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Periodic Solutions of Quasi-Monotone Semilinear Multidimensional Hyperbolic Systems

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Abstract: This paper deals with the Cauchy problem for a class of first-order semilinear hyperbolic equations of the form $\partial_t f_i + \sum_{j=1}^{d} \lambda_{ij}\partial_{x_j} f_i = Q_i(f)$. where $f_i = f_i(x,t)$ $(i = 1,\ldots,n)$ and $x = (x_1,\cdots,x_d) \in \mathbb{R}^d$ $(n \geq 2, d \geq 1)$. Under assumption of the existence of a conserved quantity $\sum_i \alpha_i f_i$ for some $\alpha_1,\ldots,\alpha_n > 0$, of (strong) quasimonotonicity and an additional assumption on the speed vectors $\Lambda_i = (\lambda_{i1},\cdots,\lambda_{id}) \in \mathbb{R}^d$—namely, span $\{\Lambda_j - \Lambda_k : j = 1,\ldots,n\} = \mathbb{R}^d$ for any k—it is proved that the set of constant steady state $\{\bar{f} \in \mathbb{R}^n : Q(\bar{f}) = 0\}$ is asymptotically stable with respect to periodic perturbations, i.e., any initial data given by an periodic L^1-perturbations of a constant steady state $\bar{f}$ leads to a solution converging to another constant steady state $\bar{g}$ (uniquely determined by the initial condition) as $t \to +\infty$.

Keywords: semilinear hyperbolic systems; stability analysis; quasi-monotonicity

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1. Introduction

In this paper, we deal with the following system of equations:

$$\partial_t f_i + \sum_{j=1}^{d} \lambda_{ij}\partial_{x_j} f_i = Q_i(f). \tag{1}$$

Here, $f = f(x,t)$, where $f = (f_1,\ldots,f_n)$ and $x = (x_1,\cdots,x_d) \in \mathbb{R}^d$ $(n \geq 2, d \geq 1)$. The vectors $\Lambda_i = (\lambda_{i1},\cdots,\lambda_{id}) \in \mathbb{R}^d$ are called *speeds*, and the function $Q = (Q_1,\ldots,Q_n)^\top \in C^1(\mathbb{R}^n,\mathbb{R}^n)$ is the *collision term*. In the following, we set $DQ = (q_{ij}) = (\partial Q_i/\partial f_j)$. We assume throughout the paper that the speeds Λ_i satisfy

$$\text{span}\,\{\Lambda_j - \Lambda_k : j = 1,\ldots,n\} = \mathbb{R}^d \qquad \forall k \in \{1,\ldots,n\} \tag{2}$$

This obviously implies that $n \geq d+1$ and that span $\{\Lambda_1,\cdots,\Lambda_n\} = \mathbb{R}^d$.

We consider the Cauchy problem for (1), given by the initial condition

$$f(x,0) = f^0(x) = (f_1^0(x),\cdots,f_n^0(x)), \tag{3}$$

where $f_0 : \mathbb{R}^d \to \mathbb{R}^n$. Precise assumptions on the initial datum f_0 will be given later.

On the collision term, we make the hypothesis:

$$\sum_{i=1}^{n} \alpha_i Q_i = 0 \qquad \text{for some } \alpha_i > 0 \qquad (\text{conservation of mass}), \tag{4}$$

$$q_{ij} = \frac{\partial Q_i}{\partial f_j} > 0 \qquad \forall i \neq j \qquad (\text{strong quasi-monotonicity}). \tag{5}$$

Condition (4) corresponds to asking for conservation of the quantity $\int\limits_{\Omega} \sum \alpha_i f_i$ for any $\Omega \subset \mathbb{R}^d$ and that any growth or decrease of it is caused by flux through the boundary $\partial\Omega$. Indeed,

$$\frac{d}{dt}\int\limits_{\Omega} \sum_{i=1}^{n} \alpha_i f_i \, dx = -\int\limits_{\Omega} \operatorname{div}\left(\sum_{i=1}^{n} \alpha_i \Lambda_i f_i\right) + \int\limits_{\Omega} \sum_{i=1}^{n} \alpha_i Q_i = -\int\limits_{\partial\Omega} \left(\sum_{i=1}^{n} \alpha_i \Lambda_i f_i, \mathbf{n}\right) ds,$$

where $\mathbf{n}$ represents the outward normal vector of $\partial\Omega$.

Concerning condition (5), let us recall that for weakly coupled quasimonotone systems, it was proved in [1] that comparison results hold. In case of system (1) the weak quasi-monotonicity condition corresponds to asking a weaker version of (5)

$$\frac{\partial Q_i}{\partial f_j}(s) \geq 0 \qquad \forall i \neq j \quad \text{(weak quasi-monotonicity)}, \tag{6}$$

Hence, under this assumption, given f^0, g^0 initial data for the Cauchy problem (1)–(3) and denoted by f and g the corresponding solutions, there holds

$$f_i^0(x) \leq g_i^0(x) \quad \forall i \qquad \Rightarrow \qquad f_i(x,t) \leq g_i(x,t) \quad \forall i$$

for almost all $(x,t) \in \mathbb{R}^d \times (0,\infty)$. In order to prove asymptotic stability of the manifold of constant states, the stronger assumption (5) is needed. In the class of quasimonotone weakly coupled systems of the form (1), this assumption is sharp, as showed by the example contained in Section 3.

