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Editorial

Editorial for the Special Issue of Axioms "Calculus of Variations, Optimal Control and Mathematical Biology: A Themed Issue Dedicated to Professor Delfim F. M. Torres on the Occasion of His 50th Birthday"

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1. Introduction

This publication is an editorial for the Special Issue of Axioms "Calculus of Variations, Optimal Control and Mathematical Biology: A Themed Issue Dedicated to Professor Delfim F. M. Torres on the Occasion of His 50th birthday". This Special Issue is dedicated to Professor Delfim F. M. Torres on the occasion of his 50th birthday, as recognition of his significant contributions to Mathematics, in particular in the calculus of variations, optimal control, and mathematical biology. Professor Torres is a distinguished University Professor, a highly cited researcher in Mathematics (in the top 1% for Mathematics in the Web of Science in the years 2015, 2016, 2017 and 2019), and a lifetime member of the American Mathematical Society. He is one of the founders of fractional variational analysis and fractional optimal control, and has made tremendous contributions to the theory of variational analysis, with applications to many other fields such as optimization, optimal control, time-scale analysis, and mathematical epidemiology and biology. Professor Torres is the recipient of several international awards, including the 325 Years of Fractional Calculus Award, which is testament to the high regard in which his achievements in the area of fractional calculus and its applications are held. He has been included in the World's Top 2% Scientists by Stanford University in the years 2020, 2021 and 2022, both as a career-long highly cited researcher and as a single calendar year highly cited researcher. He is considered by Thomson Reuters as one of the World's Most Influential Scientific Minds, has won a Publons Peer Review Award as a world's top peer reviewer, and was recognized in the top 1% of reviewers. He has also won a Sentinel of Science Award. Many of his research works have been considered top papers in their area, served as research highlights and been awarded prizes. Besides being a great scholar, he is also a great teacher who has already supervised twenty-four Ph.D. students from all over the world. This Special Issue covers many of Professor Torres' research interests, which include several areas of pure and applied mathematical sciences, such as approximations and expansions, biology and other natural sciences, calculus of variations and optimal control, optimization, difference and functional equations, fluid mechanics, functional analysis, game theory, economics, social and behavioral sciences, general measure and integration, the mechanics of deformable solids, number theory, numerical analysis, operations research, mathematical programming, ordinary differential equations, partial differential equations, quantum theory, real functions, systems and control theory, fractional calculus and its applications, integral equations and transforms, higher transcendental functions and their applications, q-series and q-polynomials, inventory modeling and optimization, dynamic equations on time scales, and mathematical modeling.

This Special Issue comprises fifteen original research papers that have been carefully reviewed. In the next section, we briefly describe the main contributions of these fifteen papers. We finalize this editorial with a brief biography of Delfim F. M. Torres.

2. Contributions

In this section, we provide an overview of the scientific contributions of the papers that constitute this Special Issue, gathering them into five mathematical research areas: Calculus of variations, Optimal control, Mathematical biology, Fractional calculus, and Differential geometry. We remark that some of the papers can be integrated in more than one of these research topics.

2.1. Calculus of Variations

In the paper *On a Non–Newtonian Calculus of Variations*, Delfim F. M. Torres presents, for the first time in the literature, a non-Newtonian calculus of variations that involves the minimization of a function defined by a non-Newtonian integral with a Lagrangian depending on the non-Newtonian derivative. The main result of this paper is a first-order necessary optimality condition of a Euler–Lagrange type that each solution of a non-Newtonian variational problem, with admissible functions taking positive values only, must satisfy. The non-Newtonian calculus of variations introduced in this paper provides a natural framework for dealing with multiplicative functions that arise in economics, physics and biology.

In the article *Weighted Generalized Fractional Integration by Parts and the Euler–Lagrange Equation*, Zine et al. introduce the right-weighted generalized fractional derivative in the Riemann–Liouville sense, and also introduce its associated integral operator, proving their main properties, and, in particular, their integration by parts formula. With the new general integration by parts formula, the authors obtain an appropriate weighted Euler–Lagrange equation for dynamic optimization, extending those existing in the literature. The paper finishes with an illustration of the obtained theoretical results in the quantum mechanics setting.

2.2. Optimal Control

The article *Maximum Principle and Second-Order Optimality Conditions in Control Problems with Mixed Constraints*, by Arutyunov et al., concerns the optimality conditions for a smooth optimal control problem with an endpoint and mixed constraints. Under the normality assumption, second-order necessary optimality conditions based on the Robinson stability theorem are derived. The main novelty is that only a local regularity with respect to the mixed constraints is required, instead of the conventional stronger global regularity hypothesis. This affects the maximum condition. Therefore, the normal set of Lagrange multipliers in question satisfies the maximum principle, albeit along with the modified maximum condition in which the maximum is taken over a reduced feasible set. In the second part of this work, the case of abnormal minimizers, that is, when the full rank of controllability matrix condition is not valid, is addressed. The same type of reduced maximum condition is obtained.

