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Exponential Stability of Hopfield Neural Network Model with Non-Instantaneous Impulsive Effects

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Abstract: We introduce a non-instantaneous impulsive Hopfield neural network model in this paper. Firstly, we prove the existence and uniqueness of an almost periodic solution of this model. Secondly, we prove that the solution of this model is exponentially stable. Finally, we give an example of this model.

Keywords: non-instantaneous impulsive; Hopfield neural network; almost periodic; exponentially stable

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1. Introduction

It is well known that neural network models have many applications in the area of parallel computing, associative memory, pattern recognition, computer vision etc. [1–5]. Therefore, more and more experts and scholars pay attention to neural network models.

The studies on neurocomputing have been improved very fast after the work of McCulloch et al. [6]. One of the neural networks model was given by Hopfield [7,8]. In the actual situation, system can be affected by short-term fluctuations in the environment. Impulses are commonly used to describe this phenomenon. For instance, according to Arbib [9] and Haykin [10], when a stimulus from the body or the external environment is received by receptors, the electrical impulses will be conveyed to the neural net and impulsive effects arise naturally in the net. Stamova and Stamov [11] proposed a Hopfield neural network with impulsive effects at fixed moments as follows

$$\begin{cases} \dot{w}_i(\iota) = \sum\limits_{j=1}^{n} a_{ij}(\iota)w_j(\iota) + \sum\limits_{j=1}^{n} b_{ij}(\iota)f_j(w_j(\iota)) + g_i(\iota),\ \iota \neq \iota_k, \\ \Delta w(\iota_k) = B_k w(\iota_k) + C_k(w(\iota_k)) + h_k,\ k \in \mathbb{N}_+, \end{cases} \tag{1}$$

where $\iota \in \mathbb{J} := \{0\} \bigcup \mathbb{R}_+$, $\mathbb{R}_+ := \{b | b \text{ is a positive real number}\}$, ι_i $(0 \leq \iota_1 < \iota_2 < \cdots)$ stand for the times which are impulses, a_{ij}, b_{ij}, $g_i \in C(\mathbb{J}, \mathbb{R})$, $f_j \in C(\mathbb{J}, \mathbb{R})$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$, $C(\mathbb{J}, \mathbb{R})$ is the space of all the continuous functions from $\mathbb{J}$ to $\mathbb{R}$, $w(\iota) = col(w_1(\iota), w_2(\iota), \cdots, w_n(\iota))$, $\Delta w(\iota_k) = w(\iota_k^+) - w(\iota_k^-)$, $w(\iota_k^+)$ is the right limits of $w(\iota_k)$ and $w(\iota_k^-)$ is the left limits of $w(\iota_k)$, $B_k \in \mathbb{R}^{n\times n}$, $C_k \in C(\mathbb{R}_+^n, \mathbb{R}^n)$, $\mathbb{R}_+^n = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n | x_i > 0,\ i = 1, 2, \cdots, n\}$, where $\mathbb{R}^n$, $n \in \mathbb{N}$ is n-dimensional Euclidean space, $h_k \in \mathbb{R}^n$, $k \in \mathbb{N}_+$, $\mathbb{N} := \{0, 1, 2, 3, \cdots\}$ and $\mathbb{N}_+ := \mathbb{N}/\{0\}$.

However, most systems do not return to normal immediately after the impulse [12]. The system stays active for a limited period of time. Therefore, Hernández et al. [13] firstly introduced the theory of non-instantaneous impulses and established the existence of solutions for a class of impulsive differential equations. After that, Wang et al. [14–16] generalized this model and carried out more in-depth research on non-instantaneous impulsive differential equations. In general, there are no impulses that happen instantaneously, that is to say, it is non-instantaneous, even if the event occurs over a short period of time. Non-instantaneous impulsive effects exist in Hopfield neural network. For instance, in implementation of electronic networks, the state of the network is often subject to non-instantaneous perturbations, which may be caused by noise instances. Moreover, many evolutionary processes, particularly some biological systems, such as biological neural networks and bursting rhythm models in pathology, might exhibit non-instantaneous impulsive effects as well. Therefore, it is beneficial to study a class of differential equations with non-instantaneous impulses.

Then, we consider the case of the model (1) with non-instantaneous impulses as follows

$$\begin{cases} \dot{w}_i(\iota) = \sum\limits_{j=1}^{n} a_{ij}(\iota)w_j(\iota) + \sum\limits_{j=1}^{n} b_{ij}(\iota)f_j(w_j(\iota)) + g_i(\iota),\ \iota \in (l_k, m_{k+1}],\ k \in \mathbb{N}, \\ w(m_k^+) = B_k w(m_k^-) + C_k(w(m_k^-)) + h_k,\ k \in \mathbb{N}_+, \\ w(\iota) = B_k w(m_k^-) + C_k(w(m_k^-)) + h_k,\ \iota \in (m_k, l_k],\ k \in \mathbb{N}_+, \\ w(l_k^+) = w(l_k^-),\ k \in \mathbb{N}_+, \end{cases} \tag{2}$$

where $0 = l_0 < m_1 < l_1 < m_2 < l_2 < \cdots < m_k < l_k < m_{k+1} < \cdots$. The solution $w(\iota) = w(\iota; \iota_0, w_0)$ of model (2) with the initial condition $w(\iota_0^+) = w_0 \in \mathbb{R}_+^n$, $\iota_0 \in \mathbb{J}$ is a piecewise continuous function with points of discontinuity of the first kind at the moments m_k, $k \in \mathbb{N}_+$, at which it is continuous from the left.

Periodic phenomenon is one of the phenomena widely existing in nature [17]. But many motion processes in the present world are approximate to periodic instead of strictly periodicity. Therefore, Danish mathematicians Bohr [18] first proposed the concept of almost periodic (AP), which is a significant generalization for practical application. Many scholars have demonstrated that it is more realistic to adopt an AP hypothesis in the process of AP study, when taking into account the impact of environmental factors, and this has certain ergodicity [19–25].

The rest of this paper is arranged as follows. In Section 2, we provide some of the necessary preliminaries for this paper. In Section 3, we prove the existence, uniqueness and exponential stability of the AP solution to (2). In Section 4, we present an Example to support our theoretical results.

2. Preliminaries

For the sequences $\{m_k\}$ and $\{l_k\}$, $k \in \mathbb{N}_+$, assume that $\lim\limits_{k\to+\infty} m_k = +\infty$, $\lim\limits_{k\to+\infty} l_k = +\infty$. Let the norm $\|j(\iota)\| = \max\{|j_1(\iota)|, |j_2(\iota)|, \cdots, |j_n(\iota)|\}$ for $j(\iota) = (j_1(\iota), j_2(\iota), \cdots, j_n(\iota))^\top$. The space $PC([0,\infty), \mathbb{R}^n) := \{w : [0,\infty) \to \mathbb{R}^n : w \in C((m_i, m_{i+1}], \mathbb{R}^n),\ w(m_i^-) = w(m_i),\ w(m_i^+)$ exist for any $i \in \mathbb{N}\}$ endowed with norm $\|w\|_{PC} = \sup\limits_{\iota\in[0,\infty)} \|w(\iota)\|$, where $C((m_i, m_{i+1}], \mathbb{R}^n)$ represents the space which is made up of all the continuous functions from $(m_i, m_{i+1}]$ to $\mathbb{R}^n$. It is obvious that $(PC([0,\infty), \mathbb{R}^n), \|\cdot\|_{PC})$ is a Banach space.