Let us introduce the following notation. Given $P = (P_1, \cdots, P_d) \in \mathbb{R}^d$, let

$$\Omega_P := [0, P_1] \times \cdots \times [0, P_d] \subset \mathbb{R}^d,$$

and, for $\phi = (\phi_1, \cdots, \phi_n) : \Omega_P \to \mathbb{R}^n$,

$$\|\phi\|_{1,\alpha} := \sum_{i=1}^{n} \alpha_i \int_{\Omega_P} |\phi_i(x)| \, dx.$$

Similar definitions can be given for the derivatives of ϕ. In what follows, the solutions of the problem are in spaces $L^1_\alpha(\Omega_P)$ or in $W^{1,1}_\alpha(\Omega_P)$ considered with the norms above defined. Finally, we will say that $\phi = (\phi_1, \cdots, \phi_n) : \mathbb{R}^d \to \mathbb{R}^n$ is a $P-$*periodic function* if $f(x+P) = f(x)$ for any $x \in \mathbb{R}^d$.

Theorem 1. *Assume (2), (4) and (5). Let $\bar{f} \in \mathbb{R}^n$ be such that $Q(\bar{f}) = 0$, $f_0(x) - \bar{f} \in L^1(\Omega_P, \mathbb{R}^n)$ and $P-$periodic for some $P \in \mathbb{R}^d$.*

Then, there is a unique global solution $f = f(x,t)$ of (1), (3) and $f \in C([0,\infty); \bar{f} + L^1_\alpha(\Omega_p, \mathbb{R}^n))$. Moreover, there exists (unique) $\bar{g} = (\bar{g}_1, \cdots, \bar{g}_n) \in \mathbb{R}^n$ with $Q(\bar{g}) = 0$ such that

$$\sum_{i=1}^{n} \alpha_i \int_{\Omega_P} (f_{0,i}(x) - \bar{g}_i) \, dx = 0 \quad \textit{and} \quad \lim_{t \to +\infty} \|f(\cdot,t) - \bar{g}\|_{1,\alpha} = 0. \tag{7}$$

The above Theorem 1 gives sufficient condition for global orbital attractivity of the equilibrium manifold $\{\bar{f} : Q(\bar{f}) = 0\}$: any initial datum that is an L^1 perturbation of an equilibrium state gives raise to a solution asymptotically converging to a constant equilibrium state.

Since the comparison property holds, it is possible to prove a result of asymptotic stability of equilibrium states, i.e., a local result. Since the localization is guaranteed by comparison, the theorem is for L^∞ perturbations.

Theorem 2. *Assume (2), (4). Let $\bar{f} \in \mathbb{R}^n$ be such that*

$$Q(\bar{f}) = 0 \quad \textit{and} \quad \frac{\partial Q_i}{\partial f_j}(\bar{f}) > 0 \quad (\forall i \neq j).$$

Consequently, there exists $\varepsilon > 0$ such that, for any $f_0 \in L^\infty(\Omega_P, \mathbb{R}^n)$ with $\|f_0 - \bar{f}\|_\infty < \varepsilon$, there exists a unique $\bar{g} \in \mathbb{R}^n$ with $Q(\bar{g}) = 0$ such that

$$\lim_{t \to +\infty} \|f(\cdot, t) - \bar{g}\|_{1,\alpha} = 0. \tag{8}$$

Before proving the result (see Section 2), we give some examples of semilinear hyperbolic systems fitting in our assumptions.

A first example fitting in the class (1) is the well-known discrete velocity Boltzmann model, introduced by Carleman,

$$\begin{cases} \partial_t f_1 - \partial_x f_1 = f_2^2 - f_1^2, \\ \partial_t f_2 + \partial_x f_2 = f_1^2 - f_2^2. \end{cases}$$

This system is clearly of the form (1) and hypothesis (4) holds for $\alpha_1 = \alpha_2 = 1$. Moreover, if we consider positive solutions, assumption (5) is satisfied and the conclusion of the theorem holds.

More results on large-time behavior of discrete velocity Boltzmann models are contained in [2]. There is considered a one-dimensional semilinear hyperbolic system with quadratic collision term. Moreover, conservation of mass, of momentum and entropy are assumed to be decreasing. On the contrary, under our assumptions, momentum cannot be conserved, and no hypothesis on entropy is made. The dissipation mechanism is encoded in the quasi-monotonicity condition (6).

