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Third-Order Neutral Differential Equation with a Middle Term and Several Delays: Asymptotic Behavior of Solutions

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Abstract: This study aims to investigate the asymptotic behavior of a class of third-order delay differential equations. Here, we consider an equation with a middle term and several delays. We obtain an iterative relationship between the positive solution of the studied equation and the corresponding function. Using this new relationship, we derive new criteria that ensure that all non-oscillatory solutions converge to zero. The new findings are an extension and expansion of relevant findings in the literature. We apply our results to a special case of the equation under study to clarify the importance of the new criteria.

Keywords: delay differential equations; third-order; asymptotic behavior; middle term; nonlinear DDEs; multi-delay equation

MSC: 34C10; 34K11

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1. Introduction

The core of the qualitative theory of delay differential equations (DDE) is the investigation of qualitative behavior such as oscillation, stability, periodicity, and others. One of the subfields of qualitative theory is oscillation theory, which focuses on the analysis of solutions' asymptotic and oscillatory behavior. The study of the oscillation theory of the DDEs began by linking the oscillatory behavior of the equations to the absence of any real solutions to the characteristic equation. The odd-order differential equations did not receive the same attention as the even-order equations. This is due to the fact that the behavior of positive solutions of odd-order differential equations is richer in possibilities than even-order equations. In addition, the characteristic equation of odd-order ordinary differential equations must have real solutions.

Functional differential equations of the sort known as DDEs account for the memory of phenomena. DDEs have numerous physical and engineering uses; for examples, see [1,2]. These uses include soil settlement, elasticity problems, and structure deflection in electrical networks with lossless transmission lines. To comprehend and analyze the behavior of these solutions, a study of the oscillatory behavior of DDE solutions needs to be developed. Half-linear equations have various applications in the study of p-Laplace equations, non-Newtonian fluid theory, porous media, and other domains; see [3–5]. In addition to the great development in the study of the qualitative aspect of solutions of differential equations, the numerical studies of solutions of differential equations have developed greatly; see, for example, [6–8].

In this study, we investigate the oscillatory behavior of third-order DDE of neutral type

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(a\cdot\left(\frac{\mathrm{d}^2}{\mathrm{d}l^2}\mathcal{B}\right)^{\kappa}\right)+\left(h\cdot\left(\frac{\mathrm{d}^2}{\mathrm{d}l^2}\mathcal{B}\right)^{\kappa}\right)+\sum_{i=1}^{j}q_i\cdot[\nu^{\kappa}\circ\eta_i]=0,\ \ l\geq l_0, \tag{1}$$

where $\kappa > 0$ is a ratio of odd integer numbers, j is a positive integer number, $\mathcal{B}(l) := \nu(l) + c_0[\nu \circ \mu](l)$, and $[\nu \circ \mu](l) = \nu(\mu(l))$. Throughout this work, we assume that a is differentiable positive real-valued function, μ, η_i, h and q_i are continuous real-valued functions on $[l_0, \infty)$, c_0, h_i and q_i are nonnegative for $i = 1, 2, \ldots, j$, $a'(l) > 0$, $\mu(l) < l$, $\eta_i(l) < l$, $\eta'(l) > 0$, $\lim_{l\to\infty}\mu(l) = \lim_{l\to\infty}\eta(l) = \infty$and

$$\int_{l_0}^{\infty}\frac{1}{a^{1/\kappa}(\mathfrak{r})}\exp\left(-\frac{1}{\kappa}\int_{l_0}^{\infty}\frac{h(\mathfrak{z})}{a(\mathfrak{z})}\mathrm{d}\mathfrak{z}\right)\mathrm{d}\mathfrak{r}=\infty, \tag{2}$$

where

$$\eta(l) := \min\{\eta_i(l),\ i = 1, 2, \ldots, j\}.$$

By a solution of (1), we mean a function $\nu \in C^2([l_*, \infty))$, $l_* \geq l_0$, which has the property $\mathcal{B} \in C^2([l_*, \infty))$, $a \cdot (\mathcal{B}'')^{\kappa} \in C^1([l_*, \infty))$, and ν satisfies (1) on $[l_*, \infty)$. We only focus on solutions of (1) that exist on $[l_0, \infty)$ and satisfy

$$\sup\{|\nu(l)| : l \geq l_*\} > 0 \text{ for every } l_* \geq l_0.$$

A solution ν of (1) is said to be oscillatory if it has arbitrary large zeros, that is, there exists a sequence of zeros $\{l_n\}_{n=0}^{\infty}$ (i.e., $x(l_n) = 0$) of v such that $\lim_{n\to\infty} l_n = \infty$. We say that (1) is almost oscillatory if any solution ν is either oscillatory or satisfies $\lim_{l\to\infty}\nu(l) = 0$. During the paper, we will need the next class:

$$\mathcal{S}_x : \text{all positive solutions of (1) whose } \mathcal{B} \text{ satisfies } \mathcal{B}(l)\mathcal{B}'(l) > 0.$$

Recently, researchers have shown an interest in the oscillatory features of DDEs. For example, it is easy to find many interesting results and improved techniques in [9–12], which focused on establishing oscillation parameters for delay and advanced equations. On the other hand, works [13–16] were concerned with extending the results of the delay equations to the neutral equations. On the other hand, the great development in the study of the asymptotic and oscillatory behavior of the solutions of difference and dynamic equations can be noted; see, for example, [17–19].

