

## Article

# Novel Method for Ranking Generalized Fuzzy Numbers Based on Normalized Height Coefficient and Benefit and Cost Areas

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**Abstract:** This paper proposes a method for ranking generalized fuzzy numbers, which guarantees that both horizontal and vertical values are important parameters affecting the final ranking score. In this method, the normalized height coefficient is introduced to evaluate the influence of the height of fuzzy numbers on the final ranking score. The higher the normalized height coefficient of a generalized fuzzy number is, the higher its ranking. The left and right areas are presented to calculate the impact of the vertical value on the final ranking score. The left area is considered the benefit area. The right area is considered the cost area. A generalized fuzzy number is preferred if the benefit area is larger and the cost area is smaller. The proposed method can be employed to rank both normal and non-normal fuzzy numbers without normalization or height minimization. Numerical examples and comparisons with other methods highlight the feasibility and robustness of the proposed method, which can overcome the shortcomings of some existing methods and can support decision-makers in selecting the best alternative.

**Keywords:** generalized fuzzy numbers; ranking; normalized height coefficient; left area; right area

MSC: 91B06



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## 1. Introduction

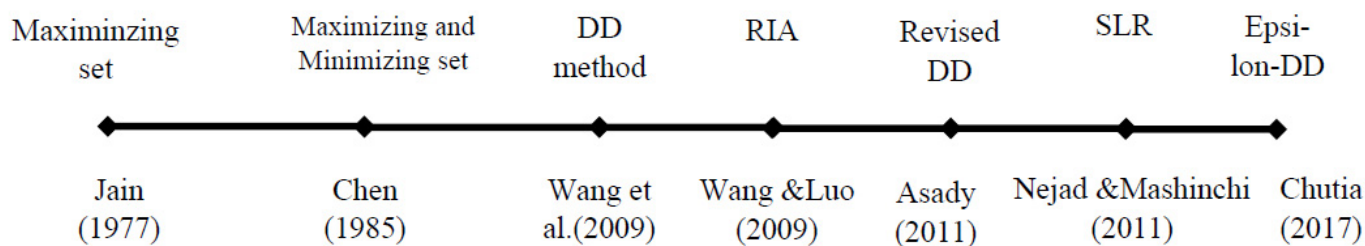
Wang and Luo [1] highlighted that ranking fuzzy numbers is a very important issue in fuzzy sets theory and applications and has been extensively researched. Some ranking methods have been reviewed and compared by [2–4]. Chou et al. [5] indicated that, nevertheless, none of these methods can always guarantee a consistent result in every situation, and some are even unintuitive and indiscriminate. Yu et al. [6] and Chi and Yu [7] emphasized that when fuzzy numbers are non-normal, some methods use height minimization ( $\min w_i$ ) or normalization, which leads to information loss. Methods using  $\min w_i$  include the maximizing and minimizing sets for ranking fuzzy numbers used by [8] and the rank and mode approach for ranking generalized fuzzy numbers proposed by [5]. The method that first normalized fuzzy numbers before ranking them is where fuzzy numbers are ranked with an integral value, introduced by [9].

It is impossible to define the boundary of the membership function of a fuzzy number in normal form. Thus, recent studies have focused on considering the height of the fuzzy number to avoid a loss of information and incorrect ranking [7]. However, such studies have some limitations. Chen and Chen [10] determined three factors that affect the ranking score: the defuzzified value, height, and spread. The defuzzified value and height of a generalized fuzzy number are the major factors determining its ranking score; the spread is only a minor factor. However, Kumar et al. [11] indicated that the ranking function proposed by Chen and Chen [10] does not satisfy the reasonable property

$\tilde{A} \succ \tilde{B} \Rightarrow (\tilde{A} - \tilde{B}) \succ (\tilde{B} - \tilde{A})$  for the ordering of fuzzy quantities, which is a contradiction according to Wang and Kerre [12] (see Example 1 in Section 2.3). Chen and Sanguansat [13] considered the areas on the positive and negative sides and the height of the generalized fuzzy numbers to evaluate the ranking score of the generalized fuzzy numbers. Xu et al. [14] pointed out that in the situation when the score is zero, the results of Chen and Sanguansat's ranking method are unreasonable.

Chi and Yu [7] proposed ranking generalized fuzzy numbers on the basis of the centroid and rank index, which prevents the truncation of heights during comparison. To avoid information loss, the original height of a given fuzzy number is retained and considered an important factor to affect the ranking of the generalized fuzzy numbers. However, this considers three factors, namely, the centroid, rank and mode, and height, as discrete factors, with height being the least influential, leading to incorrect final ranking results (see Example 2 in Section 2.3). De et al. [15] indicated that the height of fuzzy numbers plays an essential role in preventing information loss. This study considers the centroid point, rank index, and height for ranking interval type-2 fuzzy numbers. However, this method cannot be used to rank fuzzy numbers with different centroids and heights (see Example 3 in Section 2.3). Revathi and Valliathal [16] used the centroid method for ordering non-normal fuzzy numbers with more parameters, which is investigated using level analysis, which gives flexibility to the expert's opinion. Nguyen and Chu [17] proposed a DEMATEL-ANP-based fuzzy PROMETHEE II to rank startups, in which areas based on a subject's confidence level were suggested and height was not considered. He et al. [18] introduced a new fuzzy distance based on a novel interval distance that considers all points within the intervals by using the concept of integration to calculate the average distance between all points belonging to two intervals.

Jain [19] proposed the maximizing set to rank fuzzy numbers and restricted the membership function  $f_A(x)$  to the normal form. Chen [8] developed the maximizing and minimizing sets for generalized fuzzy numbers. However, this paper used height minimization, which fails to rank the same fuzzy numbers with different heights (see Example 4 in Section 2.3). Wang et al. [20] developed the deviation degree method based on the maximizing and minimizing sets. According to Chutia [21], the expectation value of the centroid points involved in the epsilon deviation degree method does not influence the heights of fuzzy numbers, which leads to the incorrect ranking of non-normal fuzzy numbers (illustrated in Example 4 in Section 2.3). Furthermore, according to Equation (11), in the case where  $\lambda = 0$  and  $1 - \lambda = 0$ , when the left and right deviation degree are multiplied by these values, they become valueless [22]. Wang and Luo [1] proposed ranking indices based on areas and considered the maximizing and minimizing sets as the positive ideal and negative ideal points, respectively. However, this study does not consider the height of fuzzy numbers and thus is not useful for ranking non-normal fuzzy numbers (as shown in Example 4 in Section 2.3). Asady [23] revised the deviation degree method with the new left and right areas. However, Hajjari and Abbasbandy [24] pointed out that Asady's revision has a shortcoming similar to the method proposed by [20]. Nejad and Mashinchi [22] proposed ranking fuzzy numbers based on the areas on the left and the right sides. To prevent the values of  $\lambda = 0$  and  $1 - \lambda = 0$ , and  $S_i^R = 0$  and  $S_i^L = 0$ , in any collection including the fuzzy number  $A_i$ ,  $i = 1, 2, \dots, n$ , two triangular fuzzy numbers,  $A_0$  and  $A_{n+1}$ , are added. Yu et al. [25] pointed out that [22,23] redefined the deviation degree of a fuzzy number to overcome the shortcomings of the method proposed by [20]. However, most methods based on the deviation degree approach exhibit the same limitations due to the neglect of the decision makers' attitude, incoherent transfer coefficient formulas, and unreliable ranking index computation. Chutia [21] proposed a method for ranking fuzzy numbers by using the value and angle in the epsilon-deviation degree method. This method also has some limitations, which are illustrated in Example 5 in Section 2.3. The historical timeline of the aforementioned research is presented in the following chart.



To overcome the aforementioned obstacles, this paper proposes a method for ranking generalized fuzzy numbers based on the left area (benefit area), the right area (cost area), and a normalized height coefficient. In this method, the left area denotes the area from  $x_{\min}$  to  $x^L$  and is bounded by the maximizing membership function  $f_M(x)$  and minimizing membership function  $f_G(x)$ . A ranking is higher if the left area is larger; therefore, the left area is considered the benefit area. The right area denotes the area from  $x_{\max}$  to  $x^R$  and is bounded by the maximizing membership function  $f_M(x)$  and minimizing membership function  $f_G(x)$ . A ranking is higher if the right area is smaller; therefore, the right area is considered the cost area. The normalized height coefficient reflects the influence of the height of generalized fuzzy numbers on their final ranking scores. The proposed method can rank both normal and non-normal fuzzy numbers without normalization or height minimization, thereby avoiding information loss and incorrect final ranking results.

The main contributions of this study to bridge these gaps are briefly as follows:

- (I) This research develops a new coefficient to calculate the impact of the height of generalized fuzzy numbers on the final ranking score.
- (II) The new areas considered as benefit and cost are introduced to reflect the influence of vertical values on the final ranking score.
- (III) A new index is proposed to guarantee that both vertical and horizontal values of a generalized fuzzy number are important parameters that impact the final ranking score.
- (IV) The proposed method can rank both normal and non-normal fuzzy numbers without normalization or height minimization, thereby avoiding information loss and incorrect final ranking results.
- (V) The proposed method can overcome the shortcomings of some existing methods and can be applied to many fuzzy MCDM models to support decision makers in selecting the most suitable alternative in the decision-making process.

This paper is organized as follows. In Section 2, some basic definitions are introduced. Section 2 also provides an overview of the deviation degree method and explores the shortcomings of recent methods. In Section 3, the proposed method of ranking generalized fuzzy numbers based on the normalized height coefficient and benefit and cost areas is presented. In Section 4, numerical examples and comparisons are presented. Finally, we provide concluding remarks in Section 5.

## 2. Preliminaries

### 2.1. Definitions and Notions

#### Definition 1. Fuzzy Sets

$A = \{(x, f_A(x)) \mid x \in U\}$ , where  $U$  is the universe of discourse,  $x$  is an element in  $U$ ,  $A$  is a fuzzy set in  $U$ , and  $f_A(x)$  is the membership function of  $A$  at  $x$  [26]. The larger  $f_A(x)$ , the stronger the grade of membership for  $x$  in  $A$ .

