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Basic Properties for Certain Subclasses of Meromorphic *p*-Valent Functions with Connected *q*-Analogue of Linear Differential Operator

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Abstract: In this paper, we define three subclasses $\mathcal{M}^{n,q}_{p,\alpha}(\eta, A, B)$, $\mathcal{I}^{n}_{p,\alpha}(\lambda, \mu, \gamma)$, $\mathrm{R}^{n,q}_{p}(\lambda, \mu, \gamma)$ connected with a q-analogue of linear differential operator $\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}$ which consist of functions $\mathcal{F}$ of the form $\mathcal{F}(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{\infty} a_j \zeta^j$ $(p \in \mathbb{N})$ satisfying the subordination condition $p - \frac{1}{\eta}\left\{ \frac{\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)'}{\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)} + p \right\} \prec p\frac{1+A\zeta}{1+B\zeta}$. Also, we study the various properties and characteristics of this subclass $\mathcal{M}^{n,q,*}_{p,\alpha}(\eta, A, B)$ such as coefficients estimate, distortion bounds and convex family. Also the concept of δ neighborhoods and partial sums of analytic functions to the class $\mathcal{M}^{n,q}_{p,\alpha}(\eta, A, B)$.

Keywords: fractional derivative; convolution; meromorphic function; q-analogue of linear differential operator; complex order; q-starlike; q-convex; neighborhoods; partial sums

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1. Introduction

Let $\mathcal{M}_p$ is the class of p-valently meromorphic functions of the form:

$$\mathcal{F}(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{\infty} a_j \zeta^j \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic in the punctured open unit disk $\Delta^* := \{\zeta \in \mathbb{C} : 0 < |\zeta| < 1\} = \Delta \backslash \{0\}$. Let $\mathcal{F}$ and $\mathcal{E}$ are analytic functions in Δ, we say that $\mathcal{F}$ is subordinate to $\mathcal{E}$ if there exists an analytic function $\varpi(\zeta)$ with $\varpi(0) = 0$ and $|\varpi(\zeta)| < 1$ $(\zeta \in \Delta)$ such that $\mathcal{F} = \mathcal{E}(\varpi(\zeta))$. We denote by $\mathcal{F} \prec \mathcal{E}$ (see [1,2]):

Let the functions $\mathcal{F}(\zeta) \in \mathcal{M}_p$ defined by (1) and $\mathcal{G}(\zeta) \in \mathcal{M}_p$ defined by

$$\mathcal{G}(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{\infty} b_j \zeta^j \quad (p \in \mathbb{N}). \tag{2}$$

The Hadamard product or convolution of $\mathcal{F}(\zeta)$ and $\mathcal{G}(\zeta)$ is defined by

$$(\mathcal{F} * \mathcal{G})(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{\infty} a_j b_j \zeta^j = (\mathcal{G} * \mathcal{F})(\zeta). \tag{3}$$

In this paper, we define some concepts of fractional derivative, for any non-negative integer j, the $q-$factorial $[j]_q!$ is defined by (see [3]):

Assume that $0 < q < 1$, the q-number $[j]_q$ are defined by (see [3–9]). where

$$[j]_q := \frac{1-q^j}{1-q} = 1 + \sum_{r=1}^{j-1} q^r. \tag{4}$$

El-Deeb et al. [10] defined the q-derivative operator for $\mathcal{F} * \mathcal{G}$ as follows (see [11])

$$\mathcal{D}_q(\mathcal{F} * \mathcal{G})(\zeta) := \begin{cases} \frac{(\mathcal{F}*\mathcal{G})(q\zeta)-(\mathcal{F}*\mathcal{G})(\zeta)}{\zeta(q-1)} & \zeta \neq 0 \\ \mathcal{F}'(0) & \zeta = 0. \end{cases} \tag{5}$$

Also, we have

$$\lim_{q\to 1^-} \mathcal{D}_q(\mathcal{F} * \mathcal{G})(\zeta) := \lim_{q\to 1^-} \frac{(\mathcal{F} * \mathcal{G})(q\zeta) - (\mathcal{F} * \mathcal{G})(\zeta)}{\zeta(q-1)} = ((\mathcal{F} * \mathcal{G})(\zeta))'.$$

From (1) and (5), we get

$$\mathcal{D}_q(\mathcal{F} * \mathcal{G})(\zeta) := -\frac{[p]_q}{q^p}\zeta^{-p-1} + \sum_{j=1-p}^{\infty} [j]_q a_j b_j \zeta^{j-1},\ \zeta \neq 0. \tag{6}$$

