$ is analytic in T^C, $\epsilon > 0$, we have that $\mathbf{G}(z)$ is analytic at $z_o \in T^C$; hence $\mathbf{G}(z)$ is analytic in T^C since z_o is an arbitrary point in T^C. Applying Theorem 1, we have $\mathbf{G}(z) \in H^p(T^C, \mathcal{H})$, $1 \leq p < 2$.

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Using Hölder's inequality, if $1 < p < 2$ and the boundedness of $\phi \in \mathcal{S}(\mathbb{R}^n)$ if $p = 1$, we have

$$\begin{aligned}&\mathcal{N}(\langle \mathbf{G}(x+iy), \phi(x)\rangle - \langle \mathbf{h}(x), \phi(x)\rangle) = \mathcal{N}\left(\int_{\mathbb{R}^n} (\mathbf{G}(x+iy) - \mathbf{h}(x))\phi(x)dx\right) \\ &\leq |\mathbf{G}(x+iy) - \mathbf{h}(x)|_p ||\phi||_{L^q(\mathbb{R}^n)}.\end{aligned}$$

By Theorem 1, $\mathbf{G}(x+iy) \to \mathbf{h}(x)$ in $L^p(\mathbb{R}^n, \mathcal{H})$, $1 \leq p < 2$, as $y \to \overline{0}$, $y \in C$; hence $\mathbf{G}(x+iy) \to \mathbf{h}(x)$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ as $y \to \overline{0}$, $y \in C$.

Now, consider $\mathbf{f}(z) - \mathbf{G}(z)$, $z \in T^C$, which is analytic in T^C. For $1 < p < 2$, we have the pointwise bound on $\mathbf{G}(z)$ for $y = Im(z)$ in any compact subcone $C' \subset C$ contained in Theorem 1. (see also ([5], (6)).) Thus, combining the bounds (1) on $\mathbf{f}(z)$ and the pointwise bound just noted on $\mathbf{G}(z)$ for $y = Im(z)$ in any compact subcone $C' \subset C$, we have the inequality

$$\begin{aligned}&\mathcal{N}(\mathbf{f}(z) - \mathbf{G}(z)) \leq P(C', r)(1 + |z|)^R, \\ &z = x + iy \in T(C', r) = \mathbb{R}^n + i(C' \setminus (C' \cap N(\overline{0}, r))),\end{aligned} \tag{17}$$

on $\mathbf{f}(z) - \mathbf{G}(z)$ for the cases $1 < p < 2$ for any compact subcone $C' \subset C$ and any $r > 0$ where $P(C', r)$ is a constant depending on $C' \subset C$ and on $r > 0$. If $p = 1$, by combining inequalities ([4], (10) and (11)) in the proof of Theorem 1 given in ([5], Lemma 3.4), we have

$$\begin{aligned}&Q(z;t) = \frac{|K(z-t)|^2}{K(2iy)} \leq (Z_n(n-1)!\delta^{-n})^2 (B(C))^{-1} |y|^{-n}, \\ &z = x + iy \in T^{C'}, C' \subset C,\end{aligned}$$

and hence

$$\begin{aligned}&\mathcal{N}(\mathbf{G}(z)) \leq |\mathbf{h}|_1 (Z_n(n-1)!\delta^{-n})^2 (B(C))^{-1} |y|^{-n}, \\ &z = x + iy \in T^{C'}, C' \subset C,\end{aligned}$$

where δ depends only on C' and not on C and Z_n is the surface area of the unit sphere in $\mathbb{R}^n$. Combining this inequality on $\mathbf{G}(z)$ with inequality (1) on $\mathbf{f}(z)$, we again have that $\mathbf{f}(z) - \mathbf{G}(z)$, $z \in T^C$, also satisfies (17) for $p = 1$. In addition, we know from the boundary values of $\mathbf{f}(z)$ and $\mathbf{G}(z)$ that

$$\lim_{y \to \overline{0}, y \in C} (\mathbf{f}(x+iy) - \mathbf{G}(x+iy)) = \mathbf{h}(x) - \mathbf{h}(x) = \Theta \tag{18}$$

in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ for $1 \leq p < 2$.