#### Definition 2. Fuzzy Numbers

A real fuzzy number  $A$  is described as any fuzzy subset of the real line  $R$  with membership function  $f_A(x)$  which possesses the following properties [27]:

- (a)  $f_A(x)$  is a continuous mapping from  $R$  to  $[0, 1]$ ;
- (b)  $f_A(x) = 0, \forall x \in (-\infty, a]$ ;

(c)  $f_A(x)$  is strictly increasing on  $[a, b]$ ;  
 (d)  $f_A(x) = 1, x \in [b, c]$ ; (e)  $f_A$  is strictly decreasing on  $[c, d]$ ;  
 (f)  $f_A(x) = 0, \forall x \in [d, \infty)$ , where  $a < b < c < d$ ,  $A$  can be denoted as  $[a, b, c, d; w]$ . A generalized trapezoidal fuzzy number  $A = (a, b, c, d; w)$  is described as any fuzzy subset of the real line  $\mathbb{R}$ . The membership function  $f_A(x)$  of the fuzzy number  $A$  can also be expressed as follows [28]:

$$f_A(x) = \begin{cases} 0, & x < a; \\ \frac{w(x-a)}{b-a}, & a \leq x \leq b; \\ w, & b \leq x \leq c; \\ \frac{w(x-d)}{c-d}, & c \leq x \leq d; \\ 0, & x > d \end{cases} \tag{1}$$

This generalized trapezoidal fuzzy number,  $A = (a, b, c, d; w), 0 \leq w \leq 1$ , as shown in Figure 1.

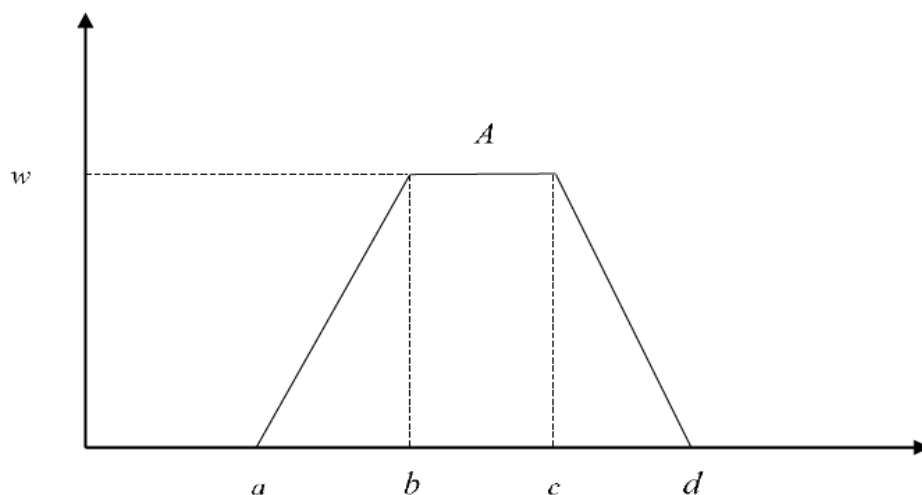


Figure 1. A generalized trapezoidal fuzzy number.

**Definition 3.** Arithmetic operations

According to Yu et al. [25], the arithmetic operations defined for two generalized trapezoidal fuzzy numbers  $A_1 = (a_1, b_1, c_1, d_1; w_1)$  and  $A_2 = (a_2, b_2, c_2, d_2; w_2)$  are as follows:

$$A_1 \oplus A_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; \min(w_1, w_2)) \tag{2}$$

$$A_1 \ominus A_2 = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; \min(w_1, w_2)) \tag{3}$$

$$A_1 \otimes A_2 = (a_1 \times a_2, b_1 \times b_2, c_1 \times c_2, d_1 \times d_2; \min(w_1, w_2)) \tag{4}$$

2.2. A Review of the Deviation Degree Method

In this section, the review method from [20] is presented. Firstly, the minimal and maximal reference sets are reviewed. Then, based on the minimal and maximal reference sets, the left and right deviation degree (L–R deviation degree) is defined. Moreover, the transfer coefficient, which measures the relative variation of the L–R deviation degree of fuzzy numbers, is quoted.

**Definition 4.** For any group of L–R fuzzy numbers  $A_1, A_2, \dots, A_n$ , let  $x_{\min}$  and  $x_{\max}$  be the infimum and supremum of the support set of these fuzzy numbers. Then,  $A_{\min}$  and  $A_{\max}$  denote

the minimal reference set and maximal reference set of these fuzzy numbers, respectively, and their membership functions are given by

$$f_G(x) = \begin{cases} \frac{x_{\max}-x}{x_{\max}-x_{\min}}, & \text{if } x \in S, \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

$$f_M(x) = \begin{cases} \frac{x-x_{\min}}{x_{\max}-x_{\min}}, & \text{if } x \in S, \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

where  $S$  is the support set of these fuzzy numbers, i.e.,  $S = \cup_{i=1}^n S(A_i)$ .

**Definition 5.** For any group of L–R fuzzy numbers  $A_1, A_2, \dots, A_n$ , let  $A_{\min}$  and  $A_{\max}$  be the minimal reference set and maximal reference set of these fuzzy numbers, respectively. Then, the left deviation degree and right deviation degree of  $A_i, i = 1, 2, \dots, n$ , are defined as follows:

$$d_i^L = \int_{x_{\min}}^{x_{A_i}^L} (f_G(x) - f_{A_i}^L(x)) dx \tag{7}$$

$$d_i^R = \int_{x_{A_i}^R}^{x_{\max}} (f_M(x) - f_{A_i}^R(x)) dx \tag{8}$$

where  $x_{A_i}^L$  and  $x_{A_i}^R, i = 1, 2, \dots, n$ , are the abscissas of the crossover points of  $f_{A_i}^L(x)$  and  $f_G(x)$  and  $f_{A_i}^R(x)$  and  $f_M(x)$ , respectively.

**Definition 6.** For a L–R fuzzy number  $A_i = (a_i, b_i, c_i, d_i, w_i)$ , its expectation value of the centroid is defined as follows:

$$M_i = \frac{\int_{a_i}^{d_i} x f_{A_i}(x) dx}{\int_{a_i}^{d_i} f_{A_i}(x) dx} \tag{9}$$

**Definition 7.** For a L–R fuzzy number  $A_i = (a_i, b_i, c_i, d_i, w_i)$ , the transfer coefficient of  $A_i, i = 1, 2, \dots, n$ , is given by

$$\lambda_i = \frac{M_i - M_{\min}}{M_{\max} - M_{\min}} \tag{10}$$

where  $M_{\max} = \max\{M_1, M_2, \dots, M_n\}$  and  $M_{\min} = \min\{M_1, M_2, \dots, M_n\}$ .

**Definition 8.** The ranking index value of a L–R fuzzy number  $A_i, i = 1, 2, \dots, n$ , is given by:

$$d_i = \begin{cases} \frac{d_i^L \lambda_i}{1+d_i^R(1-\lambda_i)}, & M_{\max} \neq M_{\min} \\ \frac{d_i^L}{1+d_i^R}, & M_{\max} = M_{\min} \end{cases} \tag{11}$$

Now, by using  $d_i$ , given in Equation (11), for any two L–R fuzzy numbers  $A_i$ , and  $A_j$ , it is possible to rank these fuzzy numbers according to the following rules:

- (1)  $A_i \succ A_j$ , if and only if  $d_i \succ d_j$
- (2)  $A_i \prec A_j$ , if and only if  $d_i \prec d_j$
- (3)  $A_i \sim A_j$ , if and only if  $d_i = d_j$

### 2.3. Limitations and Shortcomings of Existing Methods

**Example 1.** Consider two generalized trapezoidal fuzzy numbers  $A_1 = (0.2, 0.4, 0.6, 0.8; 0.35)$  and  $A_2 = (0.1, 0.2, 0.3, 0.4; 0.7)$ , adopted from [11]. According to Chen and Chen [10] approach  $A_2 \succ A_1$ . However, Kumar et al. [11] noted that  $A_2 - A_1 \prec A_1 - A_1$ , which is unreasonable and a contradiction with [12].

**Example 2.** Consider two sets; each set comprises two trapezoidal fuzzy numbers (Figures 2 and 3) as follows:

Set 1 comprises  $A_1 = (0, 0.2, 0.5, 0.7; 1)$  and  $A_2 = (0.1, 0.2, 0.6, 0.8; 1)$ .

Set 2 comprises  $A_1 = (0, 0.2, 0.5, 0.7; 1)$  and  $A_2 = (0.1, 0.2, 0.6, 0.8; 0.1)$ .

According to Chi and Yu [10], the final ranking of Set 1 and Set 2 are the same,  $A_2 \succ A_1$ , which shows that the height does not affect the final ranking. These two sets have fuzzy numbers with the same support but different heights; in Set 2, the height of  $A_2$  is 0.1.

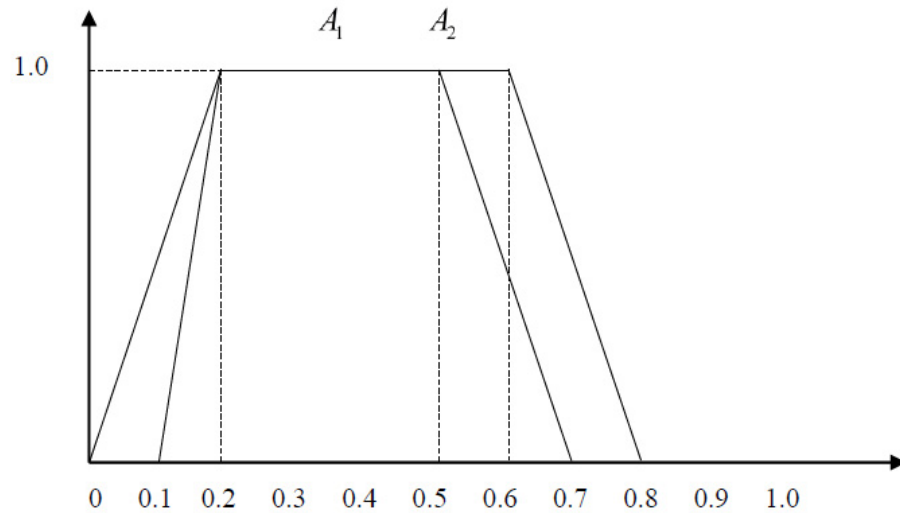


Figure 2. Fuzzy numbers  $A_1$  and  $A_2$  of Set 1 in Example 2.

