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# Ideals of Projections According to $\sigma$-Algebras and Unbounded Measurements

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**Abstract:** A theory of unbounded measures is constructed based on the quantum logics of orthogonal projections. As an analogue of the ring of sets, the projector ideal is proposed. Finite and maximal measures regarding the projector ideals are described. Analogues of a number of classical theorems of measure theory are found. A wide class of unbounded measures on projection ideals is characterized. A number of sufficient conditions are found to extend unbounded measures to an integral of the entire algebra. The problem of describing unbounded $\sigma$-finite measures in semifinite algebras using von Neumann is similar to the Gleason problem.

**Keywords:** Hilbert space; idempotent; projection; logic; measures; unbounded measure

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## 1. Introduction

The algebraic axiomatic system of quantum logic is the subject of many research papers [1]. Classical quantum logics consist of all orthogonal projections (=idempotents) of a complex Hilbert space. In quantum logic, the states (=probability measures) are described by Gleason's theorem [2], and [3].

Let $\Pi$ be a set of all orthogonal (=self-adjoint) projections in a separable Hilbert space $H$, $dimH \geq 3$. Let the function $\mu : \Pi \to \mathrm{R}^+$ be such that $\mu(\sum_i P_i) = \sum_i \mu(P_i)$. Then, there is a unique non-negative trace-class (nuclear) operator $T$ such that $\mu(P) = tr(TP)$.

Note that Gleason's theorem (countable-additive) measurements may be extended to a linear normal functional on $B(H)$-algebra. In this sense, the measurement may be called *linear*. Gleason's theorem has been generalized to orthogonal projections from von Neumann algebras (see, for example, [4]) and to real-orthogonal projections (see [5]).

The following question remains open: are there logics in Hilbert space other than orthogonal projections, which allow for one to develop a theory of quantum mechanics as efficiently as the logic of orthogonal projections?

The papers known to the author on unbounded measures of projections mainly belong to Kazan mathematicians. These results are most fully presented in the PH thesis of G.Lugovaya [6]. In this paper, the most attention is paid to measures for projections in $B(H)$. Since the algebra $B(H)$ is discrete, the methods developed by G.Lugovaya are not applicable to algebras of a continuous type, particularly algebras of type $II$.

Vector fields were studied i [7], and orthogonal vector fields were studied in article [8].

Our goal is to develop an analogue of the classical measures in orthogonal projections of Hilbert space that leads to a normal, semi-finite weight and serves as a good analogue of the integral.

Let $H$ be a separable complex Hilbert space with scalar product $(\cdot,\cdot)$; $B(H)$ is a set of all linear bounded operators on $H$; $I$ is a unity on $B(H)$. Let $A \in B(H)$. Then, $|A| = (A^*A)^{1/2}$. Using $P_{\Delta A}$, we can denote the orthogonal projection on $\overline{AH}$. Let $A, B$ be bounded self-adjoint operators in $B(H)$. We write that $A \leq B$, if $(Ax, x) \leq (Bx, x)$ for all

$x \in H$. $P^{\perp} = I - P$ is used for any $P \in \Pi$, $P \perp Q$ if $PQ = 0$. Two projections, $P$ and $Q$, are said to be in a *general position* if $P \wedge Q = P \wedge Q^{\perp} = P^{\perp} \wedge Q$.

Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be such that if $A \in \mathcal{M}$, then $A^* \in \mathcal{M}$. The set $\mathcal{M}' = \{B \in B(H) : BA = AB, \forall A \in \mathcal{M}\}$ is said to be *commutant* of $\mathcal{M}$. If $\mathcal{M} = \mathcal{M}''$, then $\mathcal{M}$ is said to be *von Neumann* algebra on $H$. Basic information about von Neumann algebras can be found in [9].

Let $\mathcal{M}^+$ be a set of all non-negative operators from $\mathcal{M}$, and let $\Pi$ be a set of all orthogonal projections from $\mathcal{M}$. Projections $P, Q \in \Pi$ are said to be equivalent (when writing $P \sim Q$), if there is a unitary operator $U \in \mathcal{M}$ such that $U^*PU = Q$.

Von Neumann algebra $\mathcal{M}$ is a type $I$ algebra if there is an abelian orthogonal projection with central cover that leads to unity $I$ on $\mathcal{M}$. For instance, $B(H)$ is a von Neumann algebra of type $I$.

Von Neumann algebra $\mathcal{M}$ is a type $II$ algebra if any projection $P$ may be "divided" into two equivalent sub-projections $P_1$, $P_2$, $P = P_1 + P_2$, and $P_1 \sim P_2$, and there is a faithful, normal, and semi-finite trace on $\mathcal{M}^+$. (In particular, for a definition Neumann algebras of of types $I$, $II$, $III$ of, see page 126 of [9]).

Note that $P = \sum_i P_i$, $P, P_i \in \Pi$ implies $P_i \perp P_j$, i.e., $P_iP_j = 0$. An operator $T$ is said to be *attached* to algebra $\mathcal{M}$, if $U^*TU = T$ for any unitary operator $U$ from the commutant $\mathcal{M}'$. Let $P \in \Pi$. Place $\mathcal{M}_P = \{PAP : A \in \mathcal{M}\}$. Let us identify operator $A \in \mathcal{M}_P$ in $PH$ with operator $AP \in \mathcal{M}$ in $H$.

A lineal $D \subseteq H$ is said to be *strongly dense* (with respect von Neumann algebra $\mathcal{M}$) if there is a sequence of orthogonal projections $P_n \in \mathcal{M}$ that increases to the unity $I$, such that $\tau(I - P_n)_{n \to +\infty} \searrow 0$ and $P_nH \subset DZ$. Here, $\tau$ is a faithful, normal, semi-finite trace on $\mathcal{M}^+$.

It is known that, for any sequence of strongly dense lineals $\{D_n\}$, the lineal $\cap_n D_n$ is strongly dense. Let $P \in \Pi$, $P \neq 0$ and let $D$ be a strongly dense lineal. Then, $PH \cap D$ is strongly dense for the $PH$ lineal.

For an unbounded self-adjoint operator $T \geq 0$ attached to the algebra $\mathcal{M}$, the value $\tau(Tp)$, $p \in \Pi$ can be determined using equation $\tau(Tp) = \lim_{\epsilon \searrow 0} \tau(T_\epsilon^{1/2} p T_\epsilon^{1/2})$. Here, $T_\epsilon = T(I + \epsilon T)^{-1}$, $\epsilon > 0$.

We can offer another option that is equivalent to the first. Let $T = \int_0^{+\infty} \lambda d(e_\lambda)$ be a spectral decomposition of $T$. Then, $\tau(Tp) = \lim_{n \to +\infty} \tau(pT_np)$. Here, $T_n = \int_0^n \lambda d(e_\lambda)$.

Let us denote the set of all integrables (square-integrable) with respect to $\tau$ operators by $\mathcal{L}_1(\tau, \mathcal{M})$ (by $\mathcal{L}_2(\tau, \mathcal{M})$).

A function $a(x, y) \to \mathrm{C}$ with $a(x + z, y) = a(x, y) + a(z, y)$, $a(x, y) = \overline{a(y, x)}$ and $a(\lambda x, y) = \lambda a(x, y)$ $\forall x, y \in H$, and $\lambda \in \mathrm{C}$ is said to be a *bilinear* form (b.f.).

The structure of the article is as follows: the first section §1, discusses sufficient conditions for extending a measure from the ideal of projections to the weight; the second section §2 demonstrates the possibility of extending an infinitely valued measure to a weight.

**Main results. Ideals of projections and their properties**

**Definition 1.** *Let $P, Q \in \Pi$. The set $\mathfrak{M} \subseteq \Pi$ is said to be* **ideal** *(of projections), if*
*$(i)$ $P \leq Q$, $Q \in \mathfrak{M}$ follow $P \in \mathfrak{M}$;*
*$(ii)$ $P, Q \in \mathfrak{M}$ and $\|PQ\| < 1$ follow $P \vee Q \in \mathfrak{M}$.*
*$(iii)$ $sup\{P : P \in \mathfrak{M}\} = I$.*

Note that condition $(iii)$ is only necessary to the uniqueness of the continued measures.