For third-order DDEs, Baculikova and Dzurina [20] presented the oscillation condition for DDE

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(a\cdot\left(\frac{\mathrm{d}^2}{\mathrm{d}l^2}\mathcal{B}\right)^{\kappa}\right)+q\cdot[\nu^{\kappa}\circ\eta]=0, \tag{3}$$

where $\mathcal{B} := \nu + p \cdot [\nu \circ \mu]$, $p(l) < c_0 < 1$ and $a'(l) > 0$, and proved that, if

$$\liminf_{l\to\infty}\int_{l}^{\infty}q(\mathfrak{z})\frac{\eta^{2\kappa}(\mathfrak{z})}{\mathfrak{z}^{\kappa}}\mathrm{d}\mathfrak{z}>\frac{(2\kappa)^{\kappa}}{(\kappa+1)^{\kappa+1}(1-c_0)^{\kappa}},$$

then $\mathcal{S}_x = \emptyset$. Baculikova and Dzurina [21] tested the oscillatory properties of (3), and it was shown that, if the first-order DDE

$$y' + q\cdot\left(\frac{(\eta-l_0)^2(1-[p\circ\eta])}{2r^{1/\kappa}}\right)^{\kappa}\cdot[y\circ\eta]=0$$

is oscillatory, then $\mathcal{S}_x = \emptyset$. Moreover, for (3), Thandapani and Li [22] proved that $\mathcal{S}_x = \emptyset$, if

$$\limsup_{l\to\infty} \int_{l_0}^{l} \left(\frac{1}{2^{\kappa-1}} \rho(\mathfrak{z}) Q(\mathfrak{z}) - \frac{(1+p^{\kappa}(l)/\mu_0)}{(\kappa+1)^{\kappa+1}} \frac{(\rho'(\mathfrak{z}))^{\kappa+1}}{(\rho(\mathfrak{z})\beta_1(\eta(\mathfrak{z}),l_0)\eta'(\mathfrak{z}))^{\kappa}} \right) = \infty,$$

where $\eta' > 0$, $\mu' \geq \mu_0 > 0$, $Q(l) := \min\{q(l), [q \circ \mu](l)\}$ and $\rho \in C([l_0,\infty),(0,\infty))$.

The oscillatory behavior of DDE

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(r_2 \cdot \frac{\mathrm{d}}{\mathrm{d}l}\left(r_1 \cdot \frac{\mathrm{d}}{\mathrm{d}l} \nu \right) \right) + p \cdot \nu' + q \cdot [f \circ \nu \circ g] = 0$$

was discussed in a number of studies; see, for example, [23–25]. Moaaz et al. [26] evaluated the oscillation of the more general third-order DDE

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(r_2 \cdot \frac{\mathrm{d}}{\mathrm{d}l}\left(r_1 \cdot \frac{\mathrm{d}}{\mathrm{d}l} \nu \right) \right) + p \cdot [f \circ \nu' \circ \mu] + q \cdot [g \circ \nu \circ \sigma] = 0.$$

The asymptotic properties of the solutions to DDEs with odd-order were addressed by Moaaz et al. [27] in several ways.

In this work, the asymptotic properties of solutions to the DDEs of third-order are investigated. We create conditions that ensure that all non-oscillatory solutions of the studied equation tend to zero. We use more than one approach to establish new criteria of an iterative nature that enables us to apply the results more than once while the previous relevant results fail.

2. Main Results

For brevity, we define

$$\omega(l) := \exp\left(\int_{l_0}^{l} \frac{h(\mathfrak{z})}{a(\mathfrak{z})} \mathrm{d}\mathfrak{z} \right),$$

$$\theta(\nu,\varrho) := \int_{\varrho}^{\nu} \frac{1}{\omega^{\frac{1}{\kappa}}(\mathfrak{z}) a^{\frac{1}{\kappa}}(\mathfrak{z})} \mathrm{d}\mathfrak{z} \ \text{ for } \varrho < \nu,$$

$$\mu^{\{0\}} = l,\ \mu^{\{m\}} = \left[\mu \circ \mu^{\{m-1\}} \right] \ \text{ for } m = 1,2,\ldots,$$

and

$$\mathcal{P}(l) := (1-c_0) \sum_{m=0}^{(n-1)/2} c_0^{2m} \left(\frac{\mu^{\{2m+1\}}(l)}{l} \right)^{2/\lambda},$$

where $l_1 \geq l_0$ and n is an odd positive integer.

Lemma 1 ([28])**.** *Let $\phi \in C^{m+1}([l_0,\infty),(0,\infty))$, and $\phi^{(k)}(l) > 0$ for $k = 0,1,2,\ldots,m$ and $\phi^{(m+1)}(l) \leq 0$. Then, $\phi(l) \geq \lambda \frac{l}{m} \phi'(l)$ for all $\lambda \in (0,1)$, and for $l \geq l_1$, where l_1 is sufficiently large.*

Lemma 2. *Let ν be a positive solution of (1). Then, $\mathcal{B}$ and $\mathcal{B}''$ are positive, $\left(\omega \cdot a \cdot (\mathcal{B}'')^{\kappa}\right)'$ is nonnegative, and $\mathcal{B}'$ is of one sign, for $l \geq l_1$, where l_1 is sufficiently large.*

Proof. Assume that ν is a positive solution of (1) on $[l_0,\infty)$. It is easy to deduce that

$$\exp\left(\int_{l_0}^{l} \frac{h(\mathfrak{z})}{a(\mathfrak{z})} \mathrm{d}\mathfrak{z} \right) \left(\left(a \cdot (\mathcal{B}'')^{\kappa} \right)' + h \cdot (\mathcal{B}'')^{\kappa} \right) = \left(\omega \cdot a \cdot (\mathcal{B}'')^{\kappa} \right)',$$

which converts Equation (1) into the form

$$\left(\omega \cdot a \cdot (\mathcal{B}'')^{\kappa} \right)' + \sum_{i=1}^{j} \omega \cdot q_i \cdot [\nu^{\kappa} \circ \eta_i] = 0.$$

Hence,

$$\left(\omega \cdot a \cdot (\mathcal{B}'')^{\kappa}\right)' = -\sum_{i=1}^{j} \omega \cdot q_i \cdot [\nu^{\kappa} \circ \eta_i] \leq 0. \tag{4}$$

Thus, $\omega \cdot a \cdot (\mathcal{B}'')^{\kappa}$ is of one sign for $l \geq l_1$, where $l_1 \geq l_0$. Hence, $\mathcal{B}''(l) < 0$ or $\mathcal{B}''(l) > 0$ for $l \geq l_1$.