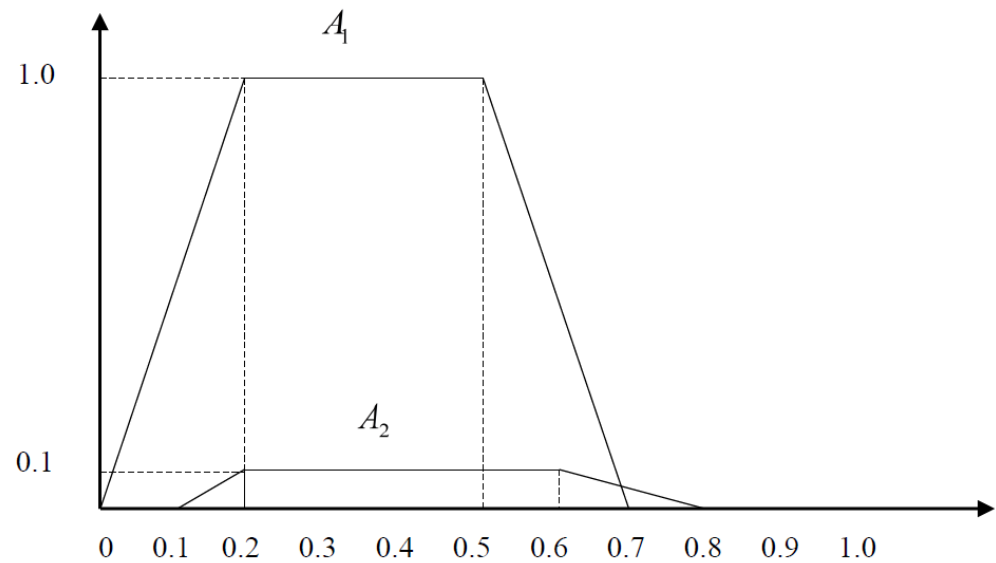


Figure 3. Fuzzy numbers  $A_1$  and  $A_2$  of Set 2 in Example 2.

**Example 3.** Consider two sets, each comprising two type-2 trapezoidal fuzzy numbers (Figures 4 and 5) as follows:

Set 3 comprises

$A_1 = ((0, 0.3, 0.5, 0.6; 1); (0.1, 0.3, 0.4, 0.5; 0.7))$  and

$A_2 = ((0.1, 0.2, 0.4, 0.8; 1); (0.2, 0.3, 0.4, 0.6; 0.8))$ .

Set 4 comprises

$A_1 = ((0, 0.3, 0.5, 0.6; 1); (0.1, 0.3, 0.4, 0.5; 0.7))$  and

$A_2 = ((0.1, 0.2, 0.4, 0.8; 0.3); (0.2, 0.3, 0.4, 0.6; 0.1))$ .

According to De et al. [15], the final rankings of Set 3 and Set 4 are the same:  $A_2 \succ A_1$ . Therefore, height does not affect the final ranking. These two sets have fuzzy numbers with the same

support but different heights; in Set 4, the heights of the upper and lower trapezoidal fuzzy numbers of  $A_2$  are only 0.3 and 0.1, respectively.

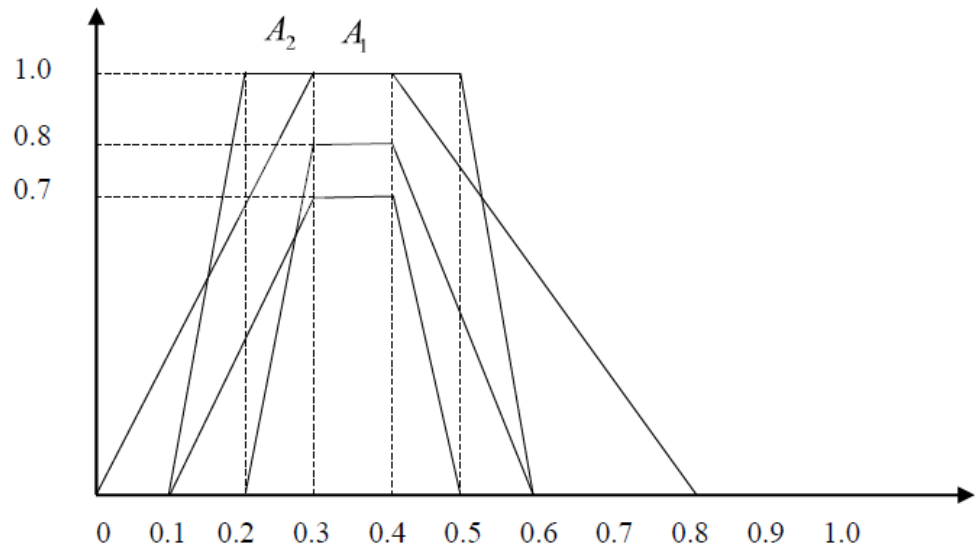


Figure 4. Fuzzy numbers  $A_1$  and  $A_2$  of Set 3 in Example 3.

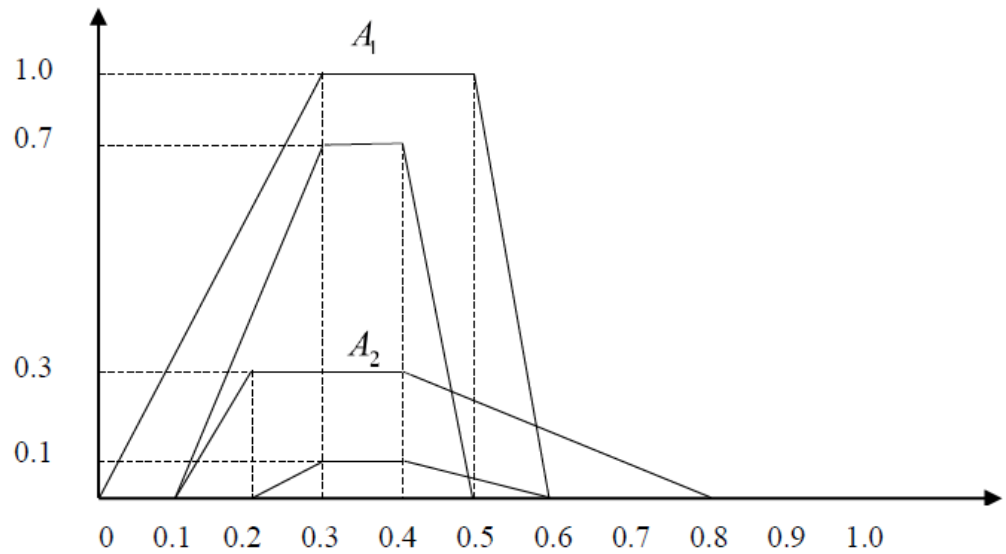


Figure 5. Fuzzy numbers  $A_1$  and  $A_2$  of Set 4 in Example 3.

**Example 4.** Consider two trapezoidal fuzzy numbers (Figure 6) as follows:

$$A_1 = (0.1, 0.3, 0.3, 0.5; 1) \text{ and } A_2 = (0.1, 0.3, 0.3, 0.5; 0.3).$$

These two fuzzy numbers have the same support, but the height of  $A_2$  is lower than that of  $A_1$ . However, the final ranking of methods proposed by [1,8] is  $A_1 \sim A_2$ , which is counterintuitive, thus illustrating a shortcoming in ranking non-normal fuzzy numbers. According to Wang et al. [20], the final ranking result  $A_1 \prec A_2$  is inconsistent with human intuition.



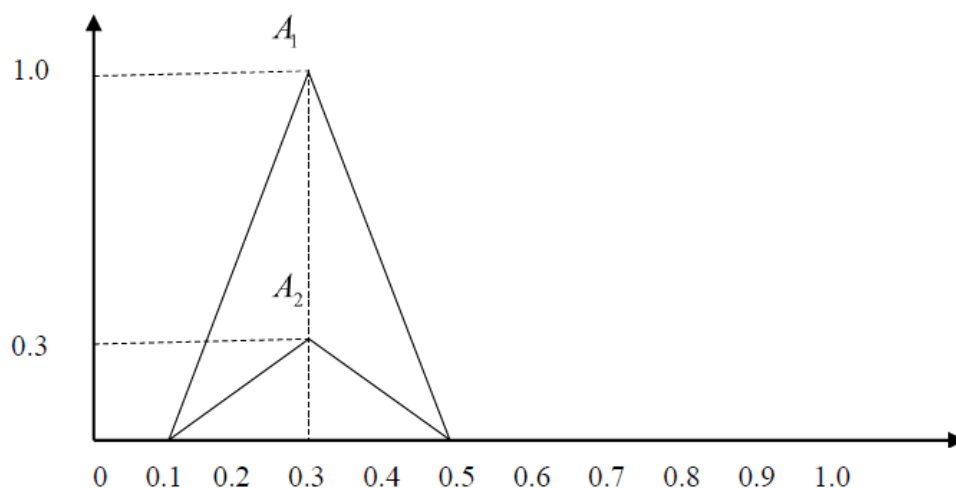


Figure 6. Fuzzy numbers  $A_1$  and  $A_2$  in Example 4.

**Example 5.** Consider three fuzzy numbers,  $A_1 = (0.3, 0.5, 0.5, 0.7; 1)$ ,  $A_2 = (0.3, 0.5, 0.5, 0.9; 1)$ , and  $A_3 = (0.3, 0.5, 0.8, 0.9; 1)$ . Seghir [29] pointed out that all the compared and proposed methods provide the correct ranking  $A_3 \succ A_2 \succ A_1$ , which is intuitive. However, the ranking  $A_3 \succ A_1 \succ A_2$  proposed by [21] is incorrect and counterintuitive.

### 3. Proposed Method

This study proposes a method that considers maximizing and minimizing sets to be reference sets, the left area to be the benefit area, and the right area to be the cost area. Additionally, the normalized height coefficient is used to determine the influence of height on the final ranking score, thus enabling the ranking of both normal and non-normal fuzzy numbers without normalization or height minimization, which avoids a loss of information and incorrect final rankings.