Another significant class of systems of the form (1) enjoying the above assumptions is considered in [3]. The limit is studied as $\varepsilon \to 0$ of the solutions to

$$\partial_t f_i + \sum_{j=1}^{d} \lambda_{ij} \partial_{x_j} f_i = \frac{1}{\varepsilon}(M_i(u) - f_i), \tag{9}$$

where $u = \sum_i f_i$. The function $M = (M_1, \ldots, M_n)$ is assumed to be such that $\sum_i M_i(s) = s$ and $0 < M_i'(s) < 1$ for any s under consideration, so that assumptions (4) and (5) are satisfied. Moreover, additional conditions of consistency are assumed with the quasilinear equation

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} A_j(u) = 0, \tag{10}$$

with $A_1, \ldots, A_d$ given flux functions. Such condition takes the form

$$\sum_{i=1}^{n} \lambda_{ij} M_i(s) = A_j(s) \qquad \forall j = 1, \ldots, d. \tag{11}$$

It is proved in [3] that the function $(f_1^\varepsilon, \ldots, f_n^\varepsilon)$ solution to the Cauchy problem for (9) converges in L^1 to some $(f_1^0, \ldots, f_n^0)$ such that $u^0 = \sum f_i^0$ is the entropy solution of the corresponding Cauchy problem for (10). See also [4] for the reduced version in the case $n = 2$.

In this context, there is an interesting connection between our result on asymptotic behavior and this singular limit result. Indeed, it is well known that the entropy solution for conservation law with initial periodic data converges to a constant as $t \to +\infty$. Since the entropy solution is approximated by solution of (9), it seems natural to ask if such asymptotic behavior is inherited by the same property of the semilinear system. This is exactly what this paper aims to achieve: to give a sufficient condition for asymptotic dissipation of periodic perturbations of constant steady states.

Let us stress that some general results on asymptotic behavior for conservation law with initial periodic data are considered in [5], proving dissipation of such perturbations of constant states. However, while in that case the dissipation is caused by the nonlinear transport effect, here, the main part of the dissipation is encoded in the structure of the zero-order term Q. Therefore, the dissipative mechanism seems rather different, at least from the point of view of differential equations. Let us stress that a discrepancy still remains: here, we also assume (2), while in [3], condition (11) is assumed.

2. Proof of Theorems 1 and 2

This section is devoted to the proof of Theorems 1 and 2. In the first part, we show the existence of the constant state $\bar{g} = (\bar{g}_1, \cdots, \bar{g}_n) \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} \alpha_i \int_{\Omega_P} (f_{0,i}(x) - \bar{g}_i)\, dx = 0.$$

In the second part, we consider the asymptotic behavior of the periodic perturbations of $\bar{g}$.

Lemma 1. *Let $Q = (Q_1, \ldots, Q_n)^\top = (q_{ij})_{i,j=1,\cdots,n}$ be a $n \times n$ matrix such that, for some $\alpha_i > 0$,*

$$\sum_{i=1}^{n} \alpha_i q_{ij} = 0 \quad j = 1, \cdots, n, \qquad \text{and} \qquad q_{ij} > 0 \quad \forall i \neq j.$$

Then, any square submatrix of order $n-1$ is nondegenerate. In particular,

$$\text{rank}\, Q = n - 1.$$

Proof. Let $e_i := (q_{i1}, \cdots, q_{in}) \in \mathbb{R}^n$ for $i = 1, \cdots, n$. Since $\sum_i \alpha_i e_i = 0$, then let $Q = 0$.

The conclusion holds if there exist $n-1$ vectors in $\{e_1, \cdots, e_n\}$—linearly independent. Suppose by contradiction that this is not the case and assume (without restriction) that there is $(\beta_1, \cdots, \beta_{n-1}) \neq (0, \cdots, 0)$ such that $\sum_{i=1}^{n-1} \beta_i e_i = 0$. Therefore,

$$\sum_{i=1}^{n-1} (k\beta_i - \alpha_i) q_{ij} = \alpha_n q_{nj} \qquad \forall j = 1, \cdots, n, \forall k \in \mathbb{R}. \tag{12}$$

If $\beta_i > 0$ for any i, then we can choose k such that $k\beta_i - \alpha_i > 0$ for any i. For $j = n$ in (2.01), we arrive at a contradiction:

$$0 > -\sum_{i=1}^{n-1} \alpha_j q_{nj} = \alpha_n q_{nn} = \sum_{i=1}^{n-1} (k\beta_i - \alpha_i) q_{in} > 0.$$

Hence, $\beta_h = \min\{\beta_1, \cdots, \beta_{n-1}\} < 0$. Let $k = \min\{\alpha_1, \cdots, \alpha_{n-1}\}/\beta_h < 0$ and $j = h$ in (2.01). Then

$$\sum_{i \neq h} (k\beta_i - \alpha_i) q_{ih} = q_{nh} > 0. \tag{13}$$

Since $\beta_h \leq \beta_i$ for any i and $k < 0$, it follows that

$$k\beta_i - \alpha_i \leq k\beta_h - \alpha_i = \min\{\alpha_1, \cdots, \alpha_n\} - \alpha_i \leq 0,$$

contradicting (13). $\square$

Proposition 1. *Let $Q = (Q_i)_{i=1,\cdots,n}$ and $\bar{f} = (\bar{f}_i)_{i=1,\cdots,n}$ be such that*

$$\sum_{i=1}^{n} \alpha_i Q_i = 0 \quad (\alpha_i > 0), \qquad rank\left(\frac{\partial Q_i}{\partial f_j}\right) = n-1 \quad and \quad Q(\bar{f}) = 0.$$

