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A Strong Maximum Principle for Nonlinear Nonlocal Diffusion Equations

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Abstract: We consider a class of nonlinear integro-differential equations that model degenerate nonlocal diffusion. We investigate whether the strong maximum principle is valid for this nonlocal equation. For degenerate parabolic PDEs, the strong maximum principle is not valid. In contrast, for nonlocal diffusion, we can formulate a strong maximum principle for nonlinearities satisfying a geometric condition related to the flux operator of the equation. In our formulation of the strong maximum principle, we find a physical re-interpretation and generalization of the standard PDE conclusion of the principle: we replace constant solutions with solutions of zero flux. We also consider nonlinearities outside the scope of our principle. For highly degenerate conductivities, we demonstrate the invalidity of the strong maximum principle. We also consider intermediate, inconclusive examples, and provide numerical evidence that the strong maximum principle is valid. This suggests that our geometric condition is sharp.

1. Introduction

"Nonlocal" interactions take place over large distances, while "local" interactions are defined by nearby vicinities. Nonlocal diffusion models have recently found diverse applications in image processing [1–3], visual saliency detection [4], population models [5,6], swarming systems [7–9], and epidemological sciences [10–12]. Nonlocal dispersive models have also been used in the description of material properties [13–16].

In addition to quantifying new natural phenomena, nonlocal models can improve upon existing models based on partial differential equations (PDEs). Solutions of parabolic PDEs become smoother with time, which can be problematic for imaging applications such as inpainting [1]. The spatial derivative ∇ in PDE models also makes it difficult to find numerical solutions when the boundary $\partial\Omega$ of a domain Ω has corners. Replacing partial derivatives with nonlocal operators allows for lower regularity solutions, while preserving much of the mathematical structure, such as variational principles [1,17], maximum principles [18], and conservation laws [19].

The objective of this paper is to determine whether the strong maximum principle works for degenerate nonlinear equations, which sheds light on which classical techniques are available for nonlocal models. We classify the nonlinearities which do, in fact, have strong maximum principles.

The strong maximum principle (SMP) is a fundamental qualitative estimate for parabolic and elliptic partial differential equations (PDEs). In the parabolic case, it asserts that solutions strictly decay from their initial, maximal values, unless they are constant

solutions. The SMP plays an important role in the qualitative analysis of solutions of PDEs, allowing for subtle uniqueness, symmetry, and asymptotic behavior results not available in other settings such as fourth order PDEs.

Degenerate parabolic PDEs do not have an SMP, in general. This can be seen from the simplest example, $\partial u / \partial t = 0$; there are stationary solutions which are not constants. A basic question is whether there are **non-stationary** solutions of a given degenerate diffusion equation which also violate the SMP. PDEs with degenerate diffusion, such as

$$\frac{\partial u}{\partial t} = (u-1)^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{or} \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left((u-1)^2 \frac{\partial u}{\partial x}\right) \tag{1}$$

have non-stationary solutions which violate the SMP; one can choose a suitable bump function as an initial condition [20].

The situation is much different for **nonlocal** diffusion, as we will show in this paper. We will exhibit large classes of degenerate diffusion equations for which the only solutions violating the strong maximum principle are stationary. We characterize such solutions as having zero flux, which generalizes the classical conclusion of solutions being constant.

Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Consider a Cauchy problem for $u \in C^1([0,\infty); L^\infty(\Omega))$:

$$\frac{\partial u}{\partial t}(t,x) = \int_\Omega k\big(u(t,x), u(t,y)\big)\big[u(t,y) - u(t,x)\big] J(x,y)\,\mathrm{d}y, \qquad (t,x) \in (0,\infty) \times \Omega,$$

$$u(0,x) = u_0(x) \in L^\infty(\Omega). \tag{2}$$

Here, kernel $J \in L^\infty(\Omega; L^1(\Omega))$ is boundedly integrable and has a positive lower bound: $J \geq \delta > 0$, which is chosen to isolate the degeneracy of the diffusion operator to the nonlinearity. Here, t is a time variable. The conductivity k satisfies that $(u,v) \mapsto k(u,v)(v-u)$ is locally Lipschitz continuous on $\mathbb{R}^2$. This is to ensure existence and uniqueness via Banach's theorem, see e.g., ([21], Chapter 2). We also assume $k \geq 0$, which ensures global existence via the (weak) maximum principle combined with iterating local existence. In defining the integral over Ω and not $\mathbb{R}^N$, we impose a "Neumann condition" [21], so no additional data are needed for the Cauchy problem.

Equation (2) describes degenerate nonlocal diffusion and generalizes recent models, which use integrable kernels and nondegenerate operators. Translation invariant kernels $J(x,y) = \mu(x-y)$ for $\mu \in L^1(\mathbb{R}^N)$ are a common example in $C(\overline{\Omega}; L^1(\Omega))$. Such models have a nonlocal divergence structure, whence (2) is a nonlocal version of Fick's law [1,18]. If we define the two-point flux function,

$$F(u,v) := k(u,v)(v-u), \qquad (u,v) \in \mathbb{R}^2,$$

then the zero flux set $F^{-1}(0)$ of F characterizes the degeneracy of the diffusion operator. Previous works considered only the diagonal case $F^{-1}(0) = \{(u,u)\}$ of minimal degeneracy.

The integro-differential Equation (2) contains, as a special case, the nonlocal *linear* diffusion equation

$$\frac{\partial u}{\partial t} = J * u - u$$

$$= \int_{\mathbb{R}^N} \big[u(t,y) - u(t,x)\big] J(x-y)\,\mathrm{d}y, \tag{3}$$

where $\int J(z)\,\mathrm{d}z = 1$ represents a probability density, and $u(t,x)$ represents a density for some quantity, such as temperature [22] or population density [6]. The first term represents the influx of species from a neighborhood surrounding the point x, while the second term accounts for population dispersion from x. A more general version was considered in [23].