Definition 1 (see [26]). *For the sequences $\{M_i\}_{i\in\mathbb{N}_+}$, $M_i \in \mathbb{R}^n$, if for any $i \in \mathbb{N}_+$ there exist $\varepsilon > 0$ and integer p such that the following inequality hold*

$$\|M_{i+p} - M_i\| < \varepsilon, \tag{3}$$

then p is called to be ε-AP of the $\{M_i\}_{i\in\mathbb{N}_+}$, $M_i \in \mathbb{R}^n$.

Definition 2 (see [27]). *$\{M_i\}_{i\in\mathbb{N}_+}$, $M_i \in \mathbb{R}^n$ are called to be AP sequences if for any $\varepsilon > 0$, there exists a relatively dense set of its ε-AP.*

Definition 3 (see [26]). *The $w \in PC([0,\infty),\mathbb{R}^n)$ is called an AP function if all of the conditions are satisfied as follows*

(i) $\{m_i^j\}$, $i,j \in \mathbb{N}_+$ *are uniformly AP sequences, where* $m_i^j = m_{i+j} - m_i$.
(ii) *For any* $\varepsilon > 0$, *there exists a number* $\delta = \delta(\varepsilon)$ *which is positive, such that if* ι_1 *and* ι_2 *are the points in the same continuous interval and* $|\iota_1 - \iota_2| < \delta$, *then* $\|w(\iota_1) - w(\iota_2)\| < \varepsilon$.
(iii) *For any* $\varepsilon > 0$, *there exists a relatively dense set* Γ *of* $\varepsilon-AP$, *such that if* $\vartheta \in \Gamma$, *then* $\|w(\iota+\vartheta) - w(\iota)\| < \varepsilon$ *for all* $\iota \in [0,\infty)$ *satisfying the condition* $|\iota - m_i| > \varepsilon$, $i \in \mathbb{N}_+$.

Together with model (2), we shall consider the linear model

$$\begin{cases} \dot{w}(\iota) = A(\iota)w(\iota),\ \iota \in (l_k, m_{k+1}],\ k \in \mathbb{N}, \\ w(m_k^+) = B_k w(m_k^-),\ k \in \mathbb{N}_+, \\ w(\iota) = B_k w(m_k^-),\ \iota \in (m_k, l_k],\ k \in \mathbb{N}_+, \\ w(l_k^+) = w(l_k^-),\ k \in \mathbb{N}_+, \end{cases} \tag{4}$$

where $A(\iota) = (a_{ij})$, $i = 1,2,3,\cdots,n$, $j = 1,2,\cdots,n$.

Let $w(\iota) = \mathcal{W}(\iota,\iota_0)w_{\iota_0}$, $0 \le \iota_0 \le \iota$ represents the solution of (4) with $w(\iota_0) = w_{\iota_0}$, where $\mathcal{W}(\iota,\iota_0)$ is the Cauchy matrix of model (4) which can be looked up on [28].

We propose some assumptions as follows.

(H_1) The sequences $\{l_k^\tau\}$, $l_k^\tau = l_{k+\tau} - l_k$ and $\{m_k^\tau\}$, $m_k^\tau = m_{k+\tau} - m_k$, $k,\tau \in \mathbb{N}_+$ are uniformly AP and $0 < l_k - m_k \le \theta < +\infty$, $0 < \varsigma \le m_{k+1} - l_k \le \bar{\theta} < +\infty$, $k \in \mathbb{N}_+$.
(H_2) The matrix function $A \in C(\mathbb{J},\mathbb{R}^{n\times n})$ is AP in the sense of Bohr.
(H_3) The sequence $\{B_k\}$, $k \in \mathbb{N}_+$ is AP.
(H_4) The functions $f_j(\iota)$ are AP in the sense of Bohr, and

$$0 < \sup_{\iota\in\mathbb{J}} |f_j(\iota)| < \infty,\ f_j(0) = 0,$$

and there exists an $L_1 > 0$ such that for ι, $s \in \mathbb{R}$,

$$\max_{j=1,2,3,\cdots,n} |f_j(\iota) - f_j(s)| < L_1|\iota - s|.$$

(H_5) The functions $b_{ij}(\iota)$ are AP in the sense of Bohr, and

$$0 < \sup_{\iota\in\mathbb{J}} |b_{ij}(\iota)| = \bar{b}_{ij} < \infty.$$

(H_6) The functions $g_i(\iota)$, $i = 1,2,3,\cdots,n$, are AP in the sense of Bohr, the sequences $\{h_k\}$, $k \in \mathbb{N}_+$ are AP and there exists a $\mathbf{C} > 0$ such that

$$\max\left\{\|g\|_{PC},\ \sup_{k\in\mathbb{N}_+}\|h_k\|\right\} \le \mathbf{C},$$

where $g(\iota) = (g_1(\iota),\ g_2(\iota),\ \cdots,\ g_n(\iota))$.
(H_7) The sequence of functions $\{C_k(x)\}$, $k \in \mathbb{N}_+$ is AP uniformly with respect to $x \in \mathbb{R}_+^n$, and there exists an $L_2 > 0$ such that

$$\|C_k(x) - C_k(y)\| \le L_2\|x - y\|,$$

for $k \in \mathbb{N}_+$, $x,y \in \mathbb{R}^n$. $C_k(x) = x$ if and only if $x = (0,0,\cdots,0)$.

Now, we need the following Lemmas.

Lemma 4 (see [28]). *Assume that* (H_1)–(H_3) *hold. Then, for the Cauchy matrix* $\mathcal{W}(\iota,\iota_0)$ *of model* (4) *there exist positive constants* $\mathbf{K} \geq 0$ *and* $\Upsilon > 0$ *such that*

$$\|\mathcal{W}(\iota,\iota_0)\| \leq \mathbf{K}e^{-\Upsilon(\iota-\iota_0)},\ 0 \leq \iota_0 \leq \iota.$$

Lemma 5 (see [28]). *For any* $\varepsilon > 0$, $0 \leq \iota_0 < \iota$, $|\iota - m_i| > \varepsilon$, $|\iota - l_i| > \varepsilon$, $|\iota_0 - m_i| > \varepsilon$ *and* $|\iota_0 - l_i| > \varepsilon$, $i \in \mathbb{N}_+$, *there exist a constant* $K > 0$ *and a relatively dense set of* Γ *of* ε*-AP such that*

$$\|\mathcal{W}(\iota + r, \iota_0 + r) - \mathcal{W}(\iota,\iota_0)\| \leq \varepsilon K e^{-\frac{1}{2}\Upsilon(\iota-\iota_0)},\ r \in \Gamma.$$

Lemma 6 (see [11]). *Let conditions* (H_1)–(H_6) *hold. Then for each* $\varepsilon > 0$, *there exist* ε_1, $0 < \varepsilon_1 < \varepsilon$, *a relatively dense set* Γ *of real numbers and a set* Q *of integers such that the following relations are fulfilled.*