Then, for any $C \in \mathbb{R}$, there exists a unique $\bar{g} = (\bar{g}_1, \cdots, \bar{g}_n) \in \mathbb{R}^n$ such that

$$Q(\bar{g}) = 0 \qquad and \qquad \sum_{i=1}^{n} \alpha_i \bar{g}_i \, dx = C.$$

Proof. First of all, we prove uniqueness. Assume that there exists $f = (f_i)$ and $g = (g_i)$ such that $Q(f) = Q(g) = 0$ and $\sum_{i=1}^{n} \alpha_i f_i = \sum_{i=1}^{n} \alpha_i g_i$. Then, it holds that

$$\begin{cases} 0 = Q_i(f) - Q_i(g) = \sum_{j=1}^{n} \frac{\partial Q_i}{\partial f_j}(\xi)(f_j - g_j), \qquad i = 1, \cdots, n, \\ \\ 0 = \sum_{j=1}^{n} \alpha_j (f_j - g_j). \end{cases} \tag{14}$$

Let $e_0 := (\alpha_1, \cdots, \alpha_n)$ and $e_n := (q_{ij})_{i,j=1,\cdots,n}$ where $q_{ij} := \partial Q_i / \partial f_j$. We claim that there are $n-1$ vectors in $\{e_1, \cdots, e_n\}$, say, for simplicity, $e_1, \cdots, e_{n-1}$, such that $e_0, \cdots, e_{n-1}$ are linearly independent. □

By Lemma 1, there are $n-1$ linearly independent vectors in $\{e_1, \cdots, e_n\}$, say, $e_1, \cdots$, e_{n-1}. Assume by contradiction that there is $(\gamma_0, \cdots, \gamma_{n-1}) \neq (0, \cdots, 0)$ such that $\sum_{i=0}^{n-1} \gamma_i e_i = 0$. Moreover, $\gamma_0 \neq 0$. Thus, there are $(\beta_1, \cdots, \beta_{n-1}) \neq (0, \cdots, 0)$ such that $e_0 = \sum_{i=1}^{n-1} \beta_i e_i$. Hence, it holds that

$$\begin{cases} \sum_{i=1}^{n-1} \beta_i q_{ij} = \alpha_i \qquad j = 1, \cdots, n, \\ \sum_{i=1}^{n-1} \alpha_i q_{ij} = -\alpha_n q_{nj} \quad j = 1, \cdots, n \end{cases}$$

Multiplying by k the first of the two equation and subtracting the other, we obtain

$$\sum_{i=1}^{n-1} (k\beta_i - \alpha_j) q_{ij} = k\alpha_i - \alpha_n q_{nj}, \qquad j = 1, \cdots, n \quad \forall k \in \mathbb{R}. \tag{15}$$

If $\beta_i \leq 0$ for any i, then $k\beta_i - \alpha_j < 0$ for any $k > 0$ and for any i. Choosing $j = n$ in (15), we obtain a contradiction.

Therefore, $\beta_h := \max\{\beta_1, \cdots, \beta_{n-1}\} > 0$. Choose $k = \max\{\alpha_1, \cdots, \alpha_{n-1}\}/\beta_h > 0$. Then,

$$k\beta_i - \alpha_j \leq k\beta_h - \alpha_j = \max\{\alpha_1, \cdots, \alpha_{n-1}\} - \alpha_j \leq 0$$

Putting $j = n$ in (15), we arrive at a contradiction. Thus, $e_0, \cdots, e_{n-1}$ are linearly independent and the conclusion follows from (14).

In order to prove existence, let us introduce the set

$$\mathcal{C} := \{C \in \mathbb{R} : \exists \bar{g} = (\bar{g}_1, \cdots, \bar{g}_n) \in \mathbb{R}^n \text{ s.t. } Q(\bar{g}) = 0 \text{ and } \sum_{i=1}^{n} \alpha_i \bar{g}_i \, dx = C\}.$$

By definition, $\mathcal{C}$ is closed and since $Q(\bar{f}) = 0$, $\mathcal{C} \neq \emptyset$. Moreover, since rank $\left(\frac{\partial Q_i}{\partial f_j}\right) = n-1$, we can apply Implicit Function Theorem and deduce that $\mathcal{C}$ is an open set. Therefore, $\mathcal{C} = \mathbb{R}$.

