axioms

Article

Exact Solutions of Maxwell Equations in Homogeneous Spaces with the Group of Motions $G_3(IX)$

Valeriy V. Obukhov [1,2]

[1] Institute of Scietific Research and Development, Tomsk State Pedagogical University (TSPU), 60 Kievskaya St., 634041 Tomsk, Russia; obukhov@tspu.edu.ru
[2] Laboratory for Theoretical Cosmology, International Center of Gravity and Cosmos, Tomsk State University of Control Systems and Radio Electronics (TUSUR), 36, Lenin Avenue, 634050 Tomsk, Russia

Abstract: This paper classifies the exact solutions of the Maxwell vacuum equations for the case when the electromagnetic fields and metrics of homogeneous spaces are invariant with respect to the motion group $G_3(IX)$. All the appropriate non-equivalent exact solutions of the Maxwell vacuum equations are found.

Keywords: Maxwell equations; Klein–Gordon–Fock equation; algebra of symmetry operators; theory of symmetry; linear partial differential equations

MSC: 70S10; 70G65; 70H20; 83C10; 83C15; 83C20; 83C50

check for updates

Citation: Obukhov, V.V. Exact Solutions of Maxwell Equations in Homogeneous Spaces with the Group of Motions $G_3(IX)$. *Axioms* **2023**, *12*, 135. https://doi.org/10.3390/axioms12020135

Academic Editors: Jiangen Liu and Xing Lü

Received: 23 December 2022
Revised: 20 January 2023
Accepted: 26 January 2023
Published: 29 January 2023

1. Introduction

All the known methods of integration of the main differential equations of mathematical physics are based on the complete reduction of these equations to a system of ordinary differential equations. The reduction is carried out using symmetry operators. For the equations of motion of a classical or quantum sample particle in external electromagnetic and gravitational fields, the symmetry operators are integrals of motion. It is known that a necessary condition for the existence of integrals of motion is the existence of the spacetime symmetry given by the Killing fields.

Thus, the problem of exact integration is closely related to the study of the spacetime symmetry. At present, two methods of the exact integration of equations of motion are known. These are the methods of commutative (CIM) and noncommutative (NCIM) integration. The first method is based on the theory of the complete separation of variables, and it is applicable in Stackel spaces. Stackel spaces admit complete sets consisting of mutually commuting Killing fields. The theory of Stackel spaces was developed in [1–7]. A description of the theory and a detailed bibliography can be found in [8–10] (see also, [11]). Solutions of the field equations, which are still used widely in the theory of gravitation, have been constructed on the basis of Stackel spaces. These solutions are often used in the study of various effects in gravitational fields (see, for example, [12–25]).

The second method (NCI method) is based on the use of noncommutative algebras of symmetry operators that are linear in moments and constructed using vector Killing fields. This method was proposed in [26]. The development of the method and its application to gravity theory can be found in [27–30].

As in Stackel spaces, in the spaces with a noncommutative group of motions, the equations of motion of a test particle admit the complete reduction to a system of ordinary differential equations. Therefore, we call the spacetime manifolds admitting noncommutative groups $G_r, r \geq 3$ post-Stackel spaces (PSS).

By analogy with Stackel spaces, we call the PSS non-isotropic, if a group G_r (or its subgroup of rank 3) acts transitively on a non-isotropic hypersurface of spacetime, or isotropic, if the hypersurface is isotropic. For non-isotropic post-Stackel spaces, we also use the term "homogeneous post-Stackel spaces (HPSS)".

The same classification problems can be considered for the PSS as for the Stackel spaces. For example, in the papers [9,10], a complete classification was given for the case when the Hamilton–Jacobi equation for a charged test particle admits the complete separation of variables in the external electromagnetic field. A similar classification problem was solved for the PSS as well. In [31], PSS with transitive four-parameter groups of motions were considered; in [32], HPSS were considered (see also, [33]); in [34], PSS with groups acting on isotropic hypersurfaces of transitivity were considered. PSS with four-parameter groups of motions were considered in [35], provided that these groups have transitive three-parameter subgroups. Thus, the potentials of all admissible electromagnetic fields have been found, for which the Hamilton–Jacobi and Klein–Gordon–Fock equations have algebras of symmetry operators given by the groups of motions of post-Stackel spaces. It was proved that the Klein–Gordon–Fock equation admits the algebra of symmetry operators given by groups of motions of PSS, if and only if the Hamilton–Jacobi equations admit the appropriate algebra of the integrals of motion.

The next classification problem is the classification of the electrovacuum solutions of the Einstein–Maxwell equations for the case when the CIM and NCIM methods are applicable. During the century-long history of general relativity, many exact solutions of the vacuum and electrovacuum Einstein equations have been found (see, for example, [36]). Nevertheless, this problem remains relevant. The main purpose of the classification is not so much to find new exact solutions but to list all the gravitational and electromagnetic fields in which the equations of motion of test particles can be exactly integrated or at least reduced to systems of ordinary differential equations. This problem is divided into two stages.

In the first stage, all the non-equivalent classes of solutions of the vacuum Maxwell equations for the potentials of admissible electromagnetic fields are found. In the second stage, the obtained classification is used to classify the corresponding electrovacuum spaces. Historically, for Stackel spaces this problem was solved before the problem of the first stage (see the bibliography given in [9,10,37]). The present article is devoted to solving the first stage of this classification problem. All the non-equivalent solutions of empty Maxwell equations in homogeneous spaces of type IX, according to Bianchi's classification, are found.