To guarantee that the vertical value is considered an important parameter that impacts the final ranking score, the left area and the right area are evaluated. Assume there are  $n$  generalized fuzzy numbers  $A_i = (a_i, b_i, c_i, d_i; w_i), i = 1, 2, \dots, n$ . The left area denotes the area from  $x_{\min}$  to  $x_{A_i}^L$  and is bounded by the maximizing membership function  $f_M(x)$  and minimizing membership function  $f_G(x)$ , where  $x_{A_i}^L$  is the intersection of the crossover point of the minimizing membership function  $f_G(x)$  and the left membership function  $f_{A_i}^L(x)$ . The left area is shown in Figure 7 and is described by Equations (12) and (13).

$$S_{A_i}^L = \int_{x_{\min}}^{x^L} (f_G(x) - f_M(x))dx \quad \text{if } x^L \leq x_I \tag{12}$$

$$S_{A_i}^L = \int_{x_{\min}}^{x_I} (f_G(x) - f_M(x))dx + \int_{x_I}^{x^L} (f_M(x) - f_G(x))dx \quad \text{if } x^L \succ x_I \tag{13}$$

$$x_{A_i}^L = \frac{w_i a_i (x_{\max} - x_{\min}) + w x_{\max} (b_i - a_i)}{w_i (x_{\max} - x_{\min}) + w (b_i - a_i)} \tag{14}$$

$$x_I = \frac{(x_{\min} + x_{\max})}{2} \tag{15}$$

$$x_{\min} = \inf a_i \tag{16}$$

In Figure 7, the left areas of generalized fuzzy number  $A_1$  and generalized fuzzy number  $A_2$  are in the case of  $x^L \leq x_I$ . Therefore, applying Equation (12), the left area of generalized fuzzy number  $A_1$  is the round dot area, and the left area of generalized fuzzy number  $A_2$  is the round dot area, adding the square dot area. The left area of generalized fuzzy



number  $A_3$  belongs to the case of  $x^L \succ x_I$ , applying Equation (13); the left area of generalized fuzzy number  $A_3$  is the round dot area, adding the square dot area and adding the long dash area.

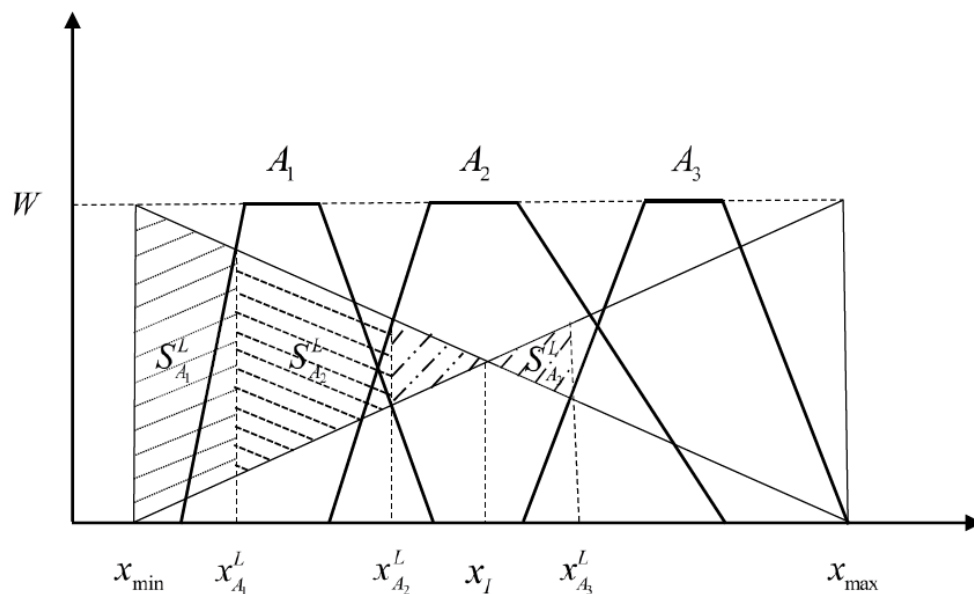


Figure 7. The left area.

The right area denotes the area from  $x_{\max}$  to  $x_{A_i}^R$  and is bounded by the maximizing membership function  $f_M(x)$  and minimizing membership function  $f_G(x)$ , where  $x_{A_i}^R$  is the intersection of the crossover point of the maximizing membership function  $f_M(x)$  and the right membership function  $f_{A_i}^R(x)$ . The right area is shown in Figure 8 and is described by Equations (17) and (18).

$$S_{A_i}^R = \int_{x^R}^{x_{\max}} (f_M(x) - f_G(x)) dx \quad \text{if } x^R \geq x_I \tag{17}$$

$$S_{A_i}^R = \int_{x^R}^{x_I} (f_G(x) - f_M(x)) dx + \int_{x_I}^{x_{\max}} (f_M(x) - f_G(x)) dx \quad \text{if } x^R < x_I \tag{18}$$

$$x_{A_i}^R = \frac{w_i d_i (x_{\max} - x_{\min}) - w x_{\min} (c_i - d_i)}{w_i (x_{\max} - x_{\min}) - w (c_i - d_i)} \tag{19}$$

$$x_{\max} = \sup d_i \tag{20}$$

In Figure 8, the right areas of generalized fuzzy number  $A_3$  and generalized fuzzy number  $A_2$  are in the case of  $x^R \geq x_I$ . Therefore, applying Equation (17), the right area of generalized fuzzy number  $A_3$  is the long dash area, and the left area of generalized fuzzy number  $A_2$  is the long dash area, adding the square dot area. The right area of generalized fuzzy number  $A_3$  belongs to the case of  $x^R < x_I$ , and applying Equation (18), the right area of generalized fuzzy number  $A_1$  is the long dash area, adding the square area, and adding the round dot area.

Herein,  $x_{\min} = \text{infa}_i$  is considered the negative ideal solution,  $x_{\max} = \sup d_i$  is considered the positive ideal solution, and  $x_I$  is the intersection of the maximizing membership function  $f_M(x)$  and minimizing membership function  $f_G(x)$ . In the proposed method, the left and right areas are new areas that are simple to calculate and provide greater consistency and robustness in comparison with other methods.

The generalized fuzzy number  $A_i$  is preferred if it is the farthest from the negative ideal solution  $x_{\min}$  and closest to the positive ideal solution  $x_{\max}$ . If  $S_{A_i}^L$  is larger, the generalized fuzzy number  $A_i$  is farther from the negative ideal solution and closer to the positive

ideal solution. Therefore,  $S_{A_i}^L$  is considered a benefit; thus, a larger  $S_{A_i}^L$  is better. Conversely, if  $S_{A_i}^R$  is smaller,  $A_i$  is farther from the negative ideal solution and closer to the positive ideal solution. Therefore,  $S_{A_i}^R$  is considered a cost; thus, a smaller  $S_{A_i}^R$  is better. In other words, a larger  $S_{A_i}^L$  and smaller  $S_{A_i}^R$  indicate a larger generalized fuzzy number,  $A_i$ .

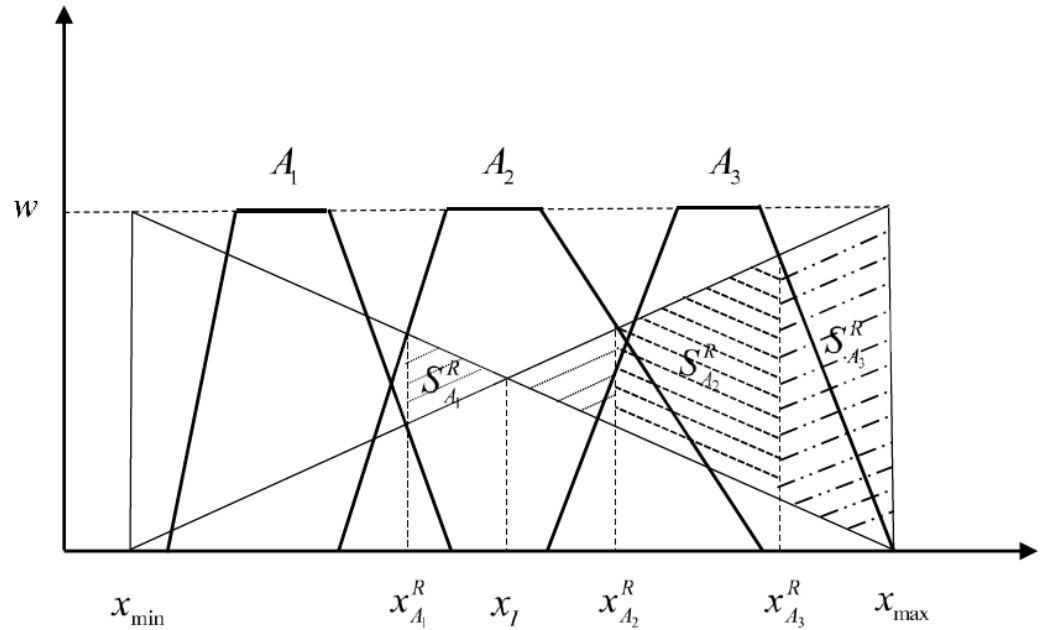


Figure 8. The right area.

To guarantee that the horizontal value is also considered an important parameter to influence the final ranking, the normalized height coefficient is defined in Equation (21) to reflect the influence of the height of a generalized fuzzy number on the final ranking score. The higher the normalized height coefficient of generalized fuzzy number  $A_i$ , the higher the ranking of  $A_i$  is. The rationale for this comes from one of four well-known normalization procedures: the linear scale transformation (sum) method. According to Chakraborty [30], this method divides the performance ratings of each attribute by the sum of the performance ratings for that attribute.

$$\zeta_{A_i} = \frac{h_{A_i}}{\sum_{i=1}^n h_{A_i}} \tag{21}$$

The final ranking score (RS) for generalized fuzzy number  $A_i$  is defined as in Equation (22). This equation is driven based on the closeness coefficient (CC) of the fuzzy technique for order preference by its similarity to an ideal solution (TOPSIS), which is determined using the distance of each alternative from the positive and negative ideal solution [31].

$$RS_{A_i} = \frac{S_{A_i}^L \zeta_{A_i}}{S_{A_i}^L \zeta_{A_i} + S_{A_i}^R (1 - \zeta_{A_i})} \tag{22}$$

If  $A_i$  and  $A_j$  are two generalized fuzzy numbers, then the ranking score leads to the following decisions:

$$\begin{aligned} &\text{If } RS_{A_i} > RS_{A_j}, \text{ then } A_i > A_j. \\ &\text{If } RS_{A_i} < RS_{A_j}, \text{ then } A_i < A_j. \\ &\text{If } RS_{A_i} = RS_{A_j}, \text{ then } A_i \sim A_j. \end{aligned} \tag{23}$$

A flowchart in Figure 9, shown below, is used to present the procedure of the proposed method.