Equation (2) also includes other nonlocal diffusion equations, such as the nonlocal *p*-Laplace equation for $p \geq 2$ covered in ([21], Chapter 6):

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} |u(t,y) - u(t,x)|^{p-2} [u(t,y) - u(t,x)] J(x-y) \, \mathrm{d}y, \tag{4}$$

and the fast diffusion equation of ([21], Chapter 5):

$$\frac{\partial u}{\partial t} \gamma(u(t,x)) = \int_{\mathbb{R}^N} [u(t,y) - u(t,x)] J(x-y) \, \mathrm{d}y.$$

Assuming γ is invertible, letting $v = \gamma(u)$ recovers (2) by selecting $k(a,b) = (b - a)/(\gamma(b) - \gamma(a))$.

Another physically interesting equation that can be obtained from (2) is analogous to the porous medium equation $u_t = \Delta |u|^m / m = \nabla \cdot (|u|^{m-1} \nabla u)$:

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} |u(t,x) + u(t,y)|^{m-1} [u(t,y) - u(t,x)] J(x-y) \, \mathrm{d}y, \tag{5}$$

where $m > 1$; however, to our knowledge, this equation has not been studied in the literature. The closest that come to it are those using fractional differential operators [24], a nonlinear diffusion PDE with a nonlocal convection term [25], and one that is considered in [26]:

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} J\left(\frac{x-y}{u^\alpha(t,y)}\right) u^{1-N\alpha}(t,y) \, \mathrm{d}y - u(t,x). \tag{6}$$

Equation (2) is also the nonlocal analogue of a classical nonlinear diffusion equation:

$$\frac{\partial u}{\partial t} = \nabla \cdot (k(u) \nabla u), \tag{7}$$

where the diffusion coefficient $k(u)$ accounts for nonlinear material properties, such as flow through a porous medium. Equation (7) can be obtained from (2) by rescaling the kernel J and taking the limit as a nonlocality "horizon" parameter vanishes, see [18,21]. It can also be viewed as the first Taylor approximation to (2) [27], provided we take $k(u,u) = k(u)$. It is worth noting that, if the equation arises from a variational principle, such that energy stability methods are available, then $k(v,u) = k(u,v)$ is symmetric. Numerical convergence was demonstrated for semi-implicit schemes of the non-local Cahn–Hilliard equation in [28].

Our work mostly concerns the strong maximum principle, a result that provides qualitative information about solutions as well as a tool for developing other a priori analytic estimates. For locally defined diffusion equations, such as (7), the usual result for classical solutions $u \in C^1([0,T]; C^2(\Omega))$ is as follows:

If u attains a global extremum inside the parabolic cylinder $[0,T] \times \Omega$, then u and its boundary values must be identically constant [29].

An analogous result holds the for the associated time-independent elliptic problem [30,31]. There are also similar results for discrete parabolic operators [32], genuinely nonlinear PDEs for semi-continuous functions [33,34], and time-fractional and elliptic fractional differential equations [24,35,36].

For integro-differential equations similar to the type in (2), there are positivity principles for the linear steady-state problem for L^2 functions [37] and strong maximum principles for linear and semilinear traveling wave equations [27,38], linear and Bellman-type-nonlinear parabolic problems [39], and degenerate parabolic and elliptic equations for semi-continuous functions involving Lévy-Itô-type nonlocal operators [40–42]. More recently, the maximum principle was employed in [23] for linear models to establish con-

vergence results as the nonlocal horizon vanishes, and L^p contraction and global existence results were considered for nonlinear equations [43].

Our work is distinguished from these by demonstrating the strong maximum principle for degenerate nonlinearities, and a novel classification of nonlinearities which have the strong maximum principle, including new conditions for which the principle is valid. In particular, we identify a new geometric condition on the equation: that the zero set of the flux $F(u, v)$ be an equivalence relation. Under this condition, we obtain Theorem 3, the strong maximum principle. This condition is novel, and may have wider applications, such as to fractional equations.

In this paper, we find degenerate conductivities $k(u, v)$, which allow for a strong maximum principle. For general $F^{-1}(0)$, we are unable to show a strong maximum principle (SMP). However, an SMP does hold when $F^{-1}(0)$ is diagonal. More generally, we can establish an SMP if $F^{-1}(0)$ is an equivalence relation and we replace *constant solutions* by the more fundamental *zero flux solutions*. Unlike in the classical case, our proof does not rely on invoking Hopf's lemma or Harnack's inequality [29], and is very similar to the proof of the weak maximum principle.

In Section 2, we present maximum principles for the cases $F^{-1}(0) = \{(u, u)\}$ diagonal, and $F^{-1}(0)$ an equivalence relation. In Section 3, we list counterexamples and consider the remaining cases from a numerical point of view in Section 4.

Notation

We denote PDE as partial differential equation and SMP as strong maximum principle.

We also list some common spaces and norms used in this article. Here, we let $T > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded open subset.

1. Norms in a single variable.

(i) $C(\overline{\Omega})$: those continuous functions of x on Ω, which are continuous up to the boundary $\partial\Omega$, with norm $\sup_\Omega |f(x)|$.

(ii) $L^p(\Omega)$: those Lebesgue measurable functions of x on Ω whose p-th power has a finite Lebesgue integral, with norm $\left(\int_\Omega |f(x)|^p dx\right)^{1/p}$. Here, $1 \le p < \infty$.

(iii) $L^\infty(\Omega)$: those Lebesgue measurable functions of x on Ω which are bounded up to a Lebesgue null set, with norm ess $\sup_\Omega |f(x)|$. Here, the essential supremum is the infimum of those constants $C \ge 0$, such that $|f(x)| \le C$ almost everywhere.