Also, we define the linear differential operator $\mathcal{D}^{n,q}_{\alpha,p,g} : \mathcal{M}_p \to \mathcal{M}_p$ as follows:

$$\mathcal{D}^{0,q}_{\alpha,p,g}\mathcal{F}(\zeta) = (\mathcal{F} * \mathcal{G})(\zeta),$$

$$\begin{aligned}
\mathcal{D}^{1,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta) &= \frac{\alpha q^p}{[p]_q}\zeta\, D_q\left(\mathcal{D}^{0,q}_{\alpha,p,g}\mathcal{F}(\zeta)\right) + (1-\alpha)(\mathcal{F} * \mathcal{G})(\zeta) + 2\alpha\zeta^{-p} \\
&= \zeta^{-p} + \sum_{j=1-p}^{\infty}\left(\frac{\alpha q^p[j]_q + (1-\alpha)[p]_q}{[p]_q}\right) a_j b_j \zeta^j \\
\mathcal{D}^{2,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta) &= \frac{\alpha q^p}{[p]_q}\zeta\, D_q\left(\mathcal{D}^{1,q}_{\alpha,p,g}\mathcal{F}(\zeta)\right) + (1-\alpha)\mathcal{D}^{1,q}_{\alpha,p,g}\mathcal{F}(\zeta) + 2\alpha\zeta^{-p} \\
&= \zeta^{-p} + \sum_{j=1-p}^{\infty}\left(\frac{\alpha q^p[j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^2 a_j b_j \zeta^j \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta) &= \frac{\alpha q^p}{[p]_q}\zeta\, D_q\left(\mathcal{D}^{n-1,q}_{\alpha,p,g}\mathcal{F}(\zeta)\right) + (1-\alpha)\mathcal{D}^{n-1,q}_{\alpha,p,g}\mathcal{F}(\zeta) + 2\alpha\zeta^{-p} \\
&= \zeta^{-p} + \sum_{j=1-p}^{\infty}\left(\frac{\alpha q^p[j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^n a_j b_j \zeta^j \qquad (7) \\
&\quad (p \in \mathbb{N},\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ 0 < q < 1,\ \alpha > 0).
\end{aligned}$$

From (7), we obtain the following relations:

$$\text{(i) } \mathcal{D}^{n+1,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta) = \frac{\alpha q^p}{[p]_q}\zeta\, D_q\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right) + (1-\alpha)\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta) + 2\alpha\zeta^{-p},\ \zeta \in \Delta^*; \tag{8}$$

$$\text{(ii) } \mathcal{I}^{n}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta) := \lim_{q\to 1^-} \mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{\infty}\left(\frac{j\alpha + p(1-\alpha)}{p}\right)^n a_j b_j \zeta^j,\ \ \zeta \in \Delta^*. \tag{9}$$

Remark 1. *(i) By taking* $\mathcal{G}(\zeta) = \frac{\zeta^{-p}}{1-\zeta}$ *(or* $b_j = 1$*) in this operator* $\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}$*, we have the linear differential operator* $\mathcal{D}^{n}_{\alpha,p,q}$ *defined by El-Deeb and El-Matary ([12], With* $A = 1$*);*

(ii) Put $\alpha = 1$ *in the operator* $\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}$*, we get the* (p,q)*-analogue of the operator* $\mathcal{D}^{n,q}_{p,\mathcal{G}}$ *defined as follows:*

$$\mathcal{D}^{n,q}_{p,\mathcal{G}}\mathcal{F}(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{\infty} \left(\frac{q^p\,[j]_q}{[p]_q}\right)^n a_j b_j \zeta^j \;\; (p \in \mathbb{N},\, n \in \mathbb{N}_0,\, 0 < q < 1,\, \zeta \in \Delta^*); \tag{10}$$

(iii) Let $\alpha = 1$ *and* $q \to 1$ *in the operator* $\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}$*, we have the operator* $\mathcal{D}^{n}_{p,\mathcal{G}}$ *defined as follows:*

$$\mathcal{D}^{n}_{p,\mathcal{G}}\mathcal{F}(\zeta) := \lim_{q\to 1^-} \mathcal{D}^{n}_{1,p,q}\mathcal{F}(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{\infty} \left(\frac{j}{p}\right)^n a_j b_j \zeta^j, \quad (p \in \mathbb{N},\, n \in \mathbb{N}_0,\, \zeta \in \Delta^*); \tag{11}$$

(iv) Taking $\alpha = 1$ *and* $\mathcal{G}(\zeta) = \frac{\zeta^{-p}}{1-\zeta}$ *(or* $b_j = 1$*) in the operator* $\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}$*, we have the* (p,q)*-analogue of Salagean operator* $\mathcal{D}^{n}_{p,q}$ *defined as follows:*

$$\mathcal{D}^{n}_{p,q}\mathcal{F}(\zeta) := \zeta^{-p} + \sum_{j=1-p}^{\infty} \left(\frac{q^p\,[j]_q}{[p]_q}\right)^n a_j \zeta^j \;\; (p \in \mathbb{N},\, n \in \mathbb{N}_0,\, 0 < q < 1,\, \zeta \in \Delta^*); \tag{12}$$