2. Admissible Electromagnetic Fields in Homogeneous Spaces

There are two definitions of homogeneous spaces. According to the first, a spacetime V_4 is homogeneous if its subspace V_3, endowed with the Euclidean space signature, admits coordinate transformations (forming the group $G_3(N)$ of motions of spaces V_4) that enable the connection of any two points in V_3 (see [38]). This definition directly implies that the metric of the V_4 in the semi-geodesic coordinate system $[u^i]$ can be represented as follows:

$$ds^2 = -du^{0^2} + \eta_{ab} l^a_\alpha l^b_\beta du^\alpha du^\beta, \quad g_{ij} = -\delta^0_i \delta^0_j + \delta^a_i \delta^b_j, \tag{1}$$

$$e^a_\alpha e^b_\beta \eta_{ab}(u^0), \quad det|\eta_{ab}| > 0 \quad e^a_{\alpha,0} = 0.$$

The coordinate indices of the variables of the semi-geodesic coordinate system are denoted by lower case Latin letters: $i, j, k = 0, 1 \ldots 3$. The coordinate indices of the variables of the local coordinate system on the hypersurface V_3 are denoted by lower case Greek letters: $\alpha, \beta, \gamma = 1, \ldots 3$. The time variable is denoted by a 0 index. The group indices and indices of nonholonomic frame are denoted by $a, b, c = 1, \ldots 3$. The summation is performed over the repeated upper and lower indices within the index range.

The 1-form $e^a_\alpha du^\alpha$ is invariant under the acting of the group $G_3(N)$. The vectors of the frame e^a_α define a nonholonomic coordinate system in V_3. The dual triplet of vectors

$$e^\alpha_a, \quad e^\alpha_a e^b_\alpha = \delta^b_a, \quad e^\alpha_a e^a_\beta = \delta^\alpha_\beta$$

defines the operators of the group algebra:

$$\hat{Y}_a = e^{\alpha}_a \partial_a, \quad [\hat{Y}_a, \hat{Y}_b] = C^c_{ab}\hat{Y}_c. \tag{2}$$

According to another definition, space–time V_4 is homogeneous if it admits a three-parameter group of motions $G_3(N)$, whose hypersurface V_3 of transitivity has the Euclidean space signature. The Killing vector fields ξ^{α}_a and their dual vector fields ξ^{a}_{α} form another frame of the space V_3 and another representation of the algebra of the group G_3. The vector fields ξ^{α}_a satisfy the Killing equations,

$$g^{\alpha\beta}_{,\gamma}\xi^{\gamma}_a = g^{\alpha\gamma}\xi^{\beta}_{a,\gamma} + g^{\beta\gamma}\xi^{\alpha}_{a,\gamma}, \tag{3}$$

and set the infinitesimal group operators of the algebra G_3 as

$$\hat{X}_a = \xi^{\alpha}_a \partial_\alpha, \quad [\hat{X}_a, \hat{X}_b] = C^c_{ab}\hat{X}_c. \tag{4}$$

Let us consider an electromagnetic field with potential A_i. For a charged test particle, moving in this external electromagnetic field, it has been proved that the Hamilton–Jacobi equation

$$g^{ij}P_iP_j = m, \quad P_i = p_i + A_i \tag{5}$$

and the Klein–Gordon–Fock equation

$$\hat{H}\varphi = (g^{ij}\hat{P}_i\hat{P}_j)\varphi = m^2\varphi, \quad \hat{P}_k = \hat{p}_k + A_k \tag{6}$$

admit the integrals of motion, which are given by the Killing vectors

$$\tilde{X}_\alpha = \xi^i_\alpha p_i \quad (or \quad \hat{\tilde{X}}_\alpha = \xi^i_\alpha \hat{p}_i),$$

if and only if the conditions

$$\xi^{\alpha}_a(\tilde{\mathbf{A}})_{,\alpha} = C^c_{ab}\tilde{\mathbf{A}} \tag{7}$$

are satisfied (see paper [32]). Here, $p_i = \partial_i\varphi$; $\hat{p}_k = -\imath\hat{\nabla}_k$; ($\hat{\nabla}_k$ is the covariant derivative operator corresponding to the partial derivative operator-$\hat{\partial}_i$ in the coordinate field u^i), and φ is a scalar function of a particle with mass m; $\tilde{\mathbf{A}}_a = \xi^{\alpha}_a A_\alpha$.

The electromagnetic field, whose potential satisfies condition (7), is called admissible. All admissible electromagnetic fields for the groups of motion $G_r(N)$ ($r \leq 4$) acting transitively on the hypersurfaces of the spacetime have been found in [32–35].

Solutions of the set of equations (7) for HPSS of type IX have the form:

$$A_\alpha = \alpha_a(u^0)l^a_\alpha \Rightarrow \mathbf{A}_a = l^\alpha_a A_\alpha = \alpha_a(u^0). \tag{8}$$

To prove this, let us find the frame vector. We use the metric tensor of IX-type space by Bianchi, found in Petrov's book [39]. As is known, the Bianchi type IX metric contains as a special case the space of constant positive curvature and, therefore, is of special interest for cosmology.

$$\begin{aligned} ds^2 = du^{1^2}[a_{11} - (a_{12}\cos 2u^3 + a_{22}\sin 2u^3)] + 2du^1du^3((a_{13}\cos u^3 - a_{23}\sin u^3) + \\ +2du^1du^2[(a_{13}\cos u^3 - a_{23}\sin u^3)\cos u^1 + (a_{12}\cos 2u^3 - a_{22}\sin 2u^3)\sin u^1] \\ +du^{2^2}[a_{33}\cos u^{1^2} + (a_{23}\cos u^3 + a_{13}\sin u^3)\sin 2u^1 + (a_{12}\sin 2u^3 + a_{22}\cos 2u^3 + a_{11})\sin u^{1^2}] \\ 2du^2du^3(a_{33}\cos u_1 + (a_{23}\cos u^3 + a_{13}\sin u^3)\sin u^1) + du^{3^2}a_{33} + edu^{0^2}. \end{aligned} \tag{9}$$

a_{ab} are arbitrary functions on u^0.

To obtain the functions l^α_a, it is sufficient to consider the components g_{13}, g_{23} from the system (3). The solution has the form:

$$l^a_\alpha = \delta^p_\alpha l^a_p(u^1, u^3) + \delta^3_\alpha \delta^a_3 \quad .$$

From the equations

$$g_{13} = a_{13}\cos u^3 - a_{23}\sin u^3 = \eta_{3a} l^a_1, \quad g_{23} = a_{33}\cos u_1 + (a_{23}\cos u^3 + a_{13}\sin u^3)\sin u^1 = \eta_{3a} l^a_1,$$

it follows that

$$l^a_\alpha = \begin{pmatrix} \cos u^3 & -\sin u^3 & 0 \\ \sin u^1 \sin u^3 & \sin u^1 \cos u^3 & \cos u^1 \\ 0 & 0 & 1 \end{pmatrix}, l^\alpha_a = \begin{pmatrix} \cos u^3 & \frac{\sin u^3}{\sin u^1} & -\frac{\cos u^1 \sin u^3}{\sin u^1} \\ -\sin u^3 & \frac{\cos u^3}{\sin u^1} & -\frac{\cos u^1 \cos u^3}{\sin u^1} \\ 0 & 0 & 1 \end{pmatrix}, \quad (10)$$

$l^a_\alpha l^\alpha_b = \delta^a_b$.