**Definition 2.** *The function $\phi : \mathcal{M}^+ \to [0, +\infty]$ with*
*(i) $\phi(A + B) = \phi(A) + \phi(B)$;*
*(ii) $\phi(\lambda A) = \lambda\phi(A)$, $\lambda \in \mathrm{C}$, $0\infty = 0$.*
*is called the* **weight**.

The weight is said to be:

**faithful**, if $\phi(A) = 0$ follows $A = 0$;
**semi-finite** if $lin\{A \in \mathcal{M}^+ : \phi(A) < +\infty\}$ is an ultra-weakly dense set on $\mathcal{M}$;
**normal** if $A_i \nearrow A \in \mathcal{M}^+$ follows $\phi(A) = \sup_i \phi(A_i)$;
**trace** if $\phi(A^*A) = \phi(AA^*)\ \forall A \in \mathcal{M}$.

Let us denote $\mathfrak{M}_\tau = \{P \in \Pi : \tau(P) < +\infty\}$. $\mathfrak{M}_\tau$ is a lattice and ideal of projections. Further, we can see that any measure is regular on this ideal for von Neumann algebra of type $II$.

First, we will study the set of projections on which any weight is finite.

**Proposition 1.** *Let $P, Q \in \Pi$ be such that $\|PQ\| < 1$. Then $\alpha(P \vee Q) \leq (P + Q)$ for same $\alpha > 0$.*

**Proof.** By $\|PQ\| < 1$, the operator $P + Q$ has a bounded inverse on $(P \vee Q)H$. Place $c = \|PQ\|$. For any $x \in H$, we have $\|(P \vee Q)x\| >(1 - c)^{1/2}\|Px\|$. According to this, $(P \vee Q - P)QH = (P \vee Q - P)H$. Therefore if $x \in (P \vee Q)H$, then there exists $x_1 \in QH$, such that $(P \vee Q - P)(x - x_1) = 0$. Thus, $x - x_1 \in PH$. Thus, we established that $(P + Q)H = (P \vee Q)H$. This proves the Proposition. □

By complicating the proof of Proposition 1, we can significantly strengthen it.

**Proposition 2.** *Let $P, Q \in \Pi$ be such that $\|PQ\| < 1$. Then,*

$$P \vee Q \leq \frac{1}{1 - \|PQ\|}(P + Q).$$

**Proof.** (1) Let us first establish the inequality that occurs when projectors $P, Q$ are one-dimensional and $dimH = 2$. Note, $P \vee Q = I$. Then, $PQP = (cos^2\alpha)P$, where $\alpha \in (0, \pi/2)$, i.e., $\|PQP\| = cos^2\alpha$. The spectral decomposition of self-adjoint operator $P + Q$ is $P + Q = (1 - cos\alpha)(I - F) + (1 + cos\alpha)F = (1 - cos\alpha)I + (2cos\alpha)F$; here, $F$ is a suitable one-dimensional projection. Thus, $P + Q \geq (1 - \|PQ\|)I$.

(2) Let the projections $P, Q$ occur in the general position and let us denote, using $\mathcal{M}(P, Q)$, the minimal von Neumann algebra generated by $P, Q$. In a separable Hilbert space $H$ $\mathcal{M}(P, Q)$, undergo a central decomposition into a direct integral of factors of type $I_2$ [9] Hapter II. $P = \int_\Lambda^\oplus p_\lambda d\lambda$ and $Q = \int_\Lambda^\oplus q_\lambda d\lambda$, and $P_\lambda, Q_\lambda$ almost everywhere in one-dimensional projectors $\|p_\lambda q_\lambda\| \leq \|PQ\|$. Furthermore,

$$P \vee Q = \int_\Lambda^\oplus P_\lambda \vee Q_\lambda d\lambda \leq \int_\Lambda^\oplus (1 - \|PQ\|)^{-1}(p_\lambda + q_\lambda)d\lambda = (1 - \|PQ\|)^{-1}(P + Q).$$

(3) Let us consider the general case of $P, Q$. Then, $P = P_1 + P_2$ and $Q = Q_1 + Q_2$, where $P_2 = P \wedge Q^\perp$, $Q_2 = Q \wedge P^\perp$, in addition to the projections $P_1, Q_1$, which are in a general position. Finally, let us apply step (2). □

**Lemma 1.** *Let $\phi$ be a weight on von Neumann algebra $\mathcal{M}$.*

*$(i)$ Let $P, Q \in \Pi$ be such that: $\phi(P) < +\infty$, $\phi(Q) < +\infty$ and $\|PQ\| < 1$. Then $\phi(P \vee Q) < +\infty$.*

*$(ii)$ Let $\phi$ be semi-finite; then, $\sup\{P : \phi(P) < +\infty\} = I$.*

**Proof.** $(i)$ The operator $P + Q$ has a bounded inverse operator in $PH + QH$. Therefore, $\alpha > 0$ occurs, such that $\alpha(P \vee Q) \leq P + Q$. Thus $\phi(\alpha(P \vee Q)) \leq \phi(P + Q)$. Hence, $\phi(P \vee Q) < +\infty$.

According to the definition of the semi-finiteness of a weight, $(ii)$ holds. □

## 2. Measures on Ideals

Let us denote, using $\mathfrak{M}_P$, the set $\{Q : Q \in \mathfrak{M}, Q \leq P\}$

**Definition 3.** *The function $\mu : \mathfrak{M} \to \mathrm{R}^+$ is said to be a* **measure** *if $\mu(\sum_i P_i) = \sum_i \mu(P_i)$, $P_i \in \mathfrak{M}$.*

Let $\mu : \mathfrak{M} \to \mathrm{R}^+$ be a measure. Let $B \geq 0$, $B \in \mathcal{M}$, $B = PBP$, $P \in \mathfrak{M}$ and $\int e_\lambda d\lambda$ be the spectral decomposition of $B$. Place $\mu'(B) = \int \mu(e_\lambda)d\lambda$.

The measure $\mu : \mathfrak{M} \to \mathrm{R}^+$ is said to be:

**finite**(=**bounded**) if $\sup\{\mu(P) : P \in \mathfrak{M}\} < +\infty$,

**infinite**(=**unbounded**) if $\sup\{\mu(P) : P \in \mathfrak{M}\} = +\infty$,

**regular** if there is weight $\phi$, such that $\mu(P) = \phi(P)$ for all $P \in \mathfrak{M}$ .

Let $\mu_1 : \mathfrak{M}_k \to \mathrm{R}^+$ and $\mu : \mathfrak{M} \to \mathrm{R}^+$ serve as measures. The measure $\mu_k$ is said to be *a continuation* of $\mu$ if $\mathfrak{M} \subset \mathfrak{M}_k$ and $\mu_1(P) = \mu(P)$ for all $P \in \mathfrak{M}$. In this case, we write $\mu \leq \mu_1$.

**Remark 1.** *Any measure $\mu$ on a finite algebra of type II in separable Hilbert space is regular. If, in addition, $\mu$ is finite, then $\mu(P) = \tau(A_\mu P)$. Here, $\tau$ is a faithful, normal, semi-finite trace and $A_\mu \in \mathcal{L}_1(\tau, \mathcal{M})$ is a unique, non-negative operator (see [4]).*

*Let $\mathcal{M}$ be a type II von Neumann algebra and $\mu : \mathfrak{M} \to \mathrm{R}^+$ be a measure. Let $P \in \Pi$. Put $\mathfrak{M}_P = \{Q \in \mathfrak{M} : Q \leq P\}$. We say that projection $P$ has a finite measure (finite $\mu$-measure) if*

$$sup\{Q : Q \in \mathfrak{M}_P\} = P \quad and \quad \overline{\mu}(P) = sup\{\mu(Q) : Q \in \mathfrak{M}_P\}$$

**Theorem 1.** *Let $\mu : \mathfrak{M} \to \mathrm{R}^+$ be a measure. Let $\overline{\mathfrak{M}}$ be the largest hereditary class of projections of finite $\mu$-measure. Then, $\overline{\mathfrak{M}}$ is the ideal of projections and there is a unique measure $\mu_1$ on $\overline{\mathfrak{M}}$, such that $\mu \leq \mu_1$.*

**Proof.** Let us show that $P, Q \in \overline{\mathfrak{M}}$ and $\|PQ\| \leq \delta < 1$ entails $P \vee Q \in \overline{\mathfrak{M}}$. Using $\|PQ\| \leq \delta < 1$, subspace $(P \vee Q)H$ is the direct sum of subspaces $PH$, $QH$. Hence, any vector $f \in (P \vee Q - Q)H$ can be represented as $f = f_P + f_Q$. Here, $f_P \in PH$, $f_Q \in QH$. The function $A : f \to f_P$ $(f \in (P \vee Q - Q)H)$ is a restricted operator (since $\|f_P\|^2 \leq (1 - \|PQ\|)^{-1}\|f\|^2$).