Proof of Theorem 1. Let f, g be solutions of (1). Then, it holds that

$$\partial_t(f_i - g_i) + \sum_{i=1}^{d} \lambda_{ij}\partial_{x_j}(f_i - g_i) = \big(Q_i(f) - Q_i(g)\big). \tag{16}$$

Multiplying (16) by $\alpha_i \text{sgn}\,(f_i - g_i)$, integrating on Ω_P and summing on i, we obtain

$$\frac{d}{dt}\|f - g\|_{1,\alpha} = \int_{\Omega_P} I(x,t)\,\mathrm{d}x, \tag{17}$$

where

$$I(x,t) := \sum_{i=1}^{n} \alpha_i \text{sgn}\,(f_i - g_i)(Q_i(f) - Q_i(g)).$$

Using Lagrange theorem on $Q_i(f) - Q_i(g)$, we obtain

$$\begin{aligned} I(x,t) &= \sum_{i,j=1}^{n} \alpha_i \text{sgn}\,(f_i - g_i)\frac{\partial Q_i}{\partial f_j}(f_j - g_j) = \\ &= \sum_{i=1}^{n}\sum_{j\neq i} \alpha_i \text{sgn}\,(f_i - g_i)\frac{\partial Q_i}{\partial f_j}(f_j - g_j) + \sum_{i=1}^{n} \alpha_i \frac{\partial Q_i}{\partial f_i}\,|f_i - g_i|. \end{aligned} \tag{18}$$

By hypothesis (4), we deduce

$$\alpha_i \frac{\partial Q_i}{\partial f_i} = -\sum_{j\neq i} \alpha_j \frac{\partial Q_j}{\partial f_i},$$

therefore (changing the order of summation in the first sum),

$$\begin{aligned} I &= \sum_{i,j=1}^{n} \alpha_i \text{sgn}\,(f_i - g_i)\frac{\partial Q_i}{\partial f_j}(f_j - g_j) - \sum_{i,j=1}^{n} \alpha_j \frac{\partial Q_j}{\partial f_i}|f_i - g_i| = \\ &= \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} \alpha_i \Big[\text{sgn}\,(f_i - g_i)\text{sgn}\,(f_j - g_j) - 1\Big]\frac{\partial Q_i}{\partial f_j}\Big)|f_j - g_j| \leq 0. \end{aligned} \tag{19}$$

From this estimate, we immediately deduce global existence and L^1-continuous dependence on the initial data of solution of (1), (3) under the assumptions of the Theorem. By (19), we deduce the result for general initial data by density argument. □

In order to obtain compactness property, we restrict our attention to initial data f_0 such that

$$\|\partial_{x_h} f_0\|_{1,\alpha} < +\infty, \qquad \forall h.$$

From (1), deriving with respect to x_h, and setting $w_{ih} := \partial_{x_h} f_i$, we obtain

$$\partial_t w_{ih} + \sum_{i=1}^{d} \lambda_{ij}\partial_{x_j} w_{ih} = \sum_{i=1}^{n} \frac{\partial Q_i}{\partial f_j}(f) w_{jh}. \tag{20}$$

Multiplying by $\alpha_i \text{sgn}\, w_{ih}$, integrating on Ω_P and summing on i, we obtain

$$\frac{d}{dt}\|w_h\|_{1,\alpha} = \int_{\Omega_P} \sum_{i,j} \alpha_i \text{sgn}\, w_{ih} \frac{\partial Q_i}{\partial f_j}(f) w_{jh}\,\mathrm{d}x,$$

where $w_h = (w_{1h}, \cdots, w_{nh})$. Proceeding as above, we obtain

$$\frac{d}{dt}\|w_h\|_{1,\alpha} = \int_{\Omega_p} J(x,t)\,\mathrm{d}x \tag{21}$$

where

$$J(x,t) := \sum_{j=1}^{n}\Big(\sum_{i=1}^{n}\alpha_i\Big[\text{sgn } w_{ih}\text{sgn } w_{jh}-1\Big]\frac{\partial Q_i}{\partial f_j}\Big)|w_{jh}| \le 0.$$

Let $\bar{f}\in\mathbb{R}^n$ be such that $Q(\bar{f})=0$ and assume f_0 be a $P-$periodic function, such that $f_0(x)-\bar{f}\in L^1(\Omega_P,\mathbb{R}^n)$ and $\|\partial_{x_h}f_0\|_{1,\alpha}<+\infty$ for any $h=1,\cdots,n$. Then, by the previous calculations, for any $t>0$,

$$\|f-\bar{f}\|_{1,\alpha}+\|\partial_{x_h}f\|_{1,\alpha}\le\|f_0-\bar{f}\|_{1,\alpha}+\|\partial_{x_h}f_0\|_{1,\alpha}.$$

These estimates provide the required compactness.

