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Article

Best Approximation Results for Fuzzy-Number-Valued Continuous Functions

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Abstract: In this paper, we study the best approximation of a fixed fuzzy-number-valued continuous function to a subset of fuzzy-number-valued continuous functions. We also introduce a method to measure the distance between a fuzzy-number-valued continuous function and a real-valued one. Then, we prove the existence of the best approximation of a fuzzy-number-valued continuous function to the space of real-valued continuous functions by using the well-known Michael selection theorem.

Keywords: best approximation; fuzzy-valued continuous function; Michael selection theorem

MSC: 41A50; 65G40

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1. Introduction

Approximation theory originated from the necessity of approximating real-valued continuous functions by a simpler class of functions, such as trigonometric or algebraic polynomials and has attracted the interest of many mathematicians for over a century. Among the most recognized results in this branch of functional analysis, we can mention the Stone–Weierstrass theorem, Korovkin type results and the approximation of functions using neural networks.

More recently, all the above results have also been addressed in the context of fuzzy functions (see, e.g., [1–6]).

Another fundamental problem in approximation theory is the study of the best approximation in spaces of continuous functions, which has a long story with famous results by Chebyshev, Haar, Young, Remez, de la Vallée-Poussin who established the existence of best approximations, as well as characterized and estimated them. In this context, the problem of the uniform approximation of a scalar-valued function continuous on a compact set by a family of continuous functions on such a compact set (see, e.g., [7,8] or [9]) should be mentioned.

The search for the best approximation of a continuous set-valued function by vector-valued ones is another important topic in approximation theory and has been studied by several authors (see, e.g., [10–13] or [14]).

In this paper, we address these two problems of the best approximation type in the context of fuzzy-number-valued continuous functions.

First, we study the best approximation of a fixed fuzzy-number-valued continuous function to a subset of fuzzy-number-valued continuous functions.

Second, we introduce a novel method to measure the distance between a fuzzy-number-valued function and a real-valued one based on the concept of nearest interval approximation of fuzzy numbers [15]. Then, we prove the existence of the best approximation of a fuzzy-number-valued continuous function to the space of real-valued continuous functions by using the well-known Michael selection theorem.

2. Preliminaries

Let $F(\mathbb{R})$ denote the family of all fuzzy subsets on the real numbers $\mathbb{R}$ (see [16]). For $\lambda \in [0,1]$ and a fuzzy set u, its λ-level set is defined as

$$[u]^\lambda := \{ x \in \mathbb{R} : u(x) \geq \lambda \}, \quad \lambda \in]0,1],$$

and $[u]^0$ stands for the closure of $\{ x \in \mathbb{R} : u(x) > 0 \}$.

The family of elements $u \in F(\mathbb{R})$ which satisfies the following properties:

1. There exists an $x_0 \in \mathbb{R}$ with $u(x_0) = 1$, that is, u is normal;
2. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0,1]$, which is to say that u is convex;
3. $[u]^0$ is a compact set in $\mathbb{R}$;
4. u is upper-semicontinuous,

is called the fuzzy number space $\mathbb{E}^1$ (see, e.g., [17]) and contains the reals. If $u \in \mathbb{E}^1$, then it is known that the λ-level set $[u]^\lambda$ of u is a compact interval for each $\lambda \in [0,1]$. We write $[u]^\lambda = [u^-(\lambda), u^+(\lambda)]$.

The following characterization of fuzzy numbers, which was proved by Goetschel and Voxman [17], is essential in the sequel:

Theorem 1. *Let $u \in \mathbb{E}^1$ and $[u]^\lambda = [u^-(\lambda), u^+(\lambda)]$, $\lambda \in [0,1]$. Then, the functions $u^-(\lambda)$ and $u^+(\lambda)$ satisfy:*

1. *$u^+(\lambda)$ is a nonincreasing bounded left continuous function on $(0,1]$;*
2. *$u^-(\lambda)$ is a nondecreasing bounded left continuous function on $(0,1]$;*
3. *$u^-(1) \leq u^+(1)$;*
4. *$u^-(\lambda)$ and $u^+(\lambda)$ are right continuous at $\lambda = 0$.*

Conversely, if two functions $\gamma(\lambda)$ and $\nu(\lambda)$ fulfill the conditions (i)–(iv), then there is a unique $u \in \mathbb{E}^1$ such that $[u]^\lambda = [\gamma(\lambda), \nu(\lambda)]$ for each $\lambda \in [0,1]$.