2. Spacetime norms. We let $C^1([0, T]; L^\infty(\Omega))$ be bounded functions of space which are continuously differentiable in time, with norm

$$\|f(t, x)\| = \sup_{[0,T]} \|f(t, x)\|_{L^\infty(\Omega)} + \sup_{[0,T]} \|\partial_t f(t, x)\|_{L^\infty(\Omega)}.$$

We define $C^1([0, T]; C(\overline{\Omega}))$ analogously. Simple examples include dyadic products $a(t)b(x)$, where $a \in C^1([0, T])$ and $b \in L^\infty(\Omega)$.

3. Two-point norms. We let $C(\overline{\Omega}; C(\overline{\Omega}))$ be continuous functions of y that continuously vary in x, both up to the boundary, with norm

$$\|J(x, y)\|_{C(\overline{\Omega}; C(\overline{\Omega}))} = \sup_{x,y \in \Omega} |J(x, y)|.$$

This is equivalent to $C(\overline{\Omega} \times \overline{\Omega})$. We denote $L^\infty(\Omega, L^1(\Omega))$ the set of integrable functions of y which vary in x in a bounded way. They have norm

$$\|J(x, y)\|_{L^\infty(\Omega; L^1(\Omega))} = \text{ess} \sup_{x \in \Omega} \int_\Omega |J(x, y)| dy.$$

Examples include translation invariant kernels $J(x, y) = \eta(x - y)$ for $\eta \in L^1(\mathbb{R}^n)$, and dyadic products $J(x, y) = a(x)b(y)$ for $a \in L^\infty(\Omega)$ and $b \in L^1(\Omega)$.

2. Strong Maximum Principles for Degenerate Diffusion Operators

Provided the nonlocal interactions determined by the flux function $F(u, v)$ are invariant under certain permutations, we can demonstrate the infinite propagation speed phenomenon via the strong maximum principle.

2.1. Space-Continuous Solutions

We will first work in the continuous category for simplicity of presentation, i.e., continuous solutions $u \in C^1([0, \infty); C(\overline{\Omega}))$, arising from data $u_0 \in C(\overline{\Omega}), J \in C(\overline{\Omega}; C(\overline{\Omega}))$.

Theorem 1. *In the continuous category, suppose $F^{-1}(0) = \{(u, u)\}$ is diagonal. If $u \leq u(T, X)$ for some $T > 0$ and $X \in \overline{\Omega}$, then $u \equiv u(T, X)$ is a constant solution.*

Proof. Since at a maximum $\partial u / \partial t(T, X) = 0$ and $u(T, y) \leq u(T, X)$ for all $y \in \Omega$, setting $(t, x) = (T, X)$ in (2) implies

$$0 = \frac{\partial u}{\partial t}(T, X) = \int k(u(T, X), u(T, y))(u(T, y) - u(T, X))J(X, y)\, dy \leq 0. \tag{8}$$

This means the integrand is identically zero, whence the solution has zero flux coming from the point $x = X$:

$$F(u(T, y), u(T, X))J(X, y) = 0, \qquad y \in \Omega. \tag{9}$$

Since $F^{-1}(0)$ is diagonal and $J > 0$, this implies $u(T, y) = u(T, X)$ for $y \in \Omega$. In other words, u is constant at time T. By uniqueness, we conclude $u \equiv u(T, X)$. □

Remark 1. *The nonlocality is relevant here: for the heat equation $u_t = \Delta u$, constancy fails for the strong maximum principle if Ω has two connected components. Theorem 1 does not assume connectedness of Ω.*

We now weaken the hypothesis that $F^{-1}(0)$ be diagonal. Inspecting the proof of Theorem 1, we see that it suffices to show that for all $U \in \mathbb{R}$, any solution $v(x)$ of the one-point functional equation

$$F(v(x), U) = 0, \qquad x \in \Omega \tag{10}$$

is constant, or more generally stationary. That is, it also solves the two-point functional equation

$$F(v(x), v(y)) = 0, \qquad x, y \in \Omega. \tag{11}$$

The following proposition characterizes this implication.

Proposition 1. *Zero flux set $F^{-1}(0)$ is an equivalence relation if and only if for every $U \in \mathbb{R}$, all solutions of (10) solve (11).*

Recall that an equivalence relation $R \subset \mathbb{R} \times \mathbb{R}$ is reflexive, symmetric, and transitive. Since $F^{-1}(0) \supset \{(u, u)\}$, reflexivity is automatic.

Proof. Suppose $F^{-1}(0)$ is an equivalence relation. Let v be a solution of (10) and $x, y \in \Omega$. Since $F(v(x), U) = 0$ and $F(v(y), U) = 0$, symmetry and transitivity imply $F(v(x), v(y)) = 0$.

Conversely, suppose that any solution u of (10) solves (11). We will show symmetry and transitivity. These properties will follow from the permutation invariance of (11): if v is a solution, then so is $v \circ \phi$ for any bijection $\phi : \Omega \to \Omega$.

1. Suppose $F(U, V) = 0$, and choose a solution $v(x)$ of (10) which satisfies $v(x_1) = U, v(x_2) = V$ for some $x_k \in \Omega$. Since v solves (11) as well, it follows that $F(V, U) = F(v(x_2), v(x_1)) = 0$.

2. Suppose that $F(U,V) = F(V,W) = 0$. Choose a solution $v(x)$ of (10) with $v(x_1) = U, v(x_2) = V, v(x_3) = W$. Since (11) holds for any permutation of the x_i, we deduce transitivity: $F(U,W) = F(v(x_1), v(x_3)) = 0$.
□

Remark 2. *Unless Ω has at least three connected components, the reverse claim does not hold in the continuous category. For example, suppose Ω is 1-connected and the flux is given by*

$$k(u,v) = |u+v|. \tag{12}$$

In this case, the non-constant solutions of (10) must map into $\{-c, c\}$ for some $c > 0$. Since Ω is 1-connected, all such continuous solutions are constants, so the choice of v in Step 1 of the proof of Proposition 1 above is not valid, and we cannot deduce symmetry. Even if Ω is 2-connected, a similar problem happens in Step 2 for transitivity.