The lower index numbers are lines. One can show that the vector fields (10) satisfy Equations (1) and (2). We present the components of the vectors ξ^α_a in the form of a matrix:

$$||\xi^\alpha_a|| = \begin{pmatrix} 0 & 1 & 0 \\ \cos u^2 & -\frac{\cos u^1 \sin u^2}{\sin u^1} & \frac{\sin u^2}{\sin u^1} \\ -\sin u^2 & -\frac{\cos u^1 \cos u^2}{\sin u^1} & \frac{\cos u^2}{\sin u^1} \end{pmatrix}.$$

The components $\tilde{\mathbf{A}}_\alpha$ can be expressed through $\mathbf{A}_\alpha$ as follows:

$$\tilde{\mathbf{A}}_a = Z^b_a \mathbf{A}_b,$$

where

$$||Z^b_a = \xi^\alpha_a l^b_\alpha|| = \begin{pmatrix} \sin u^1 \sin u^3 & \sin u^1 \cos u^3 & \cos u^1 \\ (\cos u^2 \cos u^3 - \sin u^2 \sin u^3 \cos u^1) & -(\cos u^2 \sin u^3 + \sin u^2 \cos u^3 \cos u^1) & \sin u^1 \sin u^2 \\ -(\sin u^2 \cos u^3 + \cos u^2 \sin u^3 \cos u^1) & (\sin u^2 \sin u^3 - \cos u^2 \cos u^3 \cos u^1) & \cos u^2 \sin u^1 \end{pmatrix}.$$

It can be shown by direct calculation that the elements of the matrix Z^b_a satisfy the equation:

$$Z^b_{a|c} = C^{a_1}_{ca} Z^b_{a_1}, \quad |a = l^\alpha_a \partial_\alpha. \quad (11)$$

Therefore, Equation (7) can be reduced to the form:

$$\xi^\alpha_a \mathbf{A}_{b,\alpha} = 0 \Rightarrow \mathbf{A}_a = \alpha_a(u^0). \quad (12)$$

3. Maxwell's Equations with Zero Electromagnetic Field Sources in a Homogeneous Spacetime

All the exact solutions of vacuum Maxwell equations for solvable groups have been found in papers [40,41]. In the present paper, the problem is solved for the group $G_3(IX)$.

We use the first definition of homogeneous spaces. Note, that for the spacetime with the groups of motions $G_3(I) - G_3(VI), G_3(IX)$, both definitions are equivalent.

Consider the Maxwell equations with zero electromagnetic field sources in homogeneous space in the presence of an electromagnetic field invariant with respect to the group G_r:

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}F^{ij})_{,j} = 0. \quad (13)$$

The metric tensor is defined by relations (1), and the electromagnetic potential is defined by relations (7). When $i = 0$, from the set of Equation (13), it follows that

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\alpha\beta}F_{0\beta})_\alpha = \frac{1}{l}(l l^\alpha_a \eta^{ab}\dot{\alpha}_b)_{,\alpha} = \eta^{ab}\rho_a \dot{\alpha}_b = 0. \quad (14)$$

Here, we denote $g = -\det||g_{\alpha\beta}|| = -(\eta l)^2$, where $\eta^2 = \det||\eta_{\alpha\beta}||$, $l = \det||l^a_\alpha||$, and $\rho_a = l^\alpha_{a,\alpha} + l_{|a}/l$; the dots mean the time derivatives. Let $i = \alpha$. Then, from Equation (13), it follows:

$$\frac{1}{\eta}(\eta g^{\alpha\beta}F_{0\beta})_{,0} = \frac{1}{l}(lg^{\nu\beta}g^{\alpha\gamma}F_{\beta\gamma})_{,\nu} \Rightarrow \frac{1}{\eta}(\eta\eta^{ab}l^\alpha_a\dot{\alpha}_b)_{,0} = \frac{1}{l}(ll^\nu_a l^\beta_b\eta^{ab}l^\alpha_{\tilde{a}}l^\gamma_{\tilde{b}}\eta^{\tilde{a}\tilde{b}}F_{\beta\gamma})_{,\nu} \Rightarrow \tag{15}$$

$$(\eta\eta^{ab}\dot{\alpha}_b)_{,0} = \frac{\eta l^a_\alpha}{l}(ll^\beta_b l^\alpha_{\tilde{a}_1}l^\gamma_{\tilde{b}}F_{\beta\gamma})_{|a_1}\eta^{a_1 b}\eta^{\tilde{a}\tilde{b}}. \tag{16}$$

Let us find the components of $F_{\alpha\beta}$, using the relations (8).

$$F_{\alpha\beta} = (l^a_{\beta,\alpha} - l^a_{\beta,\alpha})\alpha_a = l^c_\beta l^\gamma_c l^d_\alpha l^\nu_d(l^a_{\gamma,\nu} - l^a_{\nu,\gamma})\alpha_a = l^b_\beta l^a_\alpha l^c_\gamma(l^\gamma_{a|b} - l^\gamma_{b|a})\alpha_c = l^b_\beta l^a_\alpha C^c_{ba}\alpha_c. \tag{17}$$

Then,

$$(lF^{\alpha\beta})_{,\beta} = \eta^{ab}\eta^{\tilde{a}\tilde{b}}C^d_{\tilde{b}b}\alpha_d((ll^\alpha_a)_{|\tilde{a}} + ll^\alpha_a l^\gamma_{\tilde{a},\gamma}). \tag{18}$$