Using (18), we now proceed to complete the proof by proving $\mathbf{f}(z) = \mathbf{G}(z)$, $z \in T^C$. Put $\mathbf{F}(z) = \mathbf{f}(z) - \mathbf{G}(z)$, $z \in T^C$, which is analytic in T^C. $\mathbf{F}(z)$ satisfies (17) and (18) for each p, $1 \leq p < 2$. Consider $\mathbf{g}(z) = \mathbf{F}(z)/X_1(z)$, $z \in T^C$, where $X_1(z)$ is defined at the beginning of this proof for $\epsilon = 1$. As in obtaining (4) for $\epsilon = 1$, we have for $C' \subset C \subseteq C_v$ and $r > 0$

$$\mathcal{N}(\mathbf{g}(z)) \leq P'(C', r, 1)(1 + |z|)^{-n-2},$$
$$z = x + iy \in T(C', r) = \mathbb{R}^n + i(C' \setminus (C' \cap N(\overline{0}, r))),$$

where $P'(C', r, 1)$ is a constant. Now putting as in (5)

$$\mathbf{A}(t) = \int_{\mathbb{R}^n} \mathbf{g}(x + iy) e^{-2\pi i \langle x+iy, t\rangle} dx,\ y \in C,\ t \in \mathbb{R}^n,$$

and proceeding with the proof from (5) to (8), we have that $\mathbf{A}(t)$ is continuous, is independent of $y \in C$; has support in C^*; satisfies a growth as in (6); satisfies $e^{-2\pi\langle y,t\rangle}\mathbf{A}(t) = \mathcal{F}^{-1}[\mathbf{g}(x + iy); t],\ t \in \mathbb{R}^n,\ y \in C$, with the transform holding in both the $L^1(\mathbb{R}^n, \mathcal{H})$ and $L^2(\mathbb{R}^n, \mathcal{H})$ cases; and with $e^{-2\pi\langle y,t\rangle}\mathbf{A}(t) \in L^p(\mathbb{R}^n, \mathcal{H})$ for all p, $1 \leq p < \infty$, $y \in C$; satisfies $\mathbf{g}(x + iy) = \mathcal{F}[e^{-2\pi\langle y,t\rangle}\mathbf{A}(t); x],\ x \in \mathbb{R}^n,\ y \in C$; and satisfies

$$\mathbf{g}(x + iy) = \int_{\mathbb{R}^n} \mathbf{A}(t) e^{2\pi i\langle x+iy,t\rangle} dt,\ z = x + iy \in T^C. \tag{19}$$

For $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $y \in C$

$$\begin{aligned}
&\langle \mathbf{F}(x + iy)/X_1(x + iy), \phi(x)\rangle = \langle \mathbf{g}(x + iy), \phi(x)\rangle \\
&= \langle \mathcal{F}[e^{-2\pi\langle y,t\rangle}\mathbf{A}(t); x], \phi(x)\rangle = \langle e^{-2\pi\langle y,t\rangle}\mathbf{A}(t), \mathcal{F}[\phi(x); t]\rangle \\
&\to \langle \mathbf{A}(t), \mathcal{F}[\phi(x); t]\rangle = \langle \mathcal{F}[\mathbf{A}], \phi(x)\rangle
\end{aligned}$$

as $y \to \overline{0},\ y \in C$; and

$$\begin{aligned}
&\langle \mathbf{F}(x + iy), \phi(x)\rangle = \langle \mathbf{g}(x + iy), X_1(x + iy)\phi(x)\rangle \\
&\to \langle \mathcal{F}[\mathbf{A}], X_1(x)\phi(x)\rangle = \langle X_1(x)\mathcal{F}[\mathbf{A}], \phi(x)\rangle
\end{aligned}$$

as $y \to \overline{0},\ y \in C$. Combining this fact with (18) yields $X_1(x)\mathcal{F}[\mathbf{A}] = \Theta$, since $\mathbf{F}(z) = \mathbf{f}(z) - \mathbf{G}(z)$; hence, $\mathbf{A} = \Theta$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. Put

$$\Delta = \prod_{j=1}^{n} \left(1 - i(-1)^{v_j} \left(\frac{-1}{2\pi i}\frac{\partial}{\partial t_j}\right)\right)^{R+n+2}.$$

From (19),

$$\begin{aligned}
\mathbf{F}(z) &= X_1(z)\mathbf{g}(z) = X_1(z) \int_{\mathbb{R}^n} \mathbf{A}(t) e^{2\pi i\langle x+iy,t\rangle} dt \\
&= \langle \Delta\mathbf{A}(t), d_y(t) e^{2\pi i\langle x+iy,t\rangle}\rangle = \Theta,\ z = x + iy \in T^C.
\end{aligned}$$

Since $\mathbf{F}(z) = \mathbf{f}(z) - \mathbf{G}(z),\ z \in T^C$, and $\mathbf{G}(z)$ is given in (15), we conclude

$$\mathbf{f}(z) = \mathbf{G}(z) = \int_{\mathbb{R}^n} \mathbf{h}(t) Q(z; t) dt,\ z \in T^C;$$

and $\mathbf{f}(z) \in H^p(T^C, \mathcal{H})$, $1 \leq p < 2$, since we have previously obtained $\mathbf{G}(z) \in H^p(T^C, \mathcal{H})$, $1 \leq p < 2$, from Theorem 1. The proof of Theorem 3 is complete.