This means that for any projection $G$, $G \leq P \vee Q - Q$, the operator is $AG \in \mathcal{M}$. Let us choose a projection $F$, $F \neq 0$, such that $F \leq P_{\Delta AG}$, and projection $E$, $E \neq 0$, such that $E \leq P_{\Delta QF}$. We will obtain $F \vee E \in \mathfrak{M}$ and $0 \neq (F \vee E) \wedge G \in \mathfrak{M}$. Hence, any projection $G$, $G \leq (P \vee Q - Q)$ contains a non-zero projection from $\mathfrak{M}$. It is now clear that this similarly holds for any projection $G$, $G \leq P \vee Q$. According to Proposition 1,

$$\mu((F \vee E) \wedge G) \leq \mu(F \vee E) \leq \alpha(\overline{\mu}(P) + \overline{\mu}(Q)).$$

It follows from this inequality that every projection $G \leq P \vee Q$ has a finite $\mu$-measure. Hence, $\overline{\mathfrak{M}}$ is the ideal. The countable additivity of the function $\overline{\mu}$ is obvious. Theorem 1 is proved. □

**Definition 4.** *The measure $\overline{\mu}$ of Theorem 1 is said to be a closed measure.*

Note that the closure of a measure does not change its regularity.

Let $T = \int \lambda de_\lambda$ be non-negative self-adjoint operator associated with $\mathcal{M}$, (i.e., $e_\lambda \in \mathcal{M}$ for all $\lambda$). The function $\tau(TP) \to [0, +\infty]$ is said to be the measure *associated* with $T$.

There are projection ideals in which not every measure is regular. Therefore, the following theorem is of interest.

**Theorem 2.** *In the von Neumann algebra of type II for any measure $\nu : \mathfrak{M}_\tau \to \mathrm{R}^+$, there is a non-negative operator $T_\mu$, such that $\mu(Q) = \tau(T_\mu Q)$, $\forall\, Q \in \mathfrak{M}_\tau$.*

We first introduce some notations and provide auxiliary assertions. Let $\mu : \mathfrak{M} \to \mathrm{R}^+$ be a measurement. Put

$$\alpha_\mu(Q) = inf\{\alpha \ (\leq +\infty) : \mu(Q') < \alpha\tau(Q'), \forall Q' \in \mathfrak{M} \cap \mathfrak{M}_\tau, Q' \leq Q, Q' \neq 0\}.$$

$$N(P, a) = \{Q \in \mathfrak{M}_\tau \cap \mathfrak{M} : \mu(Q) \geq a\tau(Q), Q \leq P\}.$$

$$S_a = \{P \in \mathfrak{M}_\tau : \alpha_\mu(I-P) < a\}.$$

We note the following elementary property: If $Q_1 \perp Q_2$ then $\alpha_\mu(Q_1+Q_2) \le \alpha_\mu(Q_1) + \alpha_\mu(Q_2)$ $(< +\infty)$ We will use the following proposition

**Corollary 1.** *Let $\mu : \mathfrak{M}_\tau \to \mathrm{R}^+$ be a measure in the algebra of type II and $\alpha_\mu(I) < +\infty$. Then, operator $T_\mu \in \mathcal{M}$ is found, such that $\mu(Q) = \tau(T_\mu Q)\ \forall\ Q$.*

**Proof.** Let $\alpha_\mu(P) < c\ \forall P$. Place

$$D(\mathfrak{M}) = \cup_{Q\in\mathfrak{M}}(L_2(\mathcal{M}_P) \cap \mathcal{M}_Q),\ \ P = \vee_{Q\in\mathfrak{M}} Q.$$

Let us define the bilinear form $t(x,y) = \mu'(y^*x)$, $x, y \in D(\mathfrak{M})$. Bilinear form $t(\cdot,\cdot)$, defined on the lineal $D(\mathfrak{M})$ of operators that densely occur in a Hilbert space $\mathcal{L}_2(\mathcal{M}_P)$, is bounded, i.e., $sup_{\|x\|=1} t(x,x) < c$. According to continuity, the form $t(\cdot,\cdot)$ can be extended to the bilinear form $\bar{t}(\cdot\cdot)$ on $\mathcal{L}_2(\mathcal{M}_P)$. Operator $0 \le T_\mu \in \mathcal{M}$ is found, such that $\bar{t}(x,y) = \tau(y^*xT_\mu)$ (see [10], p. 118). Let $P \in \mathfrak{M}$. Put $P = \sum P_i$, $P_iP_j = \delta_{ij}P_i$, and $P \in D(\mathfrak{M})$. By applying Lemma 13.1 [10], we complete the proof. □

It is clear that $T_\mu \ge 0$. Let us first find out the structure of the set $S_a$, $a > 0$.

**Lemma 2.** *Let $\mu : \mathfrak{M} \to \mathrm{R}^+$ be a measure in the algebra of type II and $a > 0$. If $N$, some set of mutually orthogonal projections from $N(I,a)$ and $0 \le 2b \le \tau(\vee\{Q, Q \in N\})$, occurs, then $P \in \Pi$ can be found, such that*

$$\tau(P) = b \ \ and \ \ 2\mu(P) \ge a\tau(P) \tag{1}$$

**Proof.** Let the lemma's requirements be satisfied. Let $\mathcal{X}$ be some finite set from $N$, such that $b < \tau(\vee\{Q : (Q \in \mathcal{X})\})$. Let $\mathcal{L}_1(\tau, \mathcal{M}_P)$ be a set of all integrable by $\tau$ operators with the norm $\|B\|_1 = \tau(|B|)$. For any $P \in \mathfrak{M}$, there exists an operator $B^P_\mu$ for which $\tau(B^P_\mu) = \mu(P)$ and $\mu(Q) = \tau(B^P_\mu Q)$, $Q \le P$. Place $Z = \vee\{Q : Q \in \mathcal{X}\}$ and let $|B^Z_\mu Q| = \int \lambda de_\lambda$ be the spectral representation of $|B^Z_\mu Q|$. Then,

$$a\tau(Z) = \sum_{Q\in X} a\tau(Q) \le \sum_{Q\in X} \mu(Q) = \|B^Z_\mu\|_1.$$

If $\mu(I - e_\lambda) = \||B^Z_\mu Q|(I - e_\lambda)\|_1 < a\tau(I - e_\lambda)$, $\lambda > 0$, then $\lambda < a$ and

$$\mu(e_\lambda + Z - I) = \||B^Z_\mu Q|(e_\lambda + Z - I)\|_1 < a\tau(e_\lambda + Z - I),\ \forall\ \lambda > 0$$

Hence,

$$\mu(Z) = \mu(I - e_\lambda) + \mu(e_\lambda + Z - I) < a\tau(Z),\ \ \mu(I - e_\lambda) \ge a\tau(I - e_\lambda), \forall \lambda > 0$$

Let $b \le \tau(Z)$. If $b \ge \tau(I - e_{+0})$, then any projection $P \in \Pi$, such that $\tau(P) = b$ and $I - e_{+0} \le P \le Z$ projections will be desirable because

$$a\tau(P) \le a\tau(Z) \le \mu(Z) = \|B^Z_\mu\|_1 = \|B^Z_\mu P\|_1 = \mu(P).$$

If $b < \tau(I - e_{+0})$, then there is $\lambda > 0$ with $\tau(I - e_{\lambda+0}) \le b \le \tau(I - e_\lambda)$. Therefore, any projection $P \in \Pi$ such that $\tau(P) = b$ and $I - e_{\lambda+e} \le P \le I - e_\lambda$ will be desired. □