As usual, (see, e.g., [17]) given $u, v \in \mathbb{E}^1$ and $k \in \mathbb{R}$, we define the sum $u + v := [u^-(\lambda), u^+(\lambda)] + [v^-(\lambda), v^+(\lambda)]$ and the product $ku := k[u^-(\lambda), u^+(\lambda)]$. With these two operations, $\mathbb{E}^1$ is not a vector space, and $(\mathbb{E}^1, +)$ is not even a group.

The fuzzy number space $\mathbb{E}^1$ can be endowed with several metrics (see, e.g., [16]) but perhaps the most used is the following:

Definition 1 ([16,17])**.** *For $u, v \in \mathbb{E}^1$,*

$$d_\infty(u,v) := \sup_{\lambda \in [0,1]} \max\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}.$$

This metric on $\mathbb{E}^1$ is called the supremum metric. Indeed, $\mathbb{E}^1$ is a complete metric space with this metric. Furthermore, if we consider the Euclidean topology on $\mathbb{R}$, it can be topologically identified with the closed subspace $\tilde{R} = \{ \tilde{x} : x \in \mathbb{R} \}$ of $(\mathbb{E}^1, d_\infty)$ where $\tilde{x}^+(\lambda) = \tilde{x}^-(\lambda) = x$ for all $\lambda \in [0,1]$. We always assume that $\mathbb{E}^1$ is equipped with the supremum metric.

Proposition 1 ([3] Proposition 2.3)**.** *The following properties are satisfied by the metric space $(\mathbb{E}^1, d_\infty)$:*

1. *$d_\infty(\sum_{i=1}^m u_i, \sum_{i=1}^m v_i) \leq \sum_{i=1}^m d_\infty(u_i, v_i)$, where $u_i, v_i \in \mathbb{E}^1$ for $i = 1, ..., m$.*
2. *$d_\infty(ku, kv) = kd_\infty(u, v)$, where $u, v \in \mathbb{E}^1$ and $k > 0$.*
3. *$d_\infty(ku, \mu u) =| k - \mu | d_\infty(u, 0)$, where $u \in \mathbb{E}^1$, $k \geq 0$ and $\mu \geq 0$.*
4. *$d_\infty(ku, \mu v) \leq| k - \mu | d_\infty(u, 0) + \mu d_\infty(u, v)$, where $u, v \in \mathbb{E}^1$, $k \geq 0$ and $\mu \geq 0$.*

In $C(K, \mathbb{E}^1)$, the space of continuous functions defined on the compact Hausdorff space K which take values in $(\mathbb{E}^1, d_\infty)$, we use the following metric:

$$D(f,g) = \sup_{t \in K} d_\infty(f(t), g(t)),$$

which yields the uniform convergence topology on $C(K, \mathbb{E}^1)$.

Let us next introduce a useful tool for this section.

Definition 2. *Let M be a nonempty subset of $C(K, \mathbb{E}^1)$. We define*

$$Conv(M) := \{\varphi \in C(K, [0,1]) : \varphi f + (1-\varphi) g \in M \ \ for \ \ all \ \ f, g \in M\}.$$

Proposition 2 ([3] Proposition 3.2)**.** *Let M be a nonempty subset of $C(K, \mathbb{E}^1)$. Then, we infer:*

1. *$\phi \in Conv(M)$ implies that $1 - \phi \in Conv(M)$.*
2. *If $\phi, \varphi \in Conv(M)$, then $\phi \cdot \varphi \in Conv(M)$.*

Definition 3. *It is said that $M \subset C(K, [0,1])$ separates the points of K (or it is point-separating) if given $x, y \in K$, there exists $\psi \in M$ such that $\psi(x) \neq \psi(y)$.*

Lemma 1 ([3] Lemma 3.6)**.** *Let $M \subseteq C(K, \mathbb{E}^1)$. If $Conv(M)$ is point-separating, then, given $x_0 \in K$ and an open neighborhood $\mathcal{N}$ of x_0, there is a neighborhood $\mathcal{U}$ of x_0 contained in $\mathcal{N}$ such that for all $0 < \delta < \frac{1}{2}$, there is $\varphi \in Conv(M)$ such that*