The structural constants of a group G_3 can be present in the form:

$$C^c_{ab} = C^c_{12}\varepsilon^{12}_{\tilde{a}\tilde{b}} + C^c_{13}\varepsilon^{13}_{\tilde{a}\tilde{b}} + C^c_{23}\varepsilon^{23}_{\tilde{a}\tilde{b}}, \quad \varepsilon^{AB}_{ab} = \delta^A_a\delta^B_b - \delta^A_b\delta^B_a. \tag{19}$$

Using the notations

$$\sigma_1 = C^a_{23}\alpha_a, \quad \sigma_2 = C^a_{31}\alpha_a, \quad \sigma_3 = C^a_{12}\alpha_a,$$

$$\gamma_1 = \sigma_1\eta_{11} + \sigma_2\eta_{12} + \sigma_3\eta_{13}, \quad \gamma_2 = \sigma_1\eta_{12} + \sigma_2\eta_{22} + \sigma_3\eta_{23}, \quad \gamma_3 = \sigma_1\eta_{13} + \sigma_2\eta_{23} + \sigma_3\eta_{33},$$

let us reduce Maxwell's equations (13) to the form:

$$\eta(\eta^{ab}\dot{\alpha}_b)_{,0} = \delta^a_1(\gamma_1(C^1_{32}) - \gamma_2(C^1_{31} + \rho_3) + \gamma_3(C^1_{21} + \rho_2)) + \delta^a_2(\gamma_1(C^2_{32} + \rho_3) + \tag{20}$$
$$\gamma_2 C^2_{13} - \gamma_3(C^2_{12}\rho_1)) + \delta^a_3(-\gamma_1(C^3_{23} + \rho_2) + \gamma_2(C^3_{13} + \rho_1) + \gamma_3 C^3_{21}).$$

The order of Equation (20) can be decreased by introducing a new independent function:

$$\beta_a = \beta^a = \eta\eta^{ab}\dot{\alpha}_b \quad \Rightarrow \quad \eta\dot{\alpha}_a = \eta_{ab}\beta^b. \tag{21}$$

Let us consider the Maxwell equations for the group $G_3(IX)$. In this case, the nonzero structural constants are as follows:

$$C^3_{12} = C^2_{31} = C^1_{23} = 1.$$

The functions σ_a and γ_1 have the form

$$\sigma_1 = \alpha_1, \quad \sigma_2 = \alpha_2, \quad \sigma_3 = \alpha_3.$$

$$\gamma_1 = \alpha_1\eta_{11} + \alpha_2\eta_{12} + \alpha_3\eta_{13}, \quad \gamma_2 = \alpha_1\eta_{12} + \alpha_2\eta_{22} + \alpha_3\eta_{23}, \quad \gamma_1 = \alpha_1\eta_{13} + \alpha_2\eta_{23} + \alpha_3\eta_{33}.$$

Using these relations, we obtain Maxwell's Equations (14) and (20) as a system of linear algebraic equations on the unknown functions n_{ab}:

$$n_{ab} = \frac{\eta_{ab}}{\eta} \Rightarrow \eta = \frac{1}{\det n_{ab}}. \tag{22}$$

$$\hat{W}\hat{n} = \hat{\omega}, \tag{23}$$

where

$$\hat{W} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & \alpha_2 & \alpha_3 & 0 \\ 0 & \beta_1 & 0 & \beta_2 & \beta_3 & 0 \\ 0 & 0 & \alpha_1 & 0 & \alpha_2 & \alpha_3 \\ 0 & 0 & \beta_1 & 0 & \beta_2 & \beta_3 \end{pmatrix}, \tag{24}$$

$$\hat{n}^T = (n_{11}, n_{12}, n_{13}, n_{22}, n_{23}, n_{33}); \quad \hat{\omega}^T = (-\dot{\beta}_1, \dot{\alpha}_1, -\dot{\beta}_2, \dot{\alpha}_2, -\dot{\beta}_3, \dot{\alpha}_3).$$

Index T means the transposition of a matrix. Let us find the algebraic complement of the matrix $\hat{W}$:

$$\hat{V} = \begin{pmatrix} \beta_1 V_1^2 & -\alpha_1 V_1^2 & \beta_2 V_1^2 & -\alpha_2 V_1^2 & \beta_3 V_1^2 & -\alpha_3 V_1^2 \\ \beta_1 V_1 V_2 & -\alpha_1 V_1 V_2 & \beta_2 V_1 V_2 & -\alpha_2 V_1 V_2 & \beta_3 V_1 V_2 & -\alpha_3 V_1 V_2 \\ \beta_1 V_1 V_3 & -\alpha_1 V_1 V_3 & \beta_2 V_1 V_3 & -\alpha_2 V_1 V_3 & \beta_3 V_1 V_3 & -\alpha_3 V_1 V_3 \\ \beta_1 V_2^2 & -\alpha_1 V_2^2 & \beta_2 V_2^2 & -\alpha_2 V_2^2 & \beta_3 V_2^2 & -\alpha_3 V_2^2 \\ \beta_1 V_2 V_3 & -\alpha_1 V_2 V_3 & \beta_2 V_2 V_3 & -\alpha_2 V_2 V_3 & \beta_3 V_2 V_3 & -\alpha_3 V_2 V_3 \\ \beta_1 V_3^2 & -\alpha_1 V_3^2 & \beta_2 V_3^2 & -\alpha_2 V_3^2 & \beta_3 V_3^2 & -\alpha_3 V_3^2 \end{pmatrix}. \tag{25}$$

As $\hat{W}$ is a singular matrix, $\hat{V}$ is the annulling matrix for $\hat{W}$:

$$\hat{V}\hat{W} = 0. \tag{26}$$

Therefore, one of the equations from the system (23) can be replaced by the equation:

$$\delta^{ab}(\dot{\alpha}_a \dot{\alpha}_b + \dot{\beta}_a \dot{\beta}_b) \Rightarrow \delta^{ab}(\alpha_a \alpha_b + \beta_a \beta_b) = c^2 = const. \tag{27}$$

Depending on the rank of the matrix $\hat{W}$, one or more functions n_{ab} are independent. The remaining functions n_{ab} can be expressed through them and through the functions α_a and β_a. For classification, it is necessary to find non-equivalent solutions of the system (23). Obviously, this system is symmetric with respect to the transposition $l_1^\alpha \leftrightarrow l_2^\alpha$. Therefore, the reference indices $a = 1$ and $a = 2$ can be interchanged. Taking this observation into account, let us consider all the non-equivalent options.