Next, let us introduce the following definition:

$$\mathcal{F}_s := \{f(\cdot,t):\Omega_P\to\mathbb{R}^n : t>s\}.$$

From (19) and (21), we deduce that $\mathcal{F}_s-\bar{f}$ is a compact set of L^1_α, for any s. Thus,

$$\emptyset\neq\mathcal{A}:=\bigcap_{s>0}\mathcal{F}_s\subset\bar{f}+L^1_\alpha.$$

Let $a_0\in\mathcal{A}$ and let $a=a(x,t)$ be the solution of (1) with initial condition $f(x,0)=a_0(x)$. Then,

$$\|a(\cdot,t)-\bar{f}\|_{1,\alpha}=\text{constant}\qquad\forall\bar{f}\in\mathbb{R}^n\text{ s.t. }Q(\bar{f})=0.$$

Therefore, we deduce from (19) with $f=a$ and $g=\tilde{f}\in\mathbb{R}^n$ with $Q(\tilde{f})=0$

$$\sum_{i,j=1}^{n}\alpha_i\Big[\text{sgn }(a_i-\tilde{f}_i)\text{sgn }(a_j-\tilde{f}_j)-1\Big]\frac{\partial Q_i}{\partial f_j}|a_j-\tilde{f}_j|=0,$$

for any $t>0$ and almost all $x\in\Omega_P$. Therefore, for any $i,j=1,\ldots,n$,

$$\Big[\text{sgn }(a_i-\tilde{f}_i)\text{sgn }(a_j-\tilde{f}_j)-1\Big]\frac{\partial Q_i}{\partial f_j}|a_j-\tilde{f}_j|=0,\quad\forall t>0,\text{ a.e. in }\Omega_P. \tag{22}$$

From assumption (5), it follows that if $i\neq j$,

$$\Big[\text{sgn }(a_i-\tilde{f}_i)\text{sgn }(a_j-\tilde{f}_j)-1\Big]|a_j-\tilde{f}_j|=0,\quad\forall t>0,\text{ a.e. in }\Omega_P.$$

Hence, for any i and for any $t>0$, a.e. in Ω_P

$$a_i(x,t)\le\tilde{f}_i\quad\forall i\qquad\text{or}\qquad\tilde{f}_i\le a_i(x,t)\quad\forall i. \tag{23}$$

Note that if $\sum k_ia_i=\sum k_ib_i$ for some a_i,b_i with $a_i\le b_i$ for any i, then either $k_j\le 0$ for some $j\in\{1,\ldots,n\}$ or $a_i=b_i$ for any i. Indeed, assuming by contradiction that $k_i>0$ for any i, then $\sum_{i=1}^n k_i(a_i-b_i)=0$ implies $a_i=b_i$ for any i. For any $t>0$ and for almost any $x\in\Omega_p$, by Proposition 1, there exists a unique $\bar{g}=(\bar{g}_1,\ldots,\bar{g}_n)$ such that $Q(\bar{g})=0$ and $\sum\alpha_i\bar{g}_i=\sum\alpha_ia_i(x,t)$. Hence, by (23) and by the previous statement, we find that $a_i(x,t)=\bar{g}_i$ for any i. Therefore, we have proved that

$$Q(a(x,t))=0\qquad\forall t>0,\quad\text{a.e. in }\Omega_P.$$

Here, we stress that, since Q is Lipschitz-continuous and $a(\cdot,t)\to a_0(\cdot)$ as $t\to 0^+$ in L^p, we can deduce that any function $a_0\in\mathcal{A}$ takes values in the equilibrium manifold of Q, i.e., $Q(a_0)=0$ a.e. for any $a_0\in\mathcal{A}$. Let us note that this conclusion is a consequence of assumptions (6) and $\text{rank}\big(\partial Q_i/\partial f_j\big)=n-1$.

At this point, we have proved that $a = (a_1, \dots, a_n)$ is a solution of

$$\partial_t a_i + \sum_{i=1}^{d} \lambda_{ij} \partial_{x_j} a_i = 0, \qquad a(x,0) = a_0(x).$$

Hence, we know that a is

$$a(x,t) = (a_1(x - \Lambda_1 t), \dots, a_n(x - \Lambda_n t)),$$

where $\Lambda_i = (\lambda_{i1}, \dots, \lambda_{id})$ are the speeds defined at the very beginning.

In order to conclude the proof, we have to show that a is indeed a constant function. This is achieved by the following

Proposition 2. *Assume the same hypothesis of Theorem 1. Let $\phi = (\phi_1, \dots, \phi_n) \in L^1(\Omega_p, \mathbb{R}^n)$ be such that*

$$Q(\phi_1(x - \Lambda_1 t), \dots, \phi_n(x - \Lambda_n t)) = 0 \qquad \text{for almost any } (x,t) \in \Omega_P, \tag{24}$$

then there exists $c_i \in \mathbb{R}$ such that

$$\phi_i(x) = c_i \qquad \forall i = 1, \dots, n, \quad \text{for almost any } x \in \Omega_P.$$

Proof of Proposition 2. First of all, let us assume that $\phi \in C^1(\Omega_P, \mathbb{R}^n)$. Calculating $Q(\phi_1(x - \Lambda_1 t), \dots, \phi_n(x - \Lambda_n t))$ at $x = \Lambda_j t$ and deriving with respect to t, we obtain

$$\sum_{h=1}^{n} \sum_{k=1}^{d} \frac{\partial Q_l}{\partial f_h} \frac{\partial \phi_h}{\partial x_k} (\lambda_{jk} - \lambda_{hk}) = 0 \qquad \forall l = 1, \dots, n.$$