**Lemma 3.** *Let $\mu : \mathfrak{M}_\tau \to \mathrm{R}^+$ be a measure in the algebra of type $II_\infty$. Then, there exists $a > 0$, such that $S_a \ne \emptyset$.*

**Proof.** However, for any $a > 0$ and any $P_0 \in \mathfrak{M}_\tau$, there is a projection, $Q$, which means $Q \in \mathfrak{M}_\tau$, $Q \le I - P_0$ and $\mu(Q) > a\tau(Q)$. Let us denote, using $N$, the maximal, mutually orthogonal set from $N(I,a)$. Place $P_a = \vee\{Q : Q \in N\}$. It is clear that $P_a \notin \mathfrak{M}_\tau$. Let us choose a sequence of positive numbers, $a_n$ and $b_n$, such that $a_n \nearrow +\infty$, $\sum b_n < +\infty$ and $\sum a_n b_n = +\infty$. According to Lemma 2, there exists $P_1 \in \Pi$, such that $\tau(P_1) = b_1$ and $\mu(P_1) > a_1\tau(P_1)$.

Let us suppose that the projection $P_{n-1}$ is already chosen. Let us carry out the same reasoning as before, choosing the projection $P_n$ such that

$$\tau(P_n) = b_n, \quad P_n \in N(I - \sum_{k=1}^{k=n} P_k, a_n).$$

Place $P = \sum_n P_n$. Then, $\mu(P) = \sum \mu(P_n) = \sum_n b_n < +\infty$, i.e., $P < I$. In addition, $\mu(P) = \sum_n \mu(P_n) > \sum_n a_n \tau(P_n) > \sum_n a_n b_n = +\infty$. We obtain a contradiction. □

**Lemma 4.** *Let $\mu : \mathfrak{M} \to \mathrm{R}^+$ be a measure of the algebra of type $II$. For any $\epsilon > 0$, $a > 0$ and $P \in S_a$, such that $\tau(P) \le \epsilon$.*

The proof is easy to carry out using the opposite reasoning. We omit the proof.

**Corollary 2.** *For any $\epsilon > 0$, there exists $a > 0$ and a maximal set of mutually orthogonal projections $N$ from $N(I,a)$ with $\tau(\vee\{Q : Q \in N\}) < \epsilon$, $\forall\ \{Q : Q \in N\} \in S_a$ .*

**Lemma 5.** *Let $\mu : \mathfrak{M}_\tau \to \mathrm{R}^+$ be a measure in the algebra of type $II$. Then, there exists a sequence of projections $\{P_n\} \subset \mathfrak{M}_\tau$, such that $P_n \searrow 0$ and $\alpha_\mu(I - P_n) < +\infty\ \forall\ n$.*

**Proof.** Let the sequences $\{a_n\}$ and $Q_n \in S_{a_n}$ be such that $\tau(Q_n) < 2^{-n}$. Put $P_n = \vee_{k>n} Q_k$. Then, the sequence $\{P_n\}$ is the desired one. The lemma is proved. □

**Proof of Theorem 2.** Let $\mu : \mathfrak{M}_\tau \to \mathrm{R}^+$ be a measure of the algebra of type $II$. Let $\{P_n\}$ be the sequence obtained from Lemma 5. According to Corollary 1, the sequence operators $T^n_\mu \in \mathcal{M}$, such that $\mu(Q) = \tau(T^n_\mu Q)\ \forall\ Q \le I - P_n$. We can assume that $T^n_\mu = P_n T^n_\mu P_n$. It is clear that $T^n_\mu = P_n T^m_\mu P_n$ if $n < m$. This means that the sequence $\{T^n_\mu\}$ is fundamental in measure $\tau$. $\lim_{n\to+\infty} T^n_\mu = T_\mu$ can be obtained by measure $\mu$. It is clear that $\mu(Q) = \tau(T_\mu Q)$, $\forall\ Q$. □

A similar property, which is already in Neumann algebras of type $I$, is not true.

**Theorem 3.** *Let $\mathfrak{M}_\tau$ be the set of all finite-dimensional orthogonal projections in infinite-dimensional separable Hilbert space $H$. Let $\mu : \mathfrak{M}_\tau \to +\mathrm{R}^+$ be a measure. In addition, let $\{e_i\}_1^{+\infty}$ be the orthonormal basis in $H$, such that*

$$\mu(P) = \lim_{n\to\infty} \mu'(E_n P E_n) \tag{2}$$

*Here, $E_n$ is orthogonal projection on subspace $H_n$, generated by $e_k$ $k = 1, 2, \ldots$. Then, there exists the bounded operator $B$, such that $\mu(P) = tr(BP)\ \forall P$.*

**Proof.** Place $a(k,i) = \mu'((\cdot, e_k)e_i)$. The proof known to the author is based on the fact that the matrix $\|a(k,i)\|$ defines a closed symmetric operator $B$, which is defined everywhere. Therefore, it is bounded. Hence, $\mu(BP) = tr(PB)$. □

Note that, without condition (2), the theorem is not true

Theorem 3 was previously proved by Stinespring W.F. ([11] Theorem 2.1) He obtained this proof using the fact that equality $(S+N, T+N) = \mu'(TS^*)$, here $N = \{T \in \mathcal{F}, \mu'(T^*T) = 0\}$, and the set $\mathcal{F}$ of all finite-dimensional operators from $\mathcal{F}/\mathcal{N}$, determine the structure of a pre-Hilbert space in the left module. Having replenished this space, considering the $^*-$representation of the ring $\mathcal{F}$, Stinespringe obtained a number of necessary estimates.

Differences between the properties of measures in algebras of type $I$ and $II$ are explainedby the following:

Measures in the algebras of type $I$ are finitely additive. Therefore, singular (or, of course, additive) measures are not excluded. Such measures are identically zero on finite-dimensional projectors. Therefore, we needed an additional condition (2).

The measures in algebras of type $II$ are countably additive.

In the mathematical literature, the ideals $C_0(\Gamma)$ of measurable operators are studied as an analog of the ideal of completely continuous operators. Let $A$ be measurable operator, and $|A| = \int \lambda dE_\lambda$ is spectral decomposition of $|A|$. Then, $A \in C_0(\Gamma)$ if, and only if, for any $\lambda > 0$, $\tau(I - E_\lambda) < +\infty$.

Let $\mu : \mathfrak{M} \to \mathrm{R}^+$ be a measure of the algebra of type $II$. Place

$$N(I, \beta) = \{Q \in \mathfrak{M} : inf \frac{\mu(Q')}{\tau(Q')} \geq \beta \ \ \forall Q' \leq Q.$$

**Theorem 4.** *The measure $\mu : \mathfrak{M} \to R^+$, which is to be associated with the operator from $C_0(\Gamma)$, is sufficient for any $\beta > 0$ and every of set mutually orthogonal projections $\{P_n\}$ from $N(I, \beta)$: this shows that $\tau(\sum_n P_n) < +\infty$ .*

**Proof.** Let the condition of this theorem be satisfied. For some $\beta > 0$, we chose a maximal family $\{P_n\}$ of pairwise orthogonal projections from $W(\beta)$. Using $\tau(\sum_i P_n) < +\infty$, we obtained $\mu(Q) \leq \beta\tau(Q)$ for any $Q \in \mathfrak{M}$, such that $Q \leq I - \sum_n P_n$. According to Remark 1, we can deduce that there is a sequence of projectors $\{Q_m\} \in \mathfrak{M}$ for which $Q_m \nearrow \sum_n P_n$ and $\alpha(Q_m) < +\infty$.