If $\mathcal{B}''(l) < 0$, then there is a $M > 0$ such that

$$\omega(l)a(l)\mathcal{B}''(l)^{\kappa} \leq -M < 0.$$

Integrating this inequality from l_1 to l, we obtain

$$\mathcal{B}'(l) \leq \mathcal{B}'(l_1) - M^{\frac{1}{\kappa}} \int_{l_1}^{l} \frac{1}{a^{\frac{1}{\kappa}}(\mathfrak{z})\omega^{\frac{1}{\kappa}}(\mathfrak{z})} \mathrm{d}\mathfrak{z}.$$

Letting $l \to \infty$ and using (2), we obtain $\mathcal{B}' \to -\infty$. Thus, $\mathcal{B}'(l) < 0$ eventually. However, $\mathcal{B}''(l) < 0$ and $\mathcal{B}'(l) < 0$ eventually imply $\mathcal{B}(l) < 0$ for $l \geq l_1$, a contradiction. Then, $\mathcal{B}''(l) > 0$. □

Lemma 3. *Assume that $\nu \in \mathcal{S}_x$. Then,*

$$\nu(l) \geq (1 - c_0)\mathcal{B}(l) \sum_{m=0}^{(n-1)/2} c_0^{2m} \left(\frac{\mu^{\{2m+1\}}(l)}{l}\right)^{2/\lambda}. \tag{5}$$

for $n \in \mathbb{N}$ is an odd and $\lambda \in (0,1)$, and $l \geq l_2$, where $l_2 \geq l_1$ large enough.

Proof. Let $\nu \in \mathcal{S}_x$. From the definition of $\mathcal{B}$, we have

$$\begin{aligned} \nu &= \mathcal{B} - c_0[\nu \circ \mu] = \mathcal{B} - c_0[\mathcal{B} \circ \mu] + c_0^2\left[\nu \circ \mu^{\{2\}}\right] \\ &= \mathcal{B} - c_0[\mathcal{B} \circ \mu] + c_0^2\left[\mathcal{B} \circ \mu^{\{2\}}\right] - c_0^3\left[\nu \circ \mu^{\{3\}}\right]. \end{aligned}$$

By continuing this process, we obtain

$$\begin{aligned} \nu(l) &= \sum_{m=0}^{n} (-1)^m c_0^m \mathcal{B}\left(\mu^{\{m\}}(l)\right) + (-1)^n c_0^{n+1} \nu\left(\mu^{\{n+1\}}(l)\right) \\ &\geq \sum_{m=0}^{(n-1)/2} \left(c_0^{2m}\mathcal{B}\left(\mu^{\{2m\}}(l)\right) - c_0^{2m+1}\mathcal{B}\left(\mu^{\{2m+1\}}(l)\right)\right), \end{aligned} \tag{6}$$

for all $l \geq l_1 \geq l_0$, where l_1 is sufficiently large. Since $\mathcal{B}'(l) > 0$ and $\mu^{\{2m+1\}}(l) \leq \mu^{\{2m\}}(l)$ for all $m = 0, 1, \ldots$, inequality (6) becomes

$$\nu(l) \geq (1 - c_0) \sum_{m=0}^{(n-1)/2} c_0^{2m} \left[\mathcal{B} \circ \mu^{\{2m+1\}}\right]. \tag{7}$$

Using Lemma 1 with $\phi = \mathcal{B}$ and $m = 2$, we obtain that $\mathcal{B}(l) \geq \frac{\lambda}{2} l \mathcal{B}'(l)$ for all $\lambda \in (0,1)$ and $l \geq l_1$. By integrating this inequality from $\mu^{\{2m+1\}}$ to l , we obtain

$$\mathcal{B}\left(\mu^{\{2m+1\}}(l)\right) \geq \left(\frac{\mu^{\{2m+1\}}(l)}{l}\right)^{2/\lambda} \mathcal{B}(l),$$

for all $l \geq l_2 \geq l_1$. Thus, inequality (7) implies

$$\nu(l) \geq (1-c_0)\mathcal{B}(l) \sum_{m=0}^{\frac{(n-1)}{2}} c_0^{2m} \left(\frac{\mu^{\{2m+1\}}(l)}{l} \right)^{2/\lambda}.$$

The proof is now complete. □

2.1. Nonexistence of Solutions in $\mathcal{S}_x$

Below, we use the Riccati substitution technique to obtain a condition that guarantees no solutions in class $\mathcal{S}_x$.

Theorem 1. *If there exists a function $\rho \in C^1([l_0,\infty),(0,\infty))$ such that*

$$\limsup_{l\to\infty} \int_{l_0}^{l} \left(\rho(\mathfrak{z})\omega(\mathfrak{z}) \left(\frac{\eta(\mathfrak{z})}{\mathfrak{z}} \right)^{2/\lambda} \sum_{i=1}^{j} q_i(\mathfrak{z})\mathcal{P}^{\kappa}(\eta_i(\mathfrak{z})) - \frac{1}{\rho^{\kappa}(\mathfrak{z})\theta^{\kappa}(\mathfrak{z},l_1)} \left(\frac{\rho'(\mathfrak{z})}{\kappa+1} \right)^{\kappa+1} \right) \mathrm{d}\mathfrak{z} = \infty, \tag{8}$$

for some $\lambda \in (0,1)$, then $\mathcal{S}_x = \emptyset$.

