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On Semi-Continuous and Clisquish Functions in Generalized Topological Spaces

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Abstract: In this paper, we will focus on three types of functions in a generalized topological space, namely; lower and upper semi-continuous functions, and cliquish functions. We give some results for nowhere dense sets and for second category sets. Further, we discuss the nature of cliquish functions in generalized metric spaces and provide the characterization theorem for cliquish functions in terms of nowhere dense sets.

Keywords: Baire space; nowhere dense; lower semi-continuous function; upper semi-continuous function; cliquish function

MSC: 54A05; 54A10; 54C30

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Citation: Aponte, E.; Subramanian, V.; Macías, J.; Krishnan, M. On Semi-Continuous and Clisquish Functions in Generalized Topological Spaces. *Axioms* **2023**, *12*, 130. https://doi.org/10.3390/axioms12020130

Academic Editor: Ljubiša D. R. Kočinac

Received: 9 November 2022
Revised: 19 January 2023
Accepted: 22 January 2023
Published: 28 January 2023

1. Introduction

Generalized topological spaces were introduced by Császár in [1]. Different types of continuity in topological and generalized topological spaces were analyzed in ([2–11]). Frolik [12] characterized Baire spaces using semi-continuous functions in topological spaces. In continuation, cliquish functions have been analyzed (by Ewert [13]) in a Baire space using sequences, and these functions were introduced by H. P. Thielman [14], whose importance are discussed in ([15–17]). Using these aspects in generalized topological spaces, Korczak-Kubiak et al. [18] introduced two types of nowhere dense sets along with lower and upper semi-continuous functions, where they realized some properties of cliquish functions on Baire spaces. These served as a support to develop two sections of theory, one being a game similar to the well-known Banach–Mazur game (an infinite topological game) and the other centering on set function games.

Through these last references, we note that the theme they expose contributes to the development of topological theory. However, within this topic, it is interesting to investigate the characteristics of a function's domain in generalized topological spaces, or to determine whether or not a given set is a set nowhere dense; or whether a given map is a lower (upper) semi-continuous function, or is a cliquish function. These reasons provide the inspiration for us to present two sections of new results in generalized topological spaces.

The paper is presented as follows: Section 3, where μ, η, ζ will denote generalized topologies; and as always, we are looking for new properties, we prove on a strong generalized topological space, new properties for nowhere dense sets and second category sets. In addition, we examine the relations between μ-lower (upper) semi-continuous functions and (μ, η)-lower (upper) semi-continuous functions.

Section 4, as always it is important to distinguish the results, thus we give some characterization theorems for cliquish functions on a Baire space by using nowhere dense sets in generalized metric spaces. Furthermore, preserving theorems of cliquish functions *are* investigated.

With these results, the two classical generalized topologies $\mu^\star$ and $\mu^{\star\star}$ are considered, and it subsequently becomes possible to easily find out whether a given set is $\mu^\star$-nowhere dense ($\mu^{\star\star}$-nowhere dense) or is not in a strong Baire space. We obtain some results to check in a hyperconnected space whether a given function is η-lower (upper) semi-continuous or (η,ζ)-lower (upper) semi-continuous function, and also, whether they characterize the cliquish functions in terms of nowhere dense sets in a generalized metric space. In doing so, the results will prove useful to explore the nature of the set of all points in the domain of a given function.

2. Preliminaries

Let X be a non-null set. From [1]; a collection μ of subsets of X is a *generalized topology* on X if it contains the empty set and it closed under arbitrary union, thus the pair (X,μ) called as a *generalized topological space* (GTS). Furthermore, it is called a *strong generalized topological space* (sGTS) if $X \in \mu$. On the other hand, if $Q \in \mu$, then Q is called a *μ-open set*, and if $X - Q \in \mu$, then Q is said to be a *μ-closed set*. For $D \subset X$, the *interior of D* denoted by $i_\mu D$, is the union of all μ-open sets contained in D and the *closure of D* denoted by $c_\mu D$, is the intersection of all μ-closed sets containing D. For simplicity of notation, let us denote iD and cD to mitigate any confusion.

In what follows, we present some definitions and lemmas that are found in [18]. We denote a generalized topology by μ. In order to provide definitions, some sets will be defined beforehand.

$$\tilde{\mu} := \{Q \in \mu \mid Q \neq \emptyset\}.$$

Definition 1. *Let (X,μ) a GTS. A subset $Q \subset X$ is said to be:*

- *μ-nowhere dense, if $i_\mu c_\mu Q = \emptyset$;*
- *μ-dense, if $c_\mu Q = X$;*
- *μ-meager, if $Q = \bigcup_{n\in\mathbb{N}} P_n$ and each P_n is μ-nowhere dense.*

Note that every subset of a μ-meager set is μ-meager.

$$\mathcal{M}(\mu) := \{Q \subset X \mid Q \text{ is a } \mu\text{-meager set in } X\}.$$

Definition 2. *Let (X,μ) a GTS. A subset $Q \subset X$ is called:*

- *of μ-second category (μ-II category), if $Q \notin \mathcal{M}(\mu)$;*
- *a μ-residual set if $X - Q \in \mathcal{M}(\mu)$.*

Let;

$$\mu^\star := \{\bigcup_t (Q_1^t \cap Q_2^t \cap Q_3^t \cap \ldots \cap Q_{n_t}^t) \mid Q_1^t, Q_2^t, \ldots, Q_{n_t}^t \in \mu\}.$$
$$\mu^{\star\star} := \{P \subset X \mid P \text{ is of } \mu\text{-II category}\} \cup \{\emptyset\}.$$

It is known that $\mu^\star, \mu^{\star\star}$ are two generalized topologies and $\mu \subset \mu^\star$.

Definition 3. *The pair (X,μ) is called:*

- *a Baire space (BS) if $\tilde{\mu} \subset \mu^{\star\star}$, (that is, every non-null μ-open set is of μ-II category).*
- *a strong Baire space (sBS) if $J_1 \cap J_2 \cap \ldots \cap J_n \in \mu^{\star\star}$ for every $J_1, J_2, \ldots, J_n \in \mu$ such that:*

$$J_1 \cap J_2 \cap \ldots \cap J_n \neq \emptyset.$$

Note that if X is a BS, then $\mu \subset \mu^{\star\star}$. Moreover, $\mu^\star$ is closed under finite intersection. We denote the set of all real numbers by $\mathbb{R}$ and by

$$\mu(x) := \{Q \in \mu \mid x \in Q\}.$$

Definition 4. *Let μ, η be generalized topologies in X. The map $h : X \to \mathbb{R}$ is called:*

- *μ-lower semi-continuous (μ-l.s.c.) at a point $x_0 \in X$, if for any real number $\beta < h(x_0)$, there is $L \in \mu(x_0)$ such that $h(L) \subset (\beta, \infty)$,*
- *μ-upper semi-continuous (μ-u.s.c.) at a point $x_0 \in X$ if for any real number $\beta > h(x_0)$), there is $L \in \mu(x_0)$ such that $h(L) \subset (-\infty, \beta)$,*
- *(μ, η)-lower semi-continuous ((μ, η)-l.s.c.) at a point $x_0 \in X$ if for any real number $\beta < h(x_0)$, there is $K \in \eta(x_0)$ being a μ-residual set such that $h(K) \subset (\beta, \infty)$,*
- *(μ, η)-upper semi-continuous ((μ, η)-u.s.c.) at a point $x_0 \in X$ if for any real number $\beta > h(x_0)$, there is $K \in \eta(x_0)$ being a μ-residual set such that $h(K) \subset (-\infty, \beta)$).*

Let ζ, η be generalized topologies in X. $\mathbb{L}(\zeta)$ is the set of all ζ-lower (upper) semi-continuous functions on X. That is,

$$\mathbb{L}(\zeta) := \{h \mid h : X \to \mathbb{R} \text{ is a } \zeta\text{-l.(u.)s.c. function on } X\}.$$

Also, $\mathbb{L}(\eta, \zeta)$ is the collections of (η, ζ)-lower (upper) semi-continuous functions defined on X. That is,

$$\mathbb{L}(\eta, \zeta) := \{h \mid h : X \to \mathbb{R} \text{ is a } (\eta, \zeta)\text{-l.(u.)s.c. function on } X\}.$$

Note that $\mathbb{L}(\zeta, \eta) \subset \mathbb{L}(\eta)$ and $\mathbb{L}(\zeta) \subset \mathbb{L}(\zeta^\star)$.

Definition 5 ([19]). *Let (X, μ) be a generalized topological space and $h : X \to \mathbb{R}$ be a map. Then h is said to be continuous if and only if $r \in X$ and D is a non-null open set in $\mathbb{R}$ which contains $h(r)$ there is $Q \in \mu(r)$ such that $h(Q) \subset D$.*

The following notation is necessary to define the (μ, η)-cliquish functions. The set of μ-continuity (resp. μ-discontinuity) points of $h : X \to \mathbb{R}$, is denoted by $\mathcal{C}_\mu(h)$ (resp. $\mathcal{D}_\mu(h)$). Moreover;

$$\mathcal{D}(\mu) := \{P \subset X \mid P \text{ is } \mu\text{-dense in } X\}.$$

Definition 6. *A map $h : X \to \mathbb{R}$ is called (μ, η)-cliquish if $\mathcal{C}_\eta(h) \in \mathcal{D}(\mu)$.*

Note that if (X, η) is a BS and $\mathcal{D}_\mu(h) \in \mathcal{M}(\eta)$, then h is (η, μ)-cliquish.

Lemma 1. *Let (X, μ) be a generalized topological space and $L \subset X$ be a nowhere dense set. Then, $c_\mu L$ and any subset of L, is nowhere dense set.*

Lemma 2. *If (X, μ) is a sGTS which is sBS, then $(X, \mu^{\star\star})$ is Baire.*

Other sets to be used are:

- $\mathcal{N}(\mu) = \{Q \subset X \mid Q \text{ is } \mu\text{-nowhere dense in } X\}$,
- $\mathfrak{C}(\mu) = \{Q \subset X \mid Q \text{ is of } \mu\text{-second category in } X\}$.

Lemma 3. *If (X, μ) is a μ-II category space, then $\mathcal{N}(\mu^{\star\star}) \subset \mathcal{M}(\mu)$.*

Up to here, the preliminaries are obtained from [18].

Definition 7 ([20]). *A subset Q of a generalized topological space (X, μ) is said to be a μ-G_δ-set if $Q = \bigcap_{n \in \mathbb{N}} B_n$, where each $B_n \in \mu$.*

Definition 8 ([21]). *A space X is hyperconnected if $\tilde{\mu} \subset \mathcal{D}(\mu)$.*

Lemma 4 ([22]). *In a GTS (X, μ), countable union of a μ-meager set is μ-meager.*

3. Semi-Continuous Functions

In this section, we list and prove some properties between kinds of nowhere dense sets and second-category sets in an sGTS. These properties are useful for checking whether a set is $\mu^\star$-nowhere dense ($\mu^{\star\star}$ -nowhere dense) or not in a strong Baire space. In addition, the meaning of the collections $\mathbb{L}(\eta)$ and $\mathbb{L}(\eta,\zeta)$ in hyperconnected spaces are examined. We give the necessary condition to explore whether or not a function is in $\mathbb{L}(\eta)$, $\mathbb{L}(\eta,\zeta)$. Furthermore, we find new results for these functions in a Baire space. We start by showing that $\mu^\star \subset \mu^{\star\star}$.