(a) $\|A(\iota + r) - A(\iota)\| < \varepsilon,\ \iota \in \mathbb{J},\ r \in \Gamma$;
(b) $|b_{ij}(\iota + r) - b_{ij}(\iota)| < \varepsilon,\ \iota \in \mathbb{J},\ r \in \Gamma,\ i, j = 1, 2, 3, \cdots, n$;
(c) $|f_j(\iota + r) - f_j(\iota)| < \varepsilon,\ \iota \in \mathbb{J},\ r \in \Gamma,\ j = 1, 2, 3, \cdots, n$;
(d) $|g_j(\iota + r) - g_j(\iota)| < \varepsilon,\ \iota \in \mathbb{J},\ r \in \Gamma,\ j = 1, 2, 3, \cdots, n$;
(e) $\|B_{k+q} - B_k\| < \varepsilon,\ q \in Q,\ k \in \mathbb{R}_+$;
(f) $|h_{k+q} - h_k| < \varepsilon, q \in Q,\ k \in \mathbb{R}_+$;
(g) $|l_k^q - r| < \varepsilon_1,\ |m_k^q - r| < \varepsilon_1,\ q \in Q,\ r \in \Gamma,\ k \in \mathbb{N}_+$.

Lemma 7 (see [26]). *If the sequences* $\{m_i^j\}$, $i, j \in \mathbb{N}$ *are uniformly AP, then we can get*

(i) *There exists a constant* $\rho > 0$ *such that* $\sup\limits_{t\to+\infty} \frac{\mu(\iota+t,\iota)}{t} = \rho$ *which is uniformly with respect to* $\iota > 0$.
(ii) *For any* $p > 0$, *there exists* N *which is a positive integer such that the number of elements in the sequences* $\{m_i\}$ *on each interval of length* p *does not exceed* N. *We can choose* $N \geq \rho$.

3. Main Results

Theorem 8. *Assume that conditions* (H_1)–(H_7) *are satisfied, model* (2) *has a unique positive AP solution if*

$$\mathbf{K}\left\{\frac{L_1}{\Upsilon}\max_{i=1,2,\cdots,n}\sum_{j=1}^{n}\bar{b}_{ij} + L_2 N_1\right\} < 1.$$

Proof. Let $N_1 = \sup\limits_{\iota\in\mathbb{J}} \sum\limits_{k=1}^{\mu(\iota,0)} e^{-\Upsilon(\iota - m_k^+)}$, $N_2 = \sup\limits_{\iota\in\mathbb{J}} \sum\limits_{k=1}^{\mu(\iota,0)} e^{-\frac{1}{2}\Upsilon(\iota - m_k^+)}$, $\Omega := \{w \in PC(\mathbb{J}, \mathbb{R}_+^n)$, w is AP ($\|w(\cdot + r) - w(\cdot)\| < \varepsilon$, $r \in \Gamma$) and $\|w\|_{PC} \leq \aleph\}$, where Γ is mentioned in Lemma 5.

For $l_k < \iota < m_{k+1}$, $k \in \mathbb{N}$, let

$$\begin{aligned}\varphi &= \sum_{k=0}^{\mu(\iota,0)-1}\int_{l_k}^{m_{k+1}} \mathcal{W}(\iota,u)g(u)du + \int_{l_{\mu(\iota,0)}}^{\iota} \mathcal{W}(\iota,u)g(u)du \\ &\quad + \sum_{k=1}^{\mu(\iota,0)} \mathcal{W}(\iota, m_k^+)h_k.\end{aligned}$$

Then,

$$\begin{aligned}
\|\varphi\|_{PC} &\leq \sup_{\iota\in\mathbb{J}}\left\{\sum_{k=0}^{\mu(\iota,0)-1}\int_{l_k}^{m_{k+1}}\|\mathcal{W}(\iota,u)\|\|g(u)\|du+\int_{l_{\mu(\iota,0)}}^{\iota}\|\mathcal{W}(\iota,u)\|\|g(u)\|du\right.\\
&\quad\left.+\sum_{k=1}^{\mu(\iota,0)}\|\mathcal{W}(\iota,m_k^+)\|\|h_k\|\right\}\\
&\leq \sup_{\iota\in\mathbb{J}}\left\{\int_0^{\iota}\|\mathcal{W}(\iota,u)\|\|g(u)\|du+\sum_{k=1}^{\mu(\iota,0)}\|\mathcal{W}(\iota,m_k^+)\|\|h_k\|\right\}\\
&\leq \sup_{\iota\in\mathbb{J}}\left\{\max_{i=1,2,\cdots,n}\int_0^{\iota}\|\mathcal{W}(\iota,u)\||g_i(u)|du+\sum_{k=1}^{\mu(\iota,0)}\|\mathcal{W}(\iota,m_k^+)\|\|h_k\|\right\}\\
&\leq \sup_{\iota\in\mathbb{J}}\left\{\int_0^{\iota}\mathbf{K}e^{-\Upsilon(\iota-u)}\mathbf{C}du+\sum_{k=1}^{\mu(\iota,0)}\mathbf{K}e^{-\Upsilon(\iota-m_k^+)}\mathbf{C}\right\}\\
&\leq \frac{\mathbf{KC}}{\Upsilon}+\mathbf{K}N_1\mathbf{C}\\
&\leq \mathbf{KC}\left(\frac{1}{\Upsilon}+N_1\right)=\aleph. \qquad (5)
\end{aligned}$$

Let $r\in\Gamma$, $q\in Q$, where the sets Γ and Q are determined in Lemma 6. Then,