Setting $w_h = \sum_k \frac{\partial \phi_h}{\partial x_k} (\lambda_{jk} - \lambda_{hk}) = \nabla \phi_h \cdot (\Lambda_j - \Lambda_h)$, we arrive at the linear system

$$\sum_{h=1}^{n} q_{lh} w_h = 0.$$

Since $w_j = 0$ and any square submatrix of (q_{ij}) of order $n-1$ is on degenerate (Lemma 1), we deduce that $w_h = 0$ for any h. Rewriting

$$\nabla \phi_h \cdot (\Lambda_j - \Lambda_h) = 0 \qquad \forall h, j.$$

Since the set $\Lambda_1 - \Lambda_h, \dots, \Lambda_n - \Lambda_h$ spans all $\mathbb{R}^d$, $\nabla \phi_h = 0$ and the conclusion follows.

The general case for ϕ can be proved by the density argument. Indeed, given $\phi \in L^1(\Omega_P, \mathbb{R}^n)$ with values in a regular subset of $\mathbb{R}^n$, say, Γ, then there exists a sequence $\phi_j \in C^1(\Omega_P, \mathbb{R}^n)$ such that

$$\phi_j(\Omega_P) \subset \Gamma \qquad \forall j, \qquad \lim_{j \to \infty} \|\phi_j - \phi\|_1 = 0.$$

Therefore, $Q(\phi(x)) = 0$ implies $Q(\phi_j(x)) = 0$. By the previous analysis, ϕ_j is constant for any j, and so, passing to the limit, ϕ is constant too. □

This concludes the proof of the Proposition and, consequently, of Theorem 1. Theorem 2 can be proved following the same approach by applying at the very beginning comparison results and regularity of Q in order to guarantees that condition (5) is satisfied for any value of f under consideration.

3. Some Examples and Counterexamples

3.1. A Counterexample about the Condition on Λ_i

Here, we want to show that if for some k it holds $\mathrm{span}\{\Lambda_i - \Lambda_k\}_{i=1,\ldots,n} \neq \mathbb{R}^d$, then system (1) has nonconstant periodic traveling waves, which precludes the asymptotic stability of the set $\{\bar{f} : Q(\bar{f}) = 0\}$.

Therefore, assume that $\mathrm{span}\{\Lambda_i - \Lambda_n\}_{i=1,\ldots,n} \neq \mathbb{R}^d$, then, by changing the x variable $x \to x - \Lambda_n t$, we obtain a system of the same form with speeds $\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_n$ such that $\mathrm{span}\{\tilde{\Lambda}_i\}_{i=1,\ldots,n-1} \neq \mathbb{R}^d$ and $\tilde{\Lambda}_n = 0$.

Without restriction, assume $e_1 \equiv (1, 0, \cdots, 0) \in \mathrm{span}\{\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_{n-1}\}^{\perp}$. Then, we look for a solution to (1) in the form

$$f(x) \equiv (f_1, \ldots, f_n)(x) = (c_1, \ldots, c_n) g(x_1) \equiv c g(x_1), \tag{25}$$

with the vector $c \in \mathbb{R}^n$ and the function $g \in C^1(\mathbb{R})$ to be determined. By hypothesis on e_1, it follows $\tilde{\lambda}_{i1} = 0$ for any i, so that

$$\sum_{j=1}^{d} \tilde{\lambda}_{ij} \partial_{x_j} f_i = \sum_{j=1}^{d} \tilde{\lambda}_{ij} c_i \partial_{x_j} g(x_1) = \tilde{\lambda}_{i1} c_i g'(x_1) = 0.$$

Next, we impose that $Q(f) = 0$. In the case of linear collision term Q, that is, $Q_i(f) = \sum\limits_{j=1}^{n} q_{ij} f_j$, we obtain

$$Q_i(f_i(x)) = \left(\sum_{j=1}^{n} q_{ij} c_j \right) g(x_1) = 0 \qquad \forall x \in \mathbb{R}^d.$$

Hence, by choosing $c \in \mathbb{R}^n$ so that $\sum\limits_{j=1}^{n} q_{ij} c_j = 0$ (recall that $\mathrm{rank}\big(q_{ij}\big) = n - 1$), we find that any function of the form (25) is solution of (1) for any function $g \in C^1(\mathbb{R})$. In the nonlinear case, we can conclude the same kind of result by applying the Implicit Function Theorem close to a constant steady state. Coming back to the original variable x, we obtain a nonconstant traveling wave solution with speed of propagation Λ_k.