Lemma 5. *If (X,μ) is a sBS, then $\mu^\star \subset \mu^{\star\star}$.*

Proof. Let $P \in \mu^\star$. Then, $P = \bigcup_t (P_1^t \cap P_2^t \cap \ldots \cap P_{n_t}^t)$, where $P_1^t, P_2^t, \ldots, P_{n_t}^t \in \mu$. Take $Q_k = P_1^k \cap P_2^k \cap \ldots \cap P_{n_k}^k$ for some k such that $Q_k \neq \emptyset$. Since each $P_i^k \in \mu$ with $\cap_{i=1}^n P_i^k \neq \emptyset$ and (X,μ) is a sBS, Q_k is of μ-II category. Since $Q_k \subset P$ and superset of μ-II category is of μ-II category, we have that $P \in \mu^{\star\star}$. Therefore, $\mu^\star \subset \mu^{\star\star}$. □

Example 1 below shows that the hypothesis in Lemma 5 cannot be neglected.

Example 1. *Let $X = \{p,q,r,s\}$ and*

$$\mu = \{\emptyset, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}.$$

Take $D = \{p,q\}$ and $E = \{p,r\}$. Then $D \cap E = \{p\} \neq \emptyset$, but $i_\mu(c_\mu(\{p\})) = \emptyset$. So that, (X,μ) is not a sBS. Here,

$$\mu^\star = \{\emptyset, \{p\}, \{q\}, \{r\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\} \text{ and}$$

$$\mu^{\star\star} = \{\emptyset, \{q\}, \{p,q\}, \{q,r\}, \{q,s\}\{p,q,r\}, \{p,q,s\}, \{q,r,s\}, X\}.$$

However, $\mu^\star \nsubseteq \mu^{\star\star}$.

Moreover, $\mu^{\star\star}$ is not closed under finite intersection if (X,μ) is an sBS as shown in the following Example 2.

Example 2. *Consider the GTS (X,μ), where*

$$\mu = \{\emptyset, [0,2), (1,4], [0,2) \cup [3,4], [0,4]\}$$

and $X = [0,5]$. Then (X,μ) is a sBS and

$$\mu^{\star\star} = \{\emptyset\} \cup \{A, B \subset X \mid A \in exp((1,2)) - \{\emptyset\}, A \subset B\},$$

knowing that $exp((1,2))$ is the collection of subsets of $(1,2)$. Now, if $C = [0,1] \cup \{\frac{3}{2}\}$ and $D = [0,1] \cup \{\frac{4}{3}\}$, then $C, D \in \mu^{\star\star}$, but $C \cap D \notin \mu^{\star\star}$.

Next, we obtain some inclusion between the sets $\mathcal{N}(\mu), \mathcal{N}(\mu^\star); \mathcal{M}(\mu), \mathcal{M}(\mu^\star); \mathfrak{C}(\mu^\star), \mathfrak{C}(\mu)$ and μ-residual, $\mu^\star$-residual sets, in an sBS.

Theorem 1. *If (X,μ) is a sBS, then:*

1. $\mathcal{N}(\mu) \subset \mathcal{N}(\mu^\star)$.
2. $\mathcal{M}(\mu) \subset \mathcal{M}(\mu^\star)$.
3. $\mathfrak{C}(\mu^\star) \subset \mathfrak{C}(\mu)$.
4. *Every μ-residual set is $\mu^\star$-residual.*

Proof. *1.* Assume that $Q \in \mathcal{N}(\mu)$, whereby by Lemma 1 it follows that $c_\mu Q \in \mathcal{N}(\mu)$, and so $c_\mu Q \in \mathcal{M}(\mu)$. Suppose that $i_{\mu^\star} c_{\mu^\star} Q \neq \emptyset$, whereby $i_{\mu^\star} c_\mu Q \neq \emptyset$. Since $\mu \subset \mu^\star$, there is $L \in \tilde{\mu}^\star$ such that $L \subset c_\mu Q$.

As $L \in \tilde{\mu}^\star$, by Lemma 5, we have that $L \in \tilde{\mu}^{\star\star}$. Since superset of μ-II category set is of μ-II category, we have that $c_\mu Q \in \tilde{\mu^{\star\star}}$,; however, this is not possible. Therefore, $Q \in \mathcal{N}(\mu^\star)$. Items *2.*, *3.* and *4.* are proved in a similar manner. □

Example 3 below shows that the converse part of *1.*, in Theorem 1 is not true.

Example 3. *Consider the GTS (X, μ), where:*

$$X = \{p, q, r, s, t\} \ \text{and} \ \mu = \{\emptyset, \{q, r\}, \{p, q, r\}, \{p, q, s\}, \{p, q, r, s\}\}.$$

Thus, (X, μ) is a sBS and

$$\mu^\star = \{\emptyset, \{q\}, \{q, r\}, \{p, q\}, \{p, q, r\}, \{p, q, s\}, \{p, q, r, s\}\}.$$

Now, take $H = \{p, r, s, t\}$ so $i_{\mu^\star} c_{\mu^\star} H = i_{\mu^\star} H = \emptyset$ and so H is a $\mu^\star$-nowhere dense set in X. However, $i_\mu c_\mu H = i_\mu X = \{p, q, r, s\} \neq \emptyset$, for that H is not a μ-nowhere dense set in X.

The same results of Theorem 1 instead of an sBS are obtained in a GTS, but with an additional property.

Theorem 2. *Let (X, μ) be a GTS such that $i_\mu(J_1 \cap J_2 \cap \ldots \cap J_n) \neq \emptyset$, where $J_1, \ldots, J_n \in \mu$ with $J_1 \cap J_2 \cap \cdots \cap J_n \neq \emptyset$. Thus, the items of Theorem 1 are given.*

Proof. It is enough to prove that $\mathcal{N}(\mu) \subset \mathcal{N}(\mu^\star)$. Let $Q \in \mathcal{N}(\mu)$ and suppose that $i_{\mu^\star} c_{\mu^\star} Q \neq \emptyset$. Note that $i_\mu c_\mu Q = \emptyset$, and moreover $i_{\mu^\star} c_\mu Q \neq \emptyset$ because $\mu \subset \mu^\star$. Therefore, there exists $J \in \tilde{\mu}^\star$ such that $J \subset c_\mu Q$, for that $J = \bigcup_t (J_1^t \cap J_2^t \cap \ldots \cap J_{n_t}^t)$ where $J_1^t, J_2^t, \ldots, J_{n_t}^t \in \mu$. Take $P_k = J_1^k \cap J_2^k \cap \ldots \cap J_{n_k}^k$ for some k such that $P_k \neq \emptyset$, whereby $P_k \subset J$ and so $P_k \subset c_\mu Q$ and by hypothesis, $i_\mu P_k \neq \emptyset$. Thus, $i_\mu c_\mu Q \neq \emptyset$ which is not possible. Therefore, $Q \in \mathcal{N}(\mu^\star)$.

The rest are tested in a similar manner. □

Example 4 shows that the necessary condition in Theorem 1 cannot be dropped.

Example 4. *Consider a GTS (X, μ), where:*

$$X = \{p, q, r, s, t\} \ \text{and} \ \mu = \{\emptyset, \{p, q\}, \{q, r\}, \{p, q, r\}, \{p, r, s\}, \{p, q, r, s\}\}.$$

Let $K = \{p, q\} \in \tilde{\mu}$ so $K \in \mathcal{M}(\mu)$, whereby (X, μ) is not a BS. Therefore, (X, μ) is not an sBS. Now,

$$\mu^\star = \{\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{q, r\}, \{p, r\}, \{p, q, r\}, \{p, r, s\}, \{p, q, r, s\}\}.$$

Take $J = \{r, t\}$ so $J \in \mathcal{N}(\mu)$, but $J \notin \mathcal{N}(\mu^\star)$. This is $\mathcal{N}(\mu) \not\subseteq \mathcal{N}(\mu^\star)$.

Next, we prove other inclusions but for sets $\mathcal{N}(\mu)$, $\mathcal{N}(\mu^{\star\star})$; $\mathcal{M}(\mu)$, $\mathcal{M}(\mu^{\star\star})$; $\mathfrak{C}(\mu^{\star\star})$, $\mathfrak{C}(\mu)$, and μ-residual, $\mu^{\star\star}$-residual sets, in a Baire space.

Theorem 3. *If (X, μ) is a Baire space, then:*

1. $\mathcal{N}(\mu) \subset \mathcal{N}(\mu^{\star\star})$.
2. $\mathcal{M}(\mu) \subset \mathcal{M}(\mu^{\star\star})$.
3. $\mathfrak{C}(\mu^{\star\star}) \subset \mathfrak{C}(\mu)$.
4. *Every μ-residual set is $\mu^{\star\star}$-residual.*

Proof. We give the proof for *1.*. Let $Q \in \mathcal{N}(\mu)$ so $i_\mu c_\mu Q = \emptyset$. Assume that $i_{\mu^{\star\star}} c_{\mu^{\star\star}} Q \neq \emptyset$. As $\mu \subset \mu^{\star\star}$ then $i_{\mu^{\star\star}} c_\mu Q \neq \emptyset$, so there exists a set $J \in \tilde{\mu}^{\star\star}$ such that $J \subset c_\mu Q$.

As, $J \in \tilde{\mu}^{\star\star}$ so $c_\mu Q$ is of μ-II category set, since super set of μ-II category is of μ-II category. However, $c_\mu Q \in \mathcal{M}(\mu)$ whereby $i_{\mu^{\star\star}} c_{\mu^{\star\star}} Q = \emptyset$. Hence $Q \in \mathcal{N}(\mu^{\star\star})$. Therefore, $\mathcal{N}(\mu) \subset \mathcal{N}(\mu^{\star\star})$. The rest are tested in a similar manner. □

Example 5 below shows that the reverse implication of Theorem 3 part *1.*, is not true.

Example 5. *Consider the generalized topological space (X, μ), where $X = [0, 5]$ and*

$$\mu = \{\emptyset, [0, 3), (2, 4], [0, 4]\}.$$

Then, (X, μ) is a BS and we get that,

$$\mu^{\star\star} = \{\emptyset\} \cup \{A, B \subset X \mid A \in exp((2, 3)) - \{\emptyset\}, A \subset B\}.$$

Let $H = [0, 2] \cup [3, 5]$. So that $i_{\mu^{\star\star}} c_{\mu^{\star\star}} H = i_{\mu^{\star\star}} H = \emptyset$. Therefore, H is a $\mu^{\star\star}$-nowhere dense set in X. However, $i_\mu c_\mu H = i_\mu X = [0, 4] \neq \emptyset$, so that H is not a μ-nowhere dense set in X.

The following Theorem 4 gives the relations between different types of subsets with respect to the generalized topologies $\mu^\star$ and $\mu^{\star\star}$ in a strong Baire space.

Theorem 4. *Let (X, μ) be an sBS and sGTS. Then:*

1. $\mathcal{N}(\mu^{\star\star}) \subset \mathcal{M}(\mu^\star)$.
2. $\mathcal{M}(\mu^{\star\star}) \subset \mathcal{M}(\mu^\star)$.
3. $\mathfrak{C}(\mu^\star) \subset \mathfrak{C}(\mu^{\star\star})$.
4. *Every $\mu^{\star\star}$-residual set is $\mu^\star$-residual set.*

Proof. It is enough to prove *1.*. Let $P \in \mathcal{N}(\mu^{\star\star})$. By hypothesis and Lemma 3, $P \in \mathcal{M}(\mu)$ so that $P \in \mathcal{M}(\mu^\star)$, by Theorem 1 and the fact that (X, μ) is a sBS. □

Next, Example 6 shows that the condition "(X, μ) is an sBS" cannot be omitted in Theorem 4.