(v) Putting $q \to 1^-$ *and* $\alpha = 1$ *in the operator* $\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}$*, we get the operator in meromorphic* $\mathcal{D}^{n}_{p,\mathcal{G}}$ *defined as follows:*

$$\mathcal{D}^{n}_{p,\mathcal{G}}\mathcal{F}(\zeta) := \lim_{q\to 1^-} \mathcal{D}^{n,q}_{1,p,\mathcal{G}}\mathcal{F}(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{\infty} \left(\frac{j}{p}\right)^n a_j b_j \zeta^j, \quad (p \in \mathbb{N},\, n \in \mathbb{N}_0,\, \zeta \in \Delta^*). \tag{13}$$

A function $\mathcal{F} \in \mathcal{M}_p$ is said to be in the subclass $\mathcal{MS}^*(\gamma)$ of meromorphic starlike functions of order γ in Δ^*, if it satisfies the following condition (see [13–16]):

$$\Re\left(\frac{\zeta\mathcal{F}'(\zeta)}{\mathcal{F}(\zeta)}\right) < -\gamma \quad (\zeta \in \Delta^*;\, 0 \le \gamma < 1). \tag{14}$$

A function $\mathcal{F} \in \mathcal{M}_p$ is said to be in the subclass $MC(\gamma)$ of meromorphic convex functions of order γ in Δ^*, if it satisfies the following condition (see [17]):

$$\Re\left(1 + \frac{\zeta\mathcal{F}''(\zeta)}{\mathcal{F}'(\zeta)}\right) < -\gamma \quad (\zeta \in \Delta^*;\, 0 \le \gamma < 1). \tag{15}$$

It is easy to observe from (14) and (15) that

$$\mathcal{F} \in MC(\gamma) \Leftrightarrow -\zeta\mathcal{F}' \in \mathcal{MS}^*(\gamma). \tag{16}$$

We will generalize these classes by using the new operator $\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}$, we define the new class $\mathcal{M}^{n,q}_{p,\alpha}(\lambda,\mu,\gamma)$ and study some theorems for this class.

Definition 1. *Assume that* $\mathcal{F} \in \mathcal{M}_p$ *be in the class* $\mathcal{M}^{n,q}_{p,\alpha}(\eta, A, B)$ *if*

$$p - \frac{1}{\eta}\left\{\frac{\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)'}{\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)} + p\right\} \prec p\frac{1 + A\zeta}{1 + B\zeta} \tag{17}$$

or, equivalently, to

$$\left| \frac{\dfrac{\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)'}{\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)} + p}{B\dfrac{\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)'}{\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)} + p[(A-B)\eta + B]} \right| < 1 \tag{18}$$

$$(p \in \mathbb{N},\ n \in \mathbb{N}_0,\ 0 < q < 1,\ \alpha > 0,\ \eta \in \mathbb{C}^*,\ -1 \le B < A \le 1,\ \zeta \in \Delta^*).$$

Let $\mathcal{M}^*_p$ is subclass of $\mathcal{M}_p$ which contains functions on the form:

$$\mathcal{F}(\zeta) := \zeta^{-p} + \sum_{j=p}^{\infty} a_j \zeta^j \ \ (p \in \mathbb{N}). \tag{19}$$

Also, we can write

$$\mathcal{M}^{n,q,*}_{p,\alpha}(\eta, A, B) = \mathcal{M}^{n,q}_{p,\alpha}(\eta, A, B) \cap \mathcal{M}^*_p.$$

Remark 2. *(i) Taking $q \to 1^-$, we get $\lim\limits_{q\to 1^-} \mathcal{M}^{n,q}_{p,\alpha}(\lambda,\mu,\gamma) =: \mathcal{I}^n_{p,\alpha}(\lambda,\mu,\gamma)$, where $\mathcal{I}^n_{p,\alpha}(\lambda,\mu,\gamma)$ represents the function $\mathcal{F} \in \mathcal{M}_p$ that satisfies (18) for $\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}$ replaced with $\mathcal{I}^n_{\alpha,p,\mathcal{G}}$ given by (9);*

(ii) Putting $\alpha = 1$, we get the subclass $R^{n,q}_p(\lambda,\mu,\gamma)$ represents the function $\mathcal{F} \in \mathcal{M}_p$ that satisfies (18) for $\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}$ replaced with $\mathcal{D}^{n,q}_{p,\mathcal{G}}$ given by (10).

2. Basic Properties of the Subclass $\mathcal{M}^{n,q,*}_{p,\alpha}(\eta, A, B)$

Theorem 1. *The function $\mathcal{F}$ defined by (19) belongs to the subclass $\mathcal{M}^{n,q,*}_{p,\alpha}(\eta, A, B)$ if and only if*

$$\sum_{j=p}^{\infty}[(j+p)(1-B) - p|\eta|(A-B)]\left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^n b_j |a_j| \le p|\eta|(A-B). \tag{20}$$