In the paper *Minimum Energy Problem in the Sense of Caputo for Fractional Neutral Evolution Systems in Banach Spaces*, by Ech-chaffani et al., a class of fractional neutral evolution equations on Banach spaces involving Caputo derivatives is investigated. Conditions for the controllability of the fractional-order system and conditions for existence of a solution to an optimal control problem of minimum energy are established. The main results are proved with the help of fixed-point and semigroup theories.

2.3. Mathematical Biology

In the paper *Global Stability Condition for the Disease-Free Equilibrium Point of Fractional Epidemiological Models*, Almeida et al. present a new method to study the global asymptotic stability of dynamical systems described by fractional differential equations. The usual approach involves the determination of an appropriate Lyapunov function, very laborious computations and the application of LaSalle's invariance principle. The method proposed by the authors uses only basic results from matrix theory and some well-known results from fractional-order differential equations. To illustrate the applicability of the theoretical results, the authors exemplify the procedure with three epidemiological models: a fractional-order SEIR model with classical incidence function, a fractional-order SIRS model with a general incidence function, and a fractional-order model for HIV/AIDS.

In the research *Hybrid Method for Simulation of a Fractional COVID-19 Model with Real Case Application*, by Din et al., a mathematical analysis for the novel coronavirus responsible for COVID-19 is provided. The fractional-order analysis is carried out using a non-singular kernel type operator known as the Atangana–Baleanu–Caputo (ABC) derivative. The model, adopting available information about the disease from Pakistan in the period from 9 April to 2 June 2020, is parametrized. The authors obtain the required solution with the help of a hybrid method, which is a combination of the decomposition method and the Laplace transform. Furthermore, a sensitivity analysis is carried out to evaluate the parameters that are more sensitive to the basic reproduction number of the model. The obtained results are compared with real data of Pakistan, and numerical plots are presented at various fractional orders.

In the paper *Fractional Dynamics of a Measles Epidemic Model*, Abboubakar et al. propose a fractional mathematical model for the transmission dynamics of measles, considering a Caputo fractional derivative. The main goal of this work is to compare the dynamics of a measles epidemic model with integer and Caputo fractional derivatives. The epidemic model considers vaccination and hospitalization of infected individuals. A local and global stability analysis of the equilibrium points is proved, and the theoretical results are illustrated through numerical simulations using the Adams-type predictor–corrector iterative scheme. The numerical simulations demonstrate that the fractional model shows different quantitative behavior than the model with integer derivatives.

The paper *Pattern Formation Induced by Fuzzy Fractional-Order Model of COVID-19*, by Alnahdi et al., proposes a mathematical model for COVID-19 using a fuzzy fractional evolution equation, stated in Caputo's sense for order $\alpha \in (1, 2)$. The model considers six compartments: susceptible, exposed, totally infected, asymptotically infected, totally recovered individuals and reservoir. The existence and uniqueness of the solution of the model is proved using Schauder's fixed point theorem. Moreover, the dynamic behavior of the model is studied by combining the fuzzy Laplace approach with the Adomian decomposition transform.

In the paper *Modeling the Impact of the Imperfect Vaccination of the COVID-19 with Optimal Containment Strategy*, Benahmadi et al. propose a mathematical model applied to COVID-19, aiming to investigate the impact of vaccination in the control of the disease's spread. The compartmental model is given by a system of nine ordinary differential equations. After computing the basic reproduction number, a local and global stability analysis of the disease's free equilibrium is presented. An optimal control problem is proposed wherein the aim is to find the optimal control strategy under imperfect vaccination. Three controls are introduced in the model, representing awareness of taking the vaccine, movement restrictions for susceptible and vaccinated individuals, and improvement of the vaccine's efficacy, respectively. In the numerical simulations, the model is calibrated to real data from a vaccination campaign in Morocco between 1 February 2021 and 25 March 2021. The numerical solutions show that to reduce the impact of imperfect vaccination, a longer awareness campaign is needed to engage the population in vaccination. On the other hand, restrictions on population mobility should not be long lasting. Moreover, to ensure the full protection of the health population, vaccination efficacy must be increased by 30% in the first 50 days.

In the paper *Determining COVID-19 Dynamics Using Physics Informed Neural Networks*, by Malinzi et al., the physics informed neural networks (PINNs) framework is applied to COVID-19 epidemics, using a compartmental SIRD (susceptible–infected–recovered–death) mathematical model. The main goal of this work is to find patterns of the transmission dynamics of the disease, which involves predicting the infection, recovery and death rates and therefore, predicting the actively infected, totally recovered, susceptible and deceased individuals at any given time. First, the PINNs framework's application to the SIRD model is validated through numerical simulations using *Mathematica*, and after, the SIRD model is tested and validated using a real COVID-19 dataset.

In the paper by Francesca Acotto and Ezio Venturino, entitled *A Note on an Epidemic Model with Cautionary Response in the Presence of Asymptomatic Individuals*, the authors studied an epidemiological model, taking into consideration some demographic features, and also the case in which the illness appears in two forms, asymptomatic and symptomatic. Because of the presence of asymptomatic individuals, fear drives a reduction in social contact, lowering the overall transmission rate. With some numerical simulations, it is shown that with an increase in information regarding the propagation of the disease, the number of infected individuals decreases.