Example 6. *Let us consider again*

$$X = \{p, q, r, s\} \text{ and } \mu = \{\emptyset, \{p, r\}, \{q, r\}, \{p, s\}, \{p, q, r\}, \{p, r, s\}, \{q, r, s\}, X\}.$$

Thus,

$$\mu^\star = \{\emptyset, \{p\}, \{r\}, \{s\}, \{p, r\}, \{q, r\}, \{p, s\}, \{r, s\}, \{p, q, r\}, \{p, r, s\}, \{q, r, s\}, X\},$$

and

$$\mu^{\star\star} = \{\emptyset, \{r\}, \{p, r\}, \{q, r\}, \{r, s\}, \{p, q, r\}, \{p, r, s\}, \{q, r, s\}, X\}.$$

Here, (X, μ) is not an sBS. Take $G = \{p, r\}$ and $L = \{p, s\}$ so $G, L \in \tilde{\mu}$ and $G \cap L \neq \emptyset$, but $G \cap L \in \mathcal{M}(\mu)$. Choose $K = \{p, q, s\}$ it turns out $i_{\mu^{\star\star}}(c_{\mu^{\star\star}}(K)) = \emptyset$, so that $K \in \mathcal{N}(\mu^{\star\star})$, but $\{p\}$ and $\{s\}$ are $\mu^\star$-II category sets, which implies $K \notin \mathcal{M}(\mu^\star)$. Hence, $\mathcal{N}(\mu^{\star\star}) \nsubseteq \mathcal{M}(\mu^\star)$.

The below Theorem 5 directly follows from Lemmas 3 and 4.

Theorem 5. *Let (X, μ) be a μ-II category space. Then:*

1. *Every $\mu^{\star\star}$-meager is μ-meager.*
2. *Every $\mu^{\star\star}$-residual is μ-residual.*
3. *Every μ-II category set is of $\mu^{\star\star}$-II category.*

Now, we obtain some results between the generalized topologies, semi-continuous functions, and the hyperconnected condition.

Theorem 6. *If (X, η) is a hyperconnected space where $\eta \in \{\mu, \mu^\star, \mu^{\star\star}\}$, then J is η-residual for every $J \in \tilde{\eta}$.*

Proof. Let $J \in \tilde{\eta}$. As (X, η) is a hyperconnected space so $J \in \mathcal{D}(\eta)$. Hence, $i_\eta(X - J) = \emptyset$ and so $X - J \in \mathcal{N}(\eta)$, since $X - J$ is η-closed. Hence, $X - J \in \mathcal{M}(\eta)$. Thus, J is a η-residual set in X. □

The following Example 7 shows the necessity of hyper-connectedness in Theorem 6.

Example 7. *Consider the generalized topological space (X, μ) being*

$$X = \{p, q, r, s\} \text{ and } \mu = \{\emptyset, \{p, q\}, \{p, s\}, \{r, s\}, \{p, q, s\}, \{p, r, s\}, X\}.$$

So:

1. *(X, μ) is not a hyperconnected space. Because $\{p, q\} \in \tilde{\mu}$, but $\{p, q\} \notin \mathcal{D}(\mu)$. Take $W = \{r, s\}$ thus $W \in \tilde{\mu}$ and $X - W = \{p, q\}$. Here $\{p\}$ is of μ-second category set, so that $X - W \notin \mathcal{M}(\mu)$. Thus, W is not μ-residual.*
2. *In this part,*
$$\mu^\star = \{\emptyset, \{p\}, \{s\}, \{p, q\}, \{p, s\}, \{r, s\}, \{p, q, s\}, \{p, r, s\}, X\}.$$
Obviously, $(X, \mu^\star)$ is not a hyperconnected space. Take $D = \{p\}$, then $D \in \tilde{\mu}^\star$ and $D \notin \mathcal{D}(\mu^\star)$. Let $K = \{s\}$, thus $K \in \tilde{\mu}^\star$ and $X - K = \{p, q, r\}$. Furthermore, $\{p\}$ is of $\mu^\star$-II category, so that $X - K$ is of $\mu^\star$-II category. Therefore, K is not $\mu^\star$-residual.
3. *With*
$$\mu^{\star\star} = \{\emptyset, \{p\}, \{s\}, \{p, q\}, \{p, s\}, \{p, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X\},$$
we have that $(X, \mu^{\star\star})$ is not a hyperconnected space. Here $\{p\} \in \tilde{\mu}^{\star\star}$, but $\{p\} \notin \mathcal{D}(\mu^{\star\star})$. Take $O = \{p, q\}$ then $O \in \tilde{\mu}^{\star\star}$ and $X - O = \{r, s\}$, but $\{s\}$ is of $\mu^{\star\star}$-II category. This implies $X - O$ is not in $\mathcal{M}(\mu^{\star\star})$, which implies that O is not $\mu^{\star\star}$-residual.

The reverse implication of Theorem 6 is not true as shown by the following Example 8.

Example 8. *Let $X = \{p, q, r, s, t\}$ and*

$$\mu = \{\emptyset, \{p, q\}, \{q, r\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, \{p, q, r, s\}\}.$$

Here, each $G \in \tilde{\mu}$ is a μ-residual set in X. Let $H = \{p, q\}$, so $H \in \tilde{\mu}$ and $c_\mu H = \{p, q, t\}$, but $c_\mu H \neq X$. Therefore, H is not a μ-dense set in X and hence (X, μ) is not a hyperconnected space. In the same manner, we can prove that (X, η) is not a hyperconnected space if each $G \in \tilde{\eta}$ is a η-residual set in X, where $\eta \in \{\mu^\star, \mu^{\star\star}\}$.

The following Example 9 shows that if (X, μ) is a BS, then it is not necessarily a hyperconnected space.

Example 9. *Consider the generalized topological space (X, μ) where*

$$X = [0, 5] \text{ and } \mu = \{\emptyset, [0, 2), [2, 3], [0, 3]\}.$$

Then (X, μ) is a BS but not a hyperconnected space. Since, if $H = [0, 2)$, then $c_\mu H = [0, 2) \cup (3, 5] \neq X$.

Theorem 7. *Let (X, μ) be a μ-II category space. If (X, μ) is hyperconnected, then (X, μ) is a BS.*

Proof. Let $L \in \tilde{\mu}$. By hypothesis and Theorem 6, $X - L \in \mathcal{M}(\mu)$. Suppose $L \in \mathcal{M}(\mu)$. By Lemma 4, $(X - L) \cup L \in \mathcal{M}(\mu)$. Hence $X \in \mathcal{M}(\mu)$ which is not possible. Thus, $L \in \mathfrak{C}(\mu)$ and (X, μ) is a Baire space. □

Next, in the rest of the section with a series of theorems in a space hyperconnected, or μ-II category, or strong Baire, the essentials of lower (upper) semi-continuous functions are discussed. Further, the set theory relationship between $\mathbb{L}(\eta)$ and $\mathbb{L}(\eta, \zeta)$ is analyzed. Finally, we study the notation $\mu^{\star\star^{\star}}$ defined on [11]. We show that every continuous function is in $\mathbb{L}(\mu)$.

The below Theorem 8 is an immediate consequence of Theorem 5 and by the definition of $\mathbb{L}(\mu^{\star\star}, \eta)$.

Theorem 8. *If (X, μ) is a μ-II category space and $\eta \in \{\mu, \mu^\star, \mu^{\star\star}\}$, then $\mathbb{L}(\mu^{\star\star}, \eta) \subset \mathbb{L}(\mu, \eta)$.*

Remark 1. *Since $\zeta \subset \zeta^\star$ we have $\mathbb{L}(\zeta, \zeta) \subset \mathbb{L}(\zeta, \zeta^\star)$. Furthermore, $\mathbb{L}(\zeta^\star, \zeta) \subset \mathbb{L}(\zeta^\star, \zeta^\star)$ and $\mathbb{L}(\zeta^{\star\star}, \zeta) \subset \mathbb{L}(\zeta^{\star\star}, \zeta^\star)$. Moreover,*

$$
\begin{array}{ccc}
\mathbb{L}(\zeta^{\star\star}, \zeta) \subset \mathbb{L}(\zeta^{\star\star}, \zeta^{\star\star}) & \longleftarrow & (X, \zeta) \quad is \quad a \quad BS \\
 & & \downarrow \\
 & & \mathbb{L}(\zeta, \zeta) \subset \mathbb{L}(\zeta, \zeta^{\star\star})
\end{array}
$$

Theorem 9. *If $(X, \mu^\star)$ is a hyperconnected space, then $\mathbb{L}(\mu) \subset \mathbb{L}(\mu^\star, \mu)$.*

Proof. Consider $h \in \mathbb{L}(\mu)$ a μ-lower semi-continuous function, $t_0 \in X$ and $\beta < h(t_0)$. By assumption there is $W \in \mu(t_0)$ such that $h(W) \subset (\beta, \infty)$. Since $\mu \subset \mu^\star, W \in \mu^\star(t_0)$. By hypothesis and Theorem 6, W is a $\mu^\star$-residual set. Hence h is a $(\mu^\star, \mu)$-lower semi-continuous function at t_0 and hence h is a $(\mu^\star, \mu)$-lower semi-continuous function on X. Similar considerations apply to the case of μ-upper semi-continuous function, we get h is a $(\mu^\star, \mu)$-upper semi-continuous function on X. Therefore, $\mathbb{L}(\mu) \subset \mathbb{L}(\mu^\star, \mu)$. □

The proof of the following result is similar to that of Theorem 9.

Theorem 10. *If (X, η) is a hyperconnected space and $\eta \in \{\mu, \mu^\star, \mu^{\star\star}\}$, then $\mathbb{L}(\eta) \subset \mathbb{L}(\eta, \eta)$.*

The following Example 10 shows that the condition "$(X, \mu^\star)$ is a hyperconnected space" can not be dropped in Theorem 9.

Example 10. *Take $X = [0, 3]$ and*

$$\mu = \{\emptyset, [0, 2), [1, 3], [2, 3], [0, 3]\}.$$

Thus:

$$\mu^\star = \{\emptyset, [0, 2), [1, 2), [1, 3], [2, 3], [0, 3]\}.$$

If $H = [2, 3] \in \tilde{\mu}^\star$, then H is not a $\mu^\star$-dense set in X. Therefore, $(X, \mu^\star)$ is not a hyperconnected space.

Define a function $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & if \quad x \in [0, 1), \\ 2 & if \quad x \in [1, 2), \\ 3 & if \quad x \in [2, 3]. \end{cases}$$

For any real number $\alpha < f(x)$ for all $x \in X$, there exists $G \in \mu(x)$ such that $f(G) \subset (\alpha, \infty)$. Therefore, f is a μ-lower semi-continuous function. Take $x = 2$ and choose $\alpha = 2.9$ for which $\alpha < f(2)$. If $A = [1,3]$ and $B = [0,3]$, then $A, B \in \mu(x)$. Now,

$$i_{\mu^\star} c_{\mu^\star}(X - A) = i_{\mu^\star} c_{\mu^\star}[0,1) = i_{\mu^\star}[0,1) = \emptyset \text{ and } i_{\mu^\star} c_{\mu^\star}(X - B) = i_{\mu^\star} c_{\mu^\star}\emptyset = i_{\mu^\star}\emptyset = \emptyset.$$

So, $X - A$ and $X - B$ are $\mu^\star$-nowhere dense sets in X. Hence, $X - A$ and $X - B$ are $\mu^\star$-meager sets in X and A and B are the $\mu^\star$-residual sets in X. However, $f(A) \nsubseteq (\alpha, \infty)$ and $f(B) \nsubseteq (\alpha, \infty)$. Therefore, f is not a $(\mu^\star, \mu)$-lower semi-continuous function.