3.2. The One-Dimensional 2×2 Linear Example

It is interesting to stress with a one-dimensional 2×2 linear example fitting in the form (1) that in the class of weakly coupled quasimonotone systems, the assumptions of Theorem 1 may not be weakened. Consider the system

$$\begin{cases} \partial_t f_1 - \lambda \partial_x f_1 = -a f_1 + b f_2, \\ \partial_t f_2 + \lambda \partial_x f_2 = +a f_1 - b f_2, \end{cases} \tag{26}$$

where $a, b \in \mathbb{R}$ and $\lambda \geq 0$ (the general one-dimensional 2×2 case can be reduced to this one by a simple change of variables). The assumption (5) corresponds to $a, b > 0$, while (6) reads in this case as $a, b \geq 0$. It is also interesting to stress that

$$\mathrm{rank}\left(\frac{\partial Q_i}{\partial f_j} \right) = \mathrm{rank}\begin{pmatrix} -a & +b \\ +a & -b \end{pmatrix} = 1 \qquad \Longleftrightarrow \qquad (a, b) \neq (0, 0),$$

(for the rôle of condition $\mathrm{rank}\big(\partial Q_i / \partial f_j\big) = n - 1$, see Proposition 1).

Hence, if we choose $a > b = 0$, we have a weak quasimonotone system that is not a strong quasimonotone and that has Jacobian of the collision term Q of rank one.

Given the initial condition

$$(f_1, f_2)(x, 0) = (f_1^0, f_2^0)(x),$$

the solution is given by the explicit formula

$$\begin{cases} f_1(x,t) = f_1^0(x+\lambda t)e^{-at}, \\ \\ f_2(x,t) = f_2^0(x-\lambda t) + a\displaystyle\int_0^t f_1^0(x-\lambda(t-\tau)+\lambda\tau)e^{-a\lambda\tau}\,d\tau. \end{cases}$$

Then, it is immediate to see that if $f_1^0 \equiv 0$, the solution is

$$(f_1, f_2)(x,t) = (0, f_2^0(x-\lambda t)),$$

that does not converge to any constant state.

The same class of system can be used to show the necessity of conditions on the speeds λ_{ij}, in this case $-\lambda, \lambda$. Indeed, assume $\lambda = 0$ (so that hypothesis of Theorems 1 and 2 do not hold). Then, the system (26) reduces to a system of ordinary differential equations of the form

$$f_1' = -af_1 + bf_2, \quad f_2' = +af_1 - bf_2. \tag{27}$$

The asymptotic behavior is determined by the eigenvalues and corresponding eigenvector of the matrix $A = \begin{pmatrix} -a & +b \\ +a & -b \end{pmatrix}$. A straightforward computation reveals that the eigenvalues are 0 and $-(a+b)$, so that if the system satisfies (5), i.e., $a, b > 0$, then the solution asymptotically belongs to the set $\{(f_1, f_2) : af_1 = bf_2\}$, with no convergence to constant states for general initial data.

Finally, we conclude with some heuristics again for system (26), showing from a different point of view where the asymptotic stability comes from. Applying the Fourier analysis, we obtain the following system of ordinary differential equations

$$\begin{cases} \hat{f}_1' = (-a + i\lambda k)\hat{f}_1 + b\hat{f}_2, \\ \hat{f}_2' = +a\hat{f}_1 + (-b - i\lambda k)\hat{f}_2. \end{cases} \tag{28}$$

Stability analysis corresponds to looking for the sign of the real part of any eigenvalue μ of the matrix of coefficients in the right-hand side of (28). Setting $\mu = X + iY$, we obtain the algebraic system

$$\begin{cases} X^2 - Y^2 + (a+b)X + \lambda^2 k^2 = 0, \\ 2XY + (a+b)Y - \lambda k(b-a) = 0, \end{cases}$$

from which we deduce (for $\lambda > 0$)

$$k^2 = F(X) := -\frac{X(X+a+b)\left(X + \frac{a+b}{2}\right)^2}{\lambda^2(X+a)(X+b)}.$$

Imposing the necessary condition of stability $F(X) < 0$ for any $X > 0$, we deduce that $a, b \geq 0$, which corresponds to the weak quasimonotonicity assumption. For small X, we have for $a, b > 0$

$$k^2 = -F(X) = -\frac{(a+b)^3}{4\lambda^2 ab}X + o(X) \qquad \text{as } X \to 0,$$

so that the strong quasimonotonicity assumption corresponds to asking that the function F has finite negative slope at $X = 0$.

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References

1. Hanouzet, B.; Natalini, R. Weakly coupled systems of quasilinear hyperbolic equations. *Differ. Integral Equ.* **1996**, *9*, 1279–1292. [CrossRef]
2. Beale, J.T. Large-time behavior of discrete velocity Boltzmann equations. *Commun. Math. Phys.* **1986**, *106*, 659–678. [CrossRef]
3. Natalini, R. A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws. *J. Differ. Equ.* **1998**, *148*, 292–317. [CrossRef]
4. Natalini, R. Convergence to equilibrium for the relaxation approximations of conservation laws. *Comm. Pure Appl. Math.* **1996**, *49*, 795–823. [CrossRef]
5. Chen, G.-Q.; Frid, H. Decay of Entropy Solutions of Nonlinear Conservation Laws. *Arch. Ration. Mech. Anal.* **1999**, *146*, 95–127. [CrossRef]