1. *$\varphi(t) > 1 - \delta$, for all $t \in \mathcal{U}$;*
2. *$\varphi(t) < \delta$, for all $t \notin \mathcal{N}$.*

3. Best Approximation for Subspaces of $C(K, \mathbb{E}^1)$

Given a metric space (X, d) and a nonempty (closed) subset A of X and given an element $x \in X$, we can define

$$d(x, A) = \inf_{y \in A} d(x, y)$$

and the problem of the best approximation consists in finding an element $y_x \in A$ such that $d(x, A) = d(x, y_x)$. Although we focus on the problem of the best approximation in the space of fuzzy-valued continuous functions endowed with the crisp distance $D(f, g)$ defined above, it is worth noting that this problem has been studied for fuzzy metric spaces as well (see, e.g., [18–20]).

In this section, we get a sharper result, by obtaining that the distance is achieved at a single point.

Definition 4. *Let A be a subspace of $C(K, \mathbb{E}^1)$ and let $f \in C(K, \mathbb{E}^1)$. We define*

$$d(f, A) := \inf_{g \in A} \{\sup_{x \in K} d_\infty(f(x), g(x))\} = \inf_{g \in A} \{D(f, g)\}$$

$$d_x(f, A) := \inf_{g \in A} \{d_\infty(f(x), g(x))\}$$

Theorem 2. *Let W be a subspace of $C(K, \mathbb{E}^1)$ and assume that $Conv(W)$ separates points. For each $f \in C(K, \mathbb{E}^1)$, we have*

$$d(f, W) = d_x(f, W)$$

for some $x \in K$.

Proof. We first show that $d(f, W) = \sup_{x \in K} d_x(f, W)$. It is apparent that $d(f, W) \geq \sup_{x \in K} d_x(f, W)$ since $d(f, W) \geq d_x(f, W)$ for each $x \in K$. Let us prove that $d(f, W) \leq \sup_{x \in K} d_x(f, W)$.

To this end, fix $\varepsilon > 0$. Given $x' \in K$, we can find $f_{x'} \in W$ such that $d_\infty(f(x'), f_{x'}(x')) < \sup_{x \in K} d_x(f, W) + \varepsilon$. We next fix the following open neighborhood of x':

$$N(x') := \{t \in K : d_\infty(f(t), f_{x'}(t)) < \sup_{x \in K} d_x(f, W) + \varepsilon\}.$$

Take an open neighborhood $U(x')$ x' which satisfies the properties in Lemma 1.

Since K is compact, we can find finitely many $\{x_1, \ldots, x_m\}$ in K such that $K \subset U(x_1) \cup \ldots \cup U(x_m)$. Choose $\delta > 0$ such that $\delta < min(1, \frac{\varepsilon}{km})$, where

$$k := \max\{D(f, 0), D(f, f_{x_1}), ...D(f, f_{x_m})\}.$$

By Lemma 1, we know that there exist $\phi_1, \cdots, \phi_m \in Conv(W)$ such that for all $i = 1, \ldots, m$,

(i) $\phi_i(t) > 1 - \delta$, for all $t \in U(x_i)$;
(ii) $0 \leq \phi_i(t) < \delta, \quad if \;\; t \notin N(x_i)$.

Let us define the functions

$\psi_1 := \phi_1$,
$\psi_2 := (1 - \phi_1)\phi_2$,
$\vdots$
$\psi_m := (1 - \phi_1)(1 - \phi_2) \cdots (1 - \phi_{m-1})\phi_m$.

By Proposition 2, we know that $\psi_i \in Conv(W)$ for all $i = 1, \ldots, m$. Next, we claim that

$$\psi_1 + \ldots + \psi_j = 1 - (1 - \phi_1)(1 - \phi_2) \cdots (1 - \phi_j),$$

$j = 1, \ldots, m$. Indeed, it is clear that

$$\psi_1 + \psi_2 = \phi_1 + (1 - \phi_1)\phi_2 = 1 - (1 - \phi_1) \cdot (1 - \phi_2).$$

By induction, let us assume that it is true for a certain $j \in \{4, ..., m-1\}$. We claim

$$\psi_1 + \ldots + \psi_j + \psi_{j+1} = 1 - (1 - \phi_1)(1 - \phi_2) \cdots (1 - \phi_j)(1 - \phi_{j+1}).$$

Namely,

$$\begin{gathered} \psi_1 + \ldots + \psi_j + \psi_{j+1} = \\ = 1 - (1 - \phi_1)(1 - \phi_2) \cdots (1 - \phi_j) + (1 - \phi_1)(1 - \phi_2) \cdots (1 - \phi_j)\phi_{j+1} = \\ = 1 - (1 - \phi_1)(1 - \phi_2) \cdots (1 - \phi_j)(1 - \phi_{j+1}), \end{gathered}$$

as was to be checked.