Define a function $g : X \to \mathbb{R}$ by

$$g(x) = \begin{cases} 3 & if \quad x \in [0,1), \\ 2 & if \quad x \in [1,2), \\ 1 & if \quad x \in [2,3]. \end{cases}$$

For any real number $\alpha > g(x)$ and for all $x \in X$, there exists $G \in \mu(x)$ such that $g(G) \subset (-\infty, \alpha)$. Therefore, g is a μ-upper semi-continuous function. Now, $g(2) = 1$, choose $\alpha = 1.1 > g(2)$. If $A = [1,3]$ and $B = [0,3]$, then $A, B \in \mu(2)$. By similar manner, A and B are $\mu^\star$-residual sets in X. However, $g(A) \nsubseteq (-\infty, \alpha), g(B) \nsubseteq (-\infty, \alpha)$. Therefore, g is not a $(\mu^\star, \mu)$-upper semi-continuous function.

Theorem 11. *Let (X, μ) be a sBS and $\eta \in \{\mu, \mu^\star, \mu^{\star\star}\}$. Then:*

1. $\mathbb{L}(\mu^\star) \subset \mathbb{L}(\mu^{\star\star})$.
2. $\mathbb{L}(\eta, \mu^\star) \subset \mathbb{L}(\mu^{\star\star})$.
3. $\mathbb{L}(\mu, \eta) \subset \mathbb{L}(\mu^\star, \eta)$.

Proof. The proof directly follows from the facts that in a strong Baire space, $\mu^\star \subset \mu^{\star\star}$ and every μ-residual set is $\mu^\star$-residual. □

Theorem 12. *Let (X, μ) be μ-II category space. If $(X, \mu^{\star\star})$ is hyperconnected, then $\mathbb{L}(\mu^{\star\star}) \subset \mathbb{L}(\mu, \mu^{\star\star})$.*

Proof. By Theorem 5 part *2.*, each $\mu^{\star\star}$-residual set is a μ-residual set. If $h \in \mathbb{L}(\mu^{\star\star})$ is a $\mu^{\star\star}$-lower semi-continuous function, $t_0 \in X$ and $\beta < h(t_0)$, then there is $W \in \mu^{\star\star}(t_0)$ (with $h(W) \subset (\beta, \infty)$) such that W is $\mu^{\star\star}$-residual, whereby W is μ-residual. We deduce that h is a $(\mu, \mu^{\star\star})$-lower semi-continuous function on X. Similar considerations apply to the case of $\mu^{\star\star}$-upper semi-continuous function. We conclude that $\mathbb{L}(\mu^{\star\star}) \subset \mathbb{L}(\mu, \mu^{\star\star})$. □

The following Example 11 shows the necessity of hyper-connectedness in Theorem 12.

Example 11. *Take X, μ, f and g as given in Example 10. Then,*

$$\mu^{\star\star} = \{\emptyset\} \cup \{A, [0,1) \cup A, B \mid A \subseteq [1,3] \subseteq B\},$$

and so $(X, \mu^{\star\star})$ is not a hyperconnected space. Here, for any real number $\alpha < f(x)$ and for all $x \in X$, there exists a set $G \in \mu^{\star\star}(x)$ such that $f(G) \subset (\alpha, \infty)$. Therefore, f is a $\mu^{\star\star}$-lower semi-continuous function. For $k = 2$ and $\alpha = 2.9$, we get that $\alpha < f(2)$. If $A = [1,3]$ and $B = [0,3]$, then $A, B \in \mu^{\star\star}(k)$. Furthermore, A and B are μ-residual sets. However, $f(A) \nsubseteq (\alpha, \infty), f(B) \nsubseteq (\alpha, \infty)$. Therefore, f is not a $(\mu, \mu^{\star\star})$-lower semi-continuous function. In the same manner, we can prove that g is a $\mu^{\star\star}$-upper semi-continuous function but not a $(\mu, \mu^{\star\star})$-upper semi-continuous function.

Theorem 13. *Let (X, μ) be a sBS. If $(X, \mu^{\star\star})$ is hyperconnected, then $\mathbb{L}(\mu^\star) \subset \mathbb{L}(\mu^{\star\star}, \mu^\star)$.*

Proof. Consider $h \in \mathbb{L}(\mu^\star)$ a $\mu^\star$-lower semi-continuous function and $t_0 \in X, \beta < h(t_0)$. So, by definition and Lemma 5, we have a set $P \in \mu^{\star\star}(t_0)$ such that $h(P) \subset (\beta, \infty)$, then by

Theorem 6, P is a $\mu^{\star\star}$-residual set. We deduce that h is a $(\mu^{\star\star}, \mu^{\star})$-lower semi-continuous function on X. Similarly, it is shown that h is a $(\mu^{\star\star}, \mu^{\star})$-upper semi-continuous function. □

Theorem 14. *Let (X, μ) be an sGTS, sBS and $\eta \in \{\mu, \mu^{\star}, \mu^{\star\star}\}$. Then:*

1. $\mathbb{L}(\mu^{\star\star}, \eta) \subset \mathbb{L}(\mu^{\star}, \eta)$.
2. $\mathbb{L}(\mu^{\star\star}, \mu) \subset \mathbb{L}(\mu^{\star}, \mu^{\star})$.

Proof. *1.* Let $h \in \mathbb{L}(\mu^{\star\star}, \eta)$. Assume that h is a $(\mu^{\star\star}, \eta)$-lower semi-continuous function. Let $t_0 \in X$ and $\beta < h(t_0)$. Then, there is $Q \in \eta(t_0)$ being a $\mu^{\star\star}$-residual set such that $h(Q) \subset (\beta, \infty)$. By hypothesis and Theorem 4 part (4), Q is a $\mu^{\star}$-residual set. Thus, h is a $(\mu^{\star}, \eta)$-lower semi-continuous function at t_0. Since t_0 is an arbitrary point of X, h is a $(\mu^{\star}, \eta)$-lower semi-continuous function on X. Similar way apply to the case of upper semi-continuous function.

2. Assume that, $h \in \mathbb{L}(\mu^{\star\star}, \mu)$. By *1.*, and $\mu \subset \mu^{\star}$, the proof is completed. □

Proceeding similarly to the previous demonstration and applying Theorem 3 part *4.*, we obtain the following result.

Theorem 15. *Let (X, μ) be a BS. If $\eta \in \{\mu, \mu^{\star}, \mu^{\star\star}\}$, then $\mathbb{L}(\mu, \eta) \subset \mathbb{L}(\mu^{\star\star}, \eta)$.*

In generalized topological space, every $(\mu^{\star\star}, \mu)$-l.(u.)s.c. function is a μ-l.(u.)s.c. function. The below Theorem 16 shows a fact for the reverse implication of the above statement.

Theorem 16. *Let (X, μ) be a μ-II category space. If (X, μ) is hyperconnected, then:*

1. $\mathbb{L}(\mu) \subset \mathbb{L}(\mu^{\star\star}, \mu)$.
2. $\mathbb{L}(\mu) \subset \mathbb{L}(\mu^{\star\star}, \mu^{\star})$.

Proof. *1.* Let $h \in \mathbb{L}(\mu)$. Assume that h is a μ-lower semi-continuous function. By hypothesis and Theorem 10, h is (μ, μ)-lower semi-continuous function. Let $t_0 \in X$ and $\beta < h(t_0)$, so there is $L \in \mu(t_0)$ being μ-residual set, such that $h(L) \subset (\beta, \infty)$. By hypothesis and Theorem 7, it turns out that (X, μ) is a BS, and by Theorem 3 part *4.*, we have that L is a $\mu^{\star\star}$-residual set. Therefore, h is a $(\mu^{\star\star}, \mu)$-lower semi-continuous function at t_0 and hence h is a $(\mu^{\star\star}, \mu)$-lower semi-continuous function on X. Similar considerations apply to the case of upper semi-continuous function.

2. Assume that $h \in \mathbb{L}(\mu)$ so by *1.*, $h \in \mathbb{L}(\mu^{\star\star}, \mu)$, as $\mu \subset \mu^{\star}$, the test is followed. □

The following Example 12 shows that the condition "hyperconnectedness" on (X, μ) can not be dropped in Theorem 16.

Example 12. *Consider generalized topological space (X, μ) and the functions $f, g : X \to \mathbb{R}$, as in Example 10. Then,*

$$\mu^{\star\star} = \{\emptyset\} \cup \{A, [0,1) \cup A, B \mid A \subseteq [1,3] \subseteq B\}$$

and (X, μ) is not a hyperconnected space. Now:

1. *Clearly, f is a μ-lower semi-continuous. function on X and $f(2) = 3$. Choose $\alpha = 2.9 < f(2)$. If $U = [1,3]$ and $V = [0,3]$, then $U, V \in \mu(2)$. Now;*

$$i_{\mu^{\star\star}} c_{\mu^{\star\star}}(X - U) = i_{\mu^{\star\star}} c_{\mu^{\star\star}}[0,1) = i_{\mu^{\star\star}}[0,1) = \emptyset \text{ and}$$

$$i_{\mu^{\star\star}} c_{\mu^{\star\star}}(X - V) = i_{\mu^{\star\star}} c_{\mu^{\star\star}}\emptyset = i_{\mu^{\star\star}}\emptyset = \emptyset.$$

Therefore, $X - U$ and $X - V$ are $\mu^{\star\star}$-nowhere dense sets in X and so $X - U$ and $X - V$ are $\mu^{\star\star}$-meager sets in X. Hence, U and V are $\mu^{\star\star}$-residual sets. However, we have that $f(U) \nsubseteq (\alpha, \infty), f(V) \nsubseteq (\alpha, \infty)$. Therefore, f is not a $(\mu^{\star\star}, \mu)$-lower semi-continuous function. In the same manner, we can prove that g is a μ-upper semi-continuous function but not *a $(\mu^{\star\star}, \mu)$-upper semi-continuous function.*

2. *Here,*

$$\mu^{\star} = \{\emptyset, [0,2), [1,2), [1,3], [2,3], [0,3]\}.$$

Choose $\alpha = 2.9 < f(2) = 3$. If $U = [1,3]$ and $V = [0,3]$, then $U, V \in \mu^{\star}(2)$. Furthermore, U and V are $\mu^{\star\star}$-residual sets. However, $f(U) \nsubseteq (\alpha, \infty), f(V) \nsubseteq (\alpha, \infty)$. Therefore, f is not a $(\mu^{\star\star}, \mu^{\star})$-lower semi-continuous function. In the same manner, we can prove that g is a μ-upper semi-continuous function but not a $(\mu^{\star\star}, \mu^{\star})$-upper semi-continuous function.

Since it has been proved in a BS, every μ-residual is $\mu^{\star\star}$-residual. So, the following Theorem 17 gives the relationship between (μ, μ)-l.(u.)s.c. functions and $(\mu^{\star\star}, \mu^{\star})$-l.(u.)s.c. function in a BS.

Theorem 17. *If (X, μ) is a Baire space, then:*

1. $\mathbb{L}(\mu, \mu) \subset \mathbb{L}(\mu^{\star\star}, \mu^{\star})$.
2. $\mathbb{L}(\mu, \mu) \subset \mathbb{L}(\mu^{\star\star}, \mu^{\star\star})$.
3. *If (X, μ) is a sGTS, then $h \in \mathbb{L}(\mu, \mu^{\star}) \Leftrightarrow h \in \mathbb{L}(\mu^{\star\star}, \mu^{\star})$.*

Proof. We give the detailed proof only for *3.*. By hypothesis, X is of μ-II category. Consider $h \in \mathbb{L}(\mu, \mu^{\star})$ and assume that h is a $(\mu, \mu^{\star})$-lower semi-continuous function. Choose $t_0 \in X$ is an arbitrary point and $\beta < h(t_0)$, this implies that there is $Q \in \mu^{\star}(t_0)$ being μ-residual such that $h(Q) \subset (\beta, \infty)$. By Theorem 3 part *4.*, Q is $\mu^{\star\star}$-residual. Hence h is a $(\mu^{\star\star}, \mu^{\star})$-lower semi-continuous function on X.