Proof. Assume the contrary that $\nu \in \mathcal{S}_x$. Now, it follows from (1) that

$$\left(\omega \cdot a \cdot (\mathcal{B}'')^{\kappa}\right)' = -\sum_{i=1}^{j} \omega \cdot q_i \cdot [\nu^{\kappa} \circ \eta_i]. \tag{9}$$

From Lemma 3, we arrive at (5). Combining (1) and (4), we find

$$\begin{aligned} \left(\omega(l)a(l)(\mathcal{B}''(l))^{\kappa}\right)' &\leq -\omega(l) \sum_{i=1}^{j} q_i(l)\mathcal{P}^{\kappa}(\eta_i(l))\mathcal{B}^{\kappa}(\eta_i(l)) \\ &\leq -\omega(l)\mathcal{B}^{\kappa}(\eta(l)) \sum_{i=1}^{j} q_i(l)\mathcal{P}^{\kappa}(\eta_i(l)) \end{aligned} \tag{10}$$

Now, we define

$$\psi := \rho \cdot \frac{\omega \cdot a \cdot (\mathcal{B}'')^{\kappa}}{\mathcal{B}^{\kappa}}.$$

Clearly, $\psi(l) > 0$ for all $l \geq l_1$ and

$$\psi' = \frac{\rho'}{\rho}\psi + \rho \frac{\left(\omega \cdot a \cdot (\mathcal{B}'')^{\kappa}\right)'}{\mathcal{B}^{\kappa}} - \kappa\rho \frac{\omega \cdot a \cdot (\mathcal{B}'')^{\kappa}}{\mathcal{B}^{\kappa+1}}\mathcal{B}'. \tag{11}$$

Using Lemma 1 with $\phi = \mathcal{B}$ and $m = 2$, we obtain that $\mathcal{B}(l) \geq \frac{\lambda}{2} l\mathcal{B}'(l)$ for all $\lambda \in (0,1)$ and $l \geq l_1$. By integrating this inequality from η to l , we obtain

$$[\mathcal{B}\circ\eta] \geq \left(\frac{\eta}{l} \right)^{2/\lambda} \cdot \mathcal{B}, \tag{12}$$

Since $\left(\omega \cdot a \cdot (\mathcal{B}'')^{\kappa}\right)' \leq 0$, we obtain

$$\begin{aligned} \mathcal{B}'(l) &\geq \int_{l_1}^{l} \frac{1}{\omega^{\frac{1}{\kappa}}(\mathfrak{z})a^{\frac{1}{\kappa}}(\mathfrak{z})} \omega^{\frac{1}{\kappa}}(\mathfrak{z})a^{\frac{1}{\kappa}}(\mathfrak{z})\mathcal{B}''(\mathfrak{z})\mathrm{d}\mathfrak{z} \\ &\geq \left[\omega(l)a(l)(\mathcal{B}''(l))^{\kappa}\right]^{\frac{1}{\kappa}} \theta(l,l_1). \end{aligned} \tag{13}$$

Combining (10)–(13), we obtain

$$\begin{aligned}\psi' \quad &\leq \quad \frac{\rho'}{\rho} \cdot \psi - \rho \cdot \omega \cdot \frac{[\mathcal{B}^{\kappa} \circ \eta]}{\mathcal{B}^{\kappa}} \cdot \sum_{i=1}^{j} q_i \cdot [\mathcal{P}^{\kappa} \circ \eta_i] \\ &\quad -\kappa\rho \cdot \frac{\left[\omega \cdot a \cdot (\mathcal{B}'')^{\kappa}\right]^{\frac{\kappa+1}{\kappa}}}{\mathcal{B}^{\kappa+1}} \cdot \theta(l, l_1) \\ &\leq \quad \frac{\rho'}{\rho} \cdot \psi - \rho \cdot \omega \cdot \left(\frac{\eta}{l}\right)^{2/\lambda} \cdot \sum_{i=1}^{j} q_i \cdot [\mathcal{P}^{\kappa} \circ \eta_i] \\ &\quad -\frac{\kappa}{\rho^{\frac{1}{\kappa}}} \theta(l, l_1) \cdot \psi^{\frac{\kappa+1}{\kappa}}. \end{aligned} \tag{14}$$

Set

$$\Psi(\psi) := \frac{\rho'}{\rho}\psi - \kappa\frac{\theta}{\rho^{\frac{1}{\kappa}}}\psi^{\frac{\kappa+1}{\kappa}}.$$

We see that

$$\Psi'(\psi) = \frac{\rho'}{\rho} - (\kappa+1)\frac{\theta}{\rho^{\frac{1}{\kappa}}}\psi^{\frac{1}{\kappa}}.$$

Thus, $\Psi(\psi)$ attains its maximum value on $\mathbb{R}$ at $\psi^* = \rho\left(\frac{\rho'}{\rho(\kappa+1)\theta}\right)^{\kappa}$, and

$$\Psi(\psi) \leq \max_{\psi \in \mathbb{R}} \Psi(\psi) = \left(\frac{\rho'}{\kappa+1}\right)^{\kappa+1} \frac{1}{\rho^{\kappa}\theta^{\kappa}}.$$

Then, (14) becomes

$$\begin{aligned}\psi'(l) \quad &\leq \quad -\rho(l)\omega(l)\left(\frac{\eta(l)}{l}\right)^{2/\lambda} \sum_{i=1}^{j} q_i(l)\mathcal{P}^{\kappa}(\eta_i(l)) \\ &\quad +\left(\frac{\rho'(l)}{\kappa+1}\right)^{\kappa+1} \frac{1}{\rho^{\kappa}(l)\theta^{\kappa}(l, l_1)}.\end{aligned}$$

By integrating this inequality from $l_1 \to l$, we find

$$\int_{l_1}^{l} \left(\rho(\mathfrak{z})\omega(\mathfrak{z})\left(\frac{\eta(\mathfrak{z})}{\mathfrak{z}}\right)^{2/\lambda} \sum_{i=1}^{j} q_i(\mathfrak{z})\mathcal{P}^{\kappa}(\eta_i(\mathfrak{z})) - \frac{1}{\rho^{\kappa}(\mathfrak{z})\theta^{\kappa}(\mathfrak{z}, l_1)}\left(\frac{\rho'(\mathfrak{z})}{\kappa+1}\right)^{\kappa+1}\right) \mathrm{d}\mathfrak{z} \leq \psi(l_1),$$

which contradicts (8). □

Next, we derive a condition that ensures there are no solutions in class $\mathcal{S}_x$ using the comparison principle.

Theorem 2. *If the DDE*

$$y'(l) + \omega(l)y(\eta(l))\left(\int_{l_1}^{\eta(l)} \theta(\mathfrak{z}, l_1)\mathrm{d}\mathfrak{z}\right)^{\kappa} \sum_{i=1}^{j} q_i(l)\mathcal{P}^{\kappa}(\eta_i(l)) = 0 \tag{15}$$

is oscillatory, then $\mathcal{S}_x = \emptyset$.