Fix $x_0 \in K$. Then, there is some $i_0 \in \{1, \ldots, m\}$ such that $x_0 \in U(x_{i_0})$. Hence, $\phi_{i_0}(x_0) > 1 - \delta$ and consequently,

$$1 \geq \sum_{i=1}^{m} \psi_i(x_0) = 1 - (1 - \phi_{i_0}(x_0)) \prod_{i \neq i_0} (1 - \phi_i(x_0)) > 1 - \delta.$$

Furthermore, we clearly infer

$$\psi_i(t) < \delta \quad \text{for all } t \notin N(x_{i_0}),\ i = 1, \ldots, m. \tag{1}$$

Let

$$h := \psi_1 f_{x_1} + \psi_2 f_{x_2} + \ldots + \psi_m f_{x_m}. \tag{2}$$

It seems apparent that

$$h = \phi_1 f_{x_1} + (1 - \phi_1)[\phi_2 f_{x_2} + (1 - \phi_2)[\phi_3 f_{x_3} + \cdots + (1 - \phi_{m-1})[\phi_m f_{x_m} \cdots]].$$

Therefore, $h \in W$ since $\phi_i \in Conv(W)$ for $i = 1, ..., m$ (see Definition 2).
From Proposition 1, we know that, given $x \in K$,

$$d_\infty(f(x), h(x)) \le d_\infty\left(f(x), \sum_{i=1}^{m} \psi_i(x) f(x)\right) + d_\infty\left(\sum_{i=1}^{m} \psi_i(x) f(x), h(x)\right) \le$$

$$\le \left|1 - \sum_{i=1}^{m} \psi_i(x)\right| d_\infty(f(x), 0)) + \sum_{i=1}^{m} \psi_i(x) d_\infty(f(x), f_{x_i}(x)).$$

On the one hand, $|1 - \sum_{i=1}^{m} \psi_i(x)| d_\infty(f(x), 0)) \le \delta D(f, 0) \le \epsilon$.
On the other hand, let

$$I_x = \{1 \le i \le m : x \in N(x_i)\}$$

and

$$J_x = \{1 \le i \le m : x \notin N(x_i)\}.$$

Then, for all $i \in I_x$, we have

$$\psi_i(x) d_\infty(f(x), f_{x_i}(x)) \le \psi_i(x)(\sup_{x \in K} d_x(f, W) + \varepsilon) \le \sup_{x \in K} d_x(f, W) + \varepsilon$$

and, for all $i \in J$, inequality (1) yields

$$\psi_i(x) d_\infty(f(x), f_{x_i}(x)) \le \delta d_\infty(f(x), f_{x_i}(x)) \le \delta D(f, f_{x_i}) \le \delta k.$$

From the above two paragraphs, we can infer

$$d_\infty(f(x), h(x)) \le \epsilon + \sup_{x \in K} d_x(f, W) + \varepsilon + \delta k m \le \sup_{x \in K} d_x(f, W) + 3\varepsilon$$

and, since $x \in K$ is arbitrary,

$$D(f, h) \le \sup_{x \in K} d_x(f, W) + 3\varepsilon.$$

As a consequence, we deduce that

$$d(f, W) = \inf_{g \in W} \{D(f, g)\} \le \sup_{x \in K} d(f_x, W_x).$$

Finally, we can define a continuous function $\gamma : K \longrightarrow \mathbb{R}$ as

$$\gamma(x) := \inf_{g \in W} \{d_\infty(f(x), g(x))\}.$$

Since K is compact, we know that γ attains its supremum at some $x' \in K$. Hence, we can write $d(f, W) = d_{x'}(f, W)$. □

4. Best Approximation with Respect to Real-Valued Continuous Functions

In approximation theory, a natural question is when we can approximate a set-valued function by continuous real-valued functions. In the classical setting, Cellina's Theorem [21] is the fundamental result (see also [22–24]). In this section, we introduce a method to measure the distance between a fuzzy-number-valued function and a real-valued one. Then, we prove the existence of the best approximation of a fuzzy-number-valued continuous functions to the space of real-valued continuous functions.