The reverse inclusion follows directly from the same above argument and the fact that in a μ-II-category space, every $\mu^{\star\star}$-residual set is μ-residual. Apply similar considerations in the case of the upper semi-continuous function. □

For the following theorem, we consider that

$$\mu^{\star\star^{\star}} = \{\bigcup_t (Q_1^t \cap Q_2^t \cap \ldots \cap Q_{n_t}^t) \mid Q_1^t, Q_2^t, \ldots, Q_{n_t}^t \in \mu^{\star\star}\}.$$

Theorem 18. *If (X, μ) be a BS, then $\mathbb{L}(\mu^{\star}) \subset \mathbb{L}(\mu^{\star\star^{\star}})$.*

Proof. We present a proof for lower semi-continuous function. Let $h \in \mathbb{L}(\mu^{\star})$ and consider that h is a $\mu^{\star}$-lower semi-continuous function, $t_0 \in X$ and $\beta < h(t_0)$. Then, there is $Q \in \mu^{\star}(t_0)$ such that $h(Q) \subset (\beta, \infty)$. Now, $Q = \bigcup_t (Q_1^t \cap Q_2^t \cap Q_3^t \cap \ldots \cap Q_{n_t}^t)$, where $Q_i^t \in \mu$ for $i = 1$ to n_t. Since X is a BS, we have that $\mu \subset \mu^{\star\star}$, so that $Q_i^t \in \mu^{\star\star}$ for $i = 1$ to n_t, which implies that $Q \in \mu^{\star\star^{\star}}(t_0)$. Therefore, h is a $\mu^{\star\star^{\star}}$-lower semi-continuous function at t_0, and hence h is a $\mu^{\star\star^{\star}}$-lower semi-continuous function on X. □

Corollary 1. *Let (X, μ) be a BS, $\eta \in \{\mu, \mu^{\star}, \mu^{\star\star}\}$. Then the following set inclusions are true.*

1. $\mathbb{L}(\eta, \mu^{\star}) \subset \mathbb{L}(\mu^{\star\star^{\star}})$.
2. $\mathbb{L}(\mu) \subset \mathbb{L}(\mu^{\star\star^{\star}})$.
3. $\mathbb{L}(\eta, \mu) \subset \mathbb{L}(\mu^{\star\star^{\star}})$.

An interesting result states that every continuous real-valued function is a μ-l.(u.)s.c. function in a GTS.

Theorem 19. *Let (X, μ) be a GTS. If $h : X \to \mathbb{R}$ is a continuous map, then $h \in \mathbb{L}(\mu)$.*

Proof. For $t_0 \in X$ and $\beta < h(t_0)$, we have that $h(t_0) \in (\beta, \infty)$. Since h is continuous on X, h is continuous at t_0. Therefore, there is $P \in \mu(t_0)$ such that $h(P) \subset (\beta, \infty)$. Hence, h is a μ-lower semi-continuous function at t_0. Since t_0 is an arbitrary point of X, h is a

μ-lower semi-continuous function on X. By similar argument, we can prove h is a μ-upper semi-continuous function on X. $\square$

4. Characterizations of Cliquish Functions

In this section, cliquish functions on generalized metric spaces are examined, and several properties of these are obtained. Moreover, we characterize cliquish functions using nowhere dense sets in Baire spaces. With these results, we conclude that the set of all discontinuity points of a real valued function is either a meager set or not, but in a generalized topological space. Finally, we propose some results to examine whether a function is cliquish or not.

The pair (X, Ω) is called a *generalized metric space*[18] (briefly, GMS) if $\Omega = \{\sigma \mid \sigma$ is a metric on $Q_\sigma \subset X\}$. Denote μ_Ω, the family of Ω-*open sets* [18] in (X, Ω), more precisely, $K \in \mu_\Omega$ if and only if for each $r \in K$, there is $\sigma \in \Omega$ and $\varepsilon > 0$ such that $B_\sigma(r, \varepsilon) \subset K$, where $B_\sigma(r, \varepsilon) = \{s \in dom(\sigma) \mid \sigma(r, s) < \varepsilon\}$ and $dom(\sigma)$ means domain space of σ. So, the pair (X, μ_Ω) is a generalized topological spaces.

The following Theorem 20, is a simple method to explore whether or not a set is residual in a generalized metric space, thus reducing the computational complexity.

Theorem 20. *Let (X, η) be a GTS and Q be a η-G_δ-set, where $\eta \in \{\mu_\Omega, \mu^\star_\Omega, \mu^{\star\star}_\Omega\}$. If $Q \in \mathcal{D}(\eta)$, then Q is a η-residual set in X.*

Proof. Assume that $Q \in \mathcal{D}(\eta)$ is a G_δ-set. Then, $Q = \bigcap\limits_{n=1}^{\infty} Q_n$, where $Q_n \in \tilde{\eta}$ for every $n \in \mathbb{N}$. As, $Q \subset Q_n$ we have that each $Q_n \in \mathcal{D}(\eta)$, whereby $i_\eta(X - Q_n) = \emptyset$. Since each $Q_n \in \tilde{\eta}$, we get that $X - Q_n$ is η-closed for each $n \in \mathbb{N}$. Thus, each $X - Q_n \in \mathcal{N}(\eta)$, whereby $X - Q \in \mathcal{M}(\eta)$. Therefore, Q is a η-residual set. $\square$

The following Example 13 shows that the condition "$Q \in \mathcal{D}(\eta)$" cannot be neglected in Theorem 20.

Example 13. *Let $X = [0, 1]$ and $\Omega = \{\sigma_1, \sigma_2, \sigma_3\}$, where*

$$\sigma_1(s,t) = \begin{cases} 0 & \text{if} \quad s = t \\ 1 & \text{if} \quad s \neq t \end{cases}; \quad \sigma_2 = |s - t| \quad \text{and} \quad \sigma_3 = \min\{1, \sigma_1\}.$$

Then, (X, Ω) is a generalized topological space.
Here;

$$\mu_\Omega = \{K, L, M \subset X \mid K \in \mu_{\sigma_1}, L \in \mu_{\sigma_2}, M \in \mu_{\sigma_3}\};$$

$$\mu^\star_\Omega = \{D, J \subset X \mid D \in \tilde{\mu}_\Omega, J = \bigcup_t (Q^t_1 \cap Q^t_2 \cap Q^t_3 \cap \ldots \cap Q^t_{n_t}) \text{ with } Q^t_1, Q^t_2, \ldots, Q^t_{n_t} \in \tilde{\mu}_\Omega\};$$

$$\mu^{\star\star}_\Omega = \{H \subset X \mid H \text{ is of } \mu_\Omega - II \text{ category}\}.$$

1. *Clearly, $Q = [0, \frac{1}{2}) \in \tilde{\mu}_\Omega$, but $Q \notin \mathcal{D}(\mu_\Omega)$. Choose $D = (\frac{1}{2}, \frac{3}{4})$, thus $D \in \mu_\Omega$. As $Q \cap D = \emptyset$, we have that Q is a μ_Ω-G_δ-set, but Q is not μ_Ω-dense in X. Clearly, $X - Q = [\frac{1}{2}, 1]$ contains a μ_Ω-II category set, so that $X - Q$ is in $\mathfrak{C}(\mu_\Omega)$. Therefore, Q is a not μ_Ω-residual set.*
2. *Choose $K = [0, \frac{3}{4})$ and $L = (\frac{1}{2}, 0.825)$. Then $K, L \in \tilde{\mu}$ and $K \cap L = (\frac{1}{2}, \frac{3}{4})$. Furthermore, $K \cap L \in \tilde{\mu^\star}_\Omega$. Since, $(0.3, \frac{1}{2}) \in \tilde{\mu^\star}_\Omega$ and $(0.3, \frac{1}{2}) \cap (K \cap L) = \emptyset$, we get $K \cap L \notin \mathcal{D}(\mu^\star_\Omega)$. Here, $X - (K \cap L) = [0, \frac{1}{2}] \cup [\frac{3}{4}, 1]$, which is of $\mu^\star_\Omega$-II category, so that $K \cap L$ is not a $\mu^\star_\Omega$-residual set.*
3. *Take $K = [0.345, 1]$, then $K \in \tilde{\mu^{\star\star}}_\Omega$ and $K \in \mathcal{D}(\mu^{\star\star}_\Omega)$. Now, $X - K = [0, 0.345)$ and obviously, $X - K$ is of $\mu^{\star\star}_\Omega$-II category set. Hence, K is not $\mu^{\star\star}_\Omega$-residual.*

In a generalized metric space, we encounter some difficulty in analyzing the meaning of a set of continuity points of a given function. Theorem 21 below easily concludes the nature of this set.

Theorem 21. *Let (X, Ω) be a GMS. If $h : X \to \mathbb{R}$ is a function, then $\mathcal{C}_{\mu_\Omega}(h) = \bigcap_{n=1}^{\infty} Q_n$, where $Q_n \in \tilde{\mu}_\Omega$ for every n in $\mathbb{N}$ (that is, $\mathcal{C}_{\mu_\Omega}(h)$ is a μ_Ω-G_δ-set in X).*

Proof. Let $Q_n = \{y \mid \text{there is } \sigma \in \Omega, \delta > 0 \text{ such that } |h(s) - h(t)| < \frac{1}{n}, \text{ whenever } s, t \in B_\sigma(y, \delta)\}$. Thus, Q_n is μ_Ω-open for every $n \in \mathbb{N}$. Assume that h is continuous at $r \in X$, whereby for every $\varepsilon > 0$, there is $\delta > 0$ such that $|h(r) - h(y)| < \varepsilon$ whenever $y \in B_\sigma(r, \delta)$. Take $\varepsilon = \frac{1}{n}$ with $n \in \mathbb{N}$. Thus, $r \in Q_n$, since $r, y \in B_\sigma(r, \delta)$ for every n in $\mathbb{N}$. Therefore, $r \in \bigcap_{n=1}^{\infty} Q_n$, and

$$\mathcal{C}_{\mu_\Omega}(h) \subset \bigcap_{n=1}^{\infty} Q_n. \tag{1}$$

Conversely, assume that $y \in \bigcap_{n=1}^{\infty} Q_n$. Let $\varepsilon > 0$, so there is a positive integer m such that $\frac{1}{m} < \varepsilon$. Since $y \in \bigcap_{n=1}^{\infty} Q_n$, we get $y \in Q_m$. This implies that there is $\sigma \in \Omega, \delta > 0$ such that $|h(s) - h(t)| < \frac{1}{m}$ whenever $s, t \in B_\sigma(y, \delta)$, by definition of Q_n. Since $y \in B_\sigma(y, \delta)$ it turns out $|h(s) - h(y)| < \frac{1}{m}$. Hence, h is continuous at y and $y \in \mathcal{C}_{\mu_\Omega}(h)$. Therefore,

$$\bigcap_{n=1}^{\infty} Q_n \subset \mathcal{C}_{\mu_\Omega}(h). \tag{2}$$

From Equations (1) and (2), $\mathcal{C}_{\mu_\Omega}(h) = \bigcap_{n=1}^{\infty} Q_n$, where $Q_n \in \tilde{\mu}_\Omega$ for all $n \in \mathbb{N}$. □

Theorem 22 gives a shortcut for finding the significance of the set of all discontinuity points of a given function in a generalized metric space.