Proof. Assume the contrary that $\nu \in \mathcal{S}_x$. As in the proof of Theorem 1, we obtain that (10) and (13) hold for all $l \geq l_1 \geq l_0$. Integrating (13) from l_1 to l, we have

$$\mathcal{B}(l) \geq \left[\omega(l)a(l)\left(\mathcal{B}''(l)\right)^{\kappa}\right]^{\frac{1}{\kappa}} \int_{l_1}^{l} \theta(\mathfrak{z}, l_1)\mathrm{d}\mathfrak{z},$$

which with (10) gives

$$\begin{aligned} &\left(\omega(l)a(l)\left(\mathcal{B}''(l)\right)^{\kappa}\right)' \\ \leq\ & -\omega(l)\omega(\eta(l))a(\eta(l))\left(\mathcal{B}''(\eta(l))\right)^{\kappa}\left(\int_{l_1}^{\eta(l)} \theta(\mathfrak{z}, l_1)\mathrm{d}\mathfrak{z}\right)^{\kappa} \sum_{i=1}^{j} q_i(l)\mathcal{P}^{\kappa}(\eta_i(l)). \end{aligned}$$

Now, if we set $y := \omega \cdot a \cdot (\mathcal{B}'')^{\kappa} > 0$, then we note that $y > 0$ is a solution of

$$y'(l) + \omega(l)y(\eta(l))\left(\int_{l_1}^{\eta(l)} \theta(\mathfrak{z}, l_1)\mathrm{d}\mathfrak{z}\right)^{\kappa} \sum_{i=1}^{j} q_i(l)\mathcal{P}^{\kappa}(\eta_i(l)) \leq 0.$$

Therefore, from [29], Equation (15) also has a positive solution, which is a contradiction. □

Corollary 1. *If*

$$\liminf_{l\to\infty} \int_{\eta(l)}^{l} \omega(\mathfrak{z})\left(\int_{l_1}^{\eta(\mathfrak{z})} \theta(\mathfrak{z}, l_1)\mathrm{d}\mathfrak{z}\right)^{\kappa} \sum_{i=1}^{j} q_i(\mathfrak{z})\mathcal{P}^{\kappa}(\eta_i(\mathfrak{z}))\mathrm{d}\mathfrak{z} > \frac{1}{\mathrm{e}}, \tag{16}$$

then $\mathcal{S}_x = \emptyset$.

Proof. From Theorem 2 in [30], condition (16) guarantee that (15) is oscillatory. □

In the following theorem, by finding a condition of Hille and Nehari type, we guarantees that $\mathcal{S}_x = \emptyset$.

Theorem 3. *If*

$$\liminf_{l\to\infty} \frac{l^k}{a(l)\omega(l)} \int_{l}^{\infty} \left(\omega(\mathfrak{z})\left(\frac{\lambda}{2}\frac{\eta^2(\mathfrak{z})}{\mathfrak{z}}\right)^{\kappa} \sum_{i=1}^{j} q_i(\mathfrak{z})\mathcal{P}^{\kappa}(\eta_i(\mathfrak{z}))\right)\mathrm{d}\mathfrak{z} > \frac{\kappa^{\kappa}}{(\kappa+1)^{\kappa+1}}, \tag{17}$$

for some $\lambda \in (0,1)$, *then* $\mathcal{S}_x = \emptyset$.

Proof. Assume the contrary that $\nu \in \mathcal{S}_x$. As in the proof of Theorem 1, we obtain that (10) holds for all $l \geq l_1 \geq l_0$. Now, we define

$$\digamma := \frac{\omega \cdot a \cdot (\mathcal{B}'')^{\kappa}}{(\mathcal{B}')^{\kappa}}.$$

Clearly, $\digamma(l) > 0$ for all $l \geq l_1$ and

$$\digamma' = \frac{\left(\omega \cdot a \cdot (\mathcal{B}'')^{\kappa}\right)'}{(\mathcal{B}')^{\kappa}} - \kappa\frac{\omega \cdot a \cdot (\mathcal{B}'')^{\kappa+1}}{(\mathcal{B}')^{\kappa+1}},$$

which with (10) gives

$$\digamma'(l) \leq -\omega(l)\frac{\mathcal{B}^{\kappa}(\eta(l))}{(\mathcal{B}'(l))^{\kappa}} \sum_{i=1}^{j} q_i(l)\mathcal{P}^{\kappa}(\eta_i(l)) - \frac{\kappa}{a^{\frac{1}{\kappa}}(l)\omega^{\frac{1}{\kappa}}(l)}\digamma^{\frac{\kappa+1}{\kappa}}(l). \tag{18}$$

Using the monotonic properties of the derivatives of $\mathcal{B}'$, we obtain, from the mean value theorem that

$$\mathcal{B}''(\eta(l)) \leq \frac{\mathcal{B}'(\eta(l)) - \mathcal{B}'(l_1)}{\eta(l) - l_1}, \tag{19}$$

and

$$\mathcal{B}''(\eta(l)) \geq \frac{\mathcal{B}'(l) - \mathcal{B}'(\eta(l))}{l - \eta(l)}. \tag{20}$$

From (19), there exists a $l_2 \geq l_1$ such that

$$\frac{\mathcal{B}'(\eta(l))}{\mathcal{B}''(\eta(l))} \geq \frac{\mathcal{B}'(\eta(l)) - \mathcal{B}'(l_1)}{\mathcal{B}''(\eta(l))} \geq \eta(l) - l_1 \geq \lambda_0 \eta(l), \tag{21}$$

for all $\lambda_0 \in (0,1)$. From (20) and (21), we obtain

$$\begin{aligned} \frac{\mathcal{B}'(l)}{\mathcal{B}'(\eta(l))} &\leq (l - \eta(l))\frac{\mathcal{B}''(\eta(l))}{\mathcal{B}'(\eta(l))} + 1 \\ &\leq \frac{l - \eta(l)}{\lambda_0 \eta(l)} + 1 \\ &\leq \frac{l}{\lambda_0 \eta(l)}. \end{aligned} \tag{22}$$