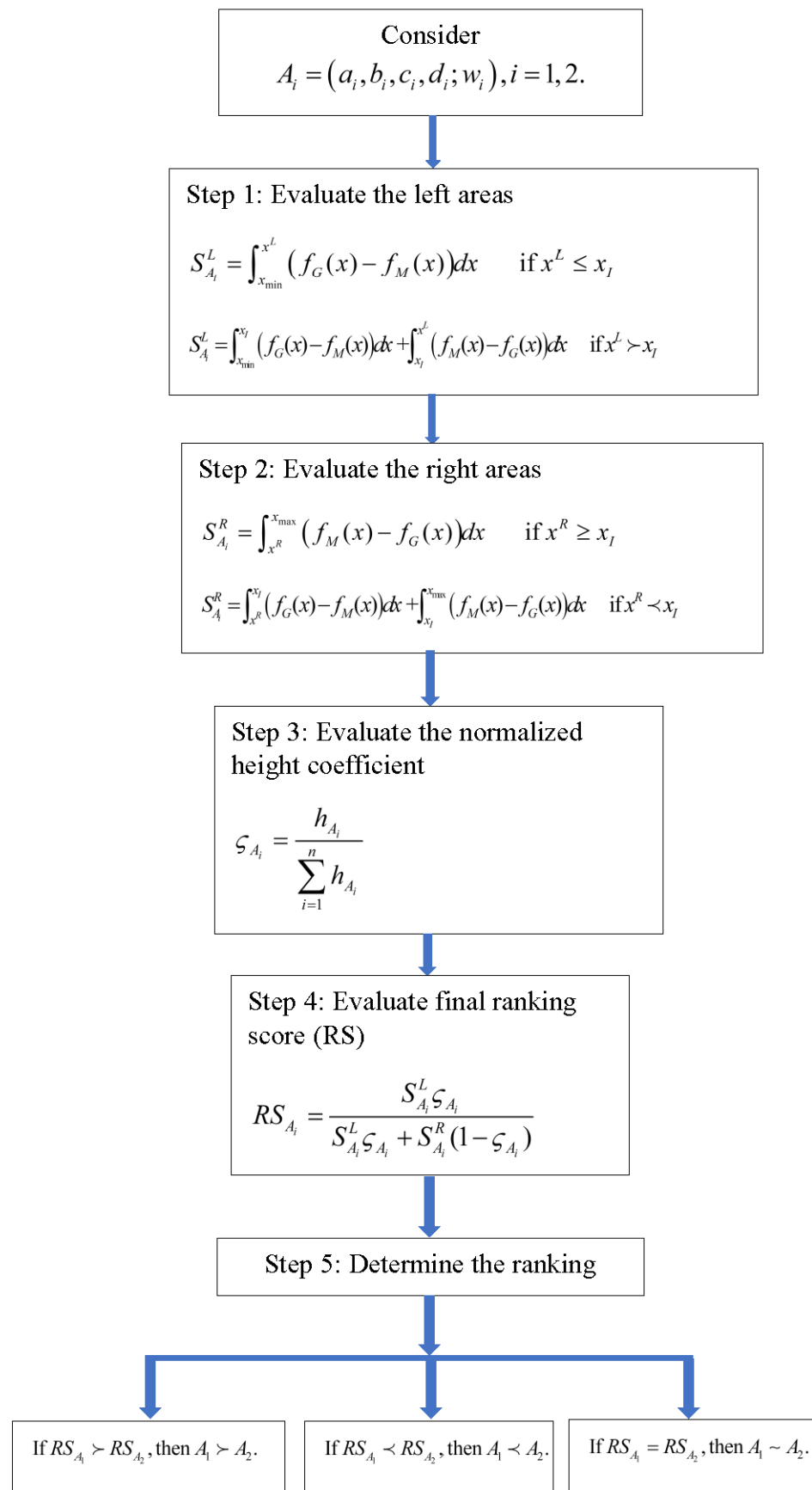


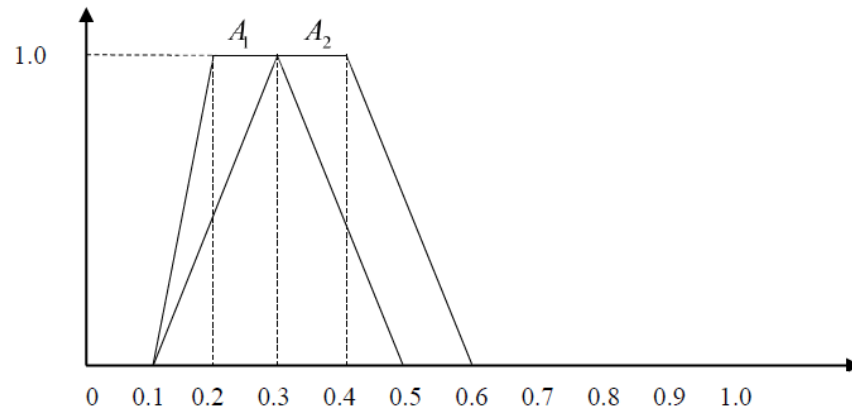
Figure 9. The graphical abstract of the proposed method.

### 4. Numerical Example and Comparative Study

#### 4.1. Examples

To highlight the advantages, consistency, and robustness of this method, numerical examples are used. Step-by-step, these examples demonstrate the simple computation and application of the proposed method.

**Example 6.** Consider two trapezoidal fuzzy numbers  $A_1 = (0.1, 0.2, 0.3, 0.5; 1)$  and  $A_2 = (0.1, 0.3, 0.4, 0.6; 1)$  (Figure 10). According to the proposed method, the final ranking is determined to be  $A_1 < A_2$  as follows:



**Figure 10.** Fuzzy numbers  $A_1$  and  $A_2$  in Example 6.

Step 1: per Equations (16) and (20),  $x_{\min} = 0.1$ ,  $x_{\max} = 0.6$ .

Step 2: Per Equations (14) and (19), the  $x_{A_i}^L$  and  $x_{A_i}^R$  of fuzzy number  $A_1$  are  $x_{A_1}^L = 0.18333$  and  $x_{A_1}^R = 0.38571$ . The  $x_{A_i}^L$  and  $x_{A_i}^R$  of fuzzy number  $A_2$  are  $x_{A_2}^L = 0.24286$  and  $x_{A_2}^R = 0.45714$ .

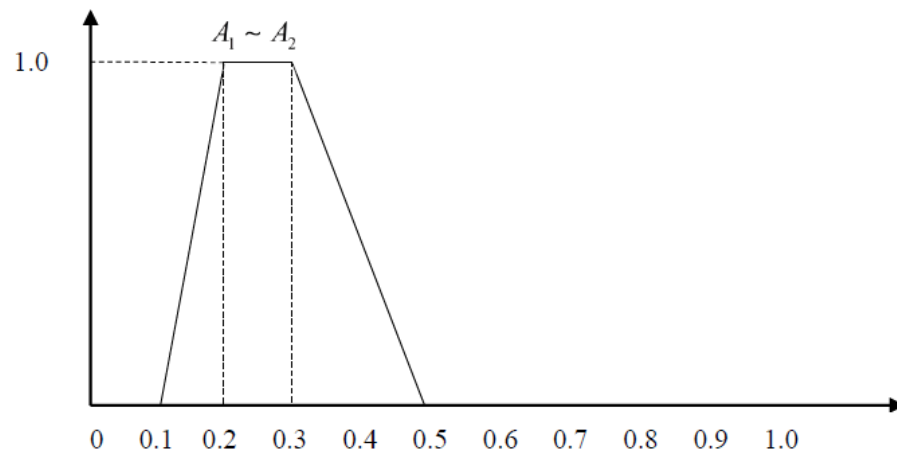
Step 3: Per Equations (12) and (17), the  $S_{A_i}^L$  and  $S_{A_i}^R$  of fuzzy number  $A_1$  are  $S_{A_1}^L = 0.06944$  and  $S_{A_1}^R = 0.12245$ . The  $S_{A_i}^L$  and  $S_{A_i}^R$  of fuzzy number  $A_2$  are  $S_{A_2}^L = 0.10204$  and  $S_{A_2}^R = 0.10204$ .

Step 4: per Equation (21), the normalized height coefficient of fuzzy number  $A_1$  and fuzzy number  $A_2$  are  $\zeta_{A_1} = 0.5$  and  $\zeta_{A_2} = 0.5$ , respectively.

Step 5: per Equation (22), the ranking score (RS) of fuzzy number  $A_1$  and fuzzy number  $A_2$  are  $RS_{A_1} = 0.36189$  and  $RS_{A_2} = 0.5000$ .

Step 6: per Equation (23), the final ranking is  $A_1 < A_2$ .

The following numerical examples (Figures 11–17) are calculated step-by-step as in Example 6; the results are shown in Table 1.