Then, for any $Q \in \mathfrak{M}$, $Q \leq (I - \sum_n P_n) + Q_n = C_n$, we have $\alpha(Q) < +\infty$. According to Remark 1, there is a sequence of bounded operators $(0 \leq) B_n \in \mathcal{M}$, such that $B_n C_n = B_n$, and for any $P \in \mathfrak{M}$, the equalities $\mu(P) = \tau(B_n P)$ and $B_n = C_n B_m C_n$ hold, if $m > n$. Hence, the sequence $\{B_n\}$ is fundamental according to the measure $\tau$. Let $B_n \to B$, by the measure $\tau$. Operator $B$ is measurable and non-negative. For any $n$, we have $C_n B C_n = B_n$. Hence, it follows that $\mu$ is associated with $B$.

Let $B = \int \lambda dE_\lambda$ be the spectral decomposition of $B$. Let us show that $\tau(I - E_\lambda) < +\infty$ for any $\lambda > 0$. Let $\{Q_n\}$ be maximal set of mutually orthogonal projections from $N(I, \lambda)$. Then, for any $Q \in \mathfrak{M}$, $Q \leq I - \sum_n Q_n$, we have $\tau(Q) < +\infty$. Thus, $B^{1/2}G$ is bounded, and $\|B^{1/2}G\| \leq \lambda^{1/2}$; here, $G = I - \sum_n Q_n$.

Let us suppose that $\tau(I - E_\lambda) = +\infty$. Then, $(I - E_\lambda) \wedge G \neq 0$. Therefore, $\|B^{1/2}G\| \geq \lambda^{1/2}$. Contradiction. The Theorem is proved. □

**Remark 2.** *The condition of Theorem* 4 *is not necessary.*

**Proof.** Let $(P^{k,n}_{k=1,n=1})$ be a set of mutually orthogonal projections of factor $\mathcal{N}$ of type $II_\infty$, such that $\tau(P^k_n) = \frac{1}{n^2}$, $k = \overline{1,n}$. Let us construct this using induction $P_{n,k}$. Place $P_{n,1} = P^1_n$. Assume $P_{n,k-1}$ has already been constructed, and let $V \in \mathcal{N}$ be a partially isometric operator, such that $VV^* = P_{n,n-1}$, $V^*_n V = P^k_n$. Put

$$P_{n,k} = \frac{1}{k} P^k_n + (1 - \frac{1}{k}) P_{n,k-1} + \sqrt{\frac{1}{k}(1 - \frac{1}{k})}(V + V^*).$$

Let us denote, using $\mu\mathfrak{M} : to\mathrm{R}^+$, the measure associated with $K \in C^\gamma_0$. Here,

$$K = \sum_1^\infty n^2 P_{n,n} + \sum_{n=1}^\infty \frac{1}{n} (\sum_{k=1}^n P^k_n - P_{n,n}).$$

According to this construction, $\tau(\sum_{n=1}^\infty \sum_{k=1}^n P^k_n) = +\infty$. At the same time, $(P^k_n)^{n,\infty}_{k=1,n=1} \in N(I, 1)$.

Therefore, we can obtain an example of a measure that is associated with a measurable operator from $C_0(\gamma)$ but does not satisfy the condition of Theorem 4. $\square$

**Bilinear forms and measures** Let $T \geq 0$ be a self-adjoining operator that is associated with $\mathcal{M}$. Let $\mathfrak{M}_T = \{P \in \Pi : \tau(TP) < +\infty\}$. Note: (a) the function $\tau(T\cdot)$ is understood in the sense of article [12]; (b) $\mathfrak{M}_I = \mathfrak{M}_\tau$. The set $\mathfrak{M}_T$ is the ideal of projections and $\mu : P \in \mathfrak{M}_T \to \tau(TP)$ is a closed measure.

Let $\mathfrak{M}$ be an ideal. Let us denote, using $\mathfrak{M}_\tau(\mathfrak{M})$, the set

$$\{P \in \mathcal{M} : \textit{ there is a finite subset } \Xi \subset \mathfrak{M} \cap \mathfrak{M}_\tau, \textit{such that } P \leq \vee_{Q \in \Xi} Q\}.$$

It is known that $\mathfrak{M}_\tau(\mathfrak{M})$ is a projection ideal and, for any $P \in \mathfrak{M}_\tau(\mathfrak{M})$, the value of the measurement $\mu$ is calculated according to $\mu(Q) = \tau(T_P Q)$, $\forall\, Q \leq P$. Here, $T_P$ is nonnegative operator attached to $\mathcal{M}$.

Let us assume that a bilinear form $a(\cdot,\cdot)$ with domain $D(a)$ is *attached* to $\mathcal{M}$ if, for any unitary operator $U \in \mathcal{M}'$, $f \in D(a)$ entails $Uf \in D(a)$ and $a(f,g) = a(Uf,Ug)$, $f$, $g \in D(a)$.

**Proposition 3.** *Let $\mu : \mathfrak{M} \to \mathrm{R}^+$ be a measure. Then, the equality*

$$a_\mu(f,g) = (T_P^{1/2} f, T_P^{1/2} g), \;\; f,\, g \in D(T_P^{1/2}).$$

*defines b.f. on the lineal $D_\mu = \bigcup_{P \in \mathfrak{M}_\tau(\mathfrak{M})} D(T_P^{1/2})$ attached to $\mathcal{M}$.*

*Coversely, let a be b.f., and let $\mathfrak{M}_a$ be the set projections from $\mathfrak{M}_\tau$ such that, for any $P \in \mathfrak{M}_a$, we have:*

*(a) The set $PH \cap D(a)$ is strongly dense with respect to $\mathcal{M}_P$;*

*(b) There is $0 \leq A_P \in \mathcal{L}_1(\tau,\mathcal{M})$, such that $a(f,g) = (A_P^{1/2} f, A_P^{1/2} g)$. Here, $f,g \in D(a) \cap PH$.*

*Then, $\mathfrak{M}_a$ is ideal and the function $\mu_a(\cdot) = \tau(A_P \cdot) : \mathfrak{M}_a \to \mathrm{R}^+$ is the measure.*

The idea of describing measures in terms of bilinear forms belongs to Sherstnev [13]. Proposition 3 is another version of the assertion that was proved by him.

Further, we assume that the set of projections satisfies the axioms of (i), (ii), and the ideal, with the exception of (iii).

Note that the equality $P \wedge Q = 0$, $\forall\, Q \in \mathfrak{M}_\tau(\mathfrak{M})$ is equivalent to

$$(\bigcup_{Q \in \mathfrak{M}_\tau(\mathfrak{M})} QH) \cap PH = 0.$$

**Proposition 4.** *Let $\mathfrak{M} \to \mathrm{R}^+$ be a measure and let this projection occur $P \in \Pi$, such that $P \wedge Q = 0$ for all $Q \in \mathfrak{M}$. Then, the measure $\mu' : \mathfrak{M}' \to \mathrm{R}^+$ is shown, such that $P \in \mathfrak{M}'$ and $\mu'$ is a continuation of $\mu$.*

**Proof.** Let the conditions of the theorem be satisfied. Let the lineal $D_\mu$ and bilinear form $a_\mu$ be constructed. Then, $D_{mu} \cap PH = 0$. Let us choose $0 \leq A \in \mathcal{L}_1(\tau,\mathcal{M})$, such that $A = AP$. Let us construct b.f. $a(f,g)$. Place

$$a(f_1 + g_1, f_2 + g_2) = (A^{1/2} f_1, A^{1/2} f_2) + a_\mu(g_1, g_2), \; \forall\, f_1, f_2 \in PH,\, g_1, g_2 \in D_\mu.$$

Let $\mathfrak{M}_a$ be ideal and $\mu_a$ be the measure constructed in Proposition 3. Then, $(\mathfrak{M}_\tau \cap \mathfrak{M}) \subset \mathfrak{M}_a$. For any $Q \in \mathfrak{M}_\tau \cap \mathfrak{M}$, we have $\mu_a(Q) = \mu(Q)$. It is clear that $P \in \overline{\mathfrak{M}_a}$ and $\mathfrak{M} \subset \overline{\mathfrak{M}_a}$. The measure $\overline{\mu} : \overline{\mathfrak{M}_a} \to \mathrm{R}^+$ is the measure that was sought. $\square$

**Definition 5.** *The measure $\mu : \mathfrak{M} \to \mathrm{R}^+$ is said to be a locally finite measure if $\forall P \in \Pi$ contains $Q \in \mathfrak{M}$, $Q \neq 0$, such that $Q \leq P$.*

Earlier, we introduced an order relation for these measures. Note that, accoding to the Zorn lemma, every measure has a maximal continuation. We can now characterize the maximum measures.