4. Solutions of Maxwell Equations

1. $a_1 V_1 \neq 0 \Rightarrow$ the minor $\hat{W}_{12}$ and its inverse matrix $\hat{\Omega} = \hat{W}_{12}^{-1}$ have the form:

$$\hat{W}_{12} = \begin{pmatrix} \alpha_2 & \alpha_3 & 0 & 0 & 0 \\ \alpha_1 & 0 & \alpha_2 & \alpha_3 & 0 \\ \beta_1 & 0 & \beta_2 & \beta_3 & 0 \\ 0 & \alpha_1 & 0 & \alpha_2 & \alpha_3 \\ 0 & \beta_1 & 0 & \beta_2 & \beta_3 \end{pmatrix}, \tag{28}$$

$$\hat{\Omega}_1 = \begin{pmatrix} -\frac{V_2}{\alpha_1 V_1} & -\frac{\alpha_3\beta_2}{\alpha_1 V_1} & \frac{\alpha_2\alpha_3}{\alpha_1 V_1} & -\frac{\alpha_3\beta_3}{\alpha_1 V_1} & \frac{\alpha_3^2}{\alpha_1 V_1} \\ -\frac{V_3}{\alpha_1 V_1} & \frac{\alpha_2\beta_2}{\alpha_1 V_1} & -\frac{\alpha_2^2}{\alpha_1 V_1} & \frac{\alpha_2\beta_3}{\alpha_1 V_1} & -\frac{\alpha_2\alpha_3}{\alpha_1 V_1} \\ -\frac{V_2^2}{\alpha_1 V_1^2} & \frac{(\alpha_3\beta_1 V_1 - \alpha_2\beta_3 V_3)}{\alpha_1 V_1^2} & \frac{\alpha_3(\alpha_2 V_2 - \alpha_1 V_1)}{\alpha_1 V_1^2} & -\frac{\alpha_3\beta_3 V_2}{\alpha_1 V_1^2} & \frac{\alpha_2^2 V_2}{\alpha_1 V_1^2} \\ -\frac{V_2 V_3}{\alpha_1 V_1^2} & \frac{\alpha_2\beta_2 V_2}{\alpha_1 V_1^2} & -\frac{\alpha_2^2 V_2}{\alpha_1 V_1^2} & -\frac{\alpha_3\beta_3 V_3}{\alpha_1 V_1^2} & \frac{\alpha_3^2 V_3}{\alpha_1 V_1^2} \\ -\frac{V_3^2}{\alpha_1 V_1^2} & \frac{\alpha_2\beta_2 V_3}{\alpha_1 V_1^2} & -\frac{\alpha_2^2 V_3}{\alpha_1 V_1^2} & \frac{(\alpha_3\beta_2 V_3 - \alpha_2\beta_1 V_1)}{\alpha_1 V_1^2} & \frac{\alpha_2(\alpha_1 V_1 - \alpha_3 V_3)}{\alpha_1 V_1^2} \end{pmatrix}. \tag{29}$$

Then, the solution of Equation (23) can be represented as:

$$\hat{n}_1 = \hat{\Omega}_1 \hat{\omega}_1, \tag{30}$$

where

$$\hat{n}_1^T = (n_{12}, n_{13}, n_{22}, n_{23}, n_{33}); \quad \hat{\omega}_1^T = (-(\dot{\beta}_1 + \alpha_1 n_{11}), -\dot{\beta}_2, \dot{\alpha}_2, -\beta_3, \dot{\alpha}_3).$$

Function n_{11}, as well as the functions α_a and β_a are arbitrary functions of u^0 that obey condition (27).

2. $\alpha_2 V_1 \neq 0, \Rightarrow \alpha_1 = 0 \Rightarrow$ the minor $\hat{W}_{14}^{-1}$ and its inverse matrix $\hat{\Omega}_2 = \hat{W}_{14}^{-1}$ have the form:

$$\hat{W}_{14} = \begin{pmatrix} \alpha_2 & \alpha_3 & 0 & 0 & 0 \\ \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_2 & \alpha_3 \\ 0 & \beta_1 & 0 & \beta_2 & \beta_3 \end{pmatrix}, \quad \hat{\Omega}_2 = \begin{pmatrix} \frac{\beta_3}{V_1} & -\frac{\alpha_3}{V_1} & 0 & 0 & 0 \\ -\frac{\beta_2}{V_1} & \frac{\alpha_2}{V_1} & 0 & 0 & 0 \\ \frac{a_3^2 \beta_1 \beta_2}{\alpha_2 V_1^2} & -\frac{\alpha_3^2 \beta_1}{V_1^2} & \frac{1}{\alpha_2} & -\frac{\alpha_3 \beta_3}{\alpha_2 V_1} & \frac{a_3^2}{\alpha_2 V_1} \\ -\frac{a_3 \beta_1 \beta_2}{V_1^2} & \frac{\alpha_2 \alpha_3 \beta_1}{V_1^2} & 0 & \frac{\beta_3}{V_1} & -\frac{a_3}{V_1} \\ \frac{a_2 \beta_1 \beta_2}{V_1^2} & -\frac{\alpha_2^2 \beta_1}{V_1^2} & 0 & -\frac{\beta_2}{V_1} & \frac{\alpha_2}{V_1} \end{pmatrix}. \tag{31}$$

The solution of Equation (23) can be represented as:

$$\hat{n}_2 = \hat{\Omega} \hat{\omega}_2, \tag{32}$$

where

$$\hat{n}_2^T = (n_{12}, n_{13}, n_{22}, n_{23}, n_{33});$$
$$\hat{\omega}_2 = (-\dot{\beta}_1, -\beta_1 n_{11}, -\dot{\beta}_2, -\dot{\beta}_3, \dot{\alpha}_3).$$

Function n_{11}, as well as the functions α_a and β_a are arbitrary functions of u^0 that obey condition (27).