**Figure 11.** Fuzzy numbers  $A_1$  and  $A_2$  in Example 7.

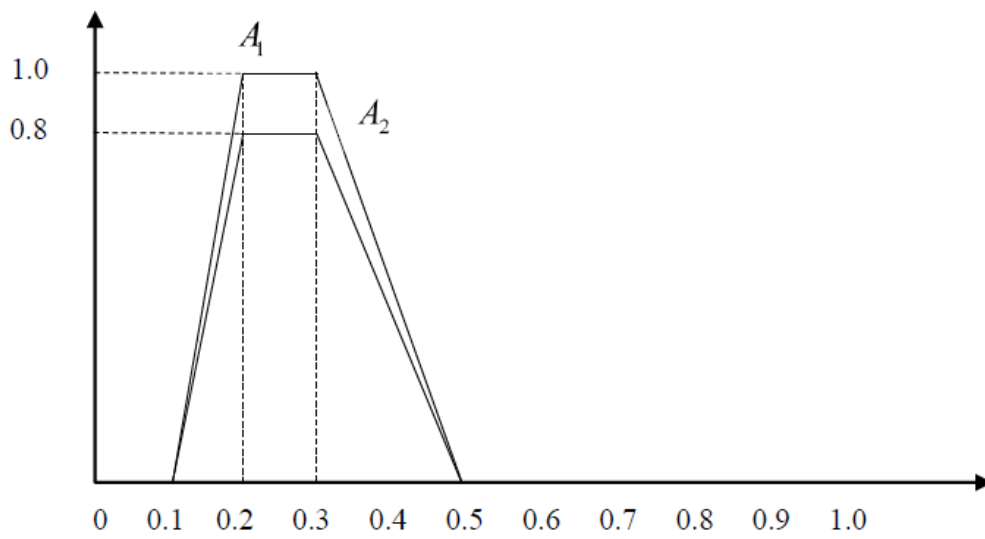


Figure 12. Fuzzy numbers  $A_1$  and  $A_2$  in Example 8.

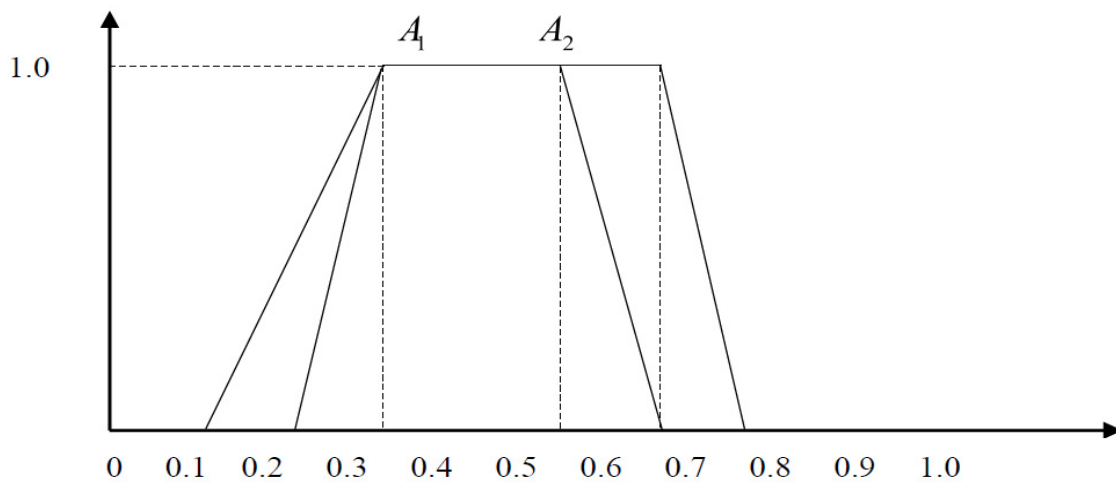


Figure 13. Fuzzy numbers  $A_1$  and  $A_2$  in Example 9.

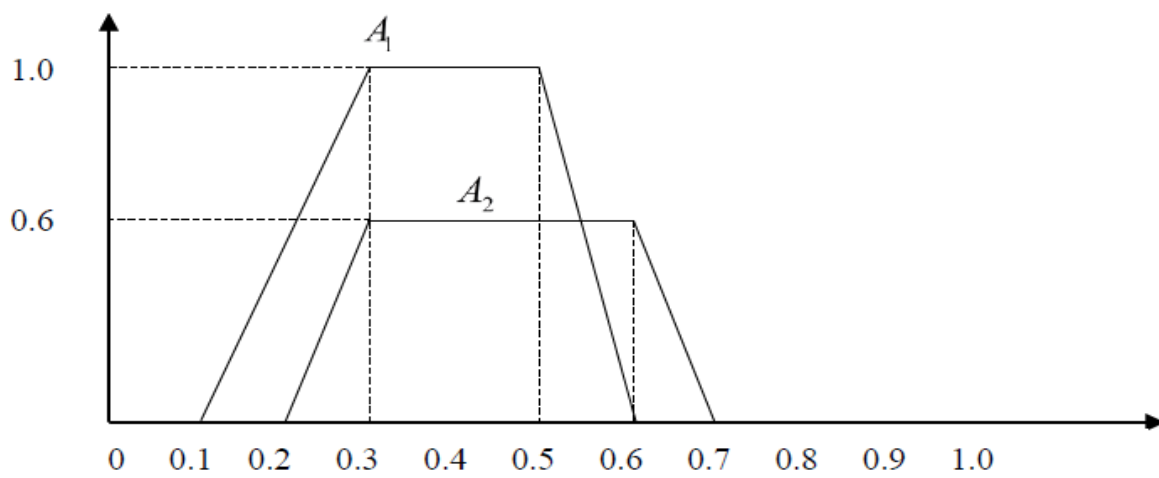


Figure 14. Fuzzy numbers  $A_1$  and  $A_2$  in Example 10.

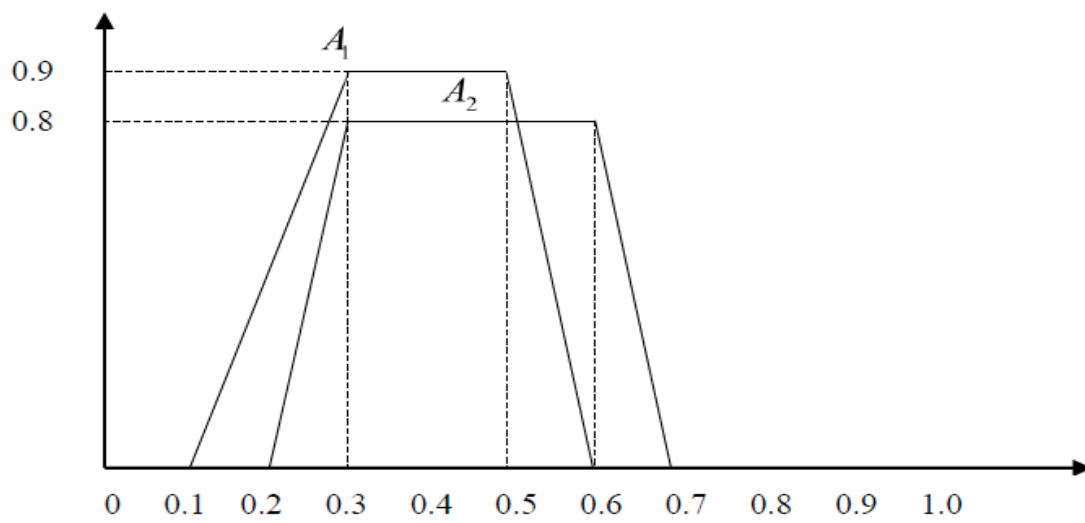


Figure 15. Fuzzy numbers  $A_1$  and  $A_2$  in Example 11.

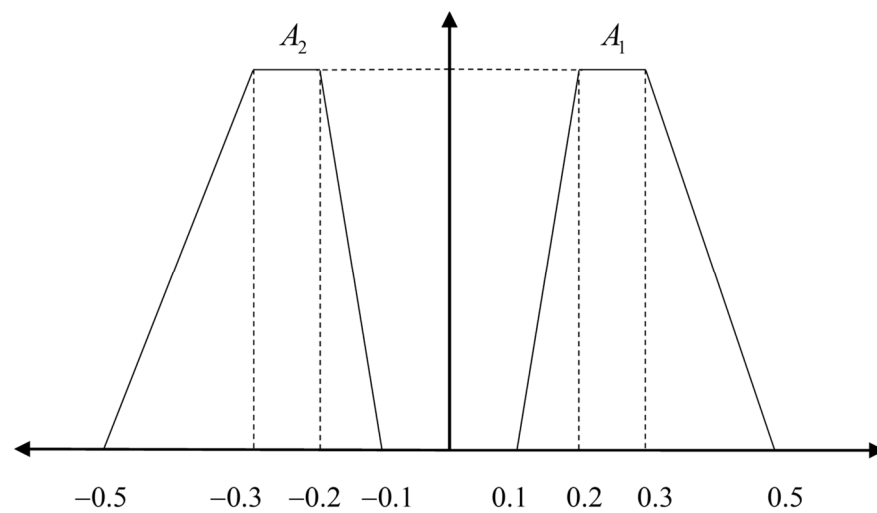


Figure 16. Fuzzy numbers  $A_1$  and  $A_2$  in Example 12.

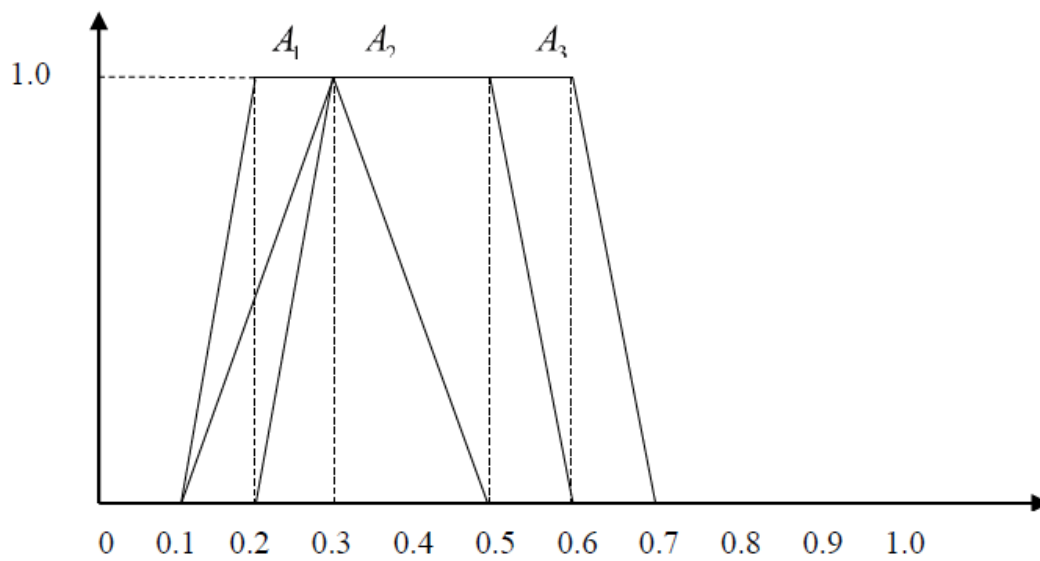


Figure 17. Fuzzy numbers  $A_1$  and  $A_2$  in Example 13.