2.4. Fractional Calculus

In *Riemann–Liouville Fractional Sobolev and Bounded Variation Spaces* by Leaci and Tomarelli, a study of fractional derivatives on Sobolev spaces is carried out. The fractional operators are given by the mean value between the left and right fractional operators. Some problems, such as embeddings or compactness properties, the Abel equation, and semigroup properties, are considered. These methods can be applied into fractional variational models for image analysis.

In the article *On Periodic Fractional (p,q)-Integral Boundary Value Problems for Sequential Fractional (p,q)-Integrodifference Equations* by Soontharanon and Sitthiwirattham, questions on the existence and uniqueness of solution for a fractional (p,q)-integrodifference equation with periodic fractional (p,q)-integral boundary condition are studied. The proofs are based on Banach and Schauder's fixed point theorems. Furthermore, some properties of the fractional (p,q)-integral are obtained. The paper ends with some illustrative examples.

The paper *Existence Results for a Multipoint Fractional Boundary Value Problem in the Fractional Derivative Banach Space* by Boucenna et al., deals with a class of nonlinear implicit fractional differential equations subject to nonlocal boundary conditions, expressed in terms of nonlinear integro-differential equations. Using the Krasnoselsky fixed-point theorem, via the Kolmogorov–Riesz criteria, the existence of solutions in a specific fractional derivative Banach space is established. Then, two numerical examples are given to illustrate the theoretical results.

2.5. Differential Geometry

The paper *Local Structure of Convex Surfaces near Regular and Conical Points*, by Plakhov, deals with the limiting behavior of a part of a surface when it is cut by a plane. More precisely, given a point on a convex surface, and a plane of support Π to the surface at this point, the author considers the plane parallel with Π cutting a part of the surface. Two cases are studied: when the point is regular, and when it is singular conical. This work follows on from a 1995 conjecture by Buttazzo, Ferone, and Kawohl, "Minimum problems over sets of concave functions and related questions", published in *Math. Nachr.*, already solved by the author under some assumptions, and continued in this current work.

3. Short Biography of Delfim F. M. Torres

Delfim Fernando Marado Torres was born 16 August 1971 in Nampula, Mozambique. Since March 2015, he has been a full Professor of Mathematics at the University of Aveiro

(UA), Director of the Research Unit CIDMA, the largest Portuguese research center in Mathematics, and Coordinator of its Systems and Control Group. He obtained a PhD in Mathematics from UA in 2002, and Habilitation in Mathematics from UA in 2011. His main research areas are calculus of variations and optimal control, optimization, fractional derivatives and integrals, dynamic equations on time scales and mathematical biology. Torres has written outstanding scientific and pedagogical publications. In particular, he has co-authored two books with Imperial College Press [1,2], three books with Springer [3–5], and edited several research books, such as [6–8]. He has strong experience in graduate and postgraduate student supervision and teaching in mathematics. Moreover, he has been team leader and member in several national and international R&D projects, including EU projects and networks. Since 2013, he has been the Director of the Doctoral Programme Consortium in Mathematics and Applications (MAP-PDMA) of the Universities of Minho, Aveiro, and Porto. Prof. Torres has been married since 2003, and has one daughter and two sons.

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References

1. Malinowska, A.B.; Torres, D.F.M. *Introduction to the Fractional Calculus of Variations*; Imperial College Press: London, UK, 2012.
2. Almeida, R.; Pooseh, S.; Torres, D.F.M. *Computational Methods in the Fractional Calculus of Variations*; Imperial College Press: London, UK, 2015.
3. Almeida, R.; Tavares, D.; Torres, D.F.M. *The Variable-Order Fractional Calculus of Variations*; SpringerBriefs in Applied Sciences and Technology; Springer: Cham, Switzerland, 2019.
4. Malinowska, A.B.; Torres, D.F.M. *Quantum Variational Calculus*; SpringerBriefs in Electrical and Computer Engineering; Springer: Cham, Switzerland, 2014.
5. Malinowska, A.B.; Odzijewicz, T.; Torres, D.F.M. *Advanced Methods in the Fractional Calculus of Variations*; SpringerBriefs in Applied Sciences and Technology; Springer: Cham, Switzerland, 2015.
6. Agarwal, P.; Nieto, J.J.; Ruzhansky, M.; Torres, D.F.M. *Analysis of Infectious Disease Problems (COVID-19) and Their Global Impact*; Infosys Science Foundation Series in Mathematical Sciences; Springer: Singapore, 2021.
7. Agarwal, P.; Nieto, J.J.; Torres, D.F.M. *Mathematical Analysis of Infectious Diseases*; Academic Press: London, UK, 2022.
8. Tchemisova, T.V.; Torres, D.F.M.; Plakhov, A. *Dynamic Control and Optimization*; Springer Nature Switzerland AG: Cham, Switzerland, 2022.