As in Theorem 1, it follows that the strong maximum principle holds in the continuous category if $F^{-1}(0)$ is an equivalence relation. The difference is, if Ω is not connected, then solutions need not be constant between components. They must, however, have zero flux. This is an effect of nonlocality.

Theorem 2. *In the continuous category, suppose that $F^{-1}(0)$ is an equivalence relation. If $u \leq u(T, X)$ for some $T > 0$ and $X \in \overline{\Omega}$, then the solution has zero flux $F(u(t,x), u(t,y)) \equiv 0$ and is stationary $u(t, x) \equiv u_0(x)$.*

Proof. We repeat Theorem 1 until Equation (9). As before, we deduce the one-point equation

$$F(u(T, X), u(T, y)) = 0, \qquad y \in \Omega. \tag{13}$$

Since $F^{-1}(0)$ is an equivalence relation, we obtain the stronger two-point equation

$$F(u(T, x), u(T, y)) = 0, \qquad x, y \in \Omega. \tag{14}$$

We claim that the function $(t, x) \mapsto u(T, x)$ is a stationary solution of (2). Indeed, since stationary solutions satisfy the nonlocal elliptic equation,

$$\int_{\Omega} F(v(x), v(y)) J(x, y) \, \mathrm{d}y = 0, \qquad x \in \Omega, \tag{15}$$

and $F(u(T, x), u(T, y)) \equiv 0$, the conclusion is immediate. By uniqueness, we conclude that $u(t, x) \equiv u(T, x)$ for all t. This means that the solution has zero flux everywhere: $F(u(t, x), u(t, y)) \equiv 0$. □

2.2. Merely Bounded Solutions

We now generalize Theorem 2 to time-continuous space-bounded solutions $u \in C^1([0, \infty); L^{\infty}(\Omega))$, arising from bounded initial data $u_0 \in L^{\infty}(\Omega)$ and bounded-integrable kernel $J \in L^{\infty}(\Omega; L^1(\Omega))$. Let us note that Equation (2) is time translation invariant and local in time, such that any $C^1([0, \infty); L^{\infty}(\Omega))$ solution restricts to a solution defined on $[0, T]$ in $C^1([0, T]; L^{\infty}(\Omega))$.

Theorem 3. *Suppose that $F^{-1}(0)$ is an equivalence relation and $u \in C^1([0, T]; L^{\infty}(\Omega))$ solves (2) for $J \in L^{\infty}(\Omega; L^1(\Omega))$. If u is maximized at the final time, or*

$$\operatorname*{ess\,sup}_{[0,T] \times \Omega} u(t, x) = \operatorname*{ess\,sup}_{\Omega} u(T, x), \tag{16}$$

then $F(u_0(x), u_0(y)) \equiv 0$ and $u(t, x) = u_0(x)$ for each $t \in [0, T]$ (the solution has zero flux and is stationary).

Proof. If we integrate (2) from $t < T$ to T, we get

$$u(T, x) - u(t, x) = \int_t^T \int_\Omega F(u(s, x), u(s, y)) J(x, y) \, \mathrm{d}y \, \mathrm{d}s. \tag{17}$$

Denote $U := \operatorname{ess\,sup} u|_{[0,T] \times \Omega}$. For all $\varepsilon > 0$, the approximate maximizing set

$$M_\varepsilon := \left\{ x \in \Omega : u(T, x) > U - \varepsilon \right\}, \tag{18}$$

has positive measure $|M_\varepsilon|$. After averaging the left side of (17) on the approximate maximizing set, we see it is approximately nonnegative:

$$\begin{aligned} A_\varepsilon &:= \frac{1}{|M_\varepsilon|} \int_{M_\varepsilon} \left[u(T, x) - u(t, x) \right] \mathrm{d}x \\ &\geq -\varepsilon + \frac{1}{|M_\varepsilon|} \int_{M_\varepsilon} \left[U - u(t, x) \right] \mathrm{d}x \\ &\geq -\varepsilon. \end{aligned} \tag{19}$$

In contrast, the right hand side will be approximately nonpositive. We first use time differentiability and F being locally Lipschitz to evaluate at $s = T$:

$$\begin{aligned} \int_t^T \int_\Omega F(u(s, x), u(s, y)) J(x, y) \, \mathrm{d}y \, \mathrm{d}s &= (T - t) \int_\Omega F(u(T, x), u(T, y)) J(x, y) \, \mathrm{d}y \\ &\quad + O(T - t)^2 \cdot L^\infty([0, T] \times \Omega). \end{aligned} \tag{20}$$

Here, $L^\infty([0, T] \times \Omega)$ denotes some bounded function of t, x on Ω, using $J \in L^\infty(\Omega; L^1(\Omega))$. Next, we evaluate at max points x:

$$\begin{aligned} \int_\Omega F(u(T, x), u(T, y)) J(x, y) \, \mathrm{d}y &= \int_\Omega F(U, u(T, y)) J(x, y) \, \mathrm{d}y \\ &\quad + \int_\Omega [F(u(T, x), u(T, y)) - F(U, u(T, y))] J(x, y) \, \mathrm{d}y. \end{aligned} \tag{21}$$