$$\begin{aligned}
\sup_{\iota\in\mathbb{J}}\|\varphi(\iota+r)-\varphi(\iota)\| &\leq \sup_{\iota\in\mathbb{J}}\left\{\sum_{k=0}^{\mu(\iota,0)-1}\int_{l_k}^{m_{k+1}}\|\mathcal{W}(\iota+r,u+r)-\mathcal{W}(\iota,u)\|\|g(u+r)\|du\right.\\
&\quad+\sum_{k=0}^{\mu(\iota,0)-1}\int_{l_k}^{m_{k+1}}\|\mathcal{W}(\iota,u)\|\|g(u+r)-g(u)\|du\\
&\quad+\int_{l_{\mu(\iota,0)}}^{\iota}\|\mathcal{W}(\iota+r,u+r)-\mathcal{W}(\iota,u)\|\|g(u+r)\|du\\
&\quad+\int_{l_{\mu(\iota,0)}}^{\iota}\|\mathcal{W}(\iota,u)\|\|g(u+r)-g(u)\|du\\
&\quad+\sum_{k=1}^{\mu(\iota,0)}\|\mathcal{W}(\iota+r,m_{k+q}^+)-\mathcal{W}(\iota,m_k^+)\|\|h_{k+q}\|\\
&\quad\left.+\sum_{k=1}^{\mu(\iota,0)}\|\mathcal{W}(\iota,m_k^+)\|\|h_{k+q}-h_k\|\right\}\\
&\leq \sup_{\iota\in\mathbb{J}}\left\{\int_0^{\iota}\|\mathcal{W}(\iota+r,u+r)-\mathcal{W}(\iota,u)\|\|g(u+r)\|du\right.\\
&\quad+\int_0^{\iota}\|\mathcal{W}(\iota,u)\|\|g(u+r)-g(u)\|du\\
&\quad+\sum_{k=1}^{\mu(\iota,0)}\|\mathcal{W}(\iota+r,m_{k+q}^+)-\mathcal{W}(\iota,m_k^+)\|\|h_{k+q}\|\\
&\quad\left.+\sum_{k=1}^{\mu(\iota,0)}\|\mathcal{W}(\iota,m_k^+)\|\|h_{k+q}-h_k\|\right\}\\
&\leq \sup_{\iota\in\mathbb{J}}\left\{\int_0^{\iota}\varepsilon Ke^{-\frac{1}{2}\Upsilon(\iota-u)}\mathbf{C}du+\int_0^{\iota}\mathbf{K}e^{-\Upsilon(\iota-u)}\varepsilon du\right.\\
&\quad\left.+\sum_{k=1}^{\mu(\iota,0)}\varepsilon Ke^{-\frac{1}{2}\Upsilon(\iota-m_k^+)}\mathbf{C}+\sum_{k=1}^{\mu(\iota,0)}\mathbf{K}e^{-\Upsilon(\iota-m_k^+)}\varepsilon\right\}\\
&\leq \varepsilon K\mathbf{C}\frac{2}{\Upsilon}+\mathbf{K}\varepsilon\frac{1}{\Upsilon}+\varepsilon KN_2\mathbf{C}+\mathbf{K}\varepsilon N_1\\
&\leq \varepsilon\left(K\mathbf{C}\frac{2}{\Upsilon}+\mathbf{K}\frac{1}{\Upsilon}+KN_2\mathbf{C}+\mathbf{K}N_1\right).
\end{aligned}$$

Set

$$F(\iota,w)=col\{F_1(\iota,w),\,F_2(\iota,w),\,\cdots,\,F_n(\iota,w)\},$$

where

$$F_i(\iota,w)=\sum_{j=1}^{n}b_{ij}(\iota)f_j(w_j(\iota)),\; i=1,\,2,\,3,\,\cdots,\,n.$$

We define in Ω an operator T,

$$\begin{aligned} T_w &= \sum_{k=0}^{\mu(\iota,0)-1} \int_{l_k}^{m_{k+1}} \mathcal{W}(\iota,u)(F(u,w(u))+g(u))du \\ &\quad + \int_{l_{\mu(\iota,0)}}^{\iota} \mathcal{W}(\iota,u)(F(u,w(u))+g(u))du \\ &\quad + \sum_{k=1}^{\mu(\iota,0)} \mathcal{W}(\iota,m_k^+)(C_k(w(m_k^-))+h_k) \end{aligned} \tag{6}$$

and consider a subset $\hat{\Omega} \subset \Omega$, where

$$\hat{\Omega} = \left\{ w \in \Omega : \|w-\varphi\|_{PC} \le \frac{R\aleph}{1-R} \right\}.$$

Consequently, for an arbitrary $w \in \hat{\Omega}$ from (5) and (6) it follows that

$$\|w\|_{PC} \le \|w-\varphi\|_{PC} + \|\varphi\|_{PC} \le \frac{R\aleph}{1-R} + \aleph = \frac{\aleph}{1-R}.$$

Now, we prove that T is a self-mapping from $\hat{\Omega}$ to $\hat{\Omega}$.
For $w \in \hat{\Omega}$ we have

$$\begin{aligned} & \|T_w-\varphi\|_{PC} \\ \le\ & \sup_{\iota\in\mathbb{J}} \Bigg\{ \sum_{k=0}^{\mu(\iota,0)-1} \int_{l_k}^{m_{k+1}} \|\mathcal{W}(\iota,u)\|\|F(u,w(u))\|du \\ & + \int_{l_{\mu(\iota,0)}}^{\iota} \|\mathcal{W}(\iota,u)\|\|F(u,w(u))\|du \\ & + \sum_{k=1}^{\mu(\iota,0)} \|\mathcal{W}(\iota,m_k^+)\|\|C_k(w(m_k^-))\| \Bigg\} \\ \le\ & \sup_{\iota\in\mathbb{J}} \Bigg\{ \int_0^{\iota} \|\mathcal{W}(\iota,u)\|\|F(u,w(u))\|du \\ & + \sum_{k=1}^{\mu(\iota,0)} \|\mathcal{W}(\iota,m_k^+)\|\|C_k(w(m_k^-))\| \Bigg\} \\ \le\ & \sup_{\iota\in\mathbb{J}} \Bigg\{ \max_{i=1,2,\cdots,n} \int_0^{\iota} \|\mathcal{W}(\iota,u)\| \sum_{j=1}^{n} \|b_{ij}(\iota)\|\|f_j(w_j(u)\|du \\ & + \sum_{k=1}^{\mu(\iota,0)} \|\mathcal{W}(\iota,m_k^+)\|\|C_k(w(m_k^-))\| \Bigg\} \\ \le\ & \sup_{\iota\in\mathbb{J}} \Bigg\{ \max_{i=1,2,\cdots,n} \int_0^{\iota} \|\mathcal{W}(\iota,u)\| \sum_{j=1}^{n} \bar{b}_{ij} L_1 \|w(u)\|du \\ & + \sum_{k=1}^{\mu(\iota,0)} \|\mathcal{W}(\iota,m_k^+)\| L_2 \|w(m_k^-)\| \Bigg\} \\ \le\ & \Bigg\{ \max_{i=1,2,\cdots,n} \int_0^{\iota} \mathbf{K} e^{-\Upsilon(\iota-u)} L_1 \sum_{j=1}^{n} \bar{b}_{ij} du \\ & + \sum_{k=1}^{\mu(\iota,0)} \mathbf{K} e^{-\Upsilon(\iota-m_k^+)} L_2 \Bigg\} \|w\|_{PC} \\ \le\ & \mathbf{K} \Bigg\{ \max_{i=1,2,\cdots,n} \frac{L_1}{\Upsilon} \sum_{j=1}^{n} \bar{b}_{ij} + N_1 L_2 \Bigg\} \|w\|_{PC} \\ =\ & R\|w\|_{PC} \le \frac{R\aleph}{1-R}. \end{aligned} \tag{7}$$