The first problem is to find a suitable definition for the distance between a fuzzy-number-valued function and a real-valued one. Following the ideas in [12], we can define a distance for each level $\lambda \in [0, 1]$.

Definition 5. *Let $f \in C(K, \mathbb{E}^1)$ and let $F \in C(K)$. The distance at level λ between f and F can be defined as*

$$D_\lambda(f, F) := \sup_{x \in K} \{ \sup\{ |F(x) - t| : t \in I_\lambda := [f(x)^-(\lambda), f(x)^+(\lambda)] \} \}.$$

Now, we need to provide how to choose the best level to measure the distance.

Bearing in mind that the intervals I_λ form a nonincreasing family, it implies that the distances form a nondecreasing family as well, that is,

$$D_\lambda(f, F) \geq D_\eta(f, F) \quad \text{if } 0 \leq \lambda \leq \eta \leq 1.$$

Thus, if we choose

$$D(f, F) := \inf_{\lambda \in [0,1]} D_\lambda(f, F) = D_1(f, F)$$

we measure the distance at the interval with the minimum length (the core), possibly single-valued.

If, on the contrary, we choose

$$D(f, F) := \sup_{\lambda \in [0,1]} D_\lambda(f, F) = D_0(f, F)$$

we get the measurement at the support. As noted in [15], these intervals are not the best to represent the fuzzy number $f(x)$.

Another choice, more accurate, is to find a level λ which represents an average of the length of the level intervals (see, e.g., [25–27]).

The function $g(\lambda) = f(x)^+(\lambda) - f(x)^-(\lambda)$ is a nonincreasing function for each $x \in K$. Theorem 2.4 in [28] can be applied to get $t \in [0, 1]$ and $\lambda \in (0, 1)$ such that

$$\int_0^1 (f(x)^+(\lambda) - f(x)^-(\lambda)) d\lambda = t \cdot g(\lambda + 0) + (1 - t) \cdot g(\lambda - 0)$$

where $g(\lambda + 0)$ and $g(\lambda - 0)$ stand for the one-sided limits of g. We choose λ_x as the minimum λ satisfying such a condition and taking $I_{\lambda_x} := [f(x)^-(\lambda_x), f(x)^+(\lambda_x)]$, we can define

$$D(f, F) := D_{\lambda_x}(f, F) = \sup_{x \in K} \left\{ \sup_{t \in I_{\lambda_x}} \{ |F(x) - t| \} \right\} \tag{3}$$

Definition 6. *Let $f \in C(K, \mathbb{E}^1)$. We can define the distance between f and $C(K)$ as*

$$D(f, C(K)) := \inf_{F \in C(K)} D(f, F).$$

where $D(f, F)$ is given by (3).

Definition 7. *Let $f \in C(K, \mathbb{E}^1)$ and $x \in K$. We can define*

$$rad(x, f) := \inf_{\alpha \in \mathbb{R}} \{ \sup\{ |\alpha - \beta| \ : \ \beta \in I_{\lambda_x} \} \}$$

It is clear that $rad(x, f)$ turns out to be the radius of the interval $[f(x)^-(\lambda_x), f(x)^+(\lambda_x)]$.

Definition 8. *Let $f \in C(K, \mathbb{E}^1)$. We define the* radius *of f as*

$$rad(f) := \sup_{x \in K} rad(x, f)$$

Remark 1. *From these definitions we infer easily that*

$$D(f,F) \geq rad(f)$$

for all $F \in C(K)$. Hence,

$$D(f,C(K)) \geq rad(f).$$

Theorem 3. *Let $f \in C(K,\mathbb{E}^1)$. Then, there exists a function $F_0 \in C(K)$ such that*

$$D(f,C(K)) = D(f,F_0).$$

Proof. Let us define a map $G : K \longrightarrow 2^{\mathbb{R}}$ such that, for each $x \in K$,

$$G(x) := \{\alpha \in \mathbb{R} : I_{\lambda_x} \subseteq [\alpha - rad(f), \alpha + rad(f)]\}$$

Let us first check that $G(x) \neq \emptyset$ for each $x \in K$. We know that

$$rad(x,f) := \inf_{\alpha\in\mathbb{R}} \sup_{\beta\in I_{\lambda_x}} |\alpha - \beta| \leq rad(f).$$

Since $rad(x,f)$ turns out to be the radius of the interval I_{λ_x}, it is clear that the center of this interval belongs to $G(x)$.