**Theorem 5.** *The measure* $\mu : \mathfrak{M} \to \mathrm{R}^+$ *is maximal if, and only if, it is a closed and locally finite measure.*

**Proof.** The maximal measure must be closed and locally finite.

Conversely, let the measure $\mu : \mathfrak{M} \to \mathrm{R}^+$ be closed and locally finite. Let the measure $\mu' : \mathfrak{M}' \to \mathrm{R}^+$ be a continuation of $\mu$. From the local finiteness of $\mu$ it follows that, for any $P \in \mathfrak{M}'$, there is $Q \le P$, $Q \in \mathfrak{M}$, i.e., $P$ is a hereditary finite $\mu$- measure projection. The closedness of $\mu$ implies $P \in \mathfrak{M}$. Hence, $\mathfrak{M}' = \mathfrak{M}$. □

**Corollary 3.** *The regular measure* $\mu_T; \mathfrak{M}_T \to \mathrm{R}^+$ *is maximal if, and only if, for any* $P \in \Pi$*, there exists such a projection* $Q \in \Pi$ *that* $QH \subset D(T^{1/2})$ *and* $T^{1/2}Q$ *is bounded.*

Note that, in factors of type I and II, the corollary is only satisfied by measurable operators. The regular maximal measures are characterized by the fact that, for the weights with which these measures are associated, the semifiniteness of the weight can be defined in the same way as for traces, i.e., for any $S \in \mathcal{M}^+$ $\varphi(S) = \sup\{\varphi(T) : T \in \mathcal{M}^+ : T \le S, \varphi(T) < +\infty\}$.

## 3. $\sigma$-Finite Measure

Let us provide one more definition of an infinite measure.

**Definition 6.** *The function* $\mu : \Pi :\to [0, +\infty]$ *with* $\mu(P) = \sum \mu(P_i)$ *when* $P = \sum P_i$ *and* $\mu(I) = +\infty$ *is said to be* $\sigma$*-a finite measure if a* $P_n \in \Pi$*, such that* $P_n \nearrow I$ *and* $\mu(P_n) < +\infty$ *for any* $n$*.*

We can offer an elementary description of $\sigma$-finite measures in finite algebras of type $II$.

**Remark 3.** *Let* $\mu$ *be a* $\sigma$*-finite measure in finite von Neumann algebra* $\mathcal{M}$ *of type* $II$*. Then, there is a unique nonnegative self-adjoint operator* $T_\mu \in \mathcal{L}_1(\tau, \mathcal{M})$ *attached to algebra* $\mathcal{M}$*, such that* $\mu(Q) = \tau(QT)$*,* $Q \in \Pi$*.*

**Proof.** Let the sequence $P_n \nearrow I$ and $\mu(P_n) < +\infty \; \forall n$. Place $Q_n = Q \wedge P_n$ for all $n$ and $Q \in \Pi$. Then, $Q_n \nearrow Q$. According to the Gleason analogy, $\mu(P) = \tau(T_n P), \forall P, P \le P_n$. Here, $0 \le T_n = P_n T_n P_n$. According to the finiteness of the algebra $\mathcal{M}$, the sequence $\{T_n\}$ is fundamental according to the measure $\tau$. Let $T_\mu = limT_n$ by $\tau$. For any $Q \in \Pi$

$$\mu(Q) = \lim_n \mu(Q_n) = \lim_n \tau(Q_n T_n) = \lim_n \tau(QT_\mu).$$

□

Our goal is to show that $\sigma$-finite measure extends to the weight. Let us start with the property taken from Lemma 1.

**Proposition 5.** *Let* $\mathcal{M}$ *be a von Neumann algebra of type* $II$*, and* $\mu$ *be the* $\sigma$*-finite measure on* $\Pi$*. Then,* $\{P \in \Pi : \mu(P) < +\infty\}$ *is the ideal of the projections.*

We need completely different arguments for the proof of Proposition 6. We will need a lemma, due to Lugovay [14]. Note that the idea of the proof of this lemma, as well as of the work [15], is inspired by the geometric idea of the work [3]. Let $B^{pr}(H)$ be the set of all orthogonal projectors.

**Lemma 6** ([14])**.** *Let* $dimH = 3$ *and let* $\mu : B^{pr}(H) \to [0, +\infty]$ *be an unbounded measure. Use the following orthogonal projections:* $p$, $\dim p = 2$, $\mu(p) < +\infty$ *and* $q$, $\dim q = 1$, *and* $\mu(q) < +\infty$. *Then,* $q < p$.

**Proof.** Without loss of generality, we can assume that:

(a) $H$ is a real space;

(b) the measure takes only two values: 0 and $+\infty$.

Assume the opposite: $p \not> q$. Let $S$ be the united sphere in $H$ and $\rho : S \to \{0, +\infty\}$ be a reper function corresponding to our measure

$$\rho(\xi) = \mu((\cdot, \xi)\xi), \;\; \xi \in S.$$

Let $K = pH \cap S$ and the vector $\eta \in S$ be such that $q = (\cdot.\eta)\eta$ and the angle $\alpha$, $(\alpha > 0)$ between the vector $\eta$ and the plane $pH$ is less than $\frac{\pi}{2}$. We introduce a rectangular coordinate system into $H$, so that the vector $\eta$ lies in the $X0Y$ plane with the angle $\frac{\pi}{2}$ to the vector $(1, 0, 0)$ if the angle is counted counterclockwise.

We will write the coordinates of the points $S$ in the spherical system coordinates $(\varphi, \theta)$, where $\varphi$ is Longitude $0 \leq \varphi < 2\pi)$ and $\theta$ is Latitude $(-\pi/2 < \theta \leq \pi/2)$. Thus, $\eta = (0, \alpha/2)$. Let $L$ be a great circle on sphere $S$, which touches the circle of Latitude $\alpha/2$ in the point $\eta$. Note that $\rho(\zeta) = 0$ for any point $\zeta \in L$. Let us first show that $\epsilon > 0$ occurs such that

$$\forall\theta\forall\zeta = (\varphi, \theta) \;\; (|\theta| \leq \epsilon \Longrightarrow \rho(\zeta) = 0). \tag{3}$$

To do this, we estimate the angle $\theta$ for which the vector $\zeta = (\varphi, \theta)$ lies on a great circle passing through two orthogonal vectors $(\varphi_1, \theta_1) \in K$, $(\varphi_2, \theta_2) \in L$. The condition ensuring that the points $(\varphi, \theta)$, $(\varphi_1, \theta_1)$, $(\varphi_2, \theta_2)$ lie on one large circle has the form:

$$\begin{vmatrix} cos\theta & cos\varphi & cos\theta & sin\varphi & sin\theta \\ cos\theta_1 & cos\varphi_1 & cos\theta_1 & sin\varphi_1 & sin\theta_1 \\ cos\theta_2 & cos\varphi_2 & cos\theta_2 & sin\varphi_2 & sin\theta_2 \end{vmatrix} = 0 \tag{4}$$

We consider the orthogonality of the vectors $(\varphi_1, \theta_1)$, $(\varphi_2, \theta_2)$ and whether the circles $K$, $L$, respectively, belong to the planes $z = -xtg(\alpha/2)$, $z = xtg(\alpha/2)$ we obtained from (4) the equation:

$$x^2(tg\theta + tg\alpha cos\varphi) - 2xsin\varphi tg\alpha + (1 - tg^2\alpha)(tg\theta - tg\alpha cos\varphi) = 0.$$

From here

$$tg^2\theta \leq sin^2\varphi tg^2\alpha/(1 - tg^2\alpha) + cos^2\varphi tg^2\alpha$$

It follows from this inequality that, for any $\varphi$ $(0 \leq \varphi < 2\pi)$, there is $\epsilon(\varphi) > 0$, such that $\rho(\zeta) = 0$ for any $\zeta = (\varphi, \theta)$, with the condition $|\theta| \leq \epsilon(\varphi)$. Hence, there is $\epsilon > 0$, ($\epsilon$ is independent of $\varphi$) for which (3).