Let $r \in \Gamma$, $q \in Q$, where the sets Γ and Q are determined in Lemma 6. Then

$$
\begin{aligned}
& \|T_w(\iota+r) - T_w(\iota)\| \\
\le\ & \sup_{\iota\in\mathbb{J}} \|(T_w(\iota+r) - T_w(\iota)) - (\varphi(\iota+r) - \varphi(\iota))\| + \sup_{\iota\in\mathbb{J}} \|\varphi(\iota+r) - \varphi(\iota)\| \\
\le\ & \sup_{\iota\in\mathbb{J}} \|(T_w(\iota+r) - \varphi(\iota+r)) - (T_w(\iota) - \varphi(\iota))\| + \sup_{\iota\in\mathbb{J}} \|\varphi(\iota+r) - \varphi(\iota)\| \\
\le\ & \sup_{\iota\in\mathbb{J}} \Bigg\{ \sum_{k=0}^{\mu(\iota,0)-1} \int_{l_k}^{m_{k+1}} \|\mathcal{W}(\iota+r, u+r) - \mathcal{W}(\iota,u)\| \|F(u+r, w(u+r))\| du \\
& + \sum_{k=0}^{\mu(\iota,0)-1} \int_{l_k}^{m_{k+1}} \|\mathcal{W}(\iota,u)\| \|F(u+r, w(u+r)) - F(u, w(u))\| du \\
& + \int_{l_{\mu(\iota,0)}}^{\iota} \|\mathcal{W}(\iota+r, u+r) - \mathcal{W}(\iota,u)\| \|F(u+r, w(u+r))\| du \\
& + \int_{l_{\mu(\iota,0)}}^{\iota} \|\mathcal{W}(\iota,u)\| \|F(u+r, w(u+r)) - F(u, w(u))\| du \\
& + \sum_{k=1}^{\mu(\iota,0)} \|\mathcal{W}(\iota+r, m_{k+q}^+) - \mathcal{W}(\iota, m_k^+)\| \|C_{k+q}(w(m_{k+q}^-))\| \\
& + \sum_{k=1}^{\mu(\iota,0)} \|\mathcal{W}(\iota, m_k^+)\| \|C_{k+q}(w(m_{k+q}^-)) - C_k(w(m_k^-))\| \Bigg\} \\
& + \sup_{\iota\in\mathbb{J}} \|\varphi(\iota+r) - \varphi(\iota)\| \\
\le\ & \sup_{\iota\in\mathbb{J}} \Bigg\{ \int_0^{\iota} \|\mathcal{W}(\iota+r, u+r) - \mathcal{W}(\iota,u)\| \|F(u+r, w(u+r))\| du \\
& + \int_0^{\iota} \|\mathcal{W}(\iota,u)\| \|F(u+r, w(u+r)) - F(u, w(u))\| du \\
& + \sum_{k=1}^{\mu(\iota,0)} \|\mathcal{W}(\iota+r, m_{k+q}^+) - \mathcal{W}(\iota, m_k^+)\| \|C_{k+q}(w(m_{k+q}^-))\| \\
& + \sum_{k=1}^{\mu(\iota,0)} \|\mathcal{W}(\iota, m_k^+)\| \|C_{k+q}(w(m_{k+q}^-)) - C_k(w(m_k^-))\| \Bigg\} \\
& + \sup_{\iota\in\mathbb{J}} \|\varphi(\iota+r) - \varphi(\iota)\| \\
\le\ & \sup_{\iota\in\mathbb{J}} \Bigg\{ \max_{i=1,2,\cdots,n} \Bigg(\int_0^{\iota} \|\mathcal{W}(\iota+r, u+r) - \mathcal{W}(\iota,u)\| \Bigg| \sum_{j=1}^{n} b_{ij}(u+r) f_j(w_j(u+r)) \Bigg| du \\
& + \int_0^{\iota} \|\mathcal{W}(\iota,u)\| \Bigg| \sum_{j=1}^{n} b_{ij}(u+r) f_j(w_j(u+r)) - \sum_{j=1}^{n} b_{ij}(u) f_j(w_j(u)) \Bigg| du \Bigg) \\
& + \sum_{k=1}^{\mu(\iota,0)} \|\mathcal{W}(\iota+r, m_{k+q}^+) - \mathcal{W}(\iota, m_k^+)\| \|C_{k+q}(w(m_{k+q}^-))\| \\
& + \sum_{k=1}^{\mu(\iota,0)} \|\mathcal{W}(\iota, m_k^+)\| \|C_{k+q}(w(m_{k+q}^-)) - C_k(w(m_k^-))\| \Bigg\} \\
& + \sup_{\iota\in\mathbb{J}} \|\varphi(\iota+r) - \varphi(\iota)\|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \sup_{\iota\in\mathbb{J}}\Bigg\{\max_{i=1,2,\cdots,n}\Bigg(\int_0^{\iota}\|\mathcal{W}(\iota+r,u+r)-\mathcal{W}(\iota,u)\|\Bigg|\sum_{j=1}^{n}b_{ij}(u+r)f_j(w_j(u+r))\Bigg|du \\
&+\int_0^{\iota}\|\mathcal{W}(\iota,u)\|\Bigg|\sum_{j=1}^{n}(b_{ij}(u+r)-b_{ij}(u))f_j(w_j(u+r)) \\
&+\sum_{j=1}^{n}b_{ij}(u)(f_j(w_j(u+r))-f_j(w_j(u)))\Bigg|du\Bigg) \\
&+\sum_{k=1}^{\mu(\iota,0)}\|\mathcal{W}(\iota+r,m_{k+q}^{+})-\mathcal{W}(\iota,m_k^{+})\|\|C_{k+q}(w(m_{k+q}^{-}))\| \\
&+\sum_{k=1}^{\mu(\iota,0)}\|\mathcal{W}(\iota,m_k^{+})\|\|C_{k+q}(w(m_{k+q}^{-}))-C_k(w(m_k^{-}))\|\Bigg\} \\
&+\sup_{\iota\in\mathbb{J}}\|\varphi(\iota+r)-\varphi(\iota)\| \\
\leq\ & \sup_{\iota\in\mathbb{J}}\Bigg\{\max_{i=1,2,\cdots,n}\Bigg(\int_0^{\iota}\varepsilon K e^{-\frac{1}{2}\Upsilon(\iota-u)}\sum_{j=1}^{n}\bar{b}_{ij}L_1|w_j(u+r))|du \\
&+\int_0^{\iota}\mathbf{K}e^{-\Upsilon(\iota-u)}\Bigg(\sum_{j=1}^{n}\varepsilon L_1|w_j(u+r)|+\sum_{j=1}^{n}\bar{b}_{ij}L_1|w_j(u+r)-w_j(u)|\Bigg)du\Bigg) \\
&+\sum_{k=1}^{\mu(\iota,0)}\varepsilon K e^{-\frac{1}{2}\Upsilon(\iota-m_k^{+})}L_2\|w(m_{k+q}^{-})\| \\
&+\sum_{k=1}^{\mu(\iota,0)}\mathbf{K}e^{-\Upsilon(\iota-m_k^{-})}L_2\|w(m_{k+q}^{-})-w(m_k^{-})\|\Bigg\} \\
&+\sup_{\iota\in\mathbb{J}}\|\varphi(\iota+r)-\varphi(\iota)\| \\
\leq\ & \sup_{\iota\in\mathbb{J}}\Bigg\{\max_{i=1,2,\cdots,n}\Bigg(\frac{\varepsilon 2K}{\Upsilon}\sum_{j=1}^{n}\bar{b}_{ij}L_1\frac{\aleph}{1-R}+\frac{\mathbf{K}}{\Upsilon}\Bigg(\varepsilon L_1\frac{\aleph}{1-R}+\sum_{j=1}^{n}\bar{b}_{ij}L_1\varepsilon\Bigg)\Bigg) \\
&+\sum_{k=1}^{\mu(\iota,0)}\varepsilon K e^{-\frac{1}{2}\Upsilon(\iota-m_k^{+})}L_2\frac{\aleph}{1-R}+\sum_{k=1}^{\mu(\iota,0)}\mathbf{K}e^{-\Upsilon(\iota-m_k^{-})}L_2\varepsilon\Bigg\} \\
&+\sup_{\iota\in\mathbb{J}}\|\varphi(\iota+r)-\varphi(\iota)\| \\
\leq\ & \varepsilon\Bigg\{\frac{L_1}{\Upsilon}\Bigg(\max_{i=1,2,\cdots,n}\Bigg(\sum_{j=1}^{n}2K\bar{b}_{ij}\frac{\aleph}{1-R}+\mathbf{K}\frac{\aleph}{1-R}+\mathbf{K}\sum_{j=1}^{n}\bar{b}_{ij}\Bigg)+KL_2N_2\frac{\aleph}{1-R}+\mathbf{K}L_2N_1\Bigg\} \\
&+\varepsilon\Bigg(K\mathbf{C}\frac{2}{\Upsilon}+\mathbf{K}\frac{1}{\Upsilon}+KN_2\mathbf{C}+\mathbf{K}N_1\Bigg) \\
\leq\ & \varepsilon\Bigg\{\frac{L_1}{\Upsilon}\Bigg(\max_{i=1,2,\cdots,n}\Bigg(\sum_{j=1}^{n}2K\bar{b}_{ij}\frac{\aleph}{1-R}+\mathbf{K}\frac{\aleph}{1-R}+\mathbf{K}\sum_{j=1}^{n}\bar{b}_{ij}\Bigg)+KL_2N_2\frac{\aleph}{1-R}+\mathbf{K}L_2N_1 \\
&+K\mathbf{C}\frac{2}{\Upsilon}+\mathbf{K}\frac{1}{\Upsilon}+KN_2\mathbf{C}+\mathbf{K}N_1\Bigg\}.
\end{aligned}
\tag{8}
$$