Proof. Let (20) holds true, we get

$$\begin{aligned}
& \left|\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)' + p\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right| - \left|B\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)' + [Bp(1-\eta) + Ap\eta]\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right| \\
= & \left|\sum_{j=p}^{\infty}(j+p)\left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^n a_j b_j \zeta^{j+p}\right| - \\
& \left|p\eta(A-B) + \sum_{j=p}^{\infty}[B(j+p) + p\eta(A-B)]\left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^n a_j b_j \zeta^{j+p}\right| \\
\le & \sum_{j=p}^{\infty}(j+p)\left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^n b_j |a_j| r^{j+p} - p\eta(A-B) \\
& - \sum_{j=p}^{\infty}[B(j+p) + p\eta(A-B)]\left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^n b_j |a_j| r^{j+p} \\
= & \sum_{j=p}^{\infty}[(1-B)(j+p) - p\eta(A-B)]\left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^n b_j |a_j| r^{j+p} - p\eta(A-B).
\end{aligned} \tag{21}$$

Since (21) holds for all $r \in (0,1)$. Letting $r \to 1^-$, we obtain

$$\begin{aligned}
&\left|\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)' + p\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right| - \left|B\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)' + [Bp(1-\eta)+Ap\eta]\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right| \\
\leq\ & \sum_{j=p}^{\infty}[(1-B)(j+p) - p\eta(A-B)]\left(\frac{\alpha q^p[j]_q+(1-\alpha)[p]_q}{[p]_q}\right)^n b_j|a_j| - p\eta(A-B) \\
\leq\ & 0 \quad (\text{by } (20)).
\end{aligned}$$

Hence, we get $\mathcal{F}(\zeta) \in \mathcal{M}^{n,q}_{p,\alpha}(\eta, A, B)$.

Conversely, Let $\mathcal{F}(\zeta)$ belongs to $\mathcal{M}^{n,q}_{p,\alpha}(\eta, A, B)$ with $\mathcal{F}(\zeta)$ of the form (19), we find from (18), that

$$\left|\frac{\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)' + p\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)}{B\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)' + [Bp(1-b)+Apb]\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)}\right|$$

$$= \left|\frac{\sum_{j=p}^{\infty}(j+p)\left(\frac{\alpha q^p[j]_q+(1-\alpha)[p]_q}{[p]_q}\right)^n a_j b_j \zeta^{j+p}}{p\eta(A-B) + \sum_{j=p}^{\infty}[B(j+p)+p\eta(A-B)]\left(\frac{\alpha q^p[j]_q+(1-\alpha)[p]_q}{[p]_q}\right)^n a_j b_j \zeta^{j+p}}\right| < 1. \tag{22}$$

Using the fact that $\Re\{\zeta\} \leq |\zeta|$ for all ζ, we get

$$\Re\left\{\frac{\frac{\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)'}{\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)} + p}{\frac{B\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)'}{\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)} + [Bp(1-\eta)+Ap\eta]}\right\} < 1,\ \zeta \in \Delta^*. \tag{23}$$

If we take ζ on real axis, so that $\frac{\zeta\left(\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)\right)'}{\mathcal{D}^{n,q}_{\alpha,p,\mathcal{G}}\mathcal{F}(\zeta)}$ is real. Upon clearing the denominator in (23) and letting $\zeta \to 1^-$, we get

$$\sum_{j=p}^{\infty}[(j+p)(1-B) - p|\eta|(A-B)]\left(\frac{\alpha q^p[j]_q+(1-\alpha)[p]_q}{[p]_q}\right)^n b_j|a_j| \leq p|\eta|(A-B), \tag{24}$$

which we've got the assertion (20) of Theorem 1. □

Corollary 1. *The function $\mathcal{F}(\zeta)$ be defined by* (19) *belongs to $\mathcal{M}^{n,q,*}_{p,\alpha}(\eta, A, B)$, then*

$$|a_j| \leq \frac{p|\eta|(A-B)}{[(j+p)(1-B) - p|\eta|(A-B)]\left(\frac{\alpha q^p[j]_q+(1-\alpha)[p]_q}{[p]_q}\right)^n b_j} \quad (j \geq p). \tag{25}$$

This result is sharp for $\mathcal{F}$ given by

$$\mathcal{F}(\zeta) = \zeta^{-p} + \frac{p|\eta|(A-B)}{[(j+p)(1-B) - p|\eta|(A-B)]\left(\frac{\alpha q^p[j]_q+(1-\alpha)[p]_q}{[p]_q}\right)^n b_j}\zeta^j \quad (j \geq p). \tag{26}$$

Theorem 2. *The function $\mathcal{F}(\zeta)$ defined by* (19) *belongs $\mathcal{M}^{n,q,*}_{p,\alpha}(\eta, A, B)$, then for $|\zeta| = r < 1$, we have*

$$\left\{\frac{(p+m-1)!}{(p-1)!}-\frac{p!|\eta|(A-B)}{[2(1-B)-|\eta|(A-B)](1+\alpha(q^p-1))^n(p-m)!b_p}r^{2p}\right\}r^{-(p+m)}$$
$$\leq\left|\mathcal{F}^{(m)}(\zeta)\right|\leq\left\{\frac{(p+m-1)!}{(p-1)!}+\frac{p!|\eta|(A-B)}{[2(1-B)-|\eta|(A-B)](1+\alpha(q^p-1))^n(p-m)!b_p}r^{2p}\right\}r^{-(p+m)}. \tag{27}$$