Let us prove that, from (3), the $\rho$ must be equal to zero in the band $|\theta| \leq \alpha$, $\alpha \geq \pi/4$. From here, the assertion of our lemma will follow. For this, we chose $\zeta_1 = (\pi/2, 0)$, $\zeta_2 = (0, \pi/4)$, $\zeta_3 = (0, -\pi/4)$. We have

$$\mu(I) = \mu(p_{\zeta_1}) + \mu(p_{\zeta_2}) + \mu(p_{\zeta_3}) = \rho(\zeta_1) + \rho(\zeta_2) + \rho(\zeta_3) = 0.$$

This will contradict the unboundedness of the measure.

Consider two orthogonal vectors $(0, \epsilon)$, $(\varphi, -\epsilon)$ on the sphere $S$, such that $cos\varphi = tg^2(\epsilon)$. The great circle passing through these points lies in the plane

$$xsin\varphi tg\epsilon - ytg\epsilon(1 + cos\varphi) - zsin\varphi = 0.$$

This great circle touches the circle of the sphere $S$ Latitude $\epsilon_1$. Here, $cose\epsilon_1 = \sqrt{cos2\epsilon}$, (such that $\epsilon_1 > \epsilon$).

Note that the reper function $\rho = 0$; therefore, this could occur at any point in the sphere $S$ lying in the strip $|\theta| \le \epsilon_1$. Repeat this procedure for angle $\epsilon_1$. We can obtain $cos\epsilon_2 = \sqrt{cos2\epsilon_1} = \sqrt{2cos2\epsilon - 1} < cos2\epsilon$. Thus, $\epsilon_2 > 2\epsilon$. □

It is not difficult to see that the proof of lemma 6 can be carried out in terms of vectors as well as projections.

Note that the proof of Lemma 6 can e generalized to the direct integral of factors of type $I_3$.

We can interpret the proof of lemma 6 as the proof of lemma 7 (this lemma will be needed to prove Proposition 5). Note that Propositions 1–3 and Lemma 7 are statements of the same order.

**Lemma 7.** *Let $dimH = 3$, and let $\mu : B^{pr}(H) \to [0, +\infty]$ be a measure. This allows for the orthogonal projections $p$, ($dimp = 2$, $\mu(p) < +\infty$), $q$, ($\dim q = 1$, $\mu(q) < +\infty$) and $\|qp\| < 1$. Then, $\mu$ is a finite measure.*

We can then proove Proposition 5.

**Proof.** It is only necessary to show that if $P, Q \in \Pi$ is such that:

$(i)$ $\mu(P) < +\infty$, $\mu(Q) < +\infty$ and $(ii)$ $\|PQ\| < 1$. Then, $\mu(P \vee Q) < +\infty$.

Let $P$, $Q$ be in a general position, i.e., $P \wedge Q = (1 - P) \wedge Q = P \wedge (I - Q) = 0$.

Note that the condition implies $P = P_{\Delta PQ}$ and $Q = P_{\Delta QP}$. This means that $P \sim Q$. It is sufficient for us to prove that $\mu(P \vee Q - P) < +\infty$.

(1) First, we assume that $P, Q \in \mathfrak{M}_\tau$. Let $(P \vee Q - P)Q = W|(P \vee Q - P)Q|$ and $QP = V|QP|$ be a polar decomposition of $(P \vee Q - P)Q$ and $QP$, respectively.

To bring the notation closer to the proof of Lemma 6, let us use $Z$ to denote the projection $P \vee Q - P$.

Let us find projections $Z_1$, $Z_2$, such that $Z_1 \sim Z_2$, $Z_1 + Z_2 = Z$ (hence, $\tau(Z_i) = (1/2)\tau(Z)$ $i = 1, 2$). Let $Q_i \le Q$ be such that $Z_i = P_{\Delta(P \wedge Q - Q)Q_i}$. Place $P_i = P_{\Delta PQ_i}$, $i = 1, 2$. Using the construction, $P_1 \sim P_2$.

Let us use $R(P_i, Q_i)$ to denote the von Neumann algebra generated by $P_i$, $Q_i$ and $VP_i$, $W^* Z_i$ with unity $P + Z_i$.

It is well-known ([9]) that there is a central representation $R(P_i, Q_i) = \int^{\oplus} L(t)dm(t)$ for the direct integral of factors of the type $I_3$, i.e., $P_i = \int^{\oplus} P(t)dm(t)$ and $Q_i = \int^{\oplus} Q(t)dm(t)$. Here, $P(t)$, $Q(t)$ are projections that occur almost everywhere in the measure $m$. It clear that $P \vee Q - P = \int^{\oplus}(I(t) - P(t))dm(t)$ and $P + X_i$ are unity operators in $R(P_i, Q_i)$.

Place $\|PQ\| = c$, $c < 1$. Then, $\|P(t)Q(t)\| \le \|Q \vee P\|$ almost everywhere in the measure $m$. This means that we can restate the proof of Lemma 6 in terms of a direct integral. By doing this, we obtain the statement of Lemma 7. Thus, $\mu(Z_i) \le +\infty$, for all $i$. Hence, $\mu(P \vee Q - P) = \mu(Z_1) + \mu(Z_2) < +\infty$ and $\mu(P \vee Q) < +\infty$. Applying Proposition 2, we obtain the following enhancement $\mu(P \vee Q) \le (1 - \|PQ\|)^{-1}(\mu(P) + \mu(Q))$

(2) Let $P, Q \notin \mathfrak{M}_\tau$. Let sequences $P_n \nearrow P$ and $Q_n \nearrow Q$ and $P_n, Q_n \subset \mathfrak{M}_\tau$. We have $\|P_n Q_n\| \le \|PQ\|$. Hence,

$$\mu(P \vee Q) = \lim_n \mu(P_n \vee Q_n) \le \lim_n (1 - \|P_n Q_n\|)^{-1} (\lim_n \mu(P_n) + \lim_n \mu(Q_n)) \le$$

$$(1 - \|PQ\|)^{-1}(\mu(P) + \mu(Q)).$$

(3) The last inequality carries over to the general case of projections $P$, $Q$. Proposition 5 is proved. □

Let us continue the study of the $\sigma$-finite measure. Our goal is to continue the measure. Considering Proposition 5, we could use the results of §1. However, we will use another method.

Let us denote, using $\mathfrak{M}_\tau(\mathfrak{M}_\mu)$, the set

$$\{P \in \Pi : \text{ there is finite subset } \Phi \subset \mathfrak{M}_\tau \cap \mathfrak{M}_\mu \text{ such that } P \leq \vee_{Q \in \Phi} Q\}$$

It is clear that the set $\mathfrak{M}_\tau(\mathfrak{M}_\mu)$ is a lattice. It is clear that the restriction of $\mu$ on the reduction algebra $\mathcal{M}_P$, $P \in \mathfrak{M}_\tau(\mathfrak{M}_\mu)$ is a $\sigma$-finite measure. Operator $T_P = T_P P$ exists, such that $\mu(Q) = \tau(T_P Q)\ \forall\ Q \leq P$. Put

$$D_\mu = \bigcup_{P \in \mathfrak{M}_\tau(\mathfrak{M}_\mu)} (D(T_P^{1/2}) \bigcap PH)$$

The set $D_\mu$ is a strongly dense set.

Here, the equality $t(f,g) = (T_P^{1/2} f, T_P^{1/2} g)$ is $P \in \mathfrak{M}_\tau(\mathfrak{M}_\mu)$ and $f, g \in PH$, defining a bilinear form.