Using Lemma 1 with $\phi = \mathcal{B}$ and $m = 2$, we obtain that $\mathcal{B}(l) \geq \frac{\lambda_1}{2} l \mathcal{B}'(l)$ for all $\lambda_1 \in (0,1)$, which with (22) implies

$$\frac{\mathcal{B}(\eta(l))}{\mathcal{B}'(l)} = \frac{\mathcal{B}'(\eta(l))}{\mathcal{B}'(l)} \frac{\mathcal{B}(\eta(l))}{\mathcal{B}'(\eta(l))} \geq \frac{\lambda}{2} \frac{\eta^2(l)}{l},$$

for all $\lambda \in (0,1)$. Therefore, from (18), we arrive at

$$\begin{aligned} \digamma'(l) &\leq -\omega(l)\left(\frac{\lambda}{2}\frac{\eta^2(l)}{l}\right)^{\kappa} \sum_{i=1}^{j} q_i(l)\mathcal{P}^{\kappa}(\eta_i(l)) - \frac{\kappa}{a^{\frac{1}{\kappa}}(l)\omega^{\frac{1}{\kappa}}(l)} \digamma^{\frac{\kappa+1}{\kappa}}(l) \\ &\leq 0. \end{aligned} \tag{23}$$

This implies that

$$\frac{1}{\kappa}\frac{\digamma'}{\digamma^{\frac{\kappa+1}{\kappa}}} < -\frac{1}{a^{\frac{1}{\kappa}} \cdot \omega^{\frac{1}{\kappa}}},$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{1}{\digamma^{\frac{1}{\kappa}}}\right) = -\frac{1}{\kappa}\frac{\digamma'}{\digamma^{\frac{\kappa+1}{\kappa}}} > \frac{1}{a^{\frac{1}{\kappa}} \cdot \omega^{\frac{1}{\kappa}}}.$$

Integrating this inequality, we find

$$\digamma(l)\theta^{\kappa}(l, l_1) < 1. \tag{24}$$

Then, $\digamma \to 0$ as $l \to \infty$. In addition, we define

$$\Omega := \liminf_{l \to \infty} \frac{l^{\kappa}\digamma(l)}{a(l)\omega(l)} > 0.$$

Then, for any $\epsilon > 0$, there is a $l_2 \geq l_1$ such that

$$\frac{l^{\kappa}\digamma(l)}{a(l)\omega(l)} \geq \Omega - \epsilon, \tag{25}$$

for all $l \geq l_2$. By integrating (23) from l to ∞, we conclude that

$$\begin{aligned} -F(l) &\leq -\int_l^\infty \left(\omega(\mathfrak{z})\left(\frac{\lambda}{2}\frac{\eta^2(\mathfrak{z})}{\mathfrak{z}}\right)^\kappa \sum_{i=1}^{j} q_i(\mathfrak{z})\mathcal{P}^\kappa(\eta_i(\mathfrak{z}))\right)\mathrm{d}\mathfrak{z} \\ &\quad -\int_l^\infty \frac{\kappa}{a^{\frac{1}{\kappa}}(\mathfrak{z})\omega^{\frac{1}{\kappa}}(\mathfrak{z})} F^{\frac{\kappa+1}{\kappa}}(\mathfrak{z})\mathrm{d}\mathfrak{z}, \end{aligned}$$

and so

$$\begin{aligned} &\frac{l^k}{a(l)\omega(l)}\int_l^\infty \left(\omega(\mathfrak{z})\left(\frac{\lambda}{2}\frac{\eta^2(\mathfrak{z})}{\mathfrak{z}}\right)^\kappa \sum_{i=1}^{j} q_i(\mathfrak{z})\mathcal{P}^\kappa(\eta_i(\mathfrak{z}))\right)\mathrm{d}\mathfrak{z} \\ \leq\; &\frac{l^k F(l)}{a(l)\omega(l)} - \frac{l^k}{a(l)\omega(l)}\int_l^\infty \frac{\kappa a(\mathfrak{z})\omega(\mathfrak{z})}{\mathfrak{z}^{\kappa+1}}\left(\frac{\mathfrak{z}^\kappa F(\mathfrak{z})}{a(\mathfrak{z})\omega(\mathfrak{z})}\right)^{\frac{\kappa+1}{\kappa}}\mathrm{d}\mathfrak{z}. \end{aligned}$$

Using (25) and the fact that $(a(l)\omega(l))' > 0$, we arrive at

$$\begin{aligned} &\frac{l^k}{a(l)\omega(l)}\int_l^\infty \left(\omega(\mathfrak{z})\left(\frac{\lambda}{2}\frac{\eta^2(\mathfrak{z})}{\mathfrak{z}}\right)^\kappa \sum_{i=1}^{j} q_i(\mathfrak{z})\mathcal{P}^\kappa(\eta_i(\mathfrak{z}))\right)\mathrm{d}\mathfrak{z} \\ \leq\; &\frac{l^k F(l)}{a(l)\omega(l)} - \frac{l^k}{a(l)\omega(l)}(\Omega-\epsilon)^{\frac{\kappa+1}{\kappa}}\int_l^\infty \frac{\kappa a(\mathfrak{z})\omega(\mathfrak{z})}{\mathfrak{z}^{\kappa+1}}\mathrm{d}\mathfrak{z}, \\ \leq\; &\frac{l^k F(l)}{a(l)\omega(l)} - l^k(\Omega-\epsilon)^{\frac{\kappa+1}{\kappa}}\int_l^\infty \frac{\kappa}{\mathfrak{z}^{\kappa+1}}\mathrm{d}\mathfrak{z}, \\ \leq\; &\frac{l^k F(l)}{a(l)\omega(l)} - (\Omega-\epsilon)^{\frac{\kappa+1}{\kappa}}. \end{aligned}$$