Consequently, after (7) and (8), we obtain that $T_w \in \hat{\Omega}$.

Let $\phi \in \hat{\Omega}$, $\xi \in \hat{\Omega}$. Then,

$$
\begin{aligned}
&\|T_\phi - T_\xi\|_{PC} \\
\leq\ & \sup_{\iota \in \mathbb{J}} \left\{ \sum_{k=0}^{\mu(\iota,0)-1} \int_{l_k}^{m_{k+1}} \|\mathcal{W}(\iota,u)\| \|F(u,\phi(u)) - F(u,\xi(u))\| du \right. \\
& + \int_{l_{\mu(\iota,0)}}^{\iota} \|\mathcal{W}(\iota,u)\| \|F(u,\phi(u)) - F(u,\xi(u))\| du \\
& + \sum_{k=1}^{\mu(\iota,0)} \|\mathcal{W}(\iota,m_k^+)\| \|C_k(\phi(m_k^-)) - C_k(\xi(m_k^-))\| \\
\leq\ & \sup_{\iota \in \mathbb{J}} \left\{ \int_0^{\iota} \|\mathcal{W}(\iota,u)\| \|F(u,\phi(u)) - F(u,\xi(u))\| du \right. \\
& \left. + \sum_{k=1}^{\mu(\iota,0)} \|\mathcal{W}(\iota,m_k^+)\| \|C_k(\phi(m_k^-)) - C_k(\xi(m_k^-))\| \right\} \\
\leq\ & \sup_{\iota \in \mathbb{J}} \left\{ \int_0^{\iota} \max_{i=1,2,\cdots,n} \mathbf{K} e^{-\Upsilon(\iota-u)} \sum_{j=1}^{n} \bar{b}_{ij} L_1 du \right. \\
& \left. + \sum_{k=1}^{\mu(\iota,0)} \mathbf{K} e^{-\Upsilon(\iota - m_k^+)} L_2 \right\} \|\phi - \xi\|_{PC} \\
\leq\ & \mathbf{K} \left\{ \frac{L_1}{\Upsilon} \max_{i=1,2,\cdots,n} \sum_{j=1}^{n} \bar{b}_{ij} + L_2 N_1 \right\} \|\phi - \xi\|_{PC}.
\end{aligned} \tag{9}
$$

Then from (9) it follows that T is a contracting operator in $\hat{\Omega}$, and there exists a unique AP solution of (2). □

Theorem 9. *Assume that all conditions in Theorem 8 and*

$$
\mathbf{K} L_1 \max_{i=1,2,\cdots,n} \sum_{j=1}^{n} \bar{b}_{ij} + N \ln(1 + \mathbf{K} L_2) < \Upsilon
$$

hold. Then, the solution of (2) is globally exponentially stable.

Proof. Let now $x(\iota)$ be an arbitrary solution of (2). Then, we obtain

$$
\begin{aligned}
\|w(\iota) - x(\iota)\| \leq\ & \mathbf{K} e^{-\Upsilon(\iota-\iota_0)} \|w(\iota_0) - x(\iota_0)\| \\
& + \int_{\iota_0}^{\iota} \max_{i=1,2,\cdots,n} \mathbf{K} e^{-\Upsilon(\iota-u)} \sum_{j=1}^{n} \bar{b}_{ij} L_1 \|w(u) - x(u)\| du \\
& + \sum_{k=\mu(\iota_0,0)+1}^{\mu(\iota,0)} \mathbf{K} e^{-\Upsilon(\iota-m_k^+)} L_2 \|w(m_k^-) - x(m_k^-)\|.
\end{aligned}
$$

Set $v(\iota) = \|w(\iota) - x(\iota)\|e^{\Upsilon\iota}$, then by means of Gronwall-Bellman's inequality, it follows that

$$\begin{aligned}
\|w(\iota) - x(\iota)\| &\leq \mathbf{K}e^{-\Upsilon(\iota-\iota_0)}\|w(\iota_0) - x(\iota_0)\| \prod_{k=\mu(\iota_0,0)+1}^{\mu(\iota,0)} (1 + \mathbf{K}L_2 e^{-\Upsilon(\iota - m_k^+)}) \\
&\quad e^{\int_{\iota_0}^{\iota} \max\limits_{i=1,2,\cdots,n} \mathbf{K}e^{-\Upsilon(\iota-u)} \sum\limits_{j=1}^{n} \bar{b}_{ij} L_1} \\
&\leq \mathbf{K}\|w(\iota_0) - x(\iota_0)\|(1 + \mathbf{K}L_2)^{\mu(\iota,\iota_0)} e^{-\Upsilon(\iota-\iota_0)} e^{\mathbf{K}L_1 \max\limits_{i=1,2,\cdots,n} \sum\limits_{j=1}^{n} \bar{b}_{ij}(\iota-\iota_0)} \\
&\leq \mathbf{K}\|w(\iota_0) - x(\iota_0)\|(1 + \mathbf{K}L_2)^{\mu(\iota,\iota_0)} e^{\left(-\Upsilon + \mathbf{K}L_1 \max\limits_{i=1,2,\cdots,n} \sum\limits_{j=1}^{n} \bar{b}_{ij}\right)(\iota-\iota_0)} \\
&\leq \mathbf{K}\|w(\iota_0) - x(\iota_0)\| e^{N(\iota-\iota_0)\ln(1+\mathbf{K}L_2)} e^{\left(-\Upsilon + \mathbf{K}L_1 \max\limits_{i=1,2,\cdots,n} \sum\limits_{j=1}^{n} \bar{b}_{ij}\right)(\iota-\iota_0)} \\
&\leq \mathbf{K}\|w(\iota_0) - x(\iota_0)\| e^{(-\Upsilon + \mathbf{K}L_1 \max\limits_{i=1,2,\cdots,n} \sum\limits_{j=1}^{n} \bar{b}_{ij} + N\ln(1+\mathbf{K}L_2))(\iota-\iota_0)}.
\end{aligned}$$