3. $a_3 V_1 \neq 0, \Rightarrow a_1 = a_2 = 0 \Rightarrow$ the minor $\hat{W}_{16}^{-1}$ and its inverse matrix $\hat{\Omega}_3 = \hat{W}_{16}^{-1}$ have the form:

$$\hat{W}_{16} = \begin{pmatrix} 0 & a_3 & 0 & 0 & 0 \\ \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 \\ \beta_1 & 0 & \beta_2 & \beta_3 & 0 \\ 0 & 0 & 0 & 0 & a_3 \end{pmatrix}, \quad \hat{\Omega}_3 = \begin{pmatrix} -\frac{\beta_3}{a_3 \beta_2} & \frac{1}{\beta_3} & 0 & 0 & 0 \\ \frac{1}{a_3} & 0 & 0 & 0 & 0 \\ \frac{\beta_1 \beta_3}{a_3 \beta_2^2} & -\frac{\beta_1}{\beta_2^2} & -\frac{\beta_3}{\beta_2 a_3} & \frac{1}{\beta_2} & 0 \\ 0 & 0 & \frac{1}{a_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a_3} \end{pmatrix}. \tag{33}$$

Then, the solution of Equation (23) can be represented as:

$$\hat{n}_3 = \hat{\Omega}_3 \hat{\omega}_3, \tag{34}$$

where

$$\hat{n}_3^T = (n_{12}, n_{13}, n_{22}, n_{23}, n_{33});$$
$$\hat{\omega}_3^T = (-\dot{\beta}_1, -\beta_1 n_{11}, -\dot{\beta}_2, 0, -\dot{\beta}_3).$$

Function n_{11}, as well as the functions α_3 and β_a are arbitrary functions of u^0 that obey condition (27).

4. $a_1 V_3 \neq 0 \Rightarrow V_1 = V_2 = 0$; otherwise, we obtain a solution equivalent to the previous ones. As $V_3 \neq 0 \Rightarrow \alpha_3 = \beta_3 = 0$. The minor $\hat{W}_{62}$ and its inverse matrix $\hat{\Omega}_4 = \hat{W}_{62}^{-1}$ have the form:

$$\hat{W}_{26} = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & \alpha_2 & 0 \\ 0 & \beta_1 & 0 & \beta_2 & 0 \\ 0 & 0 & \alpha_1 & 0 & \alpha_2 \\ 0 & 0 & \beta_1 & 0 & \beta_2 \end{pmatrix}, \quad \hat{\Omega}_4 = \begin{pmatrix} \frac{1}{\alpha_1} & -\frac{\alpha_2\beta_2}{\alpha_1 V_3} & \frac{\alpha_2^2}{\alpha_1 V_3} & 0 & 0 \\ 0 & \frac{\beta_2}{V_3} & -\frac{\alpha_2}{V_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{\beta_2}{V_3} & -\frac{\alpha_2}{V_3} \\ 0 & -\frac{\beta_1}{V_3} & \frac{\alpha_1}{V_3} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\beta_1}{V_3} & \frac{\alpha_1}{V_3} \end{pmatrix}. \tag{35}$$

Then, the solution of Equation (23) can be represented as:

$$\hat{n}_4 = \hat{\Omega}_4 \hat{\omega}_4, \tag{36}$$

where

$$\hat{n}_4^T = (n_{11}, n_{12}, n_{13}, n_{22}, n_{23});$$
$$\hat{\omega}_4^T = (-\dot{\beta}_1, -\dot{\beta}_2, \dot{\alpha}_2, 0, 0).$$

Function n_{33}, as well as the functions α_1, α_2, and β_a, are arbitrary functions of u^0 that obey the condition (27).

5. $V_a = 0$. Let us represent the system of Maxwell equations in the form:

$$\hat{Q}\hat{n}_I = \hat{\omega}_I$$

were

$$\hat{Q} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & \alpha_2 & \alpha_3 & 0 \\ 0 & 0 & \alpha_1 & 0 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & \beta_2 & \beta_3 & 0 \\ 0 & 0 & \beta_1 & 0 & \beta_2 & \beta_3 \end{pmatrix}, \tag{37}$$

$$\hat{\omega}_I = (\hat{\omega}_\beta, \hat{\omega}_\alpha); \quad \hat{\omega}_\beta = (\omega_1, \omega_3, \omega_5), \quad \hat{\omega}_\alpha = (\omega_2, \omega_4, \omega_6)$$

$$\hat{n}_I = (\hat{n}_\alpha, \hat{\beta}_\alpha); \quad \hat{n}_\alpha = (n_{11}, n_{12}, n_{13}), \quad \hat{n}_\beta = (n_{22}, n_{23}, n_{33})$$

Let us consider all possible options.

(a) $\alpha_1 \neq 0 \Rightarrow \beta_a = \frac{\alpha_a \beta_1}{\alpha_1}$. Maxwell's equations will take the form:

$$\hat{W}_I \hat{n}_\alpha = (\hat{\omega}_\beta - \hat{Q}_1 \hat{n}_\beta) \Rightarrow \hat{n}_\alpha = \hat{W}_I^{-1}(\hat{\omega}_\beta - \hat{Q}_1 \hat{n}_\beta),$$

$$\beta_1 \hat{W}_I \hat{n}_\alpha = \alpha_1 \hat{\omega}_\alpha - \beta_1 \hat{Q}_1 \hat{n}_\beta \Rightarrow \beta_1 \hat{\omega}_\beta - \alpha_1 \hat{\omega}_\alpha = 0 \Rightarrow$$

$$\begin{cases} \alpha_1 \dot{\alpha}_2 + \beta_1 \dot{\beta}_2 = 0, \\ \alpha_1 \dot{\alpha}_3 + \beta_1 \dot{\beta}_3 = 0, \\ \alpha_1 \dot{\alpha}_1 + \beta_1 \dot{\beta}_1 = 0. \end{cases} \tag{38}$$

Here:

$$\hat{W}_I = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \end{pmatrix}, \hat{W}_I^{-1} = \begin{pmatrix} \frac{1}{\alpha_1} & -\frac{\alpha_2}{\alpha_1^2} & -\frac{\alpha_3}{\alpha_1^2} \\ 0 & \frac{1}{\alpha_1} & 0 \\ 0 & 0 & \frac{1}{\alpha_1} \end{pmatrix}, \hat{Q}_I = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \alpha_3 & 0 \\ 0 & \alpha_2 & \alpha_3, \end{pmatrix}$$