With Theorem 3 proved for $1 \leq p < 2$, we now obtain this result for C being an arbitrary regular cone in $\mathbb{R}^n$; this is our desired result, which extends [5] (Theorem 4.4) to the cases $1 \leq p < 2$. The proof of the following theorem for the cases $1 \leq p < 2$ is obtained using Theorem 3 by exactly the same proof that [5] (Theorem 4.4) was proved using [5] (Theorems 4.2 and 4.3); we ask the interested reader to follow the suggested proof if desired.

Theorem 4. *Let C be a regular cone in $\mathbb{R}^n$. Let $\mathcal{H}$ be a Hilbert space. Let $\boldsymbol{f} \in \mathcal{A}(T^C, \mathcal{H})$ and satisfy* (1)*. Let the unique boundary value U of Theorem 2 be $\boldsymbol{h} \in L^p(\mathbb{R}^n, \mathcal{H})$, $1 \leq p < 2$. We have $\boldsymbol{f} \in H^p(T^C, \mathcal{H})$, $1 \leq p < 2$, and*

$$\boldsymbol{f}(z) = \int_{\mathbb{R}^n} \boldsymbol{h}(t) Q(z;t) dt, \ z \in T^C.$$

The Poisson integral representation of the function $\mathbf{f}(z)$ in Theorem 4 follows from the fact that the unique $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ boundary value $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{H})$, $1 \leq p < 2$, is obtained independently of how $y \to \bar{0}$, $y \in C$, and follows from the structure of the tubes T^{S_j}, $j = 1, ..., k$, in the referenced proof of [5] (Theorem 4.4). $\mathbf{f}(z)$ equals the Poisson integral of $\mathbf{h}$ in each of these tubes by Theorem 3 and hence in all of T^C.

In summary concerning the proofs here of Theorems 3 and 4 for $1 \leq p < 2$ and the proofs of the corresponding results in [5] for $2 \leq p \leq \infty$, we note the following. In certain places in the analysis, the products or quotients involving the boundary value h and other terms must be analyzed carefully in order for the analysis to proceed. In both restrictions on p, many times we need the product or quotient to be Fourier transformable in $L^1(\mathbb{R}^n, \mathcal{H})$ or $L^2(\mathbb{R}^n, \mathcal{H})$ or both. The properties of such products or quotients can be different depending on whether $1 \leq p < 2$ or $2 \leq p \leq \infty$; hence, the analysis must be suitably adjusted to proceed with the proof. Further, to obtain appropriate boundedness properties in the proofs the method to proceed depends on whether $p = 1$ or $1 < p < 2$ for the case $1 \leq p < 2$, and depends on whether $2 \leq p < \infty$ or $p = \infty$ for the case $2 \leq p \leq \infty$. These and other technical difficulties must be overcome for the proofs to proceed, and the difficulties depend on the two cases, $1 \leq p < 2$ or $2 \leq p \leq \infty$. Additionally, here we have stated the Poisson integral representation of $\mathbf{f}(z)$ as a conclusion in Theorem 4, but should have done so in [5] (Theorem 4.4) as well, where this conclusion is obtained by the same argument used in the paragraph below Theorem 4 above.

4. Conclusions

The primary goal of this paper was to extend the result [5] (Theorem 4.4) to the cases for $1 \leq p < 2$ in the tempered distribution setting. This we have accomplished. Combining [5] (Theorem 4.4) with Theorem 4 of this paper, we have the desired result for all p, $1 \leq p \leq \infty$. The author was motivated to obtain these results by the paper of Raina [3]. As previously described, the results of [3] are applicable in mathematical physics. In the Introduction, we have also recalled the important work of Streater and Wightman [1] and Simon [2]. Our growth estimate (1) is of the type with which Tillmann [14] characterized those analytic functions which obtain tempered distributional boundary values. We note that the results of Tillmann have been extended by Meise [15,16] in studying tempered vector-valued distributions as boundary values of vector-valued analytic functions.

Other authors have made contributions to mathematical physics in which tempered distributional boundary values of analytic functions are involved. Vladimirov [17] shows that analytic functions similar to those in this paper arise in applying the Fourier–Laplace transform to convolution equations which describe linear homogeneous processes with causality that find application in several applicable fields. The analysis of linear conjugacy of analytic functions of several variables involves tempered distributional boundary values of analytic functions represented as Fourier–Laplace integrals. Vladimirov [18] states that many problems arising in mathematical physics reduce to the problem of linear conjugacy involving tempered distributions. Considering the analysis in this paper we suggest, for example, that the linear conjugacy problem can be extended to the vector-valued case. We plan to work on the linear conjugacy problem in the vector-valued case in the future.

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