Obviously, if there exists $\mathbf{K}L_1 \max\limits_{i=1,2,\cdots,n} \sum\limits_{j=1}^{n} \bar{b}_{ij} + N\ln(1 + \mathbf{K}L_2) < \Upsilon$, then the solution of (2) is exponentially. □

4. Example

Example 10. *We shall consider the classical model of Hopfield neural networks*

$$\begin{cases}
\dot{w}_i(\iota) = -\dfrac{1}{R_i} w_i(\iota) + \sum\limits_{j=1}^{n} b_{ij} f_j(w_j(\iota)) + g_i(\iota),\ \iota \in (l_k, m_{k+1}],\ k \in \mathbb{N}, \\
w(m_k^+) = Bw(m_k^-) + C_k(w(m_k^-)) + h_k,\ k \in \mathbb{N}_+, \\
w(\iota) = Bw(m_k^-) + C_k(w(m_k^-)) + h_k,\ \iota \in (m_k, l_k],\ k \in \mathbb{N}_+, \\
w(l_k^+) = w(l_k^-), k \in \mathbb{N}_+,
\end{cases} \tag{10}$$

where $\iota \in \mathbb{J}$, $R_i > 0$, $b_{ij} \in \mathbb{R}$, $\gamma_i \in C(\mathbb{J}, \mathbb{R})$, $f_j \in C(\mathbb{R}_+, \mathbb{R})$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$, $x(\iota) = col(x_1(\iota), x_2(\iota), \cdots, x_n(\iota))$, $B = diag[b_i]$, $b_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, $C_k \in C(\mathbb{R}_+^n, \mathbb{R})$, $h_k \in \mathbb{R}^n$.

Let

$$\begin{cases}
\dot{w}_i(\iota) = -\dfrac{1}{R_i} w_i(\iota),\ \iota \in (l_k, m_{k+1}],\ k \in \mathbb{N}, \\
w(m_k^+) = Bw(m_k^-),\ k \in \mathbb{N}_+, \\
w(\iota) = Bw(m_k^-),\ \iota \in (m_k, l_k],\ k \in \mathbb{N}_+, \\
w(l_k^+) = w(l_k^-),\ k \in \mathbb{N}_+,
\end{cases}$$

be the linear part of (10).

The Cauchy matrix $\mathcal{W}(\iota,\iota_0)$ of (10) is in the form

$$\mathcal{W}(\iota,\iota_0)=\begin{cases} B^{\mu(\iota,\iota_0)+1}e^{A(\iota-l_{\mu(\iota,0)})}\prod\limits_{k=\mu(\iota,0)}^{\mu(\iota_0,0)+2}e^{A(m_k-l_{k-1})}e^{A(m_{\mu(\iota_0,0)+1}-\iota_0)}, \\ \iota_0<m_{\mu(\iota_0,0)+1}<\cdots<l_{\mu(\iota,0)}<\iota, \\ B^{\mu(\iota,\iota_0)+1}\prod\limits_{k=\mu(\iota,0)}^{\mu(\iota_0,0)+2}e^{A(m_k-l_{k-1})}e^{A(m_{\mu(\iota_0,0)+1}-\iota_0)}, \\ \iota_0<m_{\mu(\iota_0,0)+1}<\cdots<m_{\mu(\iota,0)}<\iota, \\ B^{\mu(\iota,\iota_0)+1}e^{A(\iota-l_{\mu(\iota,0)})}\prod\limits_{k=\mu(\iota,0)}^{\mu(\iota_0,0)+1}e^{A(m_k-l_{k-1})}, \\ \iota_0<l_{\mu(\iota_0,0)}<\cdots<l_{\mu(\iota,0)}<\iota, \\ B^{\mu(\iota,\iota_0)+1}\prod\limits_{k=\mu(\iota,0)}^{\mu(\iota_0,0)+1}e^{A(m_k-l_{k-1})}, \\ \iota_0<l_{\mu(\iota_0,0)}<\cdots<m_{\mu(\iota,0)}<\iota. \end{cases}$$

Then,

$$\begin{aligned} \|\mathcal{W}(\iota,\iota_0)\| &\le \left\|B^{\mu(\iota,\iota_0)+1}e^{A(\iota-\iota_0)}\right\| \\ &\le \left\|e^{\ln B^{\mu(\iota,\iota_0)+1}}e^{A(\iota-\iota_0)}\right\| \\ &\le \left\|e^{(\mu(\iota,\iota_0)+1)\ln B}e^{A(\iota-\iota_0)}\right\| \\ &\le \left\|e^{\ln B}e^{N(\iota-\iota_0)\ln B}e^{A(\iota-\iota_0)}\right\| \\ &\le \left\|e^{\ln B}e^{(A+N\ln B)(\iota-\iota_0)}\right\| \\ &\le e^{\max\limits_{i=1,2,\cdots,n}\ln b_i}e^{\left(\max\limits_{i=1,2,\cdots,n}-\frac{1}{R_i}+\max\limits_{i=1,2,\cdots,n}N\ln b_i\right)(\iota-\iota_0)} \\ &\le e^{\max\limits_{i=1,2,\cdots,n}\ln b_i}e^{-\left(\min\limits_{i=1,2,\cdots,n}\frac{1}{R_i}-\max\limits_{i=1,2,\cdots,n}N\ln b_i\right)(\iota-\iota_0)}. \end{aligned}$$

Let $\mathbf{K}=\exp\left(\max\limits_{i=1,2,\cdots,n}\ln b_i\right)$, $\Upsilon=\min\limits_{i=1,2,\cdots,n}\frac{1}{R_i}-\max\limits_{i=1,2,\cdots,n}N\ln b_i$, *then we can obtain* $\|\mathcal{W}(\iota,\iota_0)\|\le\mathbf{K}e^{-\Upsilon(\iota-\iota_0)}$.

According to the Theorems 8 and 9, assume that (H_1)–(H_7) are met and the following inequalities hold

$$\begin{aligned} &\mathbf{K}=\exp\left(\max_{i=1,2,\cdots,n}\ln b_i\right), \\ &\Upsilon=\min_{i=1,2,\cdots,n}\frac{1}{R_i}-\max_{i=1,2,\cdots,n}N\ln b_i, \\ &\mathbf{K}\left\{\frac{L_1}{\Upsilon}\max_{i=1,2,\cdots,n}\sum_{j=1}^{n}\bar{b}_{ij}+L_2N_1\right\}<1. \end{aligned}$$

Then, there exists a unique AP solution $w(\iota)$ of (10).