From the last equation of the system (38) it follows:

$$a_1 = e\sin\varphi, \quad \beta_1 = e\cos\varphi, \quad e = const,$$

Thus, $\beta_2 = \alpha_2 \frac{\cos\varphi}{\sin\varphi}, \quad \beta_3 = \alpha_3 \frac{\cos\varphi}{\sin\varphi},$ and from the previous equations it follows:

$$\alpha_a = ec_a \sin\varphi, \quad \beta_a = ec_a \cos\varphi, \quad e, c_a = const, \quad c_1 = 1.$$

Then matrices $\hat{W}_I, \hat{W}_I^{-1}, \hat{Q}_I$ and lines $\hat{\omega}^T$ take the form:

$$\hat{W}_I = \sin\varphi\hat{w}, \quad \hat{W}_I^{-1} = \frac{1}{\sin\varphi}\hat{w}^{-1}, \quad \hat{Q}_I = \sin\varphi\hat{q}.$$

$$\hat{w} = \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{w}^{-1} = \begin{pmatrix} 1 & -c_2 & -c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{q} = \begin{pmatrix} 0 & 0 & 0 \\ c_2 & c_3 & 0 \\ 0 & c_2 & c_3, \end{pmatrix}$$

$$\hat{\omega}_\beta^T = \dot{\varphi}\hat{C}^T = \dot{\varphi}\sin\varphi(c_2, 1, c_3),$$

The solution of Maxwell's equations can be represented as:

$$\hat{n}_\alpha = \hat{w}^{-1}(\dot{\varphi}\hat{C}^T - \hat{q}\hat{n}_\beta)$$

Functions n_{22}, n_{23}, and n_{33}, as well as the function φ, are arbitrary functions of u^0.

(b) $V_a = 0, \alpha_3 \neq 0$. The solutions, which are not equivalent to the previous ones, can be obtained under the conditions $a_1 = a_2 = 0 \Rightarrow \beta_1 = \beta_2 = 0$. From Maxwell's equations it follows:

$$\alpha_3 n_{13} = \alpha_3 n_{23} = 0, \quad a_3 n_{33} = -\dot{\beta}_3, \quad \beta_3 n_{33} = \dot{\alpha}_3 \Rightarrow a_3\dot{a}_3 + \beta_3\dot{\beta}_3 = 0.$$

The solution has the form:

$$n_{33} = \dot{\varphi}, \quad n_{13} = n_{23} = \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0, \quad a_3 = c\cos\varphi, \quad \beta_3 = c\sin\varphi.$$

Functions $\varphi, n_{11}, n_{12}, n_{22}$ - are arbitrary functions on u^0.

5. Conclusions

It is known that homogeneous spaces of IV and IX types according to the Bianchi classification include as special cases the spaces of constant curvature. Hence, they receive special interest in cosmology. In the universe with the metric of homogeneous space, all physical fields are invariant with respect to the group of motions of the spacetime. Therefore, these exact fields should be considered first when solving the self-consistent Einstein equations, in particular, the Einstein–Maxwell equations. The final goal of the classification of PSS with admissible electromagnetic fields is to enumerate all the electrovacuum solutions of the Einstein–Maxwell equations. In [40,41], the complete classification of the vacuum solutions of the Maxwell equations for homogeneous spaces with solvable groups of motions was carried out. In the present paper, the same problem was solved for HPSS of IX-type. For the final decision of the first stage of the classification problem, it remains to consider the HPSS $VIII$-type, which will be detailed in the next paper. The results obtained will be used in the second stage for integration of the corresponding Einstein–Maxwell equations.

Funding: This work was supported by the Russian Science Foundation, project number N 23-21-00275.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data that support the findings of this study are available within the article.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Stackel, P. Uber die intagration der Hamiltonschen differentialechung mittels separation der variablen. *Math. Ann.* **1897**, *49*, 145–147. [CrossRef]
2. Eisenhart, L.P. Separable systems of stackel. *Ann. Math.* **1934**, *35*, 284–305. [CrossRef]
3. Levi-Civita, T. Sulla Integraziome Della Equazione Di Hamilton-Jacobi Per Separazione Di Variabili. *Math. Ann.* **1904**, *59*, 383–397. [CrossRef]
4. Jarov-Jrovoy, M.S. Integration of Hamilton-Jacobi equation by complete separation of variables method. *J. Appl. Math. Mech.* **1963**, *27*, 173–219. [CrossRef]
5. Carter, B. New family of Einstein spaces. *Phys. Lett.* **1968**, *A25*, 399–400. [CrossRef]
6. Shapovalov, V.N. Symmetry and separation of variables in the Hamilton-Jacobi equation. *Sov. Phys. J.* **1978**, *21*, 1124–1132. [CrossRef]
7. Shapovalov, V.N. Stackel's spaces. *Sib. Math. J.* **1979**, *20*, 1117–1130. [CrossRef]
8. Miller, W. *Symmetry And Separation of Variables*; Cambridge University Press: Cambridge, UK, 1984; 318p.
9. Obukhov, V.V. Hamilton-Jacobi equation for a charged test particle in the Stackel space of type (2.0). *Symmetry* **2020**, *12*, 1289. [CrossRef]
10. Obukhov, V.V. Hamilton-Jacobi equation for a charged test particle in the Stackel space of type (2.1). *Int. J. Geom. Meth. Mod. Phys* **2020**, *14*, 2050186. [CrossRef]
11. Odintsov, S.D. Editorial for Special Issue Feature Papers 2020. *Symmetry* **2023**, *15*, 8. [CrossRef]
12. Oktay, C.; Salih, K. Maxwell-modified metric affine gravity. *Eur. Phys. J.* **2021**, *81*, 10. [CrossRef]
13. Mitsopoulos, A.; Tsamparlis, M.; Leon, G.A.; Paliathanasis, A. New conservation laws and exact cosmological solutions in Brans-Dicke cosmology with an extra scalar field. *Symmetry* **2021**, *13*, 1364. [CrossRef]
14. Dappiaggi, C.; Juarez-Aubry, B.A.; Ground, A.M. State for the Klein-Gordon field in anti-de Sitter spacetime with dynamical Wentzell boundary conditions. *Phys. Rev. D* **2022**, *105*, 105017. [CrossRef]
15. Astorga, F.; Salazar, J.F.; Zannias, T. On the integrability of the geodesic flow on a Friedmann-Robertson-Walker spacetime. *Phys. Scr.* **2018**, *93*, 085205. [CrossRef]
16. Capozziello, S.; De Laurentis, M.; Odintsov, D. Hamiltonian dynamics and Noether symmetries in extended gravity cosmology. *Eur. Phys. J.* **2012**, *C72*, 2068. [CrossRef]
17. Kibaroglu, S.; Cebecioglu, O. Generalized cosmological constant from gauging Maxwell-conformal algebra. *Phys. Lett. B* **2020**, *803*, 135295. [CrossRef]
18. Shin'ichi, N.; Odintsov Sergei, D.; Valerio, F. Searching for dynamical black holes in various theories of gravity. *Phys. Rev. D* **2021**, *103*, 044055. [CrossRef]
19. Epp, V.; Pervukhina, O. The Stormer problem for an aligned rotator. *MNRAS* **2018**, *474*, 5330–5339. [CrossRef]
20. Epp, V.; Masterova, M.A. Effective potential energy for relativistic particles in the field of inclined rotating magnetized sphere. *Astrophys. Space Sci.* **2014**, *353*, 473–483. [CrossRef]
21. Kumaran Y.; Ovgun, A. Deflection angle and shadow of the reissner-nordstrom black hole with higher-order magnetic correction in einstein-nonlinear-maxwell fields. *Symmetry* **2022**, *14*, 2054. [CrossRef]
22. Osetrin, K.; Osetrin, E. Shapovalov wave-like spacetimes. *Symmetry* **2020**, *12*, 1372. [CrossRef]
23. Osetrin E.; Osetrin K.; Filippov, A. Plane Gravitational Waves in Spatially-Homogeneous Models of type-(3.1) Stackel Spaces. *Russ. Phys. J.* **2019**, *62*, 292–301. [CrossRef]
24. Osetrin, K.; Osetrin, E.; Osetrina, E. Geodesic deviation and tidal acceleration in the gravitational wave of the Bianchi type IV universe. *Eur. Phys. J. Plus* **2022**, *137*, 856. [CrossRef]
25. Osetrin, K.; Osetrin, E.; Osetrina, E. Gravitational wave of the Bianchi VII universe: Particle trajectories, geodesic deviation and tidal accelerations. *Eur. Phys. J. C* **2022**, *82*, 1–16. [CrossRef]
26. Shapovalov, A.V.; Shirokov, I.V. Noncommutative integration method for linear partial differential equations. functional algebras and dimensional reduction. *Theoret. Math. Phys.* **1996**, *106*, 1–10. [CrossRef]
27. Shapovalov, A.; Breev, A. Harmonic Oscillator Coherent States from the Standpoint of Orbit Theory. *Symmetry* **2023**, *15*, 282. [CrossRef]
28. Breev, A.I.; Shapovalov, A.V. Non-commutative integration of the Dirac equation in homogeneous spaces. *Symmetry* **2020**, *12*, 1867. [CrossRef]
29. Breev, A.I.; Shapovalov, A.V. Yang–Mills gauge fields conserving the symmetry algebra of the Dirac equation in a homogeneous space. *J. Phys. Conf. Ser.* **2014**, *563*, 012004. [CrossRef]
30. Magazev, A.A.; Boldyreva, M.N. Schrodinger equations in electromagnetic fields: symmetries and noncommutative integration. *Symmetry* **2021**, *13*, 1527. [CrossRef]

31. Magazev, A.A. Integrating Klein-Gordon-Fock equations in an extremal electromagnetic field on Lie groups. *Theor. Math. Phys.* **2012**, *173*, 1654–1667. [CrossRef]
32. Obukhov, V.V. Algebra of symmetry operators for Klein-Gordon-Fock Equation. *Symmetry* **2021**, *13*, 727. [CrossRef]
33. Odintsov, S.D. Editorial for Feature Papers 2021–2022. *Symmetry* **2023**, *15*, 32. [CrossRef]
34. Obukhov, V.V. Algebra of the symmetry operators of the Klein-Gordon-Fock equation for the case when groups of motions G_3 act transitively on null subsurfaces of spacetime. *Symmetry* **2022**, *14*, 346. [CrossRef]
35. Obukhov, V.V. Algebras of integrals of motion for the Hamilton-Jacobi and Klein-Gordon-Fock equations in spacetime with a four-parameter groups of motions in the presence of an external electromagnetic field. *J. Math. Phys.* **2022**, *63*. 023505. [CrossRef]
36. Stephani, H.; Kramer, D.; MacCallum, M.; Hoenselaers, C.; Herlt, E. *Exact Solutions of Einstein's Field Equations*, 2nd ed.; Cambridge Monographs on Mathematical Physics; Cambridge University Press: Cambridge, UK, 2003. [CrossRef]
37. Obukhov, V.V. Separation of variables in Hamilton-Jacobi and Klein-Gordon-Fock equations for a charged test particle in the stackel spaces of type (1.1). *Int. J. Geom. Meth. Mod. Phys.* **2021**, *3*, 2150036. [CrossRef]
38. Landau, L.D.; Lifshits, E.M. *Theoretical Physics, Field Theory*, 7th ed.; Science, C., Ed.; Nauka: Moskow, Russia, 1988; Volume II, 512p. ISBN 5-02-014420-7.
39. Petrov, A.Z. *Einstein Spaces*; Pergamon Press: Oxford, UK, 1969.
40. Obukhov, V.V. Maxwell Equations in Homogeneous Spaces for Admissible Electromagnetic Fields. *Universe* **2022**, *8*, 245. [CrossRef]
41. Obukhov, V.V. Maxwell Equations in Homogeneous Spaces with Solvable Groups of Motions. *Symmetry* **2022**, *14*, 2595. [CrossRef]