**Table 1.** Numerical Examples.

	Fuzzy Numbers	$S^L$	$S^R$	$\mathbf{3}$	RS	Rank
Ex.6	$A_1 (0.1, 0.2, 0.3, 0.5; 1)$	0.069	0.122	0.500	0.362	2
	$A_2 (0.1, 0.3, 0.4, 0.6; 1)$	0.102	0.102	0.500	0.500	1
Ex.7	$A_1 (0.1, 0.2, 0.3, 0.5; 1)$	0.064	0.089	0.500	0.419	1
	$A_2 (0.1, 0.2, 0.3, 0.5; 1)$	0.064	0.089	0.500	0.419	1
Ex.8	$A_1 (0.1, 0.2, 0.3, 0.5; 1)$	0.044	0.065	0.556	0.460	1
	$A_2 (0.1, 0.2, 0.3, 0.5; 0.8)$	0.051	0.071	0.444	0.365	2
Ex.9	$A_1 (0.1, 0.3, 0.5, 0.6; 1)$	0.113	0.122	0.500	0.479	2
	$A_2 (0.2, 0.3, 0.6, 0.7; 1)$	0.122	0.073	0.500	0.625	1
Ex.10	$A_1 (0.1, 0.3, 0.5, 0.6; 1)$	0.050	0.066	0.625	0.558	1
	$A_2 (0.2, 0.3, 0.6, 0.7; 0.6)$	0.073	0.044	0.375	0.500	2
Ex.11	$A_1 (0.1, 0.3, 0.5, 0.6; 0.9)$	0.085	0.096	0.529	0.499	2
	$A_2 (0.2, 0.3, 0.6, 0.7; 0.8)$	0.098	0.059	0.471	0.597	1
Ex.12	$A_1 (0.1, 0.2, 0.3, 0.5; 1)$	0.231	0.139	0.500	0.625	1
	$A_2 (-0.5, -0.3, -0.2, -0.1; 1)$	0.139	0.231	0.500	0.375	2
Ex.13	$A_1 (0.1, 0.2, 0.3, 0.5; 1)$	0.073	0.150	0.333	0.197	3
	$A_2 (0.1, 0.3, 0.5, 0.6; 1)$	0.113	0.122	0.333	0.315	2
	$A_3 (0.2, 0.3, 0.6, 0.7; 1)$	0.122	0.073	0.333	0.455	1

**Example 7.** Consider two normal trapezoidal fuzzy numbers  $A_1 = (0.1, 0.2, 0.3, 0.5; 1)$  and  $A_2 = (0.1, 0.2, 0.3, 0.5; 1)$  (Figure 11). These fuzzy numbers are the same, so the ranking scores are equal, and the final ranking is  $A_1 \sim A_2$ .

**Example 8.** Consider the normal trapezoidal fuzzy number  $A_1 = (0.1, 0.2, 0.3, 0.5; 1)$  and non-normal trapezoidal fuzzy number  $A_2 = (0.1, 0.2, 0.3, 0.5; 0.8)$  (Figure 12). These fuzzy numbers share the same support, but the ranking scores are different because of different heights. The final ranking result is  $A_1 \succ A_2$ . This example indicates that the proposed method can rank both normal and non-normal fuzzy numbers.

**Example 9.** Consider two normal trapezoidal fuzzy numbers  $A_1 = (0.1, 0.3, 0.5, 0.6; 1)$  and  $A_2 = (0.2, 0.3, 0.6, 0.7; 1)$  (Figure 13). The final ranking result is  $A_1 \prec A_2$ .

**Example 10.** Consider the normal trapezoidal fuzzy number  $A_1 = (0.1, 0.3, 0.5, 0.6; 1)$  and non-normal trapezoidal fuzzy number  $A_2 = (0.2, 0.3, 0.6, 0.7; 0.6)$  (Figure 14). Fuzzy number  $A_1$  in Example 10 is the same as fuzzy number  $A_1$  in Example 9. Fuzzy number  $A_2$  in Example 10 shares the same support as the fuzzy number  $A_2$  in Example 9, but the height of fuzzy number  $A_2$  in Example 10 is 0.6. Therefore, the final ranking is  $A_1 \succ A_2$ . This example demonstrates that height is an important parameter affecting the final ranking score.

**Example 11.** Consider two non-normal trapezoidal fuzzy numbers,  $A_1 = (0.1, 0.3, 0.5, 0.6; 0.9)$  and  $A_2 = (0.2, 0.3, 0.6, 0.7; 0.8)$  (Figure 15). Fuzzy number  $A_1$  in Example 11 has the same support as fuzzy number  $A_1$  in Example 9, but the height of fuzzy number  $A_1$  in Example 11 is 0.9. Fuzzy number  $A_2$  in Example 11 shares the same support as the fuzzy number  $A_2$  in Example 9, but the height of fuzzy number  $A_2$  in Example 11 is 0.8. Therefore, the final ranking is  $A_1 \prec A_2$ . This example also indicates that the final ranking is sensitive to height.

**Example 12.** Consider two normal trapezoidal fuzzy numbers,  $A_1 = (0.1, 0.2, 0.3, 0.5; 1)$  and  $A_2 = (-0.5, -0.3, -0.2, -0.1; 1)$  (Figure 16). The final ranking is  $A_1 \succ A_2$ . This example shows that the proposed method can be used to rank positive and negative fuzzy numbers.



**Example 13.** Consider three normal trapezoidal fuzzy numbers,  $A_1 = (0.1, 0.2, 0.3, 0.5; 1)$ ,  $A_2 = (0.1, 0.3, 0.5, 0.6; 1)$ , and  $A_3 = (0.2, 0.3, 0.6, 0.7; 1)$  (Figure 17). The final ranking is  $A_1 \prec A_2 \prec A_3$  which is consistent with human intuition.

4.2. Comparison

For the objective comparison, fuzzy sets are adopted from [32]. This section presents a comparison of the proposed method based on the ranking score (RS) with the methods based on the maximizing and minimizing set method ( $U_T$ ) proposed by [8], the deviation degree (DD) proposed by [20], the area ranking based on the positive and negative ideal points (RIA) proposed by [1], the revised method of the deviation degree (RDD) proposed by [23], the areas on the left and right sides of the fuzzy number (SLR) proposed by [22], the value and angle in the epsilon-deviation degree (MEDD) proposed by [21], and the new fuzzy distance (RI) proposed by [18]. The final ranking results and comparison are presented in Tables 2 and 3, where R is the final ranking.

**Table 2.** Comparison of the proposed method with other methods.

Set	FNs	Ut	R	DD	R	RIA	R	RDD	R	SLR	R	MEDD	R	RS	R
Set 5	$A_1 (0.1, 0.3, 0.3, 0.5; 1)$	0.375	2	0.000	2	0.250	2	0.222	2	0.075	2	0.041	2	0.429	2
	$A_2 (0.3, 0.5, 0.5, 0.7; 1)$	0.625	1	0.300	1	0.750	1	0.571	1	0.303	1	24.462	1	0.571	1
Set 6	$A_1 (0.1, 0.3, 0.3, 0.5; 1)$	0.500	1	0.063	1	0.500	1	0.286	1	0.130	1	1.000	1	0.500	1
	$A_2 (0.1, 0.3, 0.3, 0.5; 1)$	0.500	1	0.063	1	0.500	1	0.286	1	0.130	1	1.000	1	0.500	1
Set 7	$A_1 (0.1, 0.3, 0.3, 0.5; 0.8)$	0.400	1	0.063	1	0.500	1	0.242	2	0.242	2	0.126	2	0.444	2
	$A_2 (0.1, 0.3, 0.3, 0.5; 1)$	0.400	1	0.061	2	0.500	1	0.286	1	0.286	1	9.488	1	0.556	1
Set 8	$A_1 (-0.5, -0.3, -0.3, -0.1; 1)$	0.250	2	0.000	2	0.125	2	0.154	2	0.035	2	0.015	2	0.333	2
	$A_2 (0.1, 0.3, 0.3, 0.5; 1)$	0.750	1	1.333	1	0.875	1	1.143	1	0.679	1	65.091	1	0.667	1
Set 9	$A_1 (0.3, 0.5, 0.5, 1.0; 1)$	0.503	1	0.327	1	0.545	1	0.514	1	0.285	1	1.185	1	0.502	1
	$A_2 (0.1, 0.6, 0.6, 0.8; 1)$	0.497	2	0.000	2	0.455	2	0.436	2	0.196	2	0.844	2	0.498	2

**Table 3.** Comparison the proposed method with other methods.

Set	FNs	Ut	R	DD	R	RIA	R	RDD	R	SLR	R	MEDD	R	RI	R	RS	R
Set 10	$A_1 (0.0, 0.4, 0.6, 0.8; 1)$	0.517	3	0.000	3	0.500	3	0.474	3	0.229	3	0.087	3	0.000	3	0.349	3
	$A_2 (0.2, 0.5, 0.5, 0.9; 1)$	0.554	2	0.313	1	0.636	2	0.600	2	0.363	1	0.498	2	0.552	2	0.363	2
	$A_3 (0.1, 0.6, 0.7, 0.8; 1)$	0.614	1	0.207	2	0.700	1	0.647	1	0.362	2	2.176	1	1.000	1	0.434	1
Set 11	$A_1 (0.4, 0.5, 0.5, 1; 1)$	0.344	3	0.000	3	0.167	3	0.222	3	0.102	3	0.043	3	0.000	3	0.198	3
	$A_2 (0.4, 0.7, 0.7, 1; 1)$	0.500	2	0.048	2	0.500	2	0.375	2	0.184	2	0.934	2	0.597	2	0.333	2
	$A_3 (0.4, 0.9, 0.9, 1; 1)$	0.656	1	0.136	1	0.833	1	0.571	1	0.296	1	3.857	1	1.000	1	0.503	1
Set 12	$A_1 (0.1, 0.2, 0.3, 0.5; 1)$	0.321	3	0.000	3	0.143	3	0.189	3	0.102	3	0.037	3	0.000	3	0.197	3
	$A_2 (0.1, 0.3, 0.5, 0.6; 1)$	0.482	2	0.037	2	0.400	2	0.333	2	0.184	2	0.974	2	0.373	2	0.315	2
	$A_3 (0.2, 0.3, 0.6, 0.7; 1)$	0.571	1	0.171	1	0.750	1	0.467	1	0.296	1	2.429	1	1.000	1	0.455	1