This result is sharp for $\mathcal{F}$ given by

$$\mathcal{F}(\zeta)=\zeta^{-p}+\frac{|\eta|(A-B)}{[2(1-B)-|\eta|(A-B)](1+\alpha(q^p-1))^n b_p}\zeta^p. \tag{28}$$

Proof. Let $\mathcal{F}(\zeta)\in\mathcal{M}_{p,\alpha}^{n,q,*}(\eta,A,B)$, then

$$\frac{p[2(1-B)-|\eta|(A-B)](1+\alpha(q^p-1))^n(p-m)!b_p}{p!}\sum_{j=p}^{\infty}\frac{j!}{(j-m)!}|a_j|$$
$$\leq\sum_{j=p}^{\infty}[(j+p)(1-B)-p|\eta|(A-B)]\left(\frac{\alpha q^p[j]_q+(1-\alpha)[p]_q}{[p]_q}\right)^n b_j\,.|a_j|\leq p|\eta|(A-B),$$

which yields

$$\sum_{j=p}^{\infty}\frac{j!}{(j-m)!}|aj|\leq\frac{|\eta|(A-B)}{[2(1-B)-|\eta|(A-B)](1+\alpha(q^p-1))^n b_p}\frac{p!}{(p-m)!}. \tag{29}$$

Differentiating both sides of (19) m times with respect to ζ, we get

$$\mathcal{F}^{(m)}(\zeta)=(-1)^m\frac{(p+m-1)!}{(p-1)!}\zeta^{-(p+m)}+\sum_{j=p}^{\infty}\frac{j!}{(j-m)!}|a_j|\zeta^{j-m}\quad(p\in\mathbb{N},\ 0\leq m<p) \tag{30}$$

and Theorem 2 follows easily from (29) and (30), and it is easy to have the bounds in (27) are attained for $\mathcal{F}$ given by (28). □

Theorem 3. *The function $\mathcal{F}$ defined by (19) belings to $\mathcal{M}_{p,\alpha}^{n,q,*}(\eta,A,B)$, then*

(i) $\mathcal{F}$ is meromorphically p-valent q-starlike of order ρ $(0\leq\rho<[p]_q)$ in the disc $|\zeta|<r_1$, that is,

$$\Re\left\{-\frac{\zeta\mathcal{D}_q\mathcal{F}(\zeta)}{\mathcal{F}(\zeta)}\right\}>\rho\quad(|\zeta|<r_1,\ 0\leq\rho<[p]_q,\ p\in\mathbb{N}), \tag{31}$$

where

$$r_1=\inf_{j\geq p}\left\{\frac{[(j+p)(1-B)-p|\eta|(A-B)]\left(\frac{\alpha q^p[j]_q+(1-\alpha)[p]_q}{[p]_q}\right)^n}{p|\eta|(A-B)}\frac{\left(\frac{[p]_q}{q^p}-\rho\right)b_j}{\left([j]_q+\rho\right)}\right\}^{\dfrac{1}{j+p}}, \tag{32}$$

(ii) $\mathcal{F}$ is meromorphically p-valent q-convex of order ρ $(0\leq\rho<[p]_q)$ in the disc $|\zeta|<r_2$, that is,

$$\Re\left\{-\left(\frac{\mathcal{D}_q(\zeta\mathcal{D}_q\mathcal{F}(\zeta))}{\mathcal{D}_q\mathcal{F}(\zeta)}\right)\right\}>\rho\quad(|\zeta|<r_2,\ 0\leq\rho<[p]_q,\ p\in\mathbb{N}), \tag{33}$$

where

$$r_2 = \inf_{j \geq p} \left\{ \frac{[(j+p)(1-B) - p|\eta|(A-B)] \left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q} \right)^n \left(\frac{[p]_q}{q^p} - \rho \right) [p]_q b_j}{p q^p [j]_q \left([j]_q + \rho \right) |\eta| (A-B)} \right\}^{\frac{1}{j+p}}. \tag{34}$$

Each of these results is sharp for the function $\mathcal{F}(\zeta)$ given by (26).