Theorem 22. *Let (X, η) be a GTS, $\eta \in \{\mu_\Omega, \mu_\Omega^\star\}$ and $h : X \to \mathbb{R}$ be a (η, μ_Ω)-cliquish. Then $\mathcal{D}_{\mu_\Omega}(h) \in \mathcal{M}(\eta)$.*

Proof. We present the proof only for $\eta = \mu_\Omega^\star$. Assume that h is (η, μ_Ω)-cliquish. By Theorem 21, it turns out that, $\mathcal{C}_{\mu_\Omega}(h) \in \mathcal{D}(\eta)$ and μ_Ω-G_δ-set in X. Since $\mu_\Omega^\star \supset \mu_\Omega$, so $\mathcal{C}_{\mu_\Omega}(h)$ is $\mu_\Omega^\star$-G_δ-set in X. Thus, $\mathcal{C}_{\mu_\Omega}(h)$ is η-G_δ-set in X. By Theorem 20, $\mathcal{C}_{\mu_\Omega}(h)$ is η-residual and therefore, $\mathcal{D}_{\mu_\Omega}(h) \in \mathcal{M}(\eta)$. □

The following two theorems are considered via a new strategy, such that:

- Theorem 23. To easily explore the meaning of a collection of all discontinuity points from a given function.
- Theorem 24. To check whether the given function is cliquish from the set of all discontinuity points of that function.

Theorem 23. *Let (X, μ_Ω) be a BS. If h is a $(\mu_\Omega^{\star\star}, \mu_\Omega)$-cliquish function, then $\mathcal{D}_{\mu_\Omega}(h)$ is $\mu_\Omega^{\star\star}$-meager and also μ_Ω-meager.*

Proof. Assume that h is $(\mu_\Omega^{\star\star}, \mu_\Omega)$-cliquish, so $\mathcal{C}_{\mu_\Omega}(h)$ is $\mu_\Omega^{\star\star}$-dense in X, by Theorem 21 is obtained that $\mathcal{C}_{\mu_\Omega}(h)$ is a μ_Ω-G_δ-set. Since (X, μ_Ω) is a BS, it turns out $\mu_\Omega \subset \mu_\Omega^{\star\star}$. Hence, $\mathcal{C}_{\mu_\Omega}(h)$ is $\mu_\Omega^{\star\star}$-G_δ-set in X. By Theorem 20 it follows that $\mathcal{C}_{\mu_\Omega}(h)$ is $\mu_\Omega^{\star\star}$-residual. Thus, $\mathcal{D}_{\mu_\Omega}(h)$ is a $\mu_\Omega^{\star\star}$-meager set.

As $\mu_\Omega \subset \mu_\Omega^{\star\star}$ and $\mathcal{C}_{\mu_\Omega}(h)$ is μ_Ω-dense in X, so $\mathcal{C}_{\mu_\Omega}(h)$ is a μ_Ω-dense set and a μ_Ω-G_δ-set in X. By Theorem 20, it is obtained that $\mathcal{C}_{\mu_\Omega}(h)$ is μ_Ω-residual. Therefore, $\mathcal{D}_{\mu_\Omega}(h)$ is a μ_Ω-meager set. □

Theorem 24. *Let (X, μ_Ω) be a μ-II category GTS, $\eta \in \{\mu_\Omega, \mu_\Omega^\star, \mu_\Omega^{\star\star}\}$ and $h : X \to \mathbb{R}$ be a map. If $\mathcal{D}_\eta(h)$ is $\mu_\Omega^{\star\star}$-meager, then h is $(\mu_\Omega^{\star\star}, \eta)$-cliquish.*

Proof. Assume that $\mathcal{D}_\eta(h)$ is $\mu_\Omega^{\star\star}$-meager. Let $K \in \tilde{\mu}_\Omega^{\star\star}$, then K is of μ_Ω-II category. By Theorem 5 part 3., it follows that K is of $\mu_\Omega^{\star\star}$-II category. So, $K \cap \mathcal{C}_\eta(h) = K - \mathcal{D}_\eta(h) \neq \emptyset$. Therefore, $\mathcal{C}_\eta(h)$ is a $\mu_\Omega^{\star\star}$-dense in X and h is $(\mu_\Omega^{\star\star}, \eta)$-cliquish. □

Theorem 25 reduces the complexity for finding whether a given function in a generalized metric space is $(\mu_\Omega^{\star\star}, \mu_\Omega)$-cliquish.

Theorem 25. *Let (X, μ_Ω) be a sGTS, BS. If h is (μ_Ω, μ_Ω)-cliquish, then, h is $(\mu_\Omega^{\star\star}, \mu_\Omega)$-cliquish.*

Proof. Assume that h is a (μ_Ω, μ_Ω)-cliquish function. By Theorem 22, $\mathcal{D}_{\mu_\Omega}(h)$ is a μ_Ω-meager set. Hence, by Theorem 3 part 2., $\mathcal{D}_{\mu_\Omega}(h)$ is a $\mu_\Omega^{\star\star}$-meager set. Since (X, μ_Ω) is a sGTS and BS, it turns out that (X, μ_Ω) is a GTS of μ_Ω-II category. By Theorem 24, it follows that h is a $(\mu_\Omega^{\star\star}, \mu_\Omega)$-cliquish function. □

Definition 9. *Let (X, μ) be a GTS and $f : X \to \mathbb{R}$ be a map. We denote by:*

- $[f = 0] = \{x \in X \mid f(x) = 0\}$,
- $[f = 1] = \{x \in X \mid f(x) = 1\}$,
- $[f < 0] = \{x \in X \mid f(x) < 0\}$,
- $[f > 0] = \{x \in X \mid f(x) > 0\}$.

Next, Theorem 26 gives a characterization of the cliquish functions in terms of nowhere dense sets.

Theorem 26. *Let (X, μ_Ω) be a GTS and $Q_0, Q_1 \subset X$. If $(X, \mu_\Omega^\star)$ is a BS, then the following are equivalent.*

1. *There is a $(\mu_\Omega^\star, \mu_\Omega)$-cliquish function $h : X \to \mathbb{R}$ such that $Q_0 = [h = 0]$ and $Q_1 = [h = 1]$.*
2. *$Q_0 \cap Q_1 = \emptyset$, the sets $c_{\mu_\Omega}(Q_0) - Q_0$, $c_{\mu_\Omega}(Q_1) - Q_1$ are $\mu_\Omega^\star$-meager and $c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1)$ is $\mu_\Omega^\star$-nowhere dense.*

Proof. *1.* $\Rightarrow$ *2.* By hypothesis, $Q_0 \cap Q_1 = \emptyset$. Let $x \in c_{\mu_\Omega}(Q_0) - Q_0$ and $V = \mathbb{R} - \{0\}$. Since $x \notin Q_0$, it turns out that $x \in h^{-1}(V)$. Now;

$$x \in c_{\mu_\Omega}(Q_0) \subset c_{\mu_\Omega}(h^{-1}(\{0\})) = X - i_{\mu_\Omega}(X - h^{-1}(\{0\})) = X - i_{\mu_\Omega}(h^{-1}(V)).$$

Thus, $x \notin i_{\mu_\Omega}(h^{-1}(V))$. Hence, $c_{\mu_\Omega}(Q_0) - Q_0 \subset \mathcal{D}_{\mu_\Omega}(h)$ and so $c_{\mu_\Omega}(Q_0) - Q_0$ is a $\mu_\Omega^\star$-meager set, see Theorem 22. Similarly, it is proven that $c_{\mu_\Omega}(Q_1) - Q_1$ is a $\mu_\Omega^\star$-meager set.

On the other hand, assume that $i_{\mu_\Omega^\star}(c_{\mu_\Omega^\star}(c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1))) \neq \emptyset$. So, there is an element $x \in i_{\mu_\Omega^\star}(c_{\mu_\Omega^\star}(c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1)))$ and we get a set $G \in \tilde{\mu}_\Omega^\star(x)$ such that

$$G \subset c_{\mu_\Omega^\star}(c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1)). \quad (3)$$

Since $(X, \mu_\Omega^\star)$ is a BS, G is of $\mu_\Omega^\star$-II category and hence $c_{\mu_\Omega^\star}(c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1))$ is of $\mu_\Omega^\star$-II category set, by (3) and the fact that superset of II-category set is of II-category.

Furthermore, $c_{\mu_\Omega^\star}(c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1)) \subset c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1)$, since $\mu_\Omega \subset \mu_\Omega^\star$. Therefore, $c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1)$ is of $\mu_\Omega^\star$-II category set. However,

$$c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1) \subset (c_{\mu_\Omega}(Q_0) - Q_0) \cup (c_{\mu_\Omega}(Q_1) - Q_1),$$

which implies that it must be $(c_{\mu_\Omega}(Q_0) - Q_0) \cup (c_{\mu_\Omega}(Q_1) - Q_1)$ of $\mu_\Omega^\star$-II category set, which is not possible. Therefore, $i_{\mu_\Omega^\star}(c_{\mu_\Omega^\star}(c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1))) = \emptyset$, and so $c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1)$ is $\mu_\Omega^\star$-nowhere dense in X.

2. $\Rightarrow$ *1.* The hypothesis implies that $c_{\mu_\Omega}(Q_i) - Q_i = \bigcup_{n=1}^{\infty} F_{i,n}$, where each $F_{i,n}$ is $\mu_\Omega^\star$-nowhere dense in X, $i \in \{0, 1\}$, $n \in \mathbb{N}$. Define a function $h : X \to \mathbb{R}$ as:

$$h(r) := \begin{cases} 0 & if \quad r \in Q_0, \\ 1 & if \quad r \in Q_1, \\ n^{-1} & if \quad r \in F_{0,n}, \\ 1 - n^{-1} & if \quad r \in F_{1,n}, \\ 2^{-1} & otherwise. \end{cases} \tag{4}$$

Thus, $Q_0 = [h = 0]$ and $Q_1 = [h = 1]$. Define $K_0 = \bigcup_{n=0}^{1} (c_{\mu_\Omega}(Q_n) - Q_n)$, so by Lemma 4 the set K_0 is a $\mu_\Omega^\star$-meager set. Let $t \in \mathcal{D}_{\mu_\Omega}(h)$ and assume that $t \notin K_0$, so $t \notin c_{\mu_\Omega}(Q_n) - Q_n$ for $n = 0, 1$. Therefore, $t \notin c_{\mu_\Omega} Q_0$ or $t \in Q_0$ and $t \notin c_{\mu_\Omega} Q_1$ or $t \in Q_1$. So, we have four cases:

Case-1: Assume that $t \notin c_{\mu_\Omega} Q_0$ and $t \notin c_{\mu_\Omega} Q_1$. Then, $t \notin c_{\mu_\Omega} Q_n$ and there is $W \in \mu_\Omega(t)$ such that $W \cap Q_n = \emptyset$ for $n = 0, 1$. This implies that $h(s) \neq 0$ and $h(s) \neq 1$ for every $s \in W$. Thus, by (4) is $h(W) \subset \{n^{-1}, 1 - n^{-1}, 2^{-1}\}$. Since $t \in W$ we get $h(t) \in \{n^{-1}, 1 - n^{-1}, 2^{-1}\}$. Hence, $h(t) = n^{-1}$ or $h(t) = 1 - n^{-1}$ or $h(t) = 2^{-1}$. Neither of this is possible.