Table 2 illustrates that the final ranking of Set 7 by the proposed method is consistent with the rankings generated by the methods based on the revised method of the deviation degree (RDD), the areas on the left and right sides of the fuzzy number (SLR), and the value and angle in the epsilon-deviation degree (MEDD); thus, our method is intuitive for ranking non-normal fuzzy numbers. The methods based on the maximizing set and minimizing set ( $U_T$ ) and the area ranking based on the positive and negative ideal points (RIA) equally rank the fuzzy numbers, which is counterintuitive because the two fuzzy numbers share the same score support but differ in height. Furthermore, the final ranking based on the deviation degree (DD) is unreasonable because the fuzzy number with a lower height has a higher ranking, making it counterintuitive. These final rankings of Sets 5, 6, 8, and 9 generated by the proposed method are consistent with those by the other methods. Thus, the final ranking generated by the proposed method is consistent with those of other methods for normal fuzzy numbers. Additionally, the proposed method can be used to rank the non-normal fuzzy numbers described in Set 7 without normalization or height minimization ( $\min w_i$ ).

Since the method of ranking fuzzy numbers on the basis of the new fuzzy distance [18], can rank a set of at least three basic ranking indices, Table 3 compares the proposed method with other methods in a set of three fuzzy numbers. Set 10 is adopted from [32], Set 11 is adopted from [21], and Set 12 is the same in Example 13. The final rankings of Set 11 and Set 12 by the proposed method are consistent with all the compared methods. The final ranking of Set 10 is the same as that generated by most other methods, namely, the maximizing set and minimizing set ( $U_T$ ), the area ranking based on the positive and negative ideal points ( $RIA$ ), the revised method of the deviation degree ( $RDD$ ), the value and angle in the epsilon-deviation degree ( $MEDD$ ), and the new fuzzy distance (RI). The final ranking of Set 10 produced by the areas on the left and right sides of the fuzzy number ( $SLR$ ) and the deviation degree ( $DD$ ) is different with other methods. Table 3 proves that the proposed method is consistent with all of the compared methods for Set 11 and Set 12, and is consistent with most of the compared methods in Set 10.

4.3. Method Reasonableness Proof

Wang and Kerre [12] proposed reasonable axioms for ranking fuzzy numbers. The reasonableness of the proposed ranking method is proven by studying it under the following axiomatic system by Wang and Kerre.

According to Wang and Kerre [12], for all fuzzy numbers,  $A$ ,  $B$ , and  $C$ , we have

$$A \succ B \Rightarrow A \oplus C \succ B \oplus C \tag{24}$$

$$A \succ B \Rightarrow A \ominus C \succ B \ominus C \tag{25}$$

$$A \sim B \Rightarrow A \oplus C \sim B \oplus C \tag{26}$$

$$A \succ B, B \succ C \Rightarrow A \oplus C \succ B \oplus C \tag{27}$$

Based on the reasonable axioms from [12], there are four generalized trapezoidal fuzzy numbers,  $A_1 = (a_1, b_1, c_1, d_1; w_1)$ ,  $A_2 = (a_2, b_2, c_2, d_2; w_2)$ ,  $A_3 = (a_3, b_3, c_3, d_3; w_3)$ ,  $A_4 = (a_4, b_4, c_4, d_4; w_4)$ , and the ranking order is determined based on the following decision:

$$A_1 \succ A_2 \text{ if } RS(A_1 \oplus A_3) \succ RS(A_2 \oplus A_3) \tag{28}$$

$$A_1 \succ A_2 \text{ if } RS(A_1 \ominus A_3) \succ RS(A_2 \ominus A_3) \tag{29}$$

$$A_1 \sim A_2 \text{ if } RS(A_1 \oplus A_3) \sim RS(A_2 \oplus A_3) \tag{30}$$

$$A_1 \succ A_2, A_3 \succ A_4 \text{ if } RS(A_1 \oplus A_3) \succ RS(A_2 \oplus A_4) \tag{31}$$

Table 4 shows the result of the method’s reasonableness proof that indicates that the proposed ranking function satisfies all reasonable properties of the fuzzy quantities proposed by [12].

The result of Set 15.1 shows the case of Equation (28) when fuzzy numbers  $A_1$  and  $A_2$  add the fuzzy number  $A_3$ ; the result satisfies the condition of Equation (28). The result of Set 15.2 shows the case of Equation (29) when fuzzy numbers  $A_1$  and  $A_2$  subtract fuzzy number  $A_3$ ; the result also satisfies the condition of Equation (29). The result of Set 16.1 shows the case of Equation (30) when the ordering of fuzzy numbers  $A_1$  and  $A_2$  is equal after adding fuzzy number  $A_3$ . The result does not change which satisfies the condition of Equation (30). The result of Set 17.3 shows the case of Equation (31) when fuzzy number  $A_2 \succ A_1$  and  $A_4 \succ A_3$ ; the result of  $A_2 \oplus A_4$  is greater than  $A_1 \oplus A_3$  which satisfies the condition of Equation (31). For the validation of the proposed ranking function, Table 4

shows that the proposed ranking function satisfies all reasonable properties of the fuzzy quantities proposed by [12].

**Table 4.** Method reasonableness proof.

	Fuzzy Numbers	$S^L$	$S^R$	<b>3</b>	<b>RS</b>	<b>Rank</b>
Set 15	$A_1 (0.1, 0.2, 0.3, 0.5; 1)$	0.102	0.122	0.500	0.455	2
	$A_2 (0.2, 0.3, 0.4, 0.6; 1)$	0.111	0.102	0.500	0.521	1
Set 15.1	$A_3 (0.3, 0.5, 0.5, 0.7; 1)$					
	$A_1 \oplus A_3 (0.4, 0.8, 0.8, 1.2; 1)$	0.192	0.213	0.500	0.474	2
	$A_2 \oplus A_3 (0.5, 0.8, 0.9, 1.3; 1)$	0.200	0.192	0.500	0.511	1
Set 15.2	$A_3 (0, 0.1, 0.1, 0.2; 1)$					
	$A_1 \ominus A_3 (-0.1, 0.2, 0.2, 0.5; 1)$	0.147	0.168	0.500	0.467	2
	$A_2 \ominus A_3 (0, 0.2, 0.3, 0.6; 1)$	0.156	0.147	0.500	0.514	1
Set 16	$A_1 (0.1, 0.3, 0.3, 0.5; 1)$	0.089	0.089	0.500	0.500	1
	$A_2 (0.1, 0.3, 0.3, 0.5; 1)$	0.089	0.089	0.500	0.500	1
Set 16.1	$A_3 (0.3, 0.5, 0.5, 0.7; 1)$					
	$A_1 \oplus A_3 (0.4, 0.8, 0.8, 1.2; 1)$	0.178	0.178	0.500	0.500	1
	$A_2 \oplus A_3 (0.4, 0.8, 0.8, 1.2; 1)$	0.178	0.178	0.500	0.500	1
Set 17.1	$A_1 (0.1, 0.3, 0.3, 0.5; 1)$	0.102	0.122	0.500	0.455	2
	$A_2 (0.2, 0.3, 0.4, 0.6; 1)$	0.111	0.102	0.500	0.521	1
Set 17.2	$A_3 (0.3, 0.5, 0.5, 0.7; 1)$	0.113	0.150	0.500	0.429	2
	$A_4 (0.5, 0.7, 0.7, 0.9; 1)$	0.150	0.113	0.500	0.571	1
Set 17.3	$A_2 \oplus A_4 (0.7, 1, 1.1, 1.5; 1)$	0.269	0.215	0.500	0.556	1
	$A_1 \oplus A_3 (0.4, 0.8, 0.8, 1.2; 1)$	0.215	0.274	0.500	0.440	2

### 5. Conclusions

This paper proposes an approach for ranking generalized fuzzy numbers based on a normalized height coefficient and benefit and cost areas. In this method, the left area denotes the area from  $x_{\min}$  to  $x_{A_i}^L$  and is bounded by the maximizing membership function  $f_M(x)$  and minimizing membership function  $f_G(x)$ . The right area denotes the area from  $x_{\max}$  to  $x_{A_i}^R$  and is bounded by the maximizing membership function  $f_M(x)$  and minimizing membership function  $f_G(x)$ .  $S_{A_i}^L$  is considered as the benefit, and larger is better.  $S_{A_i}^R$  is considered as the cost, and smaller is better. In other words, a larger  $S_{A_i}^L$  and smaller  $S_{A_i}^R$  mean a bigger generalized fuzzy number,  $A_i$ . The normalized height coefficient is designed to reflect the influence of the height of generalized fuzzy numbers on the final ranking score. The higher the normalized height coefficient of a generalized fuzzy number, the higher its ranking is. The numerical example and comparison presented herein demonstrate the feasibility and robustness of the proposed method.

The proposed ranking method can be applied to fuzzy multicriteria decision making MCDM to support decision makers in selecting the best alternative. Future research can extend this ranking method to develop other ranking methods for fuzzy numbers, including interval type-2 fuzzy numbers, intuitionistic fuzzy numbers, and hesitant fuzzy numbers, to solve more complex decision-making problems in practice. In the coming years, future research can expand this approach to ensure the consistent ordering of fuzzy Gaussian. Additionally, to assess the effectiveness of the proposed method, future studies can apply it in practical, real-world environments and evaluate its performance with real and complex applications.

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