Since F is locally Lipschitz continuous, and u is bounded, the second term is small in M_ε:

$$|F(u(T, x), u(T, y)) - F(U, u(T, y)) \leq C|u(T, x) - U| \leq C\varepsilon, \qquad x \in M_\varepsilon, \tag{22}$$

for some $C > 0$. Upon averaging over the approximate maximum set, we obtain

$$-\varepsilon \leq A_\varepsilon = (T - t) \int_\Omega F(U, u(T, y)) \widetilde{J}_\varepsilon(y) \, \mathrm{d}y + C(T - t)\varepsilon + O(T - t)^2, \tag{23}$$

where, by our positivity hypothesis on J,

$$\widetilde{J}_\varepsilon(y) = \frac{1}{|M_\varepsilon|} \int_{M_\varepsilon} J(x, y) \, \mathrm{d}x \geq \delta > 0. \tag{24}$$

Since $F(U, u_y) = k(U, u_y)(u_y - U) \leq 0$, the lower bound on $\widetilde{J}$ translates to

$$-C\varepsilon \leq (T - t) \cdot \delta \int_\Omega F(U, u(T, y)) \, \mathrm{d}y + O(T - t)^2. \tag{25}$$

Setting $\varepsilon = (T - t)^2$ and sending $t \nearrow T$, we conclude the integrand must identically vanish:

$$F(U, u(T, y)) = 0, \qquad y \in \Omega. \tag{26}$$

Because $F^{-1}(0)$ is an equivalence relation, we deduce from Proposition 1 that $x \mapsto u(T, x)$ has vanishing flux:

$$F(u(T, x), u(T, y)) = 0, \qquad x, y \in \Omega. \tag{27}$$

By uniqueness, we conclude that $u(t, x) \equiv u(T, x)$ is a stationary solution that has zero flux everywhere: $F(u(t, x), u(t, y)) \equiv 0$. $\quad\square$

2.3. Examples of Degenerate Fluxes with Maximum Principles

There is an abundance of such conductivity $k(u, v)$s whose flux functions form equivalence relation $F^{-1}(0)$s. Graphically, such relations contain and are symmetric about the diagonal, and as transitive relations, they can be thought of as having additional "rectangular" symmetry; see Figure 1. The most trivial degenerate example is $F \equiv 0$. There are at least three nontrivial ways to construct them:

Example 1. *Diagonal $F^{-1}(0) = \{(u, u)\}$. In this case, positivity $k > 0$, positive definiteness $k(u, v) = 0 \leftrightarrow u = v = 0$, and diagonality $k(u, v) = 0 \leftrightarrow u = v$ all suffice. This includes the linear case $k = 1$, the p-Laplace case $k(u, v) = |u - v|^{p-2}$ of ([21], Chapter 6), and the fast diffusion case $k(u, v) = [\gamma^{-1}(v) - \gamma^{-1}(u)] / (v - u)$ of ([21], Chapter 5) under a monotone condition on γ.*

For all these cases, solutions of zero flux are constants. This is not true for the following examples.

Example 2. *$F^{-1}(0)$ the union of the diagonal with the graph of an involutive function $f : \mathbb{R} \to \mathbb{R}$, i.e., $f^{-1} = f$. One large class of examples is of the form*

$$F(u, v) = A(u, v) \left| u - f(v) \right| (v - u) \tag{28}$$

for some positive A. For instance,

$$F(u, v) = \left| u + v \right|^{p-2} (v - u) \quad and \quad F(u, v) = \left| 1 + uv \right| (v - u) \tag{29}$$

both yield equivalence relations with involutions $f(x) = -x, 1/x$, respectively. More involutions can be generated by conjugation: $\varphi \circ f \circ \varphi^{-1}$ is an involution provided φ is a bijection and f is an involution.

The reason involutions work is because if $F(U, f(U)) = 0$ for all U, then $F(f(U), U) = F(Z, f(Z)) = 0$ for $Z = f(U)$, which shows the symmetry of $F^{-1}(0)$. To verify transitivity, if $F(U, V) = F(V, W) = 0$, then $U = V$ or $f(V)$ and $W = V$ or $f(V)$. In all cases, $F(U, W) = 0$:

$$F(V, V) = 0, \qquad F(V, f(V)) = 0, \qquad F(f(V), V) = 0, \qquad F(f(V), f(V)) = 0.$$

We now generalize this construction to incorporate more involutions.

Example 3. *$F^{-1}(0)$ generated by a group of involutions. As a concrete example, we study the flux*

$$F(u, v) = \left| (u + v)(1 + uv)(1 - uv) \right| (v - u). \tag{30}$$

We will show that $F^{-1}(0)$ is an equivalence relation; see Figure 1.

There is a general reason why: the zero set is the union of graphs of involutive functions which form a group under function composition. More specifically, $F^{-1}(0)$ is the union of diagonal (u, u) with the graphs $(u, -u), (u, 1/u)$, and $(u, -1/u)$. If we denote $f_1(x) = x, f_2(x) = -x, f_3(x) = 1/x, f_4(x) = -1/x$ and put $f_3(0), f_4(0) := 0$ for simplicity,

then the set of these functions $\{f_1, f_2, f_3, f_4\}$ is the Klein four-group ([44], pg 13) with function composition as the group operation:

$$f_2 \circ f_3 = f_4, \qquad f_3 \circ f_4 = f_2, \qquad f_4 \circ f_2 = f_3,$$

and each composition is commutative: $f_i \circ f_j = f_j \circ f_i$. Let us now verify the properties for an equivalence relation.

Symmetry: if $F(U, V) = 0$, then $V = f_i(U)$ for one such i. Since f_i is an involution, we know $U = f_i(V)$, so $F(V, U) = F(V, f_i(V)) = 0$.