Case-2: If $t \notin c_{\mu_\Omega} Q_0$ and $t \in Q_1$, by (4), $h(t) = 1$. Since h is a constant function on Q_1, t is a μ_Ω-continuity point of h, which is not possible.

Case-3: Assume that $t \in Q_0$ and $t \notin c_{\mu_\Omega} Q_1$. By (4) must be $h(t) = 0$. Since h is a constant function on Q_1, t is a μ_Ω-continuity point of h, which is a contradiction.

Case-4: Suppose $t \in Q_0$ and $t \in Q_1$. Then, $t \in Q_0 \cap Q_1$, but $Q_0 \cap Q_1 = \emptyset$.

Therefore, none of the cases are possible, whereby $t \in K_0$. So, $\mathcal{D}_{\mu_\Omega}(h) \subset K_0$ and $\mathcal{D}_{\mu_\Omega}(h)$ is $\mu_\Omega^\star$-meager, because $(X, \mu_\Omega^\star)$ is a BS. Hence, h is $(\mu_\Omega^\star, \mu_\Omega)$-cliquish. □

In a Baire space, the existence of a $(\mu_\Omega^\star, \mu_\Omega)$-cliquish function using $\mu_\Omega^\star$-nowhere dense sets can be found straightforwardly by Theorem 27.

Theorem 27. *Let (X, μ_Ω) be a GTS and $Q_0, Q_1 \subset X$. If $(X, \mu_\Omega^\star)$ is a BS, then the following are equivalent.*

1. *There is a $(\mu_\Omega^\star, \mu_\Omega)$-cliquish function $h : X \to \mathbb{R}$ such that $Q_0 \subset [h = 0]$ and $Q_1 \subset [h = 1]$.*
2. *$Q_0 \cap Q_1 = \emptyset$, and $c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1)$ is $\mu_\Omega^\star$-nowhere dense.*

Proof. *1.* $\Rightarrow$ *2.* By hypothesis, we have $Q_0 \cap Q_1 = \emptyset$. By Theorem 26 it is obtained that $c_{\mu_\Omega}([h = 0]) \cap c_{\mu_\Omega}([h = 1])$ is $\mu_\Omega^\star$-nowhere dense in X. Therefore, $c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1)$ is $\mu_\Omega^\star$-nowhere dense.

2. $\Rightarrow$ *1.* Define the sets $P_0 = Q_0 \cup (c_{\mu_\Omega}(Q_0) - c_{\mu_\Omega}(Q_1))$, $P_1 = Q_1 \cup (c_{\mu_\Omega}(Q_1) - c_{\mu_\Omega}(Q_0))$. Thus, $P_0 \cap P_1 = \emptyset$, $Q_0 \subset P_0 \subset c_{\mu_\Omega}(Q_0)$ and $Q_1 \subset P_1 \subset c_{\mu_\Omega}(Q_1)$. By definition of P_0 and P_1, is clear that $c_{\mu_\Omega}(P_0) \supset c_{\mu_\Omega}(Q_0)$ and $c_{\mu_\Omega}(P_1) \supset c_{\mu_\Omega}(Q_1)$. Hence,

$$c_{\mu_\Omega}(P_0) \cap c_{\mu_\Omega}(P_1) = c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1),$$

and so $c_{\mu_\Omega}(P_0) \cap c_{\mu_\Omega}(P_1)$ is $\mu_\Omega^\star$-nowhere dense in X. Now,

$$c_{\mu_\Omega}(Q_0) - Q_0 = c_{\mu_\Omega}(Q_0) \cap (X - Q_0) = c_{\mu_\Omega}(Q_0) \cap Q_1 \subset c_{\mu_\Omega}(Q_0) \cap c_{\mu_\Omega}(Q_1).$$

Thus, $c_{\mu_\Omega}(Q_0) - Q_0$ is $\mu_\Omega^\star$-nowhere dense. Similarly, $c_{\mu_\Omega}(Q_1) - Q_1$ is $\mu_\Omega^\star$-nowhere dense in X. Therefore, $c_{\mu_\Omega}(Q_0) - Q_0$ and $c_{\mu_\Omega}(Q_1) - Q_1$ are $\mu_\Omega^\star$-meager. Furthermore,

$$c_{\mu_\Omega}(P_0) - P_0 \subset c_{\mu_\Omega}(Q_0) - Q_0 \text{ and } c_{\mu_\Omega}(P_1) - P_1 \subset c_{\mu_\Omega}(Q_1) - Q_1.$$

Hence, $c_{\mu_\Omega}(P_0) - P_0$ and $c_{\mu_\Omega}(P_1) - P_1$ are $\mu_\Omega^\star$-meager. By Theorem 26, there is a $(\mu_\Omega^\star, \mu_\Omega)$-cliquish function $h : X \to \mathbb{R}$ such that $P_0 = [h = 0]$ and $P_1 = [h = 1]$. Therefore, there is a $(\mu_\Omega^\star, \mu_\Omega)$-cliquish function $h : X \to \mathbb{R}$ such that $Q_0 \subset [h = 0]$ and $Q_1 \subset [h = 1]$. □

Theorem 28 provides the easier route to finding the presence of $(\mu_\Omega^\star, \mu_\Omega)$-cliquish function using $\mu_\Omega^\star$-meager sets.

Theorem 28. *Let (X, μ_Ω) be a GTS and $Q^-, Q^+ \subset X$. If $(X, \mu_\Omega^\star)$ is a BS, then the following are equivalent.*

1. *There is a $(\mu_\Omega^\star, \mu_\Omega)$-cliquish function $h : X \to \mathbb{R}$ such that $Q^- = [h < 0]$ and $Q^+ = [h > 0]$.*
2. *$Q^- \cap Q^+ = \emptyset$ and the sets $Q^- - i_{\mu_\Omega}(Q^-)$, $Q^+ - i_{\mu_\Omega}(Q^+)$ are $\mu_\Omega^\star$-meager in X.*

Proof. *1.* $\Rightarrow$ *2.* By hypothesis, $Q^- \cap Q^+ = \emptyset$. Consider the function $h^- = max\{-h, 0\}$. So, h^- is $(\mu_\Omega^\star, \mu_\Omega)$-cliquish and $[h^- = 0] = X - Q^-$. By Theorem 26, it turns out that $c_{\mu_\Omega}([h^- = 0]) - [h^- = 0]$ is $\mu_\Omega^\star$-meager in X. Furthermore,

$$Q^- - i_{\mu_\Omega}(Q^-) = c_{\mu_\Omega}(X - Q^-) - (X - Q^-) = c_{\mu_\Omega}([h^- = 0]) - [h^- = 0].$$

Therefore, $Q^- - i_{\mu_\Omega}(Q^-)$ is $\mu_\Omega^\star$-meager in X. With similar considerations we can prove $Q^+ - i_{\mu_\Omega}(Q^+)$ is $\mu_\Omega^\star$-meager in X.

2. $\Rightarrow$ *1.* The hypothesis implies that the set $Q^- - i_{\mu_\Omega}(Q^-)$ can be expressed as the union of countably many $\mu_\Omega^\star$-nowhere dense subsets $\{F_n^- \mid n \in \mathbb{N}\}$, $Q^- - i_{\mu_\Omega}(Q^-) = \bigcup_{n=1}^{\infty} F_n^-$. Similarly, $Q^+ - i_{\mu_\Omega}(Q^+) = \bigcup_{n=1}^{\infty} F_n^+$ where $\{F_n^+ \mid n \in \mathbb{N}\}$ is a family of $\mu_\Omega^\star$-nowhere dense subsets of X.

Define a map $h : X \to \mathbb{R}$ as follows:

$$h(r) := \begin{cases} -1 & if \quad r \in i_{\mu_\Omega} Q^-, \\ 1 & if \quad r \in i_{\mu_\Omega} Q^+, \\ -n^{-1} & if \quad r \in F_n^-, \\ n^{-1} & if \quad r \in F_n^+, \\ 0 & otherwise. \end{cases} \tag{5}$$

So, $Q^- = [h < 0]$ and $Q^+ = [h > 0]$. Define $L_0 = Q_0 \cup Q_1$, where $Q_0 = Q^- - i_{\mu_\Omega}(Q^-)$ and $Q_1 = Q^+ - i_{\mu_\Omega}(Q^+)$. Thus, L_0 is $\mu_\Omega^\star$-meager. Let $t \in \mathcal{D}_{\mu_\Omega}(h)$ and suppose that $t \notin L_0$. Then, $t \notin Q_n$ for all $n = 0, 1$. Thus, $t \notin Q^-$ or $t \in i_{\mu_\Omega}(Q^-)$ and $t \notin Q^+$ or $t \in i_{\mu_\Omega}(Q^+)$. So, we have four cases:

Case-1: Assume that $t \notin Q^-$ and $t \notin Q^+$, Hence, $t \in [h \geq 0]$ and $t \in [h \leq 0]$ for which we get four cases.

- $h(t) > 0$ and $h(t) < 0$.
- $h(t) > 0$ and $h(t) = 0$.
- $h(t) = 0$ and $h(t) < 0$.
- $h(t) = 0$.

Thus, all the cases are not possible and our assumption is not true.

Case-2: If $t \notin Q^-$ and $t \in i_{\mu_\Omega} Q^+$, then $h(t) = 1$, by (5). Since h is a constant function on $i_{\mu_\Omega} Q^+$, t is a μ_Ω-continuity point of h, which is not possible.

Case-3: Suppose $t \in i_{\mu_\Omega} Q^-$ and $t \notin Q^+$. Then, $h(t) = -1$, by (5). Since h is a constant function on $i_{\mu_\Omega} Q^-$, t is a μ_Ω-continuity point of h, but t is a μ_Ω-discontinuity point of h.

Case-4: Assume that $t \in i_{\mu_\Omega} Q^-$ and $t \in i_{\mu_\Omega} Q^+$. Consider $t \in i_{\mu_\Omega}(Q^-)$, thus $h(t) = -1$. Now, $t \in i_{\mu_\Omega}(Q^+)$ so $h(t) = 1$. In both cases, t is a continuity point of h, which is not possible.

Thus, all the cases are not possible. Hence, $t \in L_0$ and so $\mathcal{D}_{\mu_\Omega}(h) \subset L_0$, whereby $\mathcal{D}_{\mu_\Omega}(h)$ is $\mu_\Omega^\star$-meager in X. Since $(X, \mu_\Omega^\star)$ is a BS, h is a $(\mu_\Omega^\star, \mu_\Omega)$-cliquish function on X. □

The following Theorem 29 provides an easier way to check the nature of the subsets of a domain space using the cliquish function.

Theorem 29. *Let (X, μ_Ω) be a GTS and $Q^-, Q^+ \subset X$. If $(X, \mu_\Omega^\star)$ is a BS and if there is a $(\mu_\Omega^\star, \mu_\Omega)$-cliquish function $g : X \to \mathbb{R}$ such that $Q^- \subset [g < 0]$ and $Q^+ \subset [g > 0]$, then $Q^- \cap Q^+ = \emptyset$ and the sets $Q^- \cap c_{\mu_\Omega}(Q^+)$, $Q^+ \cap c_{\mu_\Omega}(Q^-)$ are $\mu_\Omega^\star$-meager.*

Proof. Let g be a $(\mu^{\star}_{\Omega}, \mu_{\Omega})$-cliquish function such that $Q^- \subset [g < 0]$ and $Q^+ \subset [g > 0]$. Thus, $Q^- \cap Q^+ = \emptyset$. By hypothesis and Theorem 28, we have that $[g < 0] - i_{\mu_{\Omega}}([g < 0])$ and $[g > 0] - i_{\mu_{\Omega}}([g > 0])$ are $\mu^{\star}_{\Omega}$-meager set. Since $Q^- \cap cQ^+ \subset [g < 0] - i_{\mu_{\Omega}}([g < 0])$ and $Q^+ \cap cQ^- \subset [g > 0] - i_{\mu_{\Omega}}([g > 0])$, we have that $Q^- \cap c_{\mu_{\Omega}}(Q^+)$ and $Q^+ \cap c_{\mu_{\Omega}}(Q^-)$ are $\mu^{\star}_{\Omega}$-meager in X. □

The two Theorems 30 and 31 below provide shortcuts for finding the existence of $(\mu^{\star\star}_{\Omega}, \mu_{\Omega})$-cliquish function in a sBS using $\mu^{\star\star}_{\Omega}$-meager sets.