Proof. (i) From the definition (19), we easily get

$$\left| \frac{\frac{\zeta \mathcal{D}_q \mathcal{F}(\zeta)}{\mathcal{F}(\zeta)} + \frac{[p]_q}{q^p}}{\frac{\zeta \mathcal{D}_q \mathcal{F}(\zeta)}{\mathcal{F}(\zeta)} - \frac{[p]_q}{q^p} + 2\rho} \right| \leq \frac{\sum\limits_{j=p}^{\infty} ([j]_q + \frac{[p]_q}{q^p}) |a_j| |\zeta|^{j+p}}{2(\frac{[p]_q}{q^p} - \rho) - \sum\limits_{j=p}^{\infty} ([j]_q - \frac{[p]_q}{q^p} + 2\rho) |a_j| |\zeta|^{j+p}}. \tag{35}$$

We have the inequality

$$\left| \frac{\frac{\zeta \mathcal{D}_q \mathcal{F}(\zeta)}{\mathcal{F}(\zeta)} + \frac{[p]_q}{q^p}}{\frac{\zeta \mathcal{D}_q \mathcal{F}(\zeta)}{\mathcal{F}(\zeta)} - \frac{[p]_q}{q^p} + 2\rho} \right| \leq 1 \ \ (0 \leq \rho < [p]_q;\ p \in \mathbb{N}), \tag{36}$$

if

$$\sum_{j=p}^{\infty} \left(\frac{[j]_q + \rho}{\frac{[p]_q}{q^p} - \rho} \right) |a_j| |\zeta|^{j+p} \leq 1. \tag{37}$$

Hence, by Theorem 1, (37) will be true

$$\frac{\left([j]_q + \rho \right)}{\left(\frac{[p]_q}{q^p} - \rho \right)} |\zeta|^{j+p} \leq \frac{[(j+p)(1-B) - p|\eta|(A-B)] \left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q} \right)^n b_j}{p|\eta|(A-B)}$$

$$|\zeta| \leq \left\{ \frac{[(j+p)(1-B) - p|\eta|(A-B)] \left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q} \right)^n b_j}{p|\eta|(A-B)} \frac{\left(\frac{[p]_q}{q^p} - \rho \right)}{\left([j]_q + \rho \right)} \right\}^{\frac{1}{j+p}}, \tag{38}$$

the inequality leads us immediately to the disc $|\zeta| < r_1$, where r_1 is given by (32).

(ii) To prove the second assertion of Theorem 3, we get from the definition (19) that

$$\left| \frac{\frac{\mathcal{D}_q(\zeta \mathcal{D}_q \mathcal{F}(\zeta))}{\mathcal{D}_q \mathcal{F}(\zeta)} + \frac{[p]_q}{q^p}}{\frac{\mathcal{D}_q(\zeta \mathcal{D}_q \mathcal{F}(\zeta))}{\mathcal{D}_q \mathcal{F}(\zeta)} - \frac{[p]_q}{q^p} + 2\rho} \right| \leq \frac{\sum\limits_{j=p}^{\infty} [j]_q ([j]_q + \frac{[p]_q}{q^p}) |a_j| |\zeta|^{j+p}}{2\frac{[p]_q}{q^p}(\frac{[p]_q}{q^p} - \rho) - \sum\limits_{j=p}^{\infty} [j]_q ([j]_q - \frac{[p]_q}{q^p} + 2\rho) |a_j| |\zeta|^{j+p}}. \tag{39}$$

Thus, we have the desired inequality

$$\left| \frac{\frac{\mathcal{D}_q(\zeta \mathcal{D}_q \mathcal{F}(\zeta))}{\mathcal{D}_q \mathcal{F}(\zeta)} + \frac{[p]_q}{q^p}}{\frac{\mathcal{D}_q(\zeta \mathcal{D}_q \mathcal{F}(\zeta))}{\mathcal{D}_q \mathcal{F}(\zeta)} - \frac{[p]_q}{q^p} + 2\rho} \right| \leq 1 \ \ (0 \leq \rho < [p]_q,\ p \in \mathbb{N}), \tag{40}$$

if

$$\sum_{j=p}^{\infty} \frac{q^p [j]_q}{[p]_q} \left(\frac{[j]_q + \rho}{\frac{[p]_q}{q^p} - \rho} \right) |a_j||\zeta|^{j+p} \leq 1. \tag{41}$$

From Theorem 1, (41) will be true if

$$\frac{q^p [j]_q}{[p]_q} \left(\frac{[j]_q + \rho}{\frac{[p]_q}{q^p} - \rho} \right) |\zeta|^{j+p} \leq \frac{[(j+p)(1-B) - p|\eta|(A-B)] \left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q} \right)^n b_j}{p|\eta|(A-B)}. \tag{42}$$

The inequality (42) readily yields the disc $|\zeta| < r_2$, where r_2 defined by (34), and the proof of Theorem 3 is completed. □

3. Neighborhoods and Partial Sums

By following the earlier works based upon the familiar concept of neighborhoods of analytic functions by Goodman [15] and Ruscheweyh [18] and (more recently) by Altintas et al. [19–21], Liu [22], Liu and Srivastava [23] and El-Ashwah et al. [24], we introduce here the δ-neighborhoods of a function $\mathcal{F} \in \mathcal{M}_p$ has the form (1) by means of the definition given by:

$$\mathcal{N}_\delta(\mathcal{F}) = \Bigg\{ h : \; h \in \mathcal{M}_p, \; h(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{\infty} c_j z^j \text{ and}$$
$$\sum_{j=1-p}^{\infty} \frac{[(j+p)(1-B) - p|\eta|(A-B)] \left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q} \right)^n b_j}{p|\eta|(A-B)} |c_j - a_j| \leq \delta$$
$$(\, n \in \mathbb{N}_0, \; 0 < q < 1, \; \alpha > 0, \; \eta \in \mathbb{C}^*, \; -1 \leq B < A \leq 1) \Bigg\}. \tag{43}$$