It is apparent that $G(x)$ is closed for each $x \in K$ since the intervals which appear in its definition are closed.

Now we take α_1, α_2 in $G(x)$ and $k_1, k_2 \geq 0$ with $k_1 + k_2 = 1$. Then, given $\alpha \in I_{\lambda_x}$,

$$\begin{aligned} |\alpha - (k_1\alpha_1 + k_2\alpha_2)| &= |\alpha(k_1 + k_2) - (k_1\alpha_1 + k_2\alpha_2)| \\ &\leq k_1|\alpha - \alpha_1| + k_2|\alpha - \alpha_2| \leq rad(f), \end{aligned}$$

which shows that $G(x)$ is convex for each $x \in K$.

Next, we shall prove that the map G is lower semicontinuous, that is, we have to check that the set

$$\mathcal{O} := \{x \in K : G(x) \cap O \neq \emptyset\}$$

is open in K for every open set $O \subset \mathbb{R}$. To this end, fix $x_0 \in \mathcal{O}$ and take $\alpha_0 \in G(x_0) \cap O$ for a certain open set O. Let $\delta_0 > 0$ such that $(\alpha_0 - \delta_0, \alpha_0 + \delta_0) \subset O$. From the continuity of f and from the fact that

$$I_{\lambda_{x_0}} \subset [\alpha_0 - rad(f), \alpha_0 + rad(f)],$$

we infer that, given $\epsilon > 0$, as we have

$$[\alpha_0 - rad(f), \alpha_0 + rad(f)] \subset (\alpha_0 - rad(f) - \epsilon, \alpha_0 + rad(f) + \epsilon),$$

there exists an open neighborhood $Q(\epsilon)$ of x_0 such that

$$I_{\lambda_x} \subset (\alpha_0 - (rad(f) + \epsilon), \alpha_0 + rad(f) + \epsilon)$$

for all $x \in Q(\epsilon)$. Our goal is to prove that $Q(\epsilon) \subseteq \mathcal{O}$ for some $\epsilon > 0$ to get the openness of $\mathcal{O}$.

Fix $x_1 \in Q(\epsilon)$. Since $G(x_1) \neq \emptyset$, there exists $\alpha_1 \in G(x_1)$. That is,

$$I_{\lambda_{x_1}} \subset [\alpha_1 - rad(f), \alpha_1 + rad(f)].$$

Moreover, we know that

$$I_{\lambda_{x_1}} \subset (\alpha_0 - (rad(f) + \epsilon), \alpha_0 + rad(f) + \epsilon).$$

Taking $\epsilon > 0$ as small as necessary, we can find $\delta' < \delta_0$ such that

$$[\alpha_0 - (rad(f) + \epsilon), \alpha_0 + rad(f) + \epsilon] \cap [\alpha_1 - rad(f), \alpha_1 + rad(f)]$$
$$\subset [(\alpha_0 + \delta') - rad(f), (\alpha_0 + \delta') + rad(f)]$$

and consequently

$$I_{\lambda_{x_1}} \subset [(\alpha_0 + \delta') - rad(f), (\alpha_0 + \delta') + rad(f)],$$

which implies that $\alpha_0 + \delta' \in G(x_1) \cap O$. Hence, $x_1 \in \mathcal{O}$, as desired.

Gathering the information we have obtained so far, we know that G is a lower semi-continuous mapping defined between K and the closed convex subsets of $\mathbb{R}$. Hence, by the Michael selection theorem [29], we infer that there exists $F_0 \in C(K)$ such that $F_0(x) \in G(x)$ for all $x \in K$.

As a consequence of the above paragraph, we can deduce that

$$D(f, F_0) \leq rad(f),$$

which, combined with the comments before this theorem, yields $D(f, F_0) = rad(f) = D(f, C(K))$. $\square$

5. Conclusions

In this paper, we addressed two problems of the best approximation type in the context of fuzzy-number-valued continuous functions: (1) the problem of the uniform approximation of a fuzzy-number-valued function continuous on a compact set by a family of continuous functions, continuous on this compact set; and (2) the existence of the best approximation of a fuzzy-number-valued continuous function to the space of real-valued continuous functions. We obtained positive results in both cases.

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