In addition, if the following inequalities hold

$$\mathbf{K}L_1\max_{i=1,2,\cdots,n}\sum_{j=1}^{n}\bar{b}_{ij}+N\ln(1+\mathbf{K}L_2)<\Upsilon,$$

then the solution $w(\iota)$ is globally exponentially stable.

5. Conclusions

Neural network models with impulses can study many phenomena in life. We note that Stamova and Stamov [11] proposed a Hopfield neural network with impulsive effects at fixed moments. We are very interested in this work. After careful reading, we introduced the non-instantaneous impulse factor into this model and proposed a Hopfield neural network non-instantaneous impulsive model. Then, we provided conditions for the existence of a unique AP solution and the exponential stability of the solution for this model.

There are many limitations to our work. It is known that asymptotic stability of solutions to impulsive systems can be treated in both weak (convergence towards the solution depends only on the elapsed time) and strong (convergence depends on the elapsed on the elapsed time and the number of impulses) flavors [29,30]. We deal with the classical weak stability in this paper. Then, we will gradually consider the case of strong stability for the model (2) with non-instantaneous impulses in future.

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References

1. Viviani, P.; Drocco, M.; Baccega, D.; Colonnelli, I.; Aldinucci, M. Deep learning at scale. In Proceedings of the 2019 IEEE 27th Euromicro International Conference on Parallel, Distributed and Network-Based Processing (PDP), Pavia, Italy, 13–15 February 2019; pp. 124–131.
2. Wouafo, H.; Chavet, C.; Coussy, P. Clone-Based encoded neural networks to design efficient associative memories. *IEEE Trans. Neural Netw. Learn. Syst.* **2019**, *30*, 3186–3199. [CrossRef] [PubMed]
3. Allen, F.T.; Kinser, J.M.; Caulfield, H.J. A neural bridge from syntactic to statistical pattern recognition. *Neural Netw.* **1999**, *12*, 519–526. [CrossRef] [PubMed]
4. Kriegeskorte, N. Deep neural networks: A new framework for modeling biological vision and brain information processing. *Annu. Rev. Vis. Sci.* **2015**, *1*, 417–446. [CrossRef] [PubMed]
5. Calise, A.J.; Rysdyk, R.T. Nonlinear adaptive flight control using neural networks. *IEEE Control. Syst. Mag.* **1998**, *18*, 14–25.
6. McCulloch, W.S.; Pits, W. A logical calculus of the ideas immanent in nervous activity. *Bull. Math. Biophys.* **1943**, *5*, 115–133. [CrossRef]
7. Hopfield, J.J. Neural networks and physical systems with emergent collective computational abilities. *Proc. Natl. Acad. Sci. USA* **1982**, *79*, 2554–2558. [CrossRef]
8. Hopfield, J.J. Neurons with graded response have collective computational properties like those of two-state neurons. *Proc. Natl. Acad. Sci. USA* **1984**, *81*, 3088–3092. [CrossRef]
9. Arbib, M.A. *Brains, Machines and Mathematics*; Springer: New York, NY, USA, 1987.
10. Haykin, S. *Neural Networks: A Comprehensive Foundation*; Prentice-Hall: Ehglewood Cliffs, NJ, USA, 1998.
11. Stamova, I.; Stamov, G. *Applied Impulsive Mathematical Models*; Springer International Publishing: Berlin/Heidelberg, Germany, 2016.
12. Li, M.M.; Wang, J.R.; O'Regan, D. Positive almost periodic solution for a noninstantaneous impulsive Lasota-Wazewska model. *Bull. Iran. Math. Soc.* **2019**, *46*, 851–864. [CrossRef]
13. Hernández, E.; O'Regan, D.; Benxax, M.A. On a new class of abstract integral equations and applications. *Appl. Math. Comput.* **2012**, *219*, 2271–2277. [CrossRef]
14. Wang, J.R.; Fečkan, M. A general class of impulsive evolution equations. *Topol. Methods Nonlinear Anal.* **2015**, *46*, 915–933. [CrossRef]
15. Wang, J.R.; Zhou, Y.; Lin, Z. On a new class of impulsive fractional differential equations. *Appl. Math. Comput.* **2014**, *242*, 649–657. [CrossRef]
16. Wang, J.R.; Fečkan, M. *Non-Instantaneous Impulsive Differential Equations*; IOP: London, UK, 2018.

17. Guan, Y.; Fečkan, M.; Wang, J.R. Periodic solutions and Hyers-Ulam stability of atmospheric Ekman flows. *Discret. Contin. Dyn. Syst.* **2021**, *41*, 1157–1176. [CrossRef]
18. Bohr, H. Zur theorie der fast periodischen funktionen: I. eine verallgemeinerung der theorie der fourierreihen. *Acta Math.* **1925**, *45*, 29–127. [CrossRef]
19. Chen, X.X. Almost periodic solutions of nonlinear delay population equation with feedback control. *Nonlinear Anal. Real World Appl.* **2007**, *8*, 62–72.
20. Chen, X.X.; Chen, F.D. Almost-periodic solutions of a delay population equation with feedback control. *Nonlinear Anal. Real World Appl.* **2006**, *7*, 559–571.
21. Zhang, R.; Wang, L. Almost periodic solutions for cellular neural networks with distributed delays. *Acta Math. Sci.* **2011**, *31*, 422–429.
22. Menouer, M.A.; Moussaoui, A.; Dads, E.A. Existence and global asymptotic stability of positive almost periodic solution for a predator-prey system in an artificial lake. *Chaos Solitons Fractals* **2017**, *103*, 271–278. [CrossRef]
23. Zhang, T.; Gan, X. Almost periodic solutions for a discrete fishing model with feedback control and time delays. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 150–163. [CrossRef]
24. Huang, P.; Li, X.; Liu, B. Almost periodic solutions for an asymmetric oscillation. *J. Differ. Equ.* **2017**, *263*, 8916–8946. [CrossRef]
25. Zhou, H.; Wang, W.; Yang, L. Stage-structured hematopoiesis model with delays in an almost periodic environment. *Appl. Math. Lett.* **2021**, *120*, 107336. [CrossRef]
26. Samoilenko, A.M.; Perestyuk, N.A. *Impulsive Differential Equations*; World Scientific: Singapore, 1995.
27. Stamova, I. *Stability Analysis of Impulsive Functional Differential Equations*; Walter de Gruyter: Berlin, Germany, 2009.
28. Ma, R.; Wang, J.R.; Li, M.M. Almost periodic solutions for two non-instantaneous impulsive biological models. *Qual. Theory Dyn. Syst.* **2022**, *21*, 84. [CrossRef]
29. Mancilla-Aguilar, J.L.; Haimovich, H.; Feketa, P. Uniform stability of nonlinear time-varying impulsive systems with eventually uniformly bounded impulse frequency. *Nonlinear Anal. Hybrid Syst.* **2020**, *38*, 100933. [CrossRef]
30. Feketa, P.; Klinshov, V.; Lücken, L. A survey on the modeling of hybrid behaviors: how to account for impulsive jumps properly. *Commun. Nonlinear Sci. Numer. Simul.* **2021**, *103*, 105955. [CrossRef]