Using the definition (43), we will obtain the following theorem:

Theorem 4. *The function $\mathcal{F}$ defined by (1) belongs to $M_{p,\alpha}^{n,q}(\eta, A, B)$. If $\mathcal{F}$ satisfies the condition*

$$\frac{\mathcal{F}(\zeta) + \epsilon \zeta^{-p}}{1 + \epsilon} \in M_{p,\alpha}^{n,q}(\eta, A, B) \;\; (\epsilon \in \mathbb{C}, \; |\epsilon| < \delta, \; \delta > 0) \tag{44}$$

then

$$N_\delta(\mathcal{F}) \subset M_{p,\alpha}^{n,q}(\eta, A, B). \tag{45}$$

Proof. From (18), we obtain $h \in M_{p,\alpha}^{n,q}(\eta, A, B)$ if, for $\sigma \in \mathbb{C}$ with $|\sigma| = 1$, we have

$$\frac{\zeta \left(\mathcal{D}_{\alpha,p,\mathcal{G}}^{n,q} h(\zeta) \right)' + p \mathcal{D}_{\alpha,p,\mathcal{G}}^{n,q} h(\zeta)}{B\zeta \left(\mathcal{D}_{\alpha,p,\mathcal{G}}^{n,q} h(\zeta) \right)' + [Bp(1-b) + Apb] \mathcal{D}_{\alpha,p,\mathcal{G}}^{n,q} h(\zeta)} \neq \sigma \;\; (\zeta \in \Delta), \tag{46}$$

which is equivalent to

$$\frac{(h * \psi)(\zeta)}{\zeta^{-p}} \neq 0 \;\; (\zeta \in \Delta^*), \tag{47}$$

where, for convenience,

$$\psi(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{\infty} y_j \zeta^j$$
$$= \zeta^{-p} + \sum_{j=1-p}^{\infty} \frac{[(j+p)(1-B\sigma) - p|\eta|\sigma(A-B)]\left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^n b_j}{p\eta\sigma(A-B)} \zeta^j. \tag{48}$$

From (48), we get

$$|y_j| = \left| \frac{[(j+p)(1-B\sigma) - p|\eta|\sigma(A-B)]\left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^n b_j}{p\eta\sigma(A-B)} \right|$$
$$\leq \frac{[(j+p)(1+|B|) - p|\eta|(A-B)]}{p|\eta|(A-B)} \left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^n b_j \ \ (j \geq p,\ p \in \mathbb{N}). \tag{49}$$

If $\mathcal{F}(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{\infty} a_j \zeta^j \in \mathcal{M}_p$ holds the condition (44), then (47) yields

$$\left| \frac{(\mathcal{F} * \psi)(\zeta)}{\zeta^{-p}} \right| > \delta \ \ (\zeta \in \Delta^*,\ \delta > 0). \tag{50}$$

Let

$$\Phi(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{\infty} d_j \zeta^j \in N_\delta(\mathcal{F}) \tag{51}$$

we have

$$\left| \frac{[\Phi(\zeta) - \mathcal{F}(\zeta)] * \psi(\zeta)}{\zeta^{-p}} \right| = \left| \sum_{j=1-p}^{\infty} (d_j - a_j) y_j \zeta^{j+p} \right|$$

$$\leq |\zeta| \sum_{j=1-p}^{\infty} \frac{[(j+p)(1+|B|) - p|\eta|(A-B)]}{p|\eta|(A-B)} \left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^n b_j |d_j - a_j| \tag{52}$$
$$< \delta \ \ (\zeta \in \Delta,\ \delta > 0).$$

We have (47), and hence also (46) for any σ, which implies that $\Phi \in M_{p,\alpha}^{n,q}(\eta, A, B)$. This evidently proves the assertion (45) of Theorem 4. □

Theorem 5. *Let $\mathcal{F} \in \mathcal{M}_p$ defined by* (1) *and $-1 \leq B \leq 0$, the partial sums $\mathcal{S}_1(\zeta)$ and $\mathcal{S}_m(\zeta)$ are given by*

$$\mathcal{S}_1(\zeta) = \zeta^{-p} \ \ \text{and} \ \ \mathcal{S}_m(\zeta) = \zeta^{-p} + \sum_{j=1-p}^{m-1} a_j \zeta^j \ \ (m \in \mathbb{N} \backslash \{1\}). \tag{53}$$