**Definition 7.** *An operator $X \in \mathcal{M}^+$ is said to be integrable, if there exists $A_X \in \mathcal{L}_1^+(\tau, \mathcal{M})$, such that for any $Q \in \mathfrak{M}_\tau$ we have $t(X^{1/2} f, X^{1/2} g) = (A_X f, g)$. Here, $f$, $g$ derive from some strongly dense set with respect to $\mathcal{M}_Q$. Let us denote the set $H^Q_{X^{1/2}}$.*

*Let $\mathcal{M}_\mu^+$ be the set of all integrable operators.*

**Proposition 6.** *The set $\mathcal{M}_\mu^+$ is a hereditary cone. If $P \in \mathfrak{M}_\mu$ then $P \in \mathcal{M}_\mu^+$.*

**Proof.** (1) Let us establish the heredity of $\mathcal{M}_\mu^+$. Let $0 \leq Y \leq X \in \mathcal{M}_\mu^+$, $Y \in \mathcal{M}^+$. Let $A \in \mathcal{M}$, such that $Y^{1/2} = X^{1/2} A^*$. Let $Q$ be the projection of $P_{\Delta A^* P}$, meaning that $P \in \mathfrak{M}_\tau$. As is known [10], the set $H^P_{Y^{1/2}} = \{f \in PH : A^* f \in H^Q_{X^{1/2}}\}$ is strongly dense with respect to $\mathcal{M}_{\mathcal{P}}$ and for any $f, g \in H^P_{Y^{1/2}}$

$$t(Y^{1/2} f, Y^{1/2} g) = t(X^{1/2} A^* f, X^{1/2} A^* g) = (A_X A^* f, A^* g) = (A A_X A^* f, g).$$

The operator $A_Y = A A_X A^*$ is desired. The heredity of $\mathcal{M}_\mu^+$ is proven.

Note that $A_{X-Y} = B A_X B^*$, here $B \in \mathcal{M}$, such that $(X - Y)^{1/2} = X^{1/2} B^*$.

(2) Let $X$, $Y \in \mathcal{M}_\mu^+$. Let us prove that $Z = X + Y \in \mathcal{M}_\mu^+$. There exists $A$, $B \in \mathcal{M}$ with $X^{1/2} = Z^{1/2} A^*$ and $Y^{1/2} = Z^{1/2} B^*$. In addition, the operator $A^* A + B^* B$ is an orthogonal projection on $\overline{\Delta Z}$ [9] and $X^{1/2} A + Y^{1/2} B = Z^{1/2}$. For any $Q \in \mathfrak{M}_\tau$, the lineal $\{f \in QH : (X^{1/2} A + Y^{1/2} B) f \in D_\mu\}$ is strongly dense with respect to $\mathcal{M}_Q$. Hence, the lineal

$$\mathcal{N}^Q_{Z^{1/2}} = \{f \in QH : Z^{1/2} f \in Q_\mu\} \cap \{f \in QH : Af \in D(A_X),\ Bf \in D(A_Y)\}.$$

is also strongly dense. We substitute the operator $Z^{1/2}$ for $X^{1/2} A + Y^{1/2} B$. We obtain

$$t(Z^{1/2} Qf, Z^{1/2} Qf) \leq 2[t(X^{1/2} Af, X^{1/2} Af) + t(Y^{1/2} Bf, Y^{1/2} Bf)] =$$
$$= 2[(A_X Af, AF) + (A_Y BF, Bf)] = 2(A^* A_X A + B^* A_Y B) QF, Qf). \tag{5}$$

for any $f \in \mathcal{N}^Q_{Z^{1/2}}$. According to this definition, $\cup_{Q \leq P} \mathcal{N}^Q_{Z^{1/2}}$ is a strongly dense lineal in $PH$ for any $P \in \Pi$. Place $K = A^* A_X A + B^* A_Y B$. Note that $K \in \mathcal{L}_1(\tau, \mathcal{M})$. Hence, for any $\epsilon > 0$, there exists $P_\epsilon \in \Pi$ with $\tau(I - P_\epsilon) < \epsilon$ and $K P_\epsilon$ is bounded. By (2), the restriction of $t(Z)^{1/2} \cdot, Z^{1/2} \cdot)$ on $\cup_{Q \leq P_\epsilon} \mathcal{N}^Q_{Z^{1/2}}$ is bounded. Hence, operator $A_\epsilon = P_\epsilon A_\epsilon \in \mathcal{M}^+$ exists, such that

$$t(Z^{1/2} f, Z^{1/2} g) = (A_\epsilon f, g), \quad f, g \in \cup_{Q \leq P_\epsilon} \mathcal{N}^Q_{Z^{1/2}}$$

Place $\epsilon = 1/n$. We chose an increasing sequence $P_{1/n} \nearrow I$. Then, $A_{1/m} = P_{1/m} A_{1/n} P_{1/m}$, $m < n$ and $A_{1/n}$ is fundamental by $\tau$. Place $A_{1/n} \to A$ by $\tau$. Of course, $A \geq 0$. Using (3), $\tau(A_{1/n}) \leq 2\tau(K)$. Hence, $\tau(A) \leq \tau(K)$. Put $A_Z = A$.

For any $Q \in \mathfrak{M}_\tau$, the linear $D = \cup_n (Q \wedge P_{1/n})H$ is strongly dense with respect to $\mathcal{M}_Q$. Therefore, the lineals $D \cap D(A_Z)$ and $H^Q_{Z^{1/2}} = \mathcal{N}^Q_{Z^{1/2}} \cap D \cap D(A_Z)$ are similar. For any $f,g \in H^Q_{Z^{1/2}}$ there is $n$, such that $f, g \in P_{1/n}H$. Then, $t(Z^{1/2}f, Z^{1/2}g) = (A_{1/n}f, g) = (Af, g)$. We used the equality $A_{1/n} = P_{1/n}AP_{1/n}$. Therefore, $\mathcal{M}^+_\mu$ is a hereditary cone.

Let $P \in \mathfrak{M}_\mu$. Then, operator $A_P = A_P P \in \mathcal{L}_2(\tau, \mathcal{M})$ occurs, such that $\mu(Q) = \tau(A_P Q), \forall\, Q \leq P$. Analogous with the previous equation, we can see that $P \in \mathcal{M}_\mu$. The proposition is proved. □

Place $\mu(X) = +\infty$ if $X \in \mathcal{M}^+ \backslash \mathcal{M}^+_\mu$, and $\mu(X) = \tau(A_X)$, if $X \in \mathcal{M}^+_\mu$.

Let us remember that $A_Y = AA_XA^*$, $A_{X-Y} = BA_XB^*$. Operator $B^*B + A^*A = P$ is the orthogonal projection on $(B^*B + A^*A)H$ and $A_XP = A_X$. Hence,

$$\mu(X) = \tau(A_XP) = \tau(A_X(B^*B + A^*A)) = \tag{6}$$

$$\tau(BA_XB^*) + \tau(AA_XA^*) = \mu(X - Y) + \mu(Y).$$

Hence, the corollary is true

**Corollary 4.** *The function $\mu(\cdot)$ is a weight that continues the measure $\mu$.*

**Proposition 7.** *The weight $\mu(\cdot)$ is normal if $\{X_n\} \subset \mathcal{M}^+_\mu$ and $X_n \nearrow X \in \mathcal{M}^+_\mu$ have a strong operator topology. Then, $\mu(X_n) \nearrow \mu(X)$.*

**Proof.** Let the proposition conditions be fulfilled. According to (6) $\mu(X) = \mu(X_n) + \mu(X - X_n)$. Hence, $\mu(X_n) \leq \mu(X)$ and $\mu(X_n) \leq \mu(X_{n+1})$ for all $n$. To prove Proposition 6, we used the following property: $A_{X-X_n} = B_nA_XB_n^*$, here $B_n \in \mathcal{M}$, $\|B_n\| \leq 1$, such that $(X - X_n)^{1/2} = X^{1/2}B_n^* = B_nX^{1/2}$. As $X - X_n \searrow 0$, then $B_n \to 0$ is a strong operator topology. Therefore, $\mu(X - X_n) = \tau(B_nA_XB_n^*) \searrow 0$. Thus, $\mu(X_n) \nearrow \mu(X)$. □

Another example of measures of projections and the logics of projections is provided by the consideration of perfect Hilbert algebras and Tomita's theory [16].

The proofs of the corresponding assertions for perfect algebras known to the author are quite extensive and are not the aim of this paper.

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