Theorem 30. *Let (X, μ_{Ω}) be a sGTS which is sBS, and $Q_0, Q_1 \subset X$. So, the following are equivalent.*

1. *There is a $(\mu^{\star\star}_{\Omega}, \mu_{\Omega})$-cliquish function $g : X \to \mathbb{R}$ such that $Q_0 = [g = 0]$ and $Q_1 = [g = 1]$.*
2. *$Q_0 \cap Q_1 = \emptyset$, the sets $c_{\mu_{\Omega}}(Q_0) - Q_0$, $c_{\mu_{\Omega}}(Q_1) - Q_1$ are $\mu^{\star\star}_{\Omega}$-meager and $c_{\mu_{\Omega}}(Q_0) \cap c_{\mu_{\Omega}}(Q_1)$ is $\mu^{\star\star}_{\Omega}$-nowhere dense in X.*

Proof. *1.* $\Rightarrow$ *2.* Note that $Q_0 \cap Q_1 = \emptyset$. Take $r \in c_{\mu_{\Omega}}(Q_0) - Q_0$ and define the set $L = \mathbb{R} - \{0\}$. In the same way as in the demonstration of Theorem 26 part *1.* $\Rightarrow$ *2.*, we get $c_{\mu_{\Omega}}(Q_0) - Q_0 \subset \mathcal{D}_{\mu_{\Omega}}(h)$. By hypothesis and Theorem 23, $\mathcal{D}_{\mu_{\Omega}}(h) \in \mathcal{M}(\mu^{\star\star}_{\Omega})$ which implies $c_{\mu_{\Omega}}(Q_0) - Q_0 \in \mathcal{M}(\mu^{\star\star}_{\Omega})$. In a similar way we have that $c_{\mu_{\Omega}}(Q_1) - Q_1 \in \mathcal{M}(\mu^{\star\star}_{\Omega})$.

By hypothesis and Lemma 2, we get $(X, \mu^{\star\star}_{\Omega})$ is a BS. With similar considerations in the proof of Theorem 26 part (*1.* $\Rightarrow$ *2.*), we get $c_{\mu_{\Omega}}(Q_0) \cap c_{\mu_{\Omega}}(Q_1)$ is in $\mathcal{N}(\mu^{\star\star}_{\Omega})$.

2. $\Rightarrow$ *1.* The hypothesis implies that $c_{\mu_{\Omega}}(Q_i) - Q_i = \bigcup_{n=1}^{\infty} D_{i,n}$ where each $D_{i,n} \in \mathcal{N}(\mu^{\star\star}_{\Omega})$, $i \in \{0, 1\}$, $n \in \mathbb{N}$.

Define a function $g : X \to \mathbb{R}$ as:

$$g(s) := \begin{cases} 0 & if \quad s \in Q_0, \\ 1 & if \quad s \in Q_1, \\ n^{-1} & if \quad s \in D_{0,n}, \\ 1 - n^{-1} & if \quad s \in D_{1,n}, \\ 2^{-1} & otherwise. \end{cases} \tag{6}$$

Thus, $Q_0 = [g = 0]$ and $Q_1 = [g = 1]$. Define $L_0 = \bigcup_{n=0}^{1} (c_{\mu_{\Omega}}(Q_n) - Q_n)$. Thus, L_0 is in $\mathcal{M}(\mu^{\star\star}_{\Omega})$, see Lemma 4. Let $u \in \mathcal{D}_{\mu_{\Omega}}(h)$ and assume that $u \notin L_0$. So, we obtain four cases as follows:

Case-1: $t \notin c_{\mu_{\Omega}}Q_0$ and $t \notin c_{\mu_{\Omega}}Q_1$;
Case-2: $t \notin c_{\mu_{\Omega}}Q_0$ and $t \in Q_1$;
Case-3: $t \in Q_0$ and $t \notin c_{\mu_{\Omega}}Q_1$;
Case-4: $t \in Q_0$ and $t \in Q_1$;

Note that by hypothesis and Lemma 2, $(X, \mu^{\star\star}_{\Omega})$ is a Baire space. By the same arguments from the proof of Theorem 26 part 2. $\Rightarrow$ 1., we get $u \in L_0$. Hence $\mathcal{D}_{\mu_{\Omega}}(h) \in \mathcal{M}(\mu^{\star\star}_{\Omega})$. Therefore, g is a $(\mu^{\star\star}_{\Omega}, \mu_{\Omega})$-cliquish function. □

Theorem 31. *Let (X, μ_{Ω}) be a sGTS which is sBS, and $Q_0, Q_1 \subset X$. So, the following are equivalent.*

1. *There is a $(\mu^{\star\star}_{\Omega}, \mu_{\Omega})$-cliquish function $g : X \to \mathbb{R}$ such that $Q_0 \subset [g = 0]$ and $Q_1 \subset [g = 1]$.*
2. *$Q_0 \cap Q_1 = \emptyset$ and the set $c_{\mu_{\Omega}}(Q_0) \cap c_{\mu_{\Omega}}(Q_1)$ is a $\mu^{\star\star}_{\Omega}$-nowhere dense set in X.*

Proof. *1.* $\Rightarrow$ *2.* By hypothesis, $Q_0 \cap Q_1 = \emptyset$, and by Theorem 30 it turns out $c_{\mu_{\Omega}}([g = 0]) \cap c_{\mu_{\Omega}}([g = 1]) \in \mathcal{N}(\mu^{\star\star}_{\Omega})$. Since the subset of a nowhere dense set is nowhere dense, we have that $c_{\mu_{\Omega}}(Q_0) \cap c_{\mu_{\Omega}}(Q_1) \in \mathcal{N}(\mu^{\star\star}_{\Omega})$.

2. $\Rightarrow$ *1.* With same considerations in the proof of Theorem 26 part *2.* $\Rightarrow$ *1.*, and using Theorem 30, we get the proof. □

The following Theorem 32 states that in order to investigate the existence of $(\mu^{\star\star}_{\Omega}, \mu_{\Omega})$-cliquish function in a sBS, divide the domain into two disjoint sets and verify whether the particular subset of those sets is $\mu^{\star\star}_{\Omega}$-meager or not.

Theorem 32. *Let (X, μ_Ω) be an sGTS, which is sBS, and $Q^-, Q^+ \subset X$. So, the following are equivalent.*

1. *There is a $(\mu_\Omega^{\star\star}, \mu_\Omega)$-cliquish function $g : X \to R$ with $Q^- = [g < 0]$ and $Q^+ = [g > 0]$.*
2. *$Q^- \cap Q^+ = \emptyset$ and the sets $Q^- - i_{\mu_\Omega}(Q^-)$, $Q^+ - i_{\mu_\Omega}(Q^+)$ are $\mu_\Omega^{\star\star}$-meager in X.*

Proof. *1.* $\Rightarrow$ *2.* By hypothesis, $Q^- \cap Q^+ = \emptyset$. Define a function $g^- = max\{-g, 0\}$. Note that g^- is $(\mu_\Omega^{\star\star}, \mu_\Omega)$-cliquish and $[g^- = 0] = X - Q^-$. By Theorem 30, it turns out $c_{\mu_\Omega}([g^- = 0]) - [g^- = 0] \in \mathcal{M}(\mu_\Omega^{\star\star})$. Obviously,

$$Q^- - i_{\mu_\Omega}(Q^-) = c_{\mu_\Omega}(X - Q^-) - (X - Q^-) = c_{\mu_\Omega}([g^- = 0]) - [g^- = 0].$$

Therefore, $Q^- - i_{\mu_\Omega}(Q^-)$ is in $\mathcal{M}(\mu_\Omega^{\star\star})$. Similarly, we show that $Q^+ - i_{\mu_\Omega}(Q^+)$ is a$\mu_\Omega^{\star\star}$-meager set in X.

2. $\Rightarrow$ *1.* Replace $\mu^\star$ by $\mu^{\star\star}$ in the proof of Theorem 28 part *2.* $\Rightarrow$ *1.*, and use the fact that $(X, \mu_\Omega^{\star\star})$ is Baire. □

Theorem 33 states the significance of subsets of domain space, by checking whether the existence of $(\mu_\Omega^{\star\star}, \mu_\Omega)$-cliquish function or not.

Theorem 33. *Let (X, μ_Ω) be a sGTS and $Q^-, Q^+ \subset X$. If (X, μ_Ω) is a sBS and if there is a $(\mu_\Omega^{\star\star}, \mu_\Omega)$-cliquish function $g : X \to \mathbb{R}$ such that $Q^- \subset [g < 0]$ and $Q^+ \subset [g > 0]$, then $Q^- \cap Q^+ = \emptyset$ and the sets $Q^- \cap c_{\mu_\Omega}(Q^+)$, $Q^+ \cap c_{\mu_\Omega}(Q^-)$ are $\mu_\Omega^{\star\star}$-meager in X.*

Proof. Let g be a $(\mu_\Omega^{\star\star}, \mu_\Omega)$-cliquish function such that $Q^- \subset [g < 0]$ and $Q^+ \subset [g > 0]$. Thus, $Q^- \cap Q^+ = \emptyset$. By hypothesis and Theorem 32, it turns out

$$[g < 0] - i_{\mu_\Omega}([g < 0]), [g > 0] - i_{\mu_\Omega}([g > 0]) \in \mathcal{M}(\mu_\Omega^{\star\star}). \quad (7)$$

Here, $Q^- \cap cQ^+ \subset [g < 0] - i_{\mu_\Omega}([g < 0])$ and $Q^+ \cap cQ^- \subset [g > 0] - i_{\mu_\Omega}([g > 0])$. By (7) and the fact that the subset of a meager set is meager, the sets $Q^- \cap c_{\mu_\Omega}(Q^+), Q^+ \cap c_{\mu_\Omega}(Q^-)$ are in $\mathcal{M}(\mu_\Omega^{\star\star})$. □

5. Conclusions

The various properties for nowhere dense sets and for second category sets in a strong Baire space have been evaluated. The evaluation was performed with the help of interior and closure components for nowhere dense sets and similarly meager sets components for second category sets. A new relationship between $\mathbb{L}(\eta)$ and $\mathbb{L}(\eta, \zeta)$ has obtained from residual sets. Further, the necessity of meager sets in a Baire space is studied for proving some equivalent conditions for cliquish functions using nowhere dense sets. Hence, the computational complexity of a given function from the collection of continuity points is reduced using meager sets.

Author Contributions: Investigation, E.A. and V.S.; conceptualization, M.K. and J.M.; methodology, E.A., M.K. and J.M.; supervision, J.M., E.A. and V.S.; writing—original draft, V.S. and E.A. All authors have read and accepted the published version of the manuscript.

Funding: This research and APC was funded by Escuela Superior Politécnica del Litoral.

Conflicts of Interest: The authors declare no conflict of interest.

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