Also, suppose that

$$\sum_{j=1-p}^{\infty} y_{j+p} |a_j| \leq 1 \ \left(y_{j+p} = \frac{[(j+p)(1+|B|) - p|\eta|(A-B)]}{p|\eta|(A-B)} \left(\frac{\alpha q^p [j]_q + (1-\alpha)[p]_q}{[p]_q}\right)^n b_j \right), \tag{54}$$

then

(i) $\mathcal{F}(\zeta) \in M_{p,\alpha}^{n,q}(\eta, A, B)$

(ii) $Re\left\{\frac{\mathcal{F}(\zeta)}{\mathcal{S}_m(\zeta)}\right\} > 1 - \frac{1}{y_q} \qquad (\zeta \in \Delta,\ m \in \mathbb{N})$ (55)

and

(iii) $Re\left\{\frac{\mathcal{S}_m(\zeta)}{\mathcal{F}(\zeta)}\right\} > \frac{y_q}{1+y_q} \qquad (\zeta \in \Delta,\ m \in \mathbb{N}).$ (56)

The estimates in (55) *and* (56) *are sharp.*

Proof. Since $\frac{\zeta^{-p} + \varepsilon\zeta^{-p}}{1+\varepsilon} = \zeta^{-p} \in M^{n,q}_{p,\alpha}(\eta, A, B)$, $|\varepsilon| < 1$, then by Theorem 4, we have $N_\delta(\mathcal{F}) \subset M^{n,q}_{p,\alpha}(\eta, A, B)$, $p \in \mathbb{N}$. $N_1(\zeta^{-p})$ denoting the 1-neighbourhood). Now since

$$\sum_{j=1-p}^{\infty} y_j |a_j| \leq 1, \tag{57}$$

then $\mathcal{F} \in N_1(\zeta^{-p})$ and $\mathcal{F} \in M^{n,q}_{p,\alpha}(\eta, A, B)$. Since $\{y_j\}$ is an increasing sequence, we get

$$\sum_{j=1-p}^{m-p-1} |a_j| + y_m \sum_{j=m-p}^{\infty} |a_j| \leq \sum_{j=1-p}^{\infty} y_{j+p}|a_j| \leq 1, \tag{58}$$

we have used the hypothesis (54). Putting

$$h_1(\zeta) = y_m\left\{\frac{\mathcal{F}(\zeta)}{\mathcal{S}_m(\zeta)} - (1 - \frac{1}{y_m})\right\} = 1 + \frac{y_m \sum\limits_{j=m-p}^{\infty} |a_j|\zeta^{j+p}}{1 + \sum\limits_{j=1-p}^{m-p-1} |a_j|\zeta^{j+p}}$$

and applying (58), we find that

$$\left|\frac{h_1(\zeta) - 1}{h_1(\zeta) + 1}\right| \leq \frac{y_m \sum\limits_{j=m-p}^{\infty} |a_j|}{2 - 2\sum\limits_{j=1-p}^{m-p-1} |a_j| - y_m \sum\limits_{j=m-p}^{\infty} |a_j|} \leq 1 \ \ (\zeta \in \Delta), \tag{59}$$

which readily yields the assertion (55) of Theorem 5. If we take

$$\mathcal{F}(\zeta) = \zeta^{-p} - \frac{\zeta^m}{y_m}, \tag{60}$$

then

$$\frac{\mathcal{F}(\zeta)}{\mathcal{S}_m(\zeta)} = 1 - \frac{\zeta^{p+m}}{y_m} \to 1 - \frac{1}{y_m}, \ \text{ as } \zeta \to 1^-,$$

which shows that the bound in (55) is the best possible for each $m \in \mathbb{N}$.

If we put

$$h_2(\zeta) = (1 + y_m)\left\{\frac{\mathcal{S}_m(\zeta)}{\mathcal{F}(\zeta)} - \frac{y_m}{1+y_m}\right\} = 1 - \frac{(1+y_m)\sum\limits_{j=m-p}^{\infty} |a_j|\zeta^{j+p}}{1 + \sum\limits_{j=1-p}^{\infty} |a_j|\zeta^{j+p}}, \tag{61}$$

and make use of (58), we can deduce that

$$\left|\frac{h_2(\zeta)-1}{h_2(\zeta)+1}\right| \leq \frac{(1+y_m)\sum\limits_{j=m-p}^{\infty}|a_j|}{2-2\sum\limits_{j=1-p}^{m-p-1}|a_j|-(1-y_m)\sum\limits_{j=m-p}^{\infty}|a_j|} \leq 1,$$

leads us to the assertion (56) of Theorem 5. The bound in (56) is sharp. The proof of Theorem 5 is completed. □

4. Concluding Remarks and Observations

In our present investigation, we have introduced and studied the properties of some new subclasses of the class of meromorphic p-valent functions in the open unit disk Δ^* by using the combination of q-derivative and convolution and obtain the new operator $\mathcal{D}_{\alpha,p,g}^{n,q}$. Among other properties and results such as coefficients estimate, distortion bounds and convex family. Also the concept of δ neighborhoods and partial sums of analytic functions to the class $\mathcal{M}_{p,\alpha}^{n,q}(\eta, A, B)$.

Interesting results about meromorphic functions can be found in the works [25–31].

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