Taking $\liminf_{l\to\infty}$, we obtain

$$\liminf_{l\to\infty}\frac{l^k}{a(l)\omega(l)}\int_l^\infty \left(\omega(\mathfrak{z})\left(\frac{\lambda}{2}\frac{\eta^2(\mathfrak{z})}{\mathfrak{z}}\right)^\kappa \sum_{i=1}^{j} q_i(\mathfrak{z})\mathcal{P}^\kappa(\eta_i(\mathfrak{z}))\right)\mathrm{d}\mathfrak{z} \leq \Omega - \Omega^{\frac{\kappa+1}{\kappa}}$$

Using the inequality

$$B\psi - A\psi^{\frac{\kappa+1}{\kappa}} \leq \frac{\kappa^\kappa}{(\kappa+1)^{\kappa+1}}\frac{B^{\kappa+1}}{A^\kappa},$$

with $A = 1$, $B = 1$, and $\psi = \Omega$, we obtain

$$\liminf_{l\to\infty}\frac{l^k}{a(l)\omega(\mathfrak{z})}\int_l^\infty \left(\omega(\mathfrak{z})\left(\frac{\lambda}{2}\frac{\eta^2(\mathfrak{z})}{\mathfrak{z}}\right)^\kappa \sum_{i=1}^{j} q_i(\mathfrak{z})\mathcal{P}^\kappa(\eta_i(\mathfrak{z}))\right)\mathrm{d}\mathfrak{z} \leq \frac{\kappa^\kappa}{(\kappa+1)^{\kappa+1}}.$$

which contradicts (17). □

Example 1. *Consider the DDE*

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{1}{l}\frac{\mathrm{d}^2}{\mathrm{d}l^2}\mathcal{B}(l)\right) + \frac{1}{l^2}\frac{\mathrm{d}^2}{\mathrm{d}l^2}\mathcal{B}(l) + \frac{q_0}{l^4}\nu(\beta l) = 0, \tag{26}$$

where $l > 1$, $\mathcal{B}(l) = v(l) + \frac{1}{2}v(\alpha l)$, $q_0 > 0$ and $\alpha, \beta \in (0,1)$. It is easy to verify that $\mu^{\{m\}} = \alpha^m l$, for $m = 1, 2, \ldots,$ and

$$\mathcal{P}_0 = \sum_{m=0}^{(n-1)/2} \alpha^{8m+4}\left(\frac{1}{2}\right)^{2m+1}.$$

By choosing $\rho(l) = l^2$ and $\lambda = 1/2$, condition (8) reduces to

$$q_0 > \frac{1}{\mathcal{P}_0 \beta^4}. \tag{27}$$

Moreover, conditions (16) and (17) reduce to

$$q_0 > \frac{1}{2\mathcal{P}_0 \beta^2}, \tag{28}$$

and

$$q_0 > \frac{2}{\mathrm{e}\mathcal{P}_0 \beta^2 \ln \frac{1}{\beta}}, \tag{29}$$

respectively. Using the results in this section, any of conditions (27)–(29) guarantee that $\mathcal{S}_x = \emptyset$.

Remark 1. *Consider the special case of (26) when $\alpha = 0.9$. Figure 1 shows the lower bounds of the values of parameter q_0 for conditions (27)–(29). We note that these conditions are different from each other, and one of them cannot include the other along $\beta \in (0, 1)$.*

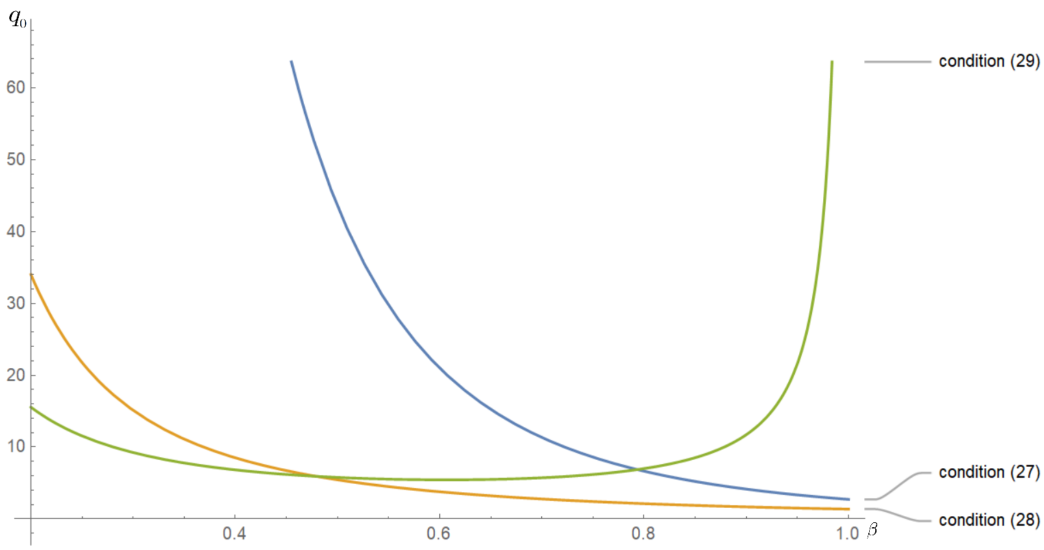

Figure 1. The minimum values of q_0 for which conditions (27)–(29) are satisfied.

2.2. Asymptotic Behavior

Theorem 4. *Let ν be an eventually positive solution of (1) and $\mathcal{B}'(l) < 0$. If*

$$\int_{l_0}^{\infty} \int_{v}^{\infty} \left(\frac{1}{\omega(u)a(u)} \int_{u}^{\infty} \omega(\mathfrak{z}) \sum_{i=1}^{j} q_i(\mathfrak{z}) \mathrm{d}\mathfrak{z} \right)^{1/\kappa} \mathrm{d}u\mathrm{d}v = \infty, \tag{30}$$

then ν converges to zero.

Proof. Suppose that ν is an eventually positive solution of (1) and $\mathcal{B}'(l) < 0$. Now, since $\mathcal{B}$ is positive and decreasing, we obtain that $\lim_{l\to\infty} \mathcal{B}(l) = \delta \geq 0$.