Transitivity: if $F(U, V) = 0$ and $F(V, W) = 0$, then $U = f_i(V)$ and $W = f_j(V)$ for some i, j. This means $F(U, W) = F(f_i(V), f_j(V))$. It suffices to show that $f_j(V) = f_k(f_i(V))$ for some k. Since f_i is an involution,

$$f_j(V) = f_j(f_i(f_i(V))) = f_k(f_i(V)),$$

where the existence of k follows from the group closure property of composition. Thus, $F(U, W) = 0$.
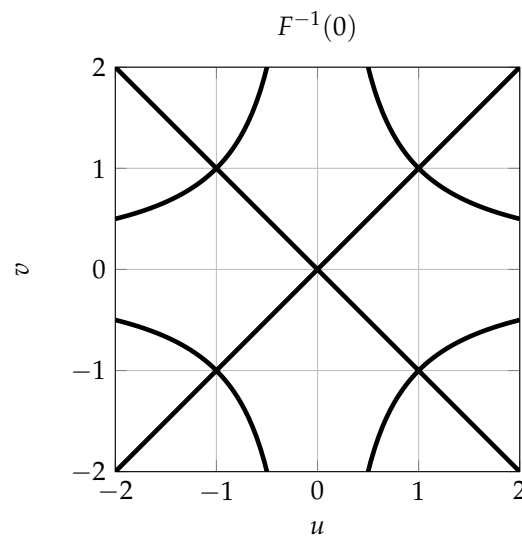
$$F^{-1}(0)$$

Figure 1. The zero set, $F^{-1}(0)$, for (30).

3. Counterexamples to the Strong Maximum Principle

Physically, the maximum principle is a consequence of the averaging effects of diffusion. If the diffusivity vanishes at a point $x = p$, it follows that the maximum principle should be violated at $x = p$. In our nonlocal framework, the diffusivity is proportional to the flux $F(u, v)$ (and is precisely $k(u, v)$), so the following result is sensible. We work in the continuous category for simplicity of presentation.

Proposition 2. *Suppose that there exists $U \in \mathbb{R}$ such that $F(u, v) = |u - U|\phi(u, v)$, where $\phi(u, v)/(v - u)$ is locally Lipschitz continuous. Then in the continuous category, there exists a solution u and $p \in \Omega$ for which $u(t, p) = \sup_\Omega u_0$ for all $t > 0$.*

If $\phi^{-1}(0)$ is also diagonal, then we can choose $u(t, x)$ to satisfy $\partial u / \partial t \not\equiv 0$.

Observe in the latter case that $F^{-1}(0) = \{(U, x) : x \in \mathbb{R}\} \cup \{(x, x) : x \in \mathbb{R}\}$ is the union of a vertical line with the diagonal, which is not symmetric and hence not an equivalence relation.

Proof. Choose a bump function u_0 so that $u_0(x) = U$ in the open ball $B(p, r)$ of radius $r > 0$ is centered at the point p, but decreases radially to the value $u_0(x) = U - 1$ as $|x - p|$

increases from r to $2r$. To verify that $u(t,p) \equiv U$, we need only observe that the $|u - U|$ portion factors out:

$$\frac{\partial u}{\partial t}(t,p) = |u(t,p) - U|\Phi(t,p),\tag{31}$$

where

$$\Phi(t,p) = \int_\Omega \phi(u(t,p), u(t,y))J(p,y)\,\mathrm{d}y\tag{32}$$

is continuous in time. This defines a Lipschitz ODE for the function $t \mapsto U(t) =: u(t,p)$ with initial condition $U(0) = U$. Since $U(t) \equiv U$ is clearly a solution, uniqueness implies that $u(t,p) \equiv U$ for all t.

Let us now suppose that $\phi^{-1}(0)$ is diagonal. Suppose that $q \in \Omega \setminus B(p,r)$ is distance $r + \varepsilon$ from p. Then we can write Φ as a small positive part plus a negative part:

$$\begin{aligned}\Phi(0,q) &= \left(\int_{B(p,r+\varepsilon)} + \int_{\Omega \setminus B(p,r+\varepsilon)}\right)\phi(u_0(q), u_0(y))J(q,y)\,\mathrm{d}y\\ &=: P + N.\end{aligned}\tag{33}$$

Indeed, since $\phi(x,y) = k(x,y)(y-x)/|x-U|$ is continuous, the first integrand behaves like $U - u_0(q) \geq 0$, while the second behaves like $(U-1) - u_0(q) < 0$. Since $u_0(q) \to U$ is $\varepsilon \to 0$, we can choose an $\varepsilon > 0$ that is sufficiently small so that $P < |N|$, i.e., $\Phi(0,q) < 0$. This verifies

$$\frac{\partial u}{\partial t}(0,q) < 0\tag{34}$$

for all $q \in \Omega \setminus B(p,r)$ sufficiently close to $B(p,r)$. $\square$

To see this also holds in the L^∞ category, let us consider the degenerate conductivity $k(u,v) = (1-u)^2$, with initial data and a positive kernel given on $[-1,1]$

$$u_0(x) = \begin{cases} 1, & |x| < 0.5, \\ 0, & \text{otherwise}, \end{cases}\tag{35}$$

$$J(x,y) = e^{-\frac{|x-y|^2}{2}}, \qquad x,y \in [-1,1].$$

We solve (2) numerically, and plot the time evolution of the solution in Figure 2. The spatial domain $[-1,1]$ is discretized using the convergent Trapezoidal rule [45] with a distance $\Delta x = 10^{-2}$ between adjacent nodes of a uniform spatial mesh. A forward Euler time stepper is used to numerically integrate over the time interval $[0,1]$ with a numerical timestep of $\Delta t = 10^{-2}$. This value is significantly lower than $\Delta t = 0.5$, which is the largest numerical time step value for which the forward Euler timestepper will be stable on the related nonlocal linear problem, as discussed in Appendix A. Other works have studied the stability of the numerical time stepping of related equations, e.g., [28]. An interesting further direction of research would be to provide a more in depth analysis regarding the numerical scheme than that we have provided here. The numerical results were generated by making using of the programming language Python. We see in Figure 2 that, although the minimum value $\inf_\Omega u(t,.)$ increases with time, the maximum value remains constant, $\sup_\Omega u(t,.) = 1$.
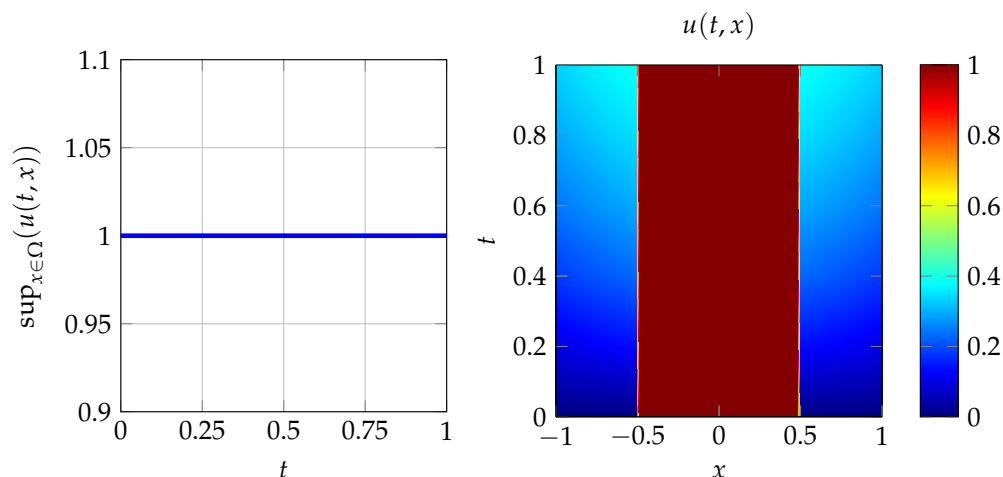
Figure 2. Maximum value and spacetime evolution of (2) with (highly) degenerate conductivity $k(u, v) = (1 - u)^2$ and data given in (35).

4. The Inconclusive Cases

The following problem remains an open issue: if $F^{-1}(0)$ is neither an equivalence relation nor highly degenerate as in Proposition 2, does a version of the strong maximum principle hold? More precisely, can we conclude that u is a zero flux stationary solution, $F(u, u) \equiv 0$ and $\partial u / \partial t \equiv 0$, if we know that $u \leq u(T, X)$ for some $T > 0, X \in \Omega$?

We conjecture the affirmative. Consider the inconclusive example $k(u, v) = (v - u^2)^2$, whose associated flux function $F(u, v)$ does not have an equivalence relation for $F^{-1}(0)$. Let us choose initial data and positive kernels of the form

$$u_0(x) = (1 - x^2)^2, \qquad x \in [-1, 1],$$

$$J(x, y) = e^{-\frac{|x-y|^2}{2}}, \qquad x, y \in [-1, 1]. \tag{36}$$

Since $k(u_0(0), u_0(0)) = 0$, or more generally $\{(u_0(x), u_0(y))\} \cap k^{-1}(0)$ is nonempty, this example exhibits (mild) degeneracy. However, as shown in Figure 3, which depicts the time evolution of a numerical solution, the supremum $\sup_\Omega u(t, .)$ visibly decreases over time.

The numerical result shown in Figure 3 is reasonable to the authors because of the following. Suppose that u_0 is nonnegative and contains a finite number of discontinuities. Further assume that u_0 attains its supremum and, for each $x \in \Omega$, the cardinality of $u_0^{-1}(u_0(x)) \subset \Omega$ is finite, which guarantees that $u_0(\Omega)$ has a nonzero measure with respect to the Lebesgue measure on $\mathbb{R}$. For any $\chi \in \Omega$, such that $u_0(\chi) = \sup_\Omega u_0$, we have

$$\frac{\partial u}{\partial t}(0, \chi) = -\int_\Omega k(u_0(\chi), u_0(y))|u_0(y) - u_0(\chi)|J(\chi, y) \, \mathrm{d}y. \tag{37}$$

Given the hypothesis, there is a set $W \subset \Omega$ of nonzero measures such that $k(u_0(\chi), u_0(y))|u_0(y) - u_0(\chi)|J(\chi, y) > 0$ for each $y \in W$. This demonstrates immediate supremum decay

$$\frac{\partial u}{\partial t}(0, \chi) < 0. \tag{38}$$

Because the diffusion is nonlocal, the initial data's oscillation $(\sup - \inf)_\Omega u_0$ contributes to diffusion even at points of (mild) degeneracy. We note that the assumptions of the argument just presented could be weakened, such as the assumption that u_0 attains its supremum, and arrive at the same conclusion.
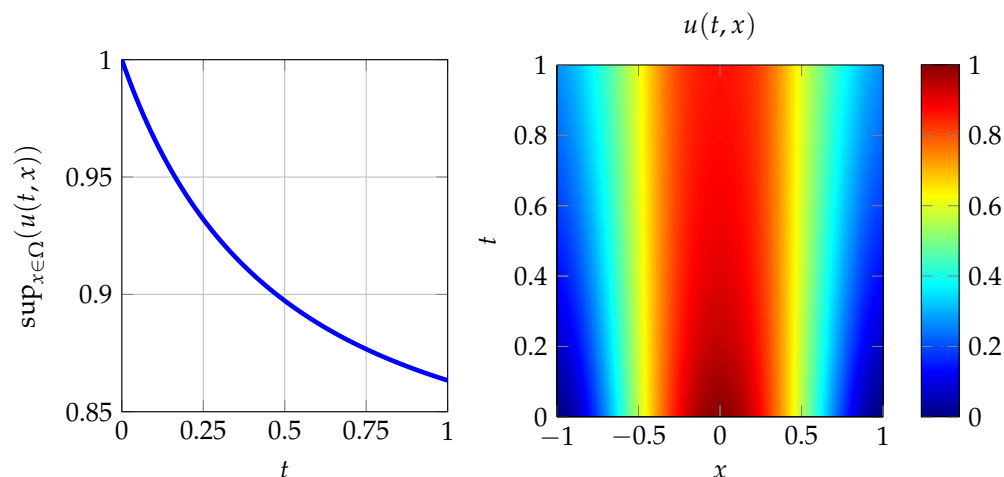
Figure 3. Maximum value and spacetime evolution of (2) with (mildly) degenerate conductivity $k(u,v) = (v - u^2)^2$ and data given in (36).

5. Conclusions

We considered whether the strong maximum principle is valid for integro-differential equations of the type (2), namely a nonlocal parabolic equation with integrable kernel and degenerate two-point nonlinear conductivity. We showed that, under a certain geometric assumption about the nonlinearity, the strong maximum principle is valid; see Theorem 3. However, the conclusion must be reformulated: instead of solutions being constants, we show that they have zero flux, and are therefore stationary. We have therefore concluded that no non-stationary solutions violate the strong maximum principle. This is in contrast to parabolic PDES with degenerate conductivity, which have non-stationary solutions which propagate the initial maximum value through time [20].

In Proposition 2, we have also considered solutions of Equation (2) with more degenerate conductivities. Such solutions are non-stationary and violate the strong maximum principle. Therefore, in some sense, our geometric conditions in Theorem 3 were necessary for the strong maximum principle to work.

For future directions, it would be interesting to ask the question of whether we can find sharp geometric conditions on the degenerate nonlinearity that allow a strong maximum principle. In Section 4, we give some evidence that the geometric condition in Theorem 3 is sharp.

Author Contributions: Methodology, T.H. and R.S. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

In [28], a comprehensive numerical and convergence analysis is conducted for the nonlocal Cahn–Hilliard equation. Here, we provide an analysis of the spectrum of the Jacobian of the discretized integral term in a nonlocal linear equation,

$$\frac{\partial u}{\partial t}(t,x) = k \int_\Omega \big[u(t,y) - u(t,x)\big] J(x,y)\,\mathrm{d}y, \qquad (t,x) \in (0,\infty) \times \Omega,$$
$$u(0,x) = u_0(x) \in L^\infty(\Omega), \tag{A1}$$

which is obtained from Equation (2) by taking $k(u, v)$ to be equal to the constant k. The purpose of said analysis is to provide insight into how the numerical time step Δt, in an explicit time stepping scheme such as forward Euler, should be chosen to obtain linearly stable discrete time steps. Here, we let Ω be a bounded open subset of $\mathbb{R}$, e.g., $(-1, 1)$ as in the numerical examples presented in Section 3. We work in the continuous category so that Equation (A1) also holds for $x \in \partial\Omega$. Upon discretizing in space on a mesh with n nodes $x_1, x_2, \ldots, x_n \in \overline{\Omega}$ and associated values of the numerical solution at said nodes $u(t) \in \mathbb{R}^n$, wherein $[u(t)]_i = u_i(t) \approx u(t, x_i)$, and integrating by quadrature with quadrature weight w_k is associated to the node x_k as

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = g(u(t)), \qquad u(0) = u_0$$

$$g_i(u) = k \sum_{\ell=1}^{n} [u_\ell - u_i] J(x_i, x_\ell) w_\ell,$$

where $u_0 \in \mathbb{R}^n$ is such that $[u_0]_i = u_0(x_i)$. The Jacobian of $g(u)$ with respect to u is relevant in the context of explicit numerical time stepping, e.g., forward Euler, as its eigenvalues define which values of the time step provide for a numerical method that is linearly stable. We next compute said Jacobian:

$$\frac{\partial g_i}{\partial u_j} = k \sum_{\ell=1}^{n} \left[\delta_{\ell,j} - \delta_{i,j} \right] J(x_i, x_\ell) w_\ell,$$

$$= k \left(J(x_i, x_j) w_j - \delta_{i,j} \sum_{\ell=1}^{n} J(x_i, x_\ell) w_\ell \right).$$

We then invoke the Gershgorin circle theorem [46], which in this context informs us that all eigenvalues of the Jacobian $\partial g / \partial u$ are contained in $(\cup_{i=1}^{n} D(a_{i,i}, R_i)) \subset \mathbb{C}$, where $D(a_{i,i}, R_i) = \{x \in \mathbb{C} \text{ such that } |x - a_{i,i}| \leq R_i\}$ is the ith Gershgorin disc with center $a_{i,i} = \partial g_i / \partial u_i$ and radius $R_i = \sum_{j \neq i} \partial g_i / \partial u_j$. For the linear nonlocal problem (A1), there is a relatively simple relationship between the center $a_{i,i}$ and radius R_i of the Gershgorin disc, namely

$$a_{i,i} = -k \sum_{\ell=1}^{n} J(x_i, x_\ell) w_\ell,$$

$$R_i = -a_{i,i}.$$

Since $a_{i,i} = -R_i < 0$, we can conclude that the eigenvalues of the Jacobian have a nonpositive real part. We can also conclude that an arbitrary eigenvalue λ of said Jacobian satisfies the bound $|\lambda| \leq 2 \max_{1 \leq i \leq n} R_i$ and, further, that

$$|\lambda| \leq 2 k \, \|J(x, y)\|_{C(\overline{\Omega}; C(\overline{\Omega}))} \sum_{\ell=1}^{n} w_\ell.$$

For the Trapezoid rule and other quadrature schemes, we have $\sum_{\ell=1}^{n} w_\ell = |\Omega|$. Having determined the eigenvalues of the Jacobian $\partial g / \partial u$, we can then determine which numerical time steps $\Delta t > 0$ will be linearly stable if we use the forward Euler numerical time integrator, that is, those time steps which satisfy

$$\Delta t < \frac{1}{k \, |\Omega| \|J(x, y)\|_{C(\overline{\Omega}; C(\overline{\Omega}))}}. \tag{A2}$$

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