Assume that $\delta > 0$. Then, for $\epsilon > 0$, there is $l_1 \geq l_0$ such that $\delta < \mathcal{B}(l) < \delta + \epsilon$, for all $l \geq l_1$. Taking $\epsilon < (\delta - \delta c_0)/c_0$. Hence, we have

$$\begin{aligned} \nu &= \mathcal{B} - c_0[\nu \circ \mu] > \delta - c_0[\mathcal{B} \circ \mu] > \delta - c_0(\delta + \epsilon) \\ &= \frac{\delta - c_0(\delta + \epsilon)}{\delta + \epsilon}(\delta + \epsilon) \\ &> \delta_0 \mathcal{B}, \end{aligned} \tag{31}$$

where $\delta_0 = \frac{\delta - c_0(\delta+\epsilon)}{\delta+\epsilon} > 0$. From (1) and (31), we obtain

$$\left(\omega \cdot a \cdot \left(\mathcal{B}''\right)^{\kappa}\right)' \leq -\delta_0^{\kappa} \omega \cdot \sum_{i=1}^{j} q_i \cdot [\mathcal{B}^{\kappa} \circ \eta_i].$$

Integrating this inequality from l to ∞, we obtain

$$\begin{aligned} \omega(l)a(l)\left(\mathcal{B}''(l)\right)^{\kappa} &> \delta_0^{\kappa} \int_l^{\infty} \omega(\mathfrak{z}) \sum_{i=1}^{j} q_i(\mathfrak{z}) \mathcal{B}^{\kappa}(\eta_i(\mathfrak{z})) \mathrm{d}\mathfrak{z} \\ &> \delta_0^{\kappa} \delta^{\kappa} \int_l^{\infty} \omega(\mathfrak{z}) \sum_{i=1}^{j} q_i(\mathfrak{z}) \mathrm{d}\mathfrak{z}, \end{aligned}$$

and

$$\mathcal{B}''(l) > \delta_0 \delta \left(\frac{1}{\omega(l)a(l)} \int_l^{\infty} \omega(\mathfrak{z}) \sum_{i=1}^{j} q_i(\mathfrak{z}) \mathrm{d}\mathfrak{z} \right)^{1/\kappa}. \tag{32}$$

Integrating (32) from l to ∞ and integrating the resulting inequality from l_1 to ∞, we obtain

$$\mathcal{B}'(l) - \lim_{l \to \infty} \mathcal{B}'(l) < -\delta_0 \delta \int_l^{\infty} \left(\frac{1}{\omega(u)a(u)} \int_u^{\infty} \omega(\mathfrak{z}) \sum_{i=1}^{j} q_i(\mathfrak{z}) \mathrm{d}\mathfrak{z} \right)^{1/\kappa} \mathrm{d}u. \tag{33}$$

From Lemma 2, we have $\mathcal{B}''(l) > 0$ for $l \geq l_1$. Since $\mathcal{B}'$ is a negative increasing function, we find that $\lim_{l \to \infty} \mathcal{B}'(l) = -L^2 < \infty$. Hence, (33) becomes

$$\mathcal{B}'(l) < -\delta_0 \delta \int_l^{\infty} \left(\frac{1}{\omega(u)a(u)} \int_u^{\infty} \omega(\mathfrak{z}) \sum_{i=1}^{j} q_i(\mathfrak{z}) \mathrm{d}\mathfrak{z} \right)^{1/\kappa} \mathrm{d}u.$$

Therefore,

$$\mathcal{B}(l_1) > \delta_0 \delta \int_{l_1}^{\infty} \int_v^{\infty} \left(\frac{1}{\omega(u)a(u)} \int_u^{\infty} \omega(\mathfrak{z}) \sum_{i=1}^{j} q_i(\mathfrak{z}) \mathrm{d}\mathfrak{z} \right)^{1/\kappa} \mathrm{d}u \mathrm{d}v,$$

which contradicts (30). This implies $\lim_{l \to \infty} \mathcal{B}(l) = 0$, and so $\lim_{l \to \infty} v(l) = 0$. □

Theorem 5. *If condition (30) and one of conditions (8), (16), or (17) are satisfied, then Equation (1) is almost oscillatory.*

Proof. Assume that ν is an eventually positive solution of (1) on $[l_0, \infty)$. It follows from Lemma 2 that $\mathcal{B}(l) > 0$ or $\mathcal{B}(l) < 0$, for $l \geq l_1$. However, any of conditions (8), (16), or (17) guarantee that $\mathcal{S}_x = \emptyset$, and thus the possibility of $\mathcal{B}(l) > 0$ is ruled out. On the other hand, condition (30) ensures that every eventually positive solution of (1) converges to zero. □

3. Conclusions

Our interest in this work is to study the asymptotic behavior of solutions of third-order differential equations that include a middle term and several delays. The study of odd-order differential equations did not obtain the same attention as the even-order differential equations. This is due to the many analytical difficulties and the many possibilities of derivatives of solutions.

After classifying the positive solutions of the studied equation, we obtained some relationships that link the derivatives of these solutions. Then, we employed these relationships to obtain an iterative relationship between the solution and its corresponding function. Using this relationship, we obtained different forms of criteria that ensure that there are no solutions in class Q. Finally, we set new conditions that ensure that all non-oscillatory solutions to Equation (1) converge to zero.

To find criteria that ensure the oscillation of all solutions of the studied equation, we need a criterion that excludes the so-called Kenser solutions, which are those that have a corresponding function whose sign differs from the sign of its first derivative. It would be interesting to find criteria that ensure that all solutions of Equation (1) oscillate. In addition, it is also interesting to obtain new oscillation criteria for the studied differential equation in the noncanonical case, that is, when

$$\int_{l_0}^{\infty} \frac{1}{a^{1/\kappa}(\mathfrak{r})} \exp\left(-\frac{1}{\kappa}\int_{l_0}^{\infty} \frac{h(\mathfrak{z})}{a(\mathfrak{z})} d\mathfrak{z}\right) d\mathfrak{